Unbounded Norm Convergence in Banach Lattices

by

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Abstract

A net (x_α) in a vector lattice X is unbounded order convergent to $x \in X$ if $|x_\alpha - x| \wedge u$ converges in order to 0 for all $u \in X_+$. Recent work by Gao *et al.* has shown that this type of convergence has many interesting theoretical and practical applications. In this thesis, we use unbounded order convergence as a tool to study a different convergence in Banach lattices. A net (x_α) in a Banach lattice X is unbounded norm convergent to $x \in X$ if $|x_\alpha - x| \wedge u$ converges in norm to 0 for all $u \in X_+$. We describe basic properties of this convergence and show that it can be viewed as a generalization of convergence in measure to the setting of Banach lattices.

Preface

The research appearing in Chapter 2 of this thesis is my contribution to the article *Unbounded norm convergence in Banach lattices* by Y. Deng, M. O'Brien and V.G. Troitsky, arXiv:1605.03538v1 [math.FA] which has been submitted for publication. All of the original results and the proofs appearing in Chapter 2 are due to myself and V.G. Troitsky. V.G. Troitsky formulated and proved Theorem 2.12, Corollary 2.13 and the results appearing in the section on un-topology.

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Contents

1. Preliminaries and Introduction

In this thesis, we will work primarily in Banach lattices, which, for the moment, can be thought of as Banach spaces with some type of ordering. Since these spaces have several structures on them, we provide a brief overview of the background material. For more details, we refer the reader to [AB06], [Sch74] and [LT79, Chapter 1]. We begin by reviewing the notion of an ordered set.

1.1. Order Structures.

Definition 1.1. A set P together with a binary relation $\preceq$ is called a ***partially ordered set*** if the following hold for each $x, y, z \in P$:

(i) $x \preceq x$ *(Reflexivity)*

(ii) If $x \preceq y$ and $y \preceq x$, then $x = y$ *(Anti-symmetry)*

(iii) If $x \preceq y$ and $y \preceq z$, then $x \preceq z$ *(Transitivity)*

In addition, we say that P is ***directed*** by $\preceq$ if for each $x, y \in P$ there exists $z \in P$ such that $x \preceq z$ and $y \preceq z$.

We may view this definition as an abstraction of $\leq$ for real numbers. Indeed, $\mathbb{R}$ with the $\leq$ relation is a directed set. We call this the usual ordering of $\mathbb{R}$. Here are some other important examples of this concept.

Example 1.2. Let X be any set. The power set of X, $\mathcal{P}(X)$, is directed by set inclusion. Explicitly, for subsets U and V of X, we write $U \preceq V$ iff $U \subseteq V$. It is straightforward to verify $\subseteq$ is a partial order on $\mathcal{P}(X)$, and that $\mathcal{P}(X)$ with this ordering is a directed set.

Example 1.3. Let (X, τ) be a topological space and let $x \in X$ be fixed. Then the set of open neighborhoods of x is directed by reverse inclusion; that is, $V \preceq U$ iff $U \subseteq V$.

We now come to the main purpose for introducing directed sets.

Definition 1.4. Let Λ be a directed set, and let X be any set. A function $x : \Lambda \to X$ is called a ***net*** in X. We write $x(\alpha) := x_\alpha$ for $\alpha \in \Lambda$

It is more common to identify a net with its range; *i.e.*, we say that $(x_\alpha)_{\alpha\in\Lambda}$ is a net in X. Sometimes we will suppress the index set and write (x_α) when the index set is clear. If X itself is partially ordered by $\preceq'$, then a net $(x_\alpha)_{\alpha\in\Lambda}$ is called ***increasing*** if $x_\alpha \preceq' x_\beta$ whenever $\alpha \preceq \beta$ in Λ; in this case, we write $x_\alpha \uparrow$. Similarly, one can define ***decreasing*** nets and these are denoted by $x_\alpha \downarrow$.

Example 1.5. Recall that a sequence in X is a function into X whose domain is the set of all natural numbers, $\mathbb{N}$. Observing that $\mathbb{N}$ is directed by $\leq$, we see that any sequence is an example of a net.

Example 1.6. If (X, τ) is a topological space and $x \in X$ is fixed, let $\mathfrak{N}_x$ denote the open neighborhoods of x. Example 1.3 yields $\mathfrak{N}_x$ is a directed set. For each $U \in \mathfrak{N}_x$, $U \neq \varnothing$, so pick some $x_U \in U$. Then $(x_U)_{U\in\mathfrak{N}_x}$ is a net in X.

For any subset A of a partially ordered set X, there is a notion of 'largest' and 'smallest' elements. An element $u \in X$ is called an ***upper bound*** for A if $a \preceq u$ for every $a \in A$. In this case, we may write $A \preceq u$ and say that A is bounded above by u. Notice that an upper bound need not belong to the set which it bounds. We say that $u \in X$ is the ***greatest element*** of A if $A \preceq u$ and $u \in A$. Notice that we called it 'the' greatest element; this is justified since a greatest element, if it exists, is unique. In a similar fashion, one can define ***lower bounds*** and a ***least element*** of A. We say that A is ***order bounded*** if it is bounded above and below; in this case, $A \subseteq [a, b] = \{x \in X : a \preceq x \preceq b\}$ where a and b are lower and upper bounds of A, respectively.

Example 1.7. Let $\mathbb{Q}$ denote the set of all rational numbers. Then $\mathbb{Q}$ with the relation $\leq$ is a partially ordered set. Consider the subset $A := \{q \in \mathbb{Q} : q^2 \leq 2\}$. Then A has many upper bounds; for example, $\frac{12}{5}$, 3 and 7 are all upper bounds for A.

This example helps to motivate the following terminology. Let $\mathcal{U}_A$ and $\mathcal{L}_A$ denote the set of all upper and lower bounds of A, respectively. If $\mathcal{U}_A$ has a least element, then we call it the ***least upper bound*** or ***supremum*** of A, and denote it by $\sup A$. If $\mathcal{L}_A$ has a greatest element, then we call it the ***greatest lower bound*** or ***infimum*** of A, and denote it by $\inf A$. It is important to remark that $\sup A$ and $\inf A$ need not exist for a general subset A of X; one such example of this is Example 1.7.

The notation $x_\alpha \uparrow u$ means that $(x_\alpha)_{\alpha \in \Lambda}$ is an increasing net and $u = \sup\{x_\alpha : \alpha \in \Lambda\}$. Similarly, $x_\alpha \downarrow u$ means $(x_\alpha)_{\alpha \in \Lambda}$ is a decreasing net and $u = \inf\{x_\alpha : \alpha \in \Lambda\}$.

Definition 1.8. Let X be a partially ordered set. For $x, y \in X$ define $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$. We say that X is a ***lattice*** if $x \vee y$ and $x \wedge y$ exist in X for every $x, y \in X$.

Here are some important examples of sets with a lattice structure.

Example 1.9. For any set X, $\mathcal{P}(X)$ is a lattice under the ordering from Example 1.2. The lattice operations are $A \vee B = A \cup B$ and $A \wedge B = A \cap B$ for any $A, B \in \mathcal{P}(X)$.

Example 1.10. Let Ω be any set. The set of all $\mathbb{R}$-valued functions on Ω, $\mathbb{R}^\Omega$, is a lattice under *pointwise* ordering of functions. That is, for $f, g \in \mathbb{R}^\Omega$ we write $f \leq g$ iff $f(\omega) \leq g(\omega)$ for every $\omega \in \Omega$. Notice that $f \vee g$ and $f \wedge g$ are defined pointwise *via*

$$(f \vee g)(\omega) = f(\omega) \vee g(\omega) \qquad (f \wedge g)(\omega) = f(\omega) \wedge g(\omega)$$

for $\omega \in \Omega$. In particular, setting $\Omega = \mathbb{N}$ yields the set of all real sequences is a lattice under pointwise operations.

Example 1.11. As a special case of the previous example, if Ω is finite, then we can identify $\mathbb{R}^\Omega$ with $\mathbb{R}^n$ (where $|\Omega| = n$) by viewing $\mathbb{R}$-valued functions on $\{1, \ldots, n\}$ as vectors in $\mathbb{R}^n$; *i.e.*, we identify $x \in \mathbb{R}^\Omega$ with the vector in $\mathbb{R}^n$ whose coordinates are given by $x_i = x(i)$ for $i \in \{1, \ldots, n\}$. It follows from Example 1.10 that $\mathbb{R}^n$ is a lattice under the pointwise ordering of functions. In this case, we say that $\mathbb{R}^n$ is ordered *coordinate-wise*; that is, for $x, y \in \mathbb{R}^n$ $x \leq y$ iff $x_i \leq y_i$ for every $i \in \{1, \ldots, n\}$. It is easy to see that the lattice operations are also computed coordinate-wise:

$$x \vee y = (x_1 \vee y_1, \ldots, x_n \vee y_n) \qquad x \wedge y = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots y_n) \in \mathbb{R}^n$.

Example 1.12. Let $C(K)$ denote the space of all continuous $\mathbb{R}$-valued functions on a compact Hausdorff space K. Again, this is a lattice under pointwise ordering of functions; *i.e.*, $f \leq g$ in $C(K)$ iff $f(t) \leq g(t)$ for all $t \in K$. We have

$$(f \vee g)(t) = \max\{f(t), g(t)\} = \frac{f(t) + g(t) + |f(t) - g(t)|}{2}$$

and

$$(f \wedge g)(t) = \min\{f(t), g(t)\} = \frac{f(t) + g(t) - |f(t) - g(t)|}{2}$$

for $t \in K$.

Example 1.13. Let (Ω, Σ, μ) be a measure space and set $L_0(\mu) = \{f : \Omega \to \mathbb{R} : f \text{ is } \Sigma\text{–measurable}\}$. Recall that we identify two functions in $L_0(\mu)$ if they are equal μ-almost everywhere (*a.e.*); hence, elements of $L_0(\mu)$ are actually equivalence classes of functions. For $f, g \in L_0(\mu)$ we set $f \leq g$ iff $f(\omega) \leq g(\omega)$ μ-a.e.. Under this order, $L_0(\mu)$ is a lattice where

$$(f \vee g)(\omega) = f(\omega) \vee g(\omega) \qquad (f \wedge g)(\omega) = f(\omega) \wedge g(\omega)$$

hold for μ-a.e. $\omega \in \Omega$. Moreover, for $1 \le p < \infty$ we let $\|f\|_p = (\int_\Omega |f(\omega)|^p d\mu)^{\frac{1}{p}}$ and define $L_p(\mu) = \{f \in L_0(\mu) : \|f\|_p < \infty\}$. Then $L_p(\mu)$ is a lattice under the order it inherits from $L_0(\mu)$.

1.2. **Vector Lattices.**

The examples at the end of the previous section illustrate how many classical function spaces have a lattice structure. In fact, it will often be helpful to view Banach lattices as function spaces. In order to develop this idea, we now consider sets with an order and a linear structure.

Definition 1.14. An ***ordered vector space*** is a $\mathbb{R}$-vector space X with an ordering $\le$ such that the following compatibility conditions hold for every $x, y, z \in X$ and $\lambda \in \mathbb{R}_+$:

(i) $x \le y$ implies $x + z \le y + z$, and

(ii) $x \le y$ implies $\lambda x \le \lambda y$.

If, in addition, X is also a lattice, then it is called a ***vector lattice***. See Example 1.10 – Example 1.13 above for examples of this concept.

Throughout the rest of this section X will always denote a vector lattice. An element $x \in X$ is called ***positive*** if $0 \le x$. The set of all positive elements of X is denoted by X_+. We say x is ***negative*** if $0 \le -x$. The following result is easy to verify.

Lemma 1.15. *For every $x, y \in X_+$ and $\lambda \in \mathbb{R}_+$ the following hold:*

(i) *$x + y \in X_+$;*

(ii) *$\lambda x \in X_+$;*

(iii) *$x \in X_+$ and $-x \in X_+$ if and only if $x = 0$.*

Lemma 1.15 says that X_+ is a ***cone*** in X; we call it the ***positive cone*** of X. We say that X is ***Archimedean*** if $\frac{1}{n}u \downarrow 0$ for any $u \in X_+$.

The following identities describe the interactions of the linear and order structure in vector lattices. We present some of their proofs to familiarize the reader with common calculations in vector lattices. For more details, see [AB06, Theorem 1.3].

Lemma 1.16. *For every $x, y, z \in X$ and $\lambda \in \mathbb{R}_+$ the following hold:*

(i) $-(x \vee y) = (-x) \wedge (-y)$ *and* $-(x \wedge y) = (-x) \vee (-y)$;

(ii) $x + y = x \wedge y + x \vee y$;

(iii) $x+(y\vee z) = (x+y)\vee(x+z)$ *and* $x+(y\wedge z) = (x+y)\wedge(x+z)$;

(iv) $\lambda(x \vee y) = (\lambda x) \vee (\lambda y)$ *and* $\lambda(x \wedge y) = (\lambda x) \wedge (\lambda y)$.

Proof. We begin by remarking that if $x \leq y$, then $y - x \geq 0$ and $-x \geq -y$.

(i) From the definition of supremum we have $x \leq x \vee y$. It follows that $-x \geq -(x \vee y)$. Similarly, $-y \geq -(x \vee y)$. This shows $-(x \vee y)$ is a lower bound for $-x$ and $-y$. Since $(-x) \wedge (-y)$ is the largest element with this property, we must have $-(x \vee y) \leq (-x) \wedge (-y)$. If $z \leq -x$ and $z \leq -y$, then $-z \geq x$ and $-z \geq y$. It follows that $-z \geq x \vee y$; hence, $z \leq -(x \vee y)$. This shows that $-(x \vee y)$ is the infimum of $-x$ and $-y$, which proves the first identity in (i); the second one follows by replacing x with $-x$ and y with $-y$ in the first identity.

(ii) Since $x\wedge y \leq x$ implies $x-x\wedge y \geq 0$, we must have $y \leq y+x-x\wedge y$. A similar argument shows $x \leq x + y - x \wedge y$, so $x + y - x \wedge y$ is an upper bound of x and y. It follows that $x \vee y \leq x + y - x \wedge y$, hence $x \wedge y + x \vee y \leq x + y$. To obtain the reverse inequality, note that

$y \le x \vee y$ which implies $y - x \vee y \le 0$, hence $x + y - x \vee y \le x$. Similarly, $x + y - x \vee y \le y$, so $x + y - x \vee y \le x \wedge y$; *i.e.*, $x + y \le x \wedge y + x \vee y$. The anti-symmetry of $\le$ yields the identity in (ii).

(iii) We leave it as an exercise to the reader to show that $(x + y) \vee (x + z) \le x + (y \vee z)$. The first identity in (iii) follows from the trick $y = -x + (x + y) \le -x + [(x + y) \vee (x + z)]$ and, similarly, $z \le -x + [(x + y) \vee (x + z)]$. The second identity can be proven in the same way.

(iv) Exercise. $\square$

Perhaps the most noticeable omission from Lemma 1.16 is the distributivity of $\vee$ and $\wedge$ over addition. In fact, $\vee$ and $\wedge$ need not be distributive over addition in a vector lattice. Nevertheless, for positive elements, we have the following inequality.

Lemma 1.17. *If $x, x_1, x_2 \in X_+$, then*

$$x \wedge (x_1 + x_2) \le x \wedge x_1 + x \wedge x_2.$$

Proof. See [AB06, Lemma 1.4] for a proof based on the definitions. $\square$

For $x \in X$ we define the ***positive part*** of x, the ***negative part*** of x, and the ***modulus*** of x with the following identities.

$$x^+ = x \vee 0 \qquad x^- = (-x) \vee 0 \qquad |x| = x \vee (-x)$$

Remark 1.18. The modulus of x, $|x|$, should not be confused with the absolute value over $\mathbb{R}$, though, for $\mathbb{R}$ with the usual ordering, they coincide.

Example 1.19. In Example 1.10-Example 1.13 we showed that the lattice operations $\vee$ and $\wedge$ are computed pointwise. Similarly, for an element f in any of these vector lattices, f^+, f^- and $|f|$ can be computed pointwise with the appropriate modifications for each case. For instance, if $f \in L_0(\mu)$ then $|f| \in L_0(\mu)$ is computed pointwise μ-a.e. by

$$|f|(\omega) = |f(\omega)|$$

for μ-a.e. $\omega \in \Omega$.

One can derive the following basic results using Definition 1.14 and Lemma 1.16.

Lemma 1.20. *The following hold for every $x \in X$:*

(i) $x^+, x^-, |x| \in X_+$;

(ii) $x = x^+ - x^-$;

(iii) $|x| = x^+ + x^-$.

We call the mappings that send x to $x^+, x^-, |x|, x \vee y$, or $x \wedge y$ the ***lattice operations*** on X. It is of interest to note that all of the lattice operations can be expressed in terms of each other.

Lemma 1.21. *The following identities hold for $x, y \in X$:*

(i) $x^+ = \frac{1}{2}(|x| + x)$ *and* $x^- = \frac{1}{2}(|x| - x)$,

(ii) $x \vee y = \frac{x+y+|x-y|}{2}$ *and* $x \wedge y = \frac{x+y-|x-y|}{2}$, *and*

(iii) $x \vee y = x + (y-x)^+$ *and* $x \wedge y = x - (x-y)^+$.

We say that $x, y \in X$ are ***disjoint*** if $|x| \wedge |y| = 0$. In this case, we write $x \perp y$.

We will often use the following inequality for making estimates in vector lattices.

Lemma 1.22. *For every $x, y \in X$,*

$$\big||x| - |y|\big| \leq |x \pm y| \leq |x| + |y|$$

Moreover, if $x \perp y$ then $|x + y| = |x| + |y|$.

Remark 1.23. In light of Remark 1.18, it is peculiar that the modulus, which is only defined in-terms of the order structure, also satisfies the triangle inequality. This alludes to a deeper connection between the modulus and absolute value. Indeed, it can be shown that every inequality that is valid for real numbers is also true for vectors in a Banach lattice; see, for example, [LT79, Theorem 1.d.1].

In order to understand the structure of a vector lattice, we need to understand both its vector subspaces and 'lattice subspaces'. Let Y be a vector subspace of X. If, in addition, Y is closed under each of the lattice operations, then we say that Y is a ***sublattice*** of X. By Lemma 1.21, it is enough to show that Y is closed under any single lattice operation. For example, if Y is closed under the modulus operation, *i.e.*, $x \in Y$ implies $|x| \in Y$ for every $x \in Y$, then Y is closed under all lattice operations since they are all expressed *via* each other.

Another important class of sublattices are the *order ideals*. We say that Y is an ***order ideal*** of X, or just an ***ideal***, if Y is a sublattice of X and $0 \leq x \leq y$ implies $x \in Y$ whenever $y \in Y$. For $S \subseteq X$, the intersection of all ideals containing S is, again, an ideal containing S;

this is the smallest such ideal, so we call it the ***ideal generated by S***, and denote it by I_S. If $S = \{e\}$, then we write I_e instead of $I_{\{e\}}$. It is important to note that we can describe I_S for any subset S of X.

Lemma 1.24. *For any subset S of a vector lattice X, we have*

$$I_S = \Big\{x : |x| \leq \sum_{i=1}^{k} \lambda_i |x_i|; k \in \mathbb{N}, \lambda_1, \ldots, \lambda_k \in \mathbb{R}_+, x_1, \ldots, x_k \in S\Big\}$$

If B is an ideal of X and $x_\alpha \uparrow$ in B_+ implies $\sup_\alpha\{x_\alpha\} \in B$ for any net (x_α) in B_+, then we call B a ***band***. As with ideals, given a subset S of a vector lattice, one may speak about the ***band generated by S***, B_S. In the case where $S = \{e\}$, we write B_e for the band generated by e. The following is a useful characterization of B_e.

Lemma 1.25. *If $e, x \in X_+$, then*

$$x \in B_e \iff x = \sup_{n \in \mathbb{N}}\{x \wedge ne\}$$

Remark 1.26. It is interesting to contrast I_e with B_e. Suppose $x \in X_+$. On the one hand, by Lemma 1.24, $x \in I_e$ iff there exists some $n \in \mathbb{N}$ such that $x \leq ne$; *i.e.*, x is eventually dominated by a multiple of e. Said differently, $x \in I_e$ iff there exists some $n_0 \in \mathbb{N}$ such that $x = x \wedge ne$ for all $n \geq n_0$. On the other hand, $(x \wedge ne)_{n \in \mathbb{N}}$ is a positive increasing net in B_e, so $x \in B_e$ iff $x \wedge ne \uparrow x$.

The following is a standard fact from the theory of vector lattices. For a proof, see [AB06, Theorem 1.20].

Theorem 1.27 (The Riesz Decomposition Property)**.** *Let $x_1, ..., x_n$ and $y_1, ..., y_m$ be positive vectors in a vector lattice X. If*

$$\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j$$

then there is a finite set $\{z_{ij} : i = 1, ..., n; j = 1, ..., m\}$ of positive vectors such that

$$x_i = \sum_{j=1}^{m} z_{ij}$$

for each $i = 1, ..., n$ and

$$y_j = \sum_{i=1}^{n} z_{ij}$$

for each $j = 1, ..., m$.

Lemma 1.28. *Let $|x| = u + v$ for some vector x and some positive vectors u and v in a vector lattice. Then there exist y and z such that $x = y + z$, $|y| = u$, and $|z| = v$.*

Proof. Applying Theorem 1.27 to the equality $x^+ + x^- = u + v$, we find four positive vectors vectors a, b, c, and d such that $u = a + b$, $v = c + d$, $x^+ = a + c$, and $x^- = b + d$. Put $y = a - b$ and $z = c - d$. Then $y + z = x^+ - x^- = x$. It follows from $0 \leq a \leq x^+$ and $0 \leq b \leq x^-$ that $a \perp b$ and, therefore, $|y| = |a - b| = a + b = u$. Similarly, $c \perp d$, and, therefore, $|z| = v$. □

The following concepts are useful for studying the structure of vector lattices.

Definition 1.29. A vector $e \in X_+$ is called a

(i) ***weak unit*** if for every $x \in X$, $x \perp e$ implies $x = 0$;

(ii) ***strong unit*** if for every $x \in X$, there is a $\lambda \in \mathbb{R}_+$ such that $|x| \leq \lambda e$.

In Remark 1.26 we highlighted a similarity between I_e and B_e for an arbitrary $e \in X_+$. We can now deepen this connection through the following simple facts.

Proposition 1.30. *$e \in X_+$ is a strong unit iff $I_e = X$. If X is Archimedean, then e is a weak unit iff $B_e = X$.*

Remark 1.31. The condition that X is Archimedean is not a very restrictive one. Indeed, any vector lattice with a compatible norm structure is an Archimedean vector lattice. In particular, every Banach lattice is Archimedean; for clarification of these definitions, see Section 1.3 below.

We now introduce a concept of convergence in vector lattices.

Definition 1.32. A net $(x_\alpha)_{\alpha \in A}$ in a vector lattice X is said to be ***order convergent*** to $x \in X$ if

(i) there is a net $(z_\beta)_{\beta \in B}$ in X such that $z_\beta \downarrow 0$, and

(ii) for every $\beta \in B$, there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \leq z_\beta$ whenever $\alpha \geq \alpha_0$.

For short, we will denote this convergence by $x_\alpha \xrightarrow{o} x$ and write that x_α is o-convergent to x.

Definition 1.32 is quite abstract. It is best to think of order convergence as bounded a.e. convergence in $L_p(\mu)$, as the following example demonstrates.

Example 1.33. A sequence (f_n) in $L_p(\mu)$ converges in order to $f \in L_p(\mu)$ if and only if (f_n) is order bounded and $f_n \xrightarrow{\text{a.e.}} f$.

The next fact can often be used to simplify arguments involving order convergence.

Lemma 1.34. *If (x_α) is a net in a vector lattice X such that $x_\alpha \uparrow x$, then $x_\alpha \xrightarrow{\text{o}} x$. The same is true if we replace $\uparrow$ with $\downarrow$, or replace nets with sequences. In particular, a band is an ideal that is closed with respect to order convergence.*

A function on a vector lattice is said to be ***order continuous*** if $f(x_\alpha) \xrightarrow{\text{o}} f(x)$ whenever $x_\alpha \xrightarrow{\text{o}} x$ in X.

Proposition 1.35. *The lattice operations are order continuous.*

1.3. **Banach Lattices.**

We are now ready to introduce the formal definition of a Banach lattice. A ***normed lattice*** is a vector lattice with a norm that satisfies the following conditions:

(i) $x \leq y$ implies $\|x\| \leq \|y\|$ for every $x, y \in X_+$;
(ii) $\big\||x|\big\| = \|x\|$ for every $x \in X$.

A norm that satisfies these criteria is called a ***lattice norm***. If, in addition to being a normed lattice, X is complete with respect to its lattice norm, then we call it a ***Banach lattice***. Unless stated otherwise, we will always assume that X is a Banach lattice. Since X is also a Banach space, we may discuss norm-convergence in X; for a net (x_α) and $x \in X$, we write $x_\alpha \to x$ when $\|x_\alpha - x\| \to 0$. We may also consider the norm dual of X, X^*. The next results show that the lattice operations are compatible with taking limits.

Proposition 1.36. *The lattice operations on X are continuous.*

Proof. Suppose $x_\alpha \to x$ in X. Then

$$\big\| |x_\alpha| - |x| \big\| \leq \big\| |x_\alpha - x| \big\| = \|x_\alpha - x\| \to 0.$$

Thus, $|x_\alpha| \to |x|$; that is, the modulus operation is continuous. The desired result is now an immediate consequence of Lemma 1.21. □

The order structure on X interacts with limits in the following way.

Corollary 1.37. *If $(x_\alpha), (y_\alpha)$ are nets in X such that $x_\alpha \to x, y_\alpha \to y$ and $x_\alpha \leq y_\alpha$ for every α, then $x \leq y$.*

Proof. Our goal is to show that $y - x \in X_+$. Since $x_\alpha \leq y_\alpha$, we have $z_\alpha = y_\alpha - x_\alpha \in X_+$ for every α. As $z_\alpha \to y - x$, it suffices to show that X_+ is norm closed. To this end, suppose $w_\beta \to w$ for any net (w_β) in X_+ and some $w \in X$. By Proposition 1.36, $w_\beta = w_\beta^+ \to w^+$. As the norm topology is Hausdorff, the limit is unique and we obtain $w = w^+ \in X_+$. □

Here are some important examples of Banach lattices.

Example 1.38. $\mathbb{R}$ with the usual order and absolute value is a Banach lattice.

Example 1.39. The following sequence spaces are all sublattices of $\mathbb{R}^{\mathbb{N}}$ under the pointwise order given in Example 1.10. In addition, they are all Banach lattices with their given norms.

- ℓ_∞, the space of all bounded sequences, with norm $\|x\|_\infty = \sup_n |x_n|$ for $x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty$;
- c, the space of all convergent sequences, with $\|\cdot\|_\infty$;
- c_0, the space of all sequences converging to zero, with $\|\cdot\|_\infty$; and
- ℓ_p, the space of all p-summable sequences, where $1 \leq p < \infty$ and the norm is given by $\|x\|_p = (\sum_{n=1}^\infty |x_n|^p)^{\frac{1}{p}}$ for $x = (x_n)_{n\in\mathbb{N}} \in \ell_p$.

Example 1.40. In Example 1.12 we introduced a lattice structure on $C(K)$ for a compact Hausdorff space K. In fact, it is a Banach lattice. As $C(K)$ with the sup-norm $\|f\| = \sup_{x\in K}\{|f(x)|\}$ is a Banach space and a vector lattice under pointwise ordering of functions, it remains to show that the sup-norm is a lattice norm. Indeed, suppose $f \leq g$ in $C(K)$. Then $|f(x)| \leq |g(x)|$ for every $x \in K$. It follows that $\|f\| \leq \|g\|$. Also, $\||f|\| = \sup_{x\in K}\{|f|(x)\} = \sup_{x\in K}\{|f(x)|\} = \|f\|$.

Example 1.41. If X is a locally compact Hausdorff space, then we denote by $C_0(X)$ the set of all continuous $\mathbb{R}$-valued functions on X that vanish at infinity; *i.e.* it is the set of all continuous functions $f : X \to \mathbb{R}$ such that for every $\epsilon > 0$ there is a compact subset K of X with $f(x) < \epsilon$ whenever $x \in X \setminus K$. With the sup-norm from the previous example, $C_0(X)$ is a Banach lattice.

Example 1.42. For a measure space (Ω, Σ, μ), recall that a function $f \in L_0(\mu)$ is said to be ***essentially bounded*** if there exists a positive real number M such that $\mu(\{\omega : |f(\omega)| > M\}) = 0$. Taking the infimum over all such M defines a norm on all essentially bounded functions; we denote this space by $L_\infty(\mu)$ where $\|f\|_\infty = \inf_{M \in \mathbb{R}^+} \big\{M : \mu(\{\omega : |f(\omega)| > M\}) = 0\big\}$. It is a basic fact from functional analysis that $L_\infty(\mu)$ is a Banach space. Moreover, it is a Banach lattice under the order introduced in Example 1.13.

Example 1.43. Let (Ω, Σ, μ) be a measure space. For $1 \leq p < \infty$, we have mentioned that the space $L_p(\mu)$ is a vector lattice under the pointwise μ-a.e. order. Recall that two functions $f, g \in L_p(\mu)$ that agree μ-a.e. are identified with each other. In this case, $\|f\|_p = (\int_\Omega |f(\omega)|^p d\mu)^{\frac{1}{p}}$ is a lattice norm and $L_p(\mu)$ is a Banach lattice.

Given that X has the structure of both a vector lattice and a Banach space, it is natural to ask about the relation between order and norm convergence in X. Unfortunately, these two convergences may not be directly related in general. If $x_\alpha \xrightarrow{o} x$ implies $x_\alpha \to x$ for any net (x_α) and $x \in X$, then we say that X is ***order continuous***. If this property holds for sequences instead of nets, then we say that X is σ-***order continuous***. Clearly, order continuity implies σ-order continuity.

Example 1.44. $L_p(\mu)$ for $1 \leq p < \infty$ is order continuous.

Example 1.45. $C([0,1])$ is not even σ-order continuous.

The following is a standard result that is useful for extracting order convergent sequences from norm convergent ones; see [AA02, Exercise 13]

Proposition 1.46. *Every norm convergent sequence in a Banach lattice has a subsequence that converges in order to the same limit.*

1.4. **Representations of Banach Lattices.**

In this section we give a more rigorous discussion on how one may view Banach lattices as spaces of functions. We begin with a few definitions.

Let T be a linear operator between two Banach lattices $T : X \to Y$. T is said to be ***positive*** if $T(X_+) \subset Y_+$. We call T a ***lattice homomorphism*** if $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in X$.

Remark 1.47. In the definition above, we made no assumptions about the continuity of T. In fact, positive operators between Banach lattices are continuous; see, for example, [AB06, Theorem 4.3]. If T is a lattice homomorphism, then T preserves all lattice operations *via* Lemma 1.21. In particular, if $x \in X_+$ then $Tx = T|x| = |Tx| \geq 0$; that is, T is positive and, therefore, continuous.

If T is an injective lattice homomorphism, then we call T a ***lattice isomorphism***. Notice that we do not require T to be surjective. If there is a surjective lattice isomorphism between X and Y, then we say that X and Y are lattice isomorphic. Similarly, a lattice isomorphism

T is called a ***lattice isometry*** if $\|Tx\| = \|x\|$ for every $x \in X$, and X is lattice isometric to Y if there is a lattice isometry from X onto Y.

If $e \in X_+$, then one can consider $I_e = \{x : |x| \le \lambda e \text{ for some } \lambda \in [0, \infty)\}$ as in Lemma 1.24. We can define a lattice norm on I_e *via*

$$\|x\|_e = \inf\{\lambda \ge 0 : |x| \le \lambda e\}$$

For a Banach lattice, $(I_e, \|\cdot\|_e)$ is complete for every choice of $e \in X_+$. In addition, if e is a strong unit, then $I_e = X$ with $\|\cdot\|_e$ is a Banach lattice. Since any two norms turning X into a Banach lattice are equivalent, $\|\cdot\|_e$ is equivalent to the original norm on X. This leads to the following result. For a proof, see [LT79, Chapter 1b].

Theorem 1.48. *For every $e \in X_+$ $(I_e, \|\cdot\|_e)$ is lattice isometric to $C(K)$ for some compact Hausdorff space K. Moreover, through this correspondence we may identify $e \in I_e$ with the constant function $\mathbb{1}$ in $C(K)$; i.e. $\mathbb{1}(x) = 1$ for every $x \in K$.*

In particular, if e is a strong unit in X, then Theorem 1.48 says that X is lattice isomorphic to a $C(K)$ space. Therefore, from the point of view of the lattice structure, Banach lattices with strong units are $C(K)$ spaces.

In this thesis, we will be interested in studying the relationships between different convergences in Banach lattices. Since we will often deal with order and norm convergence, it is natural to restrict our attention to order continuous Banach lattices. However, in Example 1.45 we showed that $C(K)$ spaces are not necessarily order continuous; hence, $C(K)$ spaces may not be sufficient for the purposes of representing order continuous spaces. In light of this, it will be useful to have a representation of a Banach lattice on a space that is always order continuous. We can do this with the following construction.

A ***strictly positive functional*** is an element $h \in X^*$ such that $h(x) > 0$ whenever $x > 0$. This concept is useful because it allows us to define the following *norm*. If X admits a strictly positive functional, h, define

$$\|x\|_h = h(|x|)$$

It is easy to verify that $(X, \|\cdot\|_h)$ is a normed lattice, but it need not be complete with respect to $\|\cdot\|_h$. Any norm satisfying

$$\|x + y\| = \|x\| + \|y\| \text{ whenever } x \perp y \tag{1}$$

is called an ***AL-norm*** and a Banach lattice with a norm satisfying (1) is called an ***AL-space***. It is straightforward to check that $\|\cdot\|_h$ satisfies (1); hence, if X is a Banach lattice that admits a strictly positive functional, we can apply the above construction to obtain an AL-norm on X. It follows that the completion of X with respect to $\|\cdot\|_h$, $\tilde{X}$, is an AL-space. At this point, we draw on the following powerful result due to [Kak41].

Theorem 1.49 (*Kakutani's AL-representation Theorem*)**.** *Every AL-space is lattice isometric to $L_1(\mu)$ for some measure μ.*

Since $\tilde{X}$ can be identified with an L_1-space *via* a lattice isometry, we will often write $\tilde{X} = L_1(\mu)$ and say that $L_1(\mu)$ is an ***AL-representation*** of X.

Remark 1.50. In the AL-representation of X, we will view X as a sublattice of $L_1(\mu)$. To justify this, let $V : \tilde{X} \to L_1(\mu)$ be a lattice isometry and let $\iota : X \to \tilde{X}$ be the inclusion map. If we let $T = V \circ \iota$, then $T : X \to L_1(\mu)$ is a lattice isomorphism; hence, the range of T is a sublattice of $L_1(\mu)$. Now X is lattice isometric to $T(X)$.

Theorem 1.51. *Let X be an order continuous Banach lattice with a weak unit, e. Then X is a dense ideal in $L_1(\mu)$ for a finite measure μ. Moreover, $e \in X$ can be identified with $\mathbb{1} \in L_1(\mu)$ and the inclusion of X into $L_1(\mu)$ is continuous.*

Remark 1.52. The proof of this result is a direct consequence of Theorem 1.49. We point out a few subtleties. The conditions that X is order continuous and has a weak unit guarantee that there is an order continuous strictly positive functional on X; see, e.g. [LT79, Proposition 1.b.15]. This leads to the fact that X is an ideal in $L_1(\mu)$. The fact that there is a weak unit in X allows one to choose the measure in Theorem 1.49 to be finite. Finally, the continuity of the inclusion of X into $L_1(\mu)$ follows from $\|x\|_{L_1(\mu)} = \|x\|_h = h(|x|) \leq \|h\|\|x\|$ for some $h \in X^*$ that is strictly positive.

1.5. **Unbounded Order Convergence.**

The variety of structures on Banach lattices make them host to a number of interesting convergences. In addition, one can study how these convergences are related amongst each other. For example, in Banach lattices one may study order convergence and how it relates with the norm convergence. Of course, these convergences need not be directly related in general, but there is a rich theory that follows from the cases when they are; for further reading in this area, see [AB06, Chapter 4.1] on the theory of order continuous Banach lattices. Still, even in vector lattices, the order convergence is lacking some desirable properties. For instance, if Y is a sublattice of a vector lattice X, then a net (y_α) in Y which converges in order to some $y \in Y$ need not converge in order in X; this is in sharp contrast with the behavior of norm convergence in subspaces. Of particular interest to this thesis, [GTX] show that we can resolve this issue if we replace order convergence with *uo-convergence* and use *regular* sublattices.

Definition 1.53. A sublattice Y of a vector lattice X is called ***regular*** if $x_\alpha \downarrow 0$ in Y implies $x_\alpha \downarrow 0$ in X. In particular, every ideal in a vector lattice is a regular sublattice.

Definition 1.54. A net (x_α) in a vector lattice X is said to be ***unbounded order convergent*** to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{\text{o}} 0$ for every $u \in X_+$; we denote this convergence by $x_\alpha \xrightarrow{\text{uo}} x$ and say x_α *uo-converges* to x.

Theorem 1.55 (*GTX, Theorem 3.2*)**.** *Let Y be a sublattice of a vector lattice X. The following are equivalent:*

(i) *Y is regular;*

(ii) *For any net (y_α) in Y, $y_\alpha \xrightarrow{\text{uo}} 0$ in Y implies $y_\alpha \xrightarrow{\text{uo}} 0$ in X;*

(iii) *For any net (y_α) in Y, $y_\alpha \xrightarrow{\text{uo}} 0$ in Y if and only if $y_\alpha \xrightarrow{\text{uo}} 0$ in X.*

Uo-convergence first appeared in [Nak48] and was formally introduced by [Wic77]. Since this time, there have been several theoretical and practical applications of uo-convergence; see, for example, [Gao14] and [GX]. For a detailed account of many interesting properties of uo-convergence, we refer the reader to [GX14], [Gao14] and [GTX]. In particular, it is straightforward to verify that this is a linear convergence; hence it suffices to study uo-convergence of nets to 0. One property that will be particularly useful in this thesis is the following.

Lemma 1.56. [GTX, Corollary 3.5] *Let X be a vector lattice with a weak unit, e. Then for a net (x_α) in X, $x_\alpha \xrightarrow{\text{uo}} 0$ in X if and only if* $|x_\alpha| \wedge e \xrightarrow{\text{o}} 0$

Thus, if a weak unit exists, it suffices to check the uo-convergence at a single point.

It is of theoretical interest to note that one may view uo-convergence in Banach lattices as a generalization of almost everywhere convergence in L_p-spaces.

Theorem 1.57. [GTX] *Let X be an order continuous Banach lattice with a weak unit, and let $L_1(\mu)$ be an AL-representation of X. Then $x_n \xrightarrow{\text{uo}} 0$ in X if and only if $x_n \xrightarrow{\text{a.e.}} 0$ in $L_1(\mu)$.*

2. Unbounded Norm Convergence

In this thesis, we will use uo-convergence as a tool to study another convergence in Banach lattices. A net (x_α) in X is ***unbounded norm convergent*** to $x \in X$ if $|x_\alpha - x| \wedge u \to 0$ for all $u \in X_+$; in this case, we write $x_\alpha \xrightarrow{\text{un}} x$ and say x_α *un-converges* to x. This concept was first introduced in [Tro04], where it was shown that un-convergence in c_0 is the same as coordinate-wise convergence. Additionally, it was shown in [Tro04] that un-convergence agrees with convergence in measure in $L_p(\mu)$ for a finite measure μ and $1 \leq p < \infty$. Given the relationship between a.e. convergence and convergence in measure in L_p-spaces, it is natural to ask about how uo- and un-convergence are related in general. As their definitions only differ by the use of order and norm convergence, the appropriate setting for this question is in order continuous spaces. This leads to the following characterization of un-convergence in terms of uo-convergence for order continuous Banach lattices: a sequence (x_n) un-converges to x if and only if each subsequence has a further subsequence that uo-converges to the same limit. We will also demonstrate that un-convergence can be viewed as a generalization of convergence in measure to Banach lattices.

2.1. **Basic Properties.**

A version of the results presented in this chapter can be found in [DOT].

Unless stated otherwise, we will assume that X is a Banach lattice and all nets and vectors lie in X. The first result highlights some basic properties of un-convergence.

Lemma 2.1.

(i) $x_\alpha \xrightarrow{\text{un}} x$ *iff* $(x_\alpha - x) \xrightarrow{\text{un}} 0$*;*

(ii) *If* $x_\alpha \xrightarrow{\text{un}} x$*, then* $|x_\alpha| \xrightarrow{\text{un}} |x|$*;*

(iii) *If* $x_\alpha \xrightarrow{\text{un}} x$ *and* $y_\alpha \xrightarrow{\text{un}} y$*, then* $ax_\alpha + by_\alpha \xrightarrow{\text{un}} ax + by$ *for any* $a, b \in \mathbb{R}$*;*

(iv) *If* $x_\alpha \xrightarrow{\text{un}} x$ *and* $x_\alpha \xrightarrow{\text{un}} y$*, then* $x = y$*;*

(v) *If* $x_n \xrightarrow{\text{un}} x$*, then* $x_{n_k} \xrightarrow{\text{un}} x$ *for any subsequence* (x_{n_k}) *of* (x_n)*.*

Proof. (i) Suppose $x_\alpha \xrightarrow{\text{un}} x$. Then $|(x_\alpha - x) - 0| \wedge u = |x_\alpha - x| \wedge u \to 0$ for each $u \in X_+$, so $(x_\alpha - x) \xrightarrow{\text{un}} 0$. The proof of the converse is similar.

(ii) As $\big||x_\alpha| - |x|\big| \leq |x_\alpha - x|$, it follows that

$$\big||x_\alpha| - |x|\big| \wedge u \leq |x_\alpha - x| \wedge u \to 0$$

for every $u \in X_+$; that is, $|x_\alpha| \xrightarrow{\text{un}} |x|$.

(iii) Suppose $x_\alpha \xrightarrow{\text{un}} x$ and $y_\alpha \xrightarrow{\text{un}} y$. Applying the triangle inequality, we obtain

$$|(x_\alpha + y_\alpha) - (x + y)| \wedge u \leq (|x_\alpha - x| + |y_\alpha - y|) \wedge u \leq |x_\alpha - x| \wedge u + |y_\alpha - y| \wedge u$$

for each α and $u \in X_+$. It follows that $x_\alpha + y_\alpha \xrightarrow{\text{un}} x + y$.

Next, fix $a \in \mathbb{R}$ and let $u \in X_+$. Observe that $|ax_\alpha - ax| \wedge u = (|a| \cdot |x_\alpha - x|) \wedge u$. If $|a| \leq 1$, then

$$|a| \cdot |x_\alpha - x| \wedge u \leq |x_\alpha - x| \wedge u \to 0.$$

If $|a| \geq 1$, then $u \leq |a| \cdot u$ and

$$|a| \cdot |x_\alpha - x| \wedge u \leq |a| \cdot |x_\alpha - x| \wedge |a| \cdot u = |a| \cdot (|x_\alpha - x| \wedge u) \to 0.$$

In either case, $ax_\alpha \xrightarrow{\text{un}} ax$.

(iv) Observe that $|x-y| \leq |x-x_\alpha| + |y-x_\alpha|$ for every α. Let $u = |x-y|$ and notice that

$$|x-y| = |x-y| \wedge u \leq |x-x_\alpha| \wedge u + |y-x_\alpha| \wedge u \to 0.$$

(v) Suppose $x_n \xrightarrow{\text{un}} x$ and let (x_{n_k}) be any subsequence of (x_n). For any $\epsilon > 0$ and $u \in X_+$ there is a $n_0 \in \mathbb{N}$ such that $\|(x_n - x) \wedge u\| < \epsilon$ whenever $n \geq n_0$. Then for $k \geq n_k \geq n_0$ we must have $\|(x_{n_k} - x) \wedge u\| < \epsilon$. As ϵ was arbitrary, we have $x_{n_k} \xrightarrow{\text{un}} x$. $\square$

Remark 2.2. Recall that each of the lattice operations can be expressed *via* each other. In light of this, Lemma 2.1 (ii) shows that un-convergence preserves all the lattice operations.

Remark 2.3. Combining Lemma 2.1 (i) and (ii) we obtain $x_\alpha \xrightarrow{\text{un}} 0$ if and only if $|x_\alpha - x| \xrightarrow{\text{un}} 0$. This often allows us to reduce un-convergence of nets to un-convergence of positive nets to zero.

The next result justifies the name *unbounded* norm convergence.

Proposition 2.4. *If $x_\alpha \to 0$ then $x_\alpha \xrightarrow{\text{un}} 0$. If (x_α) is order bounded, then the converse is also true.*

Proof. Without loss of generality, suppose $x_\alpha \geq 0$ for all α. Then

$$x_\alpha \wedge u \leq x_\alpha \to 0$$

for any $u \in X_+$. Conversely, if $x_\alpha \xrightarrow{\text{un}} 0$ and there is some $b \in X_+$ such that $x_\alpha \leq b$ for each α, then

$$x_\alpha = x_\alpha \wedge b \to 0.$$

□

We will show that the condition that (x_α) is order bounded in Proposition 2.4 can be weakened. First, we need the following relation between uo- and un-convergence.

Proposition 2.5. *In an order continuous Banach lattice, uo-convergence implies un-convergence.*

Proof. In order continuous spaces, order convergence implies norm convergence. Thus, if $x_\alpha \xrightarrow{\text{uo}} 0$ we must have $|x_\alpha| \wedge u \xrightarrow{\text{o}} 0$ for each $u \in X_+$. Order continuity gives $|x_\alpha| \wedge u \to 0$ for each $u \in X_+$. □

Example 2.6. Let (e_n) denote the standard unit sequence in ℓ_∞; that is, for each $n \in \mathbb{N}$ let $e_n = (0, 0, ..., 0, 0, 1, 0, 0, ...)$ be the sequence with a 1 in the n^{th} coordinate and 0 elsewhere. Note that $(0, 0, 0, ...) \leq e_n \leq (1, 1, 1, ...)$ for each n; hence, (e_n) is order bounded. However, $\|e_n\| = 1$ for each n, so $e_n \not\to 0$ in ℓ_∞. It follows from Proposition 2.4 that $e_n \overset{\text{un}}{\not\longrightarrow} 0$ in ℓ_∞. Since it is known that ℓ_∞ is not order continuous and $e_n \xrightarrow{\text{uo}} 0$ in ℓ_∞ (see [Gao14, Lemma 1.1]), the order continuity assumption in Proposition 2.5 cannot be dropped.

Example 2.7. Example 2.6 also shows that Theorem 1.55 fails for un-convergence. Indeed, (e_n) converges coordinate-wise to $(0, 0, 0, ...)$; hence, by [Tro04, Example 21], $e_n \xrightarrow{\text{un}} 0$ in c_0, but not in ℓ_∞.

We now use Proposition 2.5 to weaken the condition that (x_α) is order bounded in Proposition 2.4. A subset A of X is ***almost order bounded*** if for every $\varepsilon > 0$ there exists $u \in X_+$ such that $A \subseteq [-u, u] + \varepsilon B_X$ where B_X denotes the unit ball of X.

Lemma 2.8. *If $x_\alpha \xrightarrow{\text{un}} x$ and (x_α) is almost order bounded then $x_\alpha \to x$.*

Proof. The proof is identical to [GX14, Lemma 3.7] after applying Proposition 2.5 to reduce uo-convergence to un-convergence. However, unlike [GX14], we do not require the space to be order continuous. $\square$

We also have the following estimate on the limit of a un-convergent net.

Lemma 2.9. *If $x_\alpha \xrightarrow{\text{un}} x$ then $|x_\alpha| \wedge |x| \to |x|$ and $\|x\| \leq \liminf_\alpha \|x_\alpha\|$.*

Proof. Again, the proof follows immediately from [GX14, Lemma 3.6] and Proposition 2.5. $\square$

Our next goal is to reduce the task of checking un-convergence at *every* positive vector to a single 'special' vector as in Lemma 1.56. A vector $e \in X_+$ is called a ***quasi-interior point*** if $x \wedge ne \to x$ for every $x \in X_+$. This leads to the following result.

Lemma 2.10. *Let X be a Banach lattice with a quasi-interior point e. Then $x_\alpha \xrightarrow{\text{un}} 0$ if and only if $|x_\alpha| \wedge e \to 0$.*

Proof. The forward implication is immediate. For the reverse implication, let $u \in X_+$ be arbitrary and fix $\varepsilon > 0$. Note that

$$|x_\alpha| \wedge u = |x_\alpha| \wedge (u - u \wedge me + u \wedge me)$$

for every α and each $m \in \mathbb{N}$. It follows that

$$|x_\alpha| \wedge u \leq |x_\alpha| \wedge (u - u \wedge me) + |x_\alpha| \wedge (u \wedge me) \leq (u - u \wedge me) + m\big(|x_\alpha| \wedge e\big)$$

and, therefore,

$$\big\||x_\alpha| \wedge u\big\| \leq \|u - u \wedge me\| + m\big\||x_\alpha| \wedge e\big\|$$

for all α and all $m \in \mathbb{N}$. Since e is quasi-interior, we can find m such that $\|u - u \wedge me\| < \varepsilon$. Furthermore, it follows from $|x_\alpha| \wedge e \to 0$ that there exists α_0 such that $\big\||x_\alpha| \wedge e\big\| < \frac{\varepsilon}{m}$ whenever $\alpha \geq \alpha_0$. It follows that $\big\||x_\alpha| \wedge u\big\| < \varepsilon + m\frac{\varepsilon}{m} = 2\varepsilon$. Therefore, $|x_\alpha| \wedge u \to 0$. ☐

In order continuous spaces, we can obtain a version of Lemma 1.56 for un-convergence.

Corollary 2.11. *Let X be an order continuous Banach lattice with a weak unit e. Then $x_\alpha \xrightarrow{\text{un}} 0$ if and only if $|x_\alpha| \wedge e \to 0$.*

Proof. If X is order continuous, then e is a weak unit if and only if e is a quasi-interior point. ☐

In [GTX, Corollary 3.6], it was shown that every disjoint sequence is uo-null. Example 2.6 shows that this fact is not true for un-convergence. The following result says that un-null sequences are "almost" disjoint.

Theorem 2.12. *Let (x_α) be a net in X such that $x_\alpha \xrightarrow{\mathrm{un}} 0$. Then there exists an increasing sequence of indices (α_k) and a disjoint sequence (d_k) such that $x_{\alpha_k} - d_k \to 0$.*

Proof. Assume first that $x_\alpha \geq 0$ for every α. Pick any α_1. Suppose that $\alpha_1, \dots, \alpha_{k-1}$ have been constructed. Note that $x_\alpha \wedge x_{\alpha_i} \to 0$ for every $i = 1, \dots, k-1$. Choose $\alpha_k > \alpha_{k-1}$ so that $\big\| x_{\alpha_k} \wedge x_{\alpha_i} \big\| \leq \frac{1}{2^{k+i}}$ for every $i = 1, \dots, k-1$. This produces an increasing sequence of indices (α_k) such that $\|z_{ik}\| \leq \frac{1}{2^{k+i}}$ where $z_{ik} = x_{\alpha_i} \wedge x_{\alpha_k}$, $1 \leq i < k$.

For every k, put $v_k = \sum_{i=1}^{k-1} z_{ik} + \sum_{j=k+1}^{\infty} z_{kj}$. Clearly, v_k is defined and $\|v_k\| < \frac{1}{2^k}$. Put $d_k = (x_{\alpha_k} - v_k)^+$. It is easy to see that $0 \leq x_{\alpha_k} - d_k \leq v_k$, so that $\|x_{\alpha_k} - d_k\| \to 0$ as $k \to \infty$. It is left to show that the sequence (d_k) is disjoint. Let $k < m$. Then

$$\begin{aligned} d_k &= (x_{\alpha_k} - v_k)^+ \leq (x_{\alpha_k} - z_{km})^+ = x_{\alpha_k} - x_{\alpha_k} \wedge x_{\alpha_m}, \text{ and} \\ d_m &= (x_{\alpha_m} - v_m)^+ \leq (x_{\alpha_m} - z_{km})^+ = x_{\alpha_m} - x_{\alpha_k} \wedge x_{\alpha_m}. \end{aligned}$$

It follows that $d_k \perp d_m$.

For the general case, we apply the first part of the proof to the net $\big(|x_\alpha|\big)$ and produce an increasing sequence of indices (α_k) and two positive sequences (w_k) and (h_k) such that $|x_{\alpha_k}| = w_k + h_k$, (w_k) is disjoint, and $h_k \to 0$. By Lemma 1.28, we can find sequences (d_k) and (g_k) in X with $|d_k| = w_k$, $|g_k| = h_k$ and $x_{\alpha_k} = d_k + g_k$. It follows that (d_k) is a disjoint sequence and $g_k \to 0$. Thus, $x_{\alpha_k} - d_k \to 0$. $\square$

It is well-known that a topology induced by a norm is sequential in nature; *i.e.*, the norm topology can be described by prescribing convergent *sequences* instead of convergent nets. For order continuous Banach lattices, un-convergence is also sequential in nature; that is, one can always work with un-convergent sequences instead of nets.

Corollary 2.13. *Let (x_α) be a net in an order continuous Banach lattice X such that $x_\alpha \xrightarrow{\text{un}} 0$. Then there exists an increasing sequence of indices (α_k) such that $x_{\alpha_k} \xrightarrow{\text{uo}} 0$ and $x_{\alpha_k} \xrightarrow{\text{un}} 0$.*

Proof. Let (α_k) and (d_k) be as in Theorem 2.12. Since (d_k) is disjoint, we have $d_k \xrightarrow{\text{uo}} 0$ and, therefore, $d_k \xrightarrow{\text{un}} 0$. It now follows from $x_{\alpha_k} - d_k \to 0$ that $x_{\alpha_k} - d_k \xrightarrow{\text{un}} 0$ and, therefore, $x_{\alpha_k} \xrightarrow{\text{un}} 0$. Furthermore, since $x_{\alpha_k} - d_k \to 0$, passing to a further subsequence, we may assume that $x_{\alpha_k} - d_k \xrightarrow{\text{o}} 0$ and, therefore, $x_{\alpha_k} - d_k \xrightarrow{\text{uo}} 0$. This yields $x_{\alpha_k} \xrightarrow{\text{uo}} 0$. $\square$

2.2. **Generalized Convergence in Measure.**

In this section, we characterize un-convergence in terms of uo-convergence. As a consequence, we obtain a generalization of convergence in measure in Banach lattices.

Proposition 2.14. *If $x_n \xrightarrow{\text{un}} 0$ then there is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \xrightarrow{\text{uo}} 0$.*

Proof. Define $e := \sum_{n=1}^{\infty} \frac{|x_n|}{2^n \|x_n\|}$. Let B_e be the band generated by e in X. It follows from $x_n \xrightarrow{\text{un}} 0$ that $|x_n| \wedge e \to 0$ in X. As B_e is norm closed, $|x_n| \wedge e \to 0$ in B_e. By Proposition 1.46, there exists a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k}| \wedge e \xrightarrow{\text{o}} 0$ in B_e. Since e is a weak unit in B_e, we have $x_{n_k} \xrightarrow{\text{uo}} 0$ in B_e. Finally, as B_e is an ideal in X, it follows from Theorem 1.55 that $x_{n_k} \xrightarrow{\text{uo}} 0$ in X. $\square$

It was observed in [Tro04, Example 23] that for sequences in $L_p(\mu)$, where μ is a finite measure, un-convergence agrees with convergence in measure. We now provide an alternative proof of this fact based on Proposition 2.14.

Corollary 2.15. [Tro04] *Let (f_n) be a sequence in $L_p(\mu)$ where $1 \leq p < \infty$ and μ is a finite measure. Then $f_n \xrightarrow{\text{un}} 0$ if and only if $f_n \xrightarrow{\mu} 0$.*

Proof. Without loss of generality, $f_n \geq 0$ for all n. Suppose $f_n \xrightarrow{\mu} 0$. It is easy to see that $f_n \wedge \mathbb{1} \to 0$ in the norm of $L_p(\mu)$. Since $L_p(\mu)$ is order continuous and $\mathbb{1}$ is a weak unit, it follows from Corollary 2.11 that $f_n \xrightarrow{\text{un}} 0$.

Conversely, suppose that $f_n \xrightarrow{\text{un}} 0$. Then every subsequence (f_{n_k}) is still un-null and, therefore, has a further subsequence $(f_{n_{k_i}})$ such that $f_{n_{k_i}} \xrightarrow{\text{uo}} 0$ by Proposition 2.14. Since uo-convergence agrees with a.e. convergence in $L_p(\mu)$, we have $f_{n_{k_i}} \xrightarrow{\text{a.e.}} 0$. In summary, (f_n) is a sequence in $L_p(\mu)$ with the property that every subsequence has a further subsequence that converges a.e. to the same limit. Since μ is finite, this yields $f_n \xrightarrow{\mu} 0$. $\square$

Remark 2.16. In the preceding proof, we used the fact that if (Ω, Σ, μ) is measure space with a finite measure and (f_n) is a sequence of Σ-measurable functions on Ω, then $f_n \xrightarrow{\mu} 0$ iff every subsequence (f_{n_k})

has a further subsequence $(f_{n_{k_i}})$ such that $f_{n_{k_i}} \xrightarrow{\text{a.e.}} 0$; see, for example, [Cohn13, Exercise 6, p. 84]. It is of interest to note that Proposition 2.14 may be viewed as an extension of the forward direction of this result to general Banach lattices. Our next result shows that the converse is true in order continuous Banach lattices.

Theorem 2.17. *A sequence in an order continuous Banach lattice X is un-null if and only if every subsequence has a further subsequence which is uo-null.*

Proof. The forward implication is Proposition 2.14. To show the converse, assume that $x_n \overset{\text{un}}{\not\to} 0$. Then there exist $\delta > 0$, $u \in X_+$, and a subsequence (x_{n_k}) such that $\big\||x_{n_k}| \wedge u\big\| > \delta$ for all k. By assumption, there is a subsequence $(x_{n_{k_i}})$ of (x_{n_k}) such that $x_{n_{k_i}} \xrightarrow{\text{uo}} 0$; therefore, $x_{n_{k_i}} \xrightarrow{\text{un}} 0$ by Proposition 2.5. This yields $|x_{n_{k_i}}| \wedge u \to 0$, which is a contradiction. □

Remark 2.18. Again, Example 2.6 shows that the order continuity assumption in Theorem 2.17 cannot be dropped.

These techniques allow us to generalize convergence in measure to the setting of order continuous Banach lattices with a weak unit.

Theorem 2.19. *Let X be an order continuous Banach lattice with a weak unit, e, and let $L_1(\mu)$ be an AL-representation of X. For a sequence (x_n) in X, we have $x_n \xrightarrow{\text{un}} 0$ in X if and only if $x_n \xrightarrow{\mu} 0$ in $L_1(\mu)$.*

Proof. Recall that, by Theorem 1.51, these conditions allow us to represent X as a dense ideal in $L_1(\mu)$ for a finite measure μ, where e is identified with $\mathbb{1}$, and T, the inclusion of X into $L_1(\mu)$, is a continuous lattice isomorphism. We also view X as sitting inside $L_1(\mu)$ by Remark 1.50.

Without loss of generality, we take $x_n \geq 0$ for all $n \in \mathbb{N}$. For the forward direction, suppose $x_n \xrightarrow{\text{un}} 0$ in X. Since we view X as being contained inside $L_1(\mu)$ and $e \in X$ is identified with $\mathbb{1}$, Corollary 2.11 and the continuity of T gives

$$x_n \wedge \mathbb{1} \to 0.$$

in $L_1(\mu)$. As $\mathbb{1}$ is a weak unit in $L_1(\mu)$ when μ is finite, Corollary 2.11 yields that $x_n \xrightarrow{\text{un}} 0$ in $L_1(\mu)$. By Corollary 2.15, this is equivalent to $x_n \xrightarrow{\mu} 0$ in $L_1(\mu)$.

Conversely, suppose (x_n) is a sequence in X whose representation in $L_1(\mu)$ satisfies $x_n \xrightarrow{\mu} 0$. Let (x_{n_k}) be any subsequence of (x_n). Then there exists a further subsequence $(x_{n_{k_i}})$ such that $x_{n_{k_i}} \xrightarrow{\text{a.e.}} 0$; so $x_{n_{k_i}} \xrightarrow{\text{uo}} 0$ in $L_1(\mu)$. Since we view X as an ideal in $L_1(\mu)$, Theorem 1.55 gives us that $x_{n_{k_i}} \xrightarrow{\text{uo}} 0$ in X. Now apply Theorem 2.17 to obtain $x_n \xrightarrow{\text{un}} 0$ in X. $\square$

2.3. **Un-Topology.**

Recall that a topology on a set X is determined by its open subsets. Equivalently, one can describe a topology by prescribing convergent nets or by describing a system of neighborhoods.

It is known that a.e. convergence is not given by a topology [Ord66]; *i.e.* there is no topology for which the associated convergence is a.e. convergence. In particular, this shows that uo-convergence is generally not given by a topology. We show that un-convergence is always given by a topology and define its neighborhoods explicitly. In light of Lemma 2.1 and Remark 2.3, we expect that this will be a linear topology; hence, we begin by describing a base of neighborhoods of zero. Given an $\varepsilon > 0$ and a non-zero $u \in X_+$, we set

$$V_{u,\varepsilon} = \bigl\{x \in X \ : \ \bigl\||x| \wedge u\bigr\| < \varepsilon\bigr\}.$$

Set $\mathcal{N}_0 = \{V_{u,\epsilon} : \epsilon > 0; u \in X_+ \setminus \{0\}\}$.

Proposition 2.20. *The collection $\mathcal{N}_0$ is a base of neighborhoods of zero for a linear topology.*

Proof. We apply [KN76, Theorem 5.1].

First, every set in $\mathcal{N}_0$ trivially contains zero.

Second, we need to show that the intersection of any two sets in $\mathcal{N}_0$ contains another set in $\mathcal{N}_0$. Take V_{u_1,ε_1} and V_{u_2,ε_2} in $\mathcal{N}_0$. Put $\varepsilon = \varepsilon_1 \wedge \varepsilon_2$ and $u = u_1 \vee u_2$. We claim that $V_{u,\varepsilon} \subseteq V_{u_1,\varepsilon_1} \cap V_{u_2,\varepsilon_2}$. Indeed, take any $x \in V_{u,\varepsilon}$. Then $\bigl\||x| \wedge u\bigr\| < \varepsilon$. It follows from $|x| \wedge u_1 \leq |x| \wedge u$ that

$$\bigl\||x| \wedge u_1\bigr\| \leq \bigl\||x| \wedge u\bigr\| < \varepsilon \leq \varepsilon_1,$$

so that $x \in V_{u_1,\varepsilon_1}$. Similarly, $x \in V_{u_2,\varepsilon_2}$.

It is easy to see that $V_{u,\varepsilon} + V_{u,\varepsilon} \subseteq V_{u,2\varepsilon}$. This immediately implies that for every U in $\mathcal{N}_0$ there exists $V \in \mathcal{N}_0$ such that $V + V \subseteq U$. It is also easy to see that for every $U \in \mathcal{N}_0$ and every scalar λ with $|\lambda| \leq 1$ we have $\lambda U \subseteq U$.

Next, we need to show that for every $U \in \mathcal{N}_0$ and every $y \in U$, there exists $V \in \mathcal{N}_0$ such that $y + V \subseteq U$. Let $y \in V_{u,\varepsilon}$ for some $\varepsilon > 0$ and a non-zero $u \in X_+$. We need to find $\delta > 0$ and a non-zero $v \in X_+$ such that $y + V_{v,\delta} \subseteq V_{u,\varepsilon}$. Put $v := u$. It follows from $y \in V_{u,\varepsilon}$ that $\big\||y| \wedge u\big\| < \varepsilon$; take $\delta := \varepsilon - \big\||y| \wedge u\big\|$. We claim that $y + V_{v,\delta} \subseteq V_{u,\varepsilon}$. Let $x \in V_{v,\delta}$; it suffices show that $y + x \in V_{u,\varepsilon}$. Indeed, $|y + x| \wedge u \leq |y| \wedge u + |x| \wedge u$, so that

$$\big\||y + x| \wedge u\big\| \leq \big\||y| \wedge u\big\| + \big\||x| \wedge u\big\| < \big\||y| \wedge u\big\| + \delta = \varepsilon.$$

□

Now we can describe all the neighborhoods in this topology: *a subset U of X is a neighborhood of y if $y + V \subseteq U$ for some $V \in \mathcal{N}_0$.*

The next result shows that un-convergence is topological.

Proposition 2.21. *Un-convergence in a Banach lattice is the same as the convergence in the topology whose base neighborhoods of zero are given by $\mathcal{N}_0$.*

Proof. If $x_\alpha \xrightarrow{\text{un}} 0$, then for every $\epsilon > 0$ and $u \in X_+$ there is some α_0 such that $\big\||x_\alpha| \wedge u\big\| < \epsilon$ whenever $\alpha \succeq \alpha_0$. Said differently, for any $V_{u,\epsilon} \in \mathcal{N}_0$, there is an α_0 such that $x_\alpha \in V_{u,\epsilon}$ whenever $\alpha \succeq \alpha_0$. The converse follows immediately. □

Thus, the natural convergence in this topology is exactly un-convergence. Notice that this topology must also be Hausdorff by Lemma 2.1 (iv).

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