University of Alberta

Multiscale Method for Linear and Semi linear elliptic equations

by

Peimin Chen ©

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Abstract

This thesis deals with two problems on multiscale methods for elliptic equations. One is to discuss super-convergent techniques in multiscale methods. The other one is to deal with a class of nonlinear multiscale problems based on the theory of upper and lower solutions.

For many problems of fundamental and practical importance in science and engineering, which have multiple-scale solutions, it is well known that the calculation of numerical methods for these problems is very huge, even by using some multi-scale methods. So, it is necessary for us to find an efficient method to deal with them. In this thesis, superconvergent techniques are used in existing multi-scale methods to reduce the calculation. Furthermore, by comprehensive analysis, the order of the error estimates between the numerical approximation and the exact solution is verified to be improved reasonably.

At present, for some nonlinear problems with microstructure, there are many papers, based on the multiscale expansion and homogenization theory, to deal with them. But there is no systemic method to solve all of nonlinear partial differential equations since for different nonlinear problems, the multiscale expansion is different and some parameters are also different, which lead to the process of homogenization also being different. In this thesis, a systematic method based on the theory of upper and lower solution is provided. It can deal with a class of nonlinear problems just as that in solving linear problems. In addition, in the last part, numerical computations are also presented to support our theoretical analysis.

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Chapter 1

Introduction

1.1 Sobolev Space

In this section, we provide some definitions of Sobolev space (see [1]) and state some important and useful theorems, which, for simplicity, are not proved in this thesis.

Set Ω to be an open set in R^n, and Γ its boundary. Assume that $I(\Omega)$ is a linear space constructed by infinitely differentiable functions with compact support set in Ω, and

$$I(\overline{\Omega}) = \{\varphi| : \varphi \in I(R^n)\}.$$

Set $I'(\Omega)$ to be the dual space of $I(\Omega)$. If f is a local Lebesgue integrable function, then the distribution associated with it is

$$< f, \varphi >= \int_\Omega f(x)\varphi(x)dx, \quad \forall \varphi \in I(\Omega).$$

Let Z_+^n be the n times positive integer space, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in Z_+^n$, moreover define $|\alpha| = \sum_{i=1}^{n} \alpha_i$, then, for any $u \in I'(\Omega)$, define $\partial^n u \in I'(\Omega)$ by

$$< \partial^n u, \varphi >= (-1)^{|\alpha|} < u, \partial^n \varphi >, \quad \forall \varphi \in I(\Omega).$$

Set $L^p(\Omega)$ to be the p-times Lebesgue integrable space with norm

$$\|u\|_{0,p} = (\int_\Omega |u|^p dx)^{1/p},$$

then for $m \in N, p \in R, 1 \le p \le \alpha$, define the Sobolev space

$$W^{m,p}(\Omega) = \{v | v \in L^p(\Omega), \quad \partial^\alpha v \in L^p(\Omega), \quad \forall |\alpha| \le m\}$$

with norm

$$\|u\|_{m,p,\Omega} = (\sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{0,p,\Omega}^p)^{1/p}, \quad p < \infty,$$

$$\|u\|_{m,\infty,\Omega} = \sup_{|\alpha|\leq m} \|\partial^\alpha u\|_{0,\infty,\Omega}, \quad p = \infty.$$

For the space $W^{m,p}(\Omega)$, define

$$|u|_{m,p,\Omega} = (\sum_{|\alpha|=m} \|\partial^\alpha u\|_{0,p,\Omega}^p)^{1/p}, \quad p < \infty,$$

$$|u|_{m,\infty,\Omega} = \sup_{|\alpha|=m} \|\partial^\alpha u\|_{0,\infty,\Omega}, \quad p = \infty.$$

If $p = 2$, the Sobolev space $W^{m,2}$ is written as $H^m(\Omega)$.

Theorem 1.1.1 *(Imbedding Theorem) (see [1]) If $n = 2$, we have*

$$\|w\|_{m,p,\Omega} \leq C\|w\|_{m+1,\Omega} \quad p \in [1, \infty),$$

$$\|w\|_{m,\infty,\Omega} \leq C\|w\|_{m+1,p,\Omega} \quad p > 2.$$

Set T_h be regular partition of Ω with elements e with size h_e, and define $h := \max_{e\in T_h} h_e$, then we have

Theorem 1.1.2 *(Trace Theorem) (see [1])*

$$\|\nu_0 w\|_{m-1,\partial\Omega} \leq \|\nu_0 w\|_{m-1/2,\partial\Omega} \leq C\|w\|_{m,\Omega},$$

where C is independent of Ω. Especially, for any $e \in T_h$, we have

$$\|\nu_0 w\|_{m-1,\partial e} \leq \|\nu_0 w\|_{m-1/2,\partial e} \leq Ch^{-1/2}\|w\|_{m,e}.$$

1.2 Finite Element Method

Let P_k be the space of polynomials with degree no more than k. We define the finite element space to be ([4])

$$V_h := \{v \in H_0^1(D) : \quad v|_e \in P_1(e) \quad \forall\, e \in T_h\},$$

then for any $v \in V_h$, we have that

$$\|v\|_{m,p,\Omega} \leq Ch^{l-m+n(1/p-1/q)}\|v\|_{l,q,\Omega},$$

where e and T_h are defined in section 1.1.

In addition, we also have (see [4])

$$|u - u_p^I|_{m,q,\Omega} \leq Ch^{k+1-m}|u|_{k+1,q,\Omega},$$

where u_p^I is the p-interpolation of u, $0 \leq m \leq k+1, 1 \leq k \leq p$.

In the following part, some important theorems are presented.

Theorem 1.2.1 *(Lax-Milgram Theorem) (see [4], [2]) Let V be a Hilbert space, $a(u, v)$ a bounded bilinear function in V, $F(v)$ a bounded linear function in V. If there exists a constant $\alpha > 0$ such that*

$$a(u, u) \geq \alpha\|u\|^2, \quad \forall u \in V,$$

then, there exists a unique $u \in V$, such that

$$a(u, v) = F(v), \quad \forall v \in V,$$

moreover,

$$\|u\| \leq \frac{1}{\alpha}\|F\|,$$

where $\|F\|$ is the norm of F.

Theorem 1.2.2 *(Poincaré-Friedrichs inequality) (see [4]) There exists a constant c, independent of Ω, such that*

$$\|u\|_0 \leq C|u|_1, \quad \forall u \in H_0^1(\Omega).$$

Theorem 1.2.3 *(Céa Theorem) (see [4]) If V is a Hilbert space, V_h is a linear subspace in V, and $u \in V$, $u_h \in V_h$ are the solutions of the following equations, respectively*

$$a(u, v) = F(v), \quad \forall v \in V,$$

$$a(u_h, v) = F(v), \quad \forall v \in V_h,$$

where the bilinear function a and the function F satisfy the conditions of Lax-Milgram theorem. Then there exists a constant C independent of V_h, such that

$$\|u - u_h\| \leq C \inf_{v_h \in V_h} \|u - v_h\|.$$

Theorem 1.2.4 *(Strang Theorem) (see [4]) If V and V_h are Hilbert spaces, and $u \in V$, $u_h \in V_h$, with u_h the solution of the following problem*

$$a_h(u_h, v) = F(v), \quad \forall v \in V_h,$$

where F is a linear function in V_h, a_h is a bilinear function in $V_h + V$. Set $\| \cdot \|_h$ to be the norm in $V_h + V$, and assume that there exists M and $\alpha > 0$, such that

$$a_h(u, u) \geq \alpha \|u\|_h^2 \quad \forall u \in V_h,$$

$$a_h(u, v) \leq M \|u\|_h \cdot \|v\|_h \quad \forall u, v \in V_h + V,$$

then there exists constant C, dependent on M, and α, such that

$$\|u - u_h\|_h \leq C(\inf_{v_h \in V_h} \|u - v_h\|_h + \sup_{0 \neq w_h \in V_h} \frac{|a_h(u, w_h) - F(w_h)|}{\|w_h\|_h}).$$

1.3 Multiscale problems

Let $\Omega \subset R^n$ be an open bounded set, we introduce a differential operator A^ϵ (see [3]). Assume that A^ϵ oscillates with period ϵ. Define

$$Y = \Pi_{j=0}^n]0, Y_j^0[\subset R^n.$$

Let a_{ij}, $i, j = 1, 2, ..., n$, satisfy

$$a_{ij}(y) \in R, \quad a_{ij} : Y - periodic, \quad a_{ij} \in L^\infty(R^n),$$

$$a_{ij}(y)\xi_i\xi_j \geq \alpha \xi_i \xi_i, \quad \alpha > 0,$$

where α is a constant and the Einstein notation is used. Also, consider a_0, such that

$$a_0 \in L^\infty(R^n), \quad a_0 : Y - periodic,$$

$$a_0(y) \geq a_0 \geq 0.$$

Then, we set the operator A^ε to be given by

$$A^\varepsilon = -\frac{\partial}{\partial x_i}(a_{ij}(\frac{x}{\varepsilon})\frac{\partial}{\partial x_j}) + a_0(\frac{x}{\varepsilon}),$$

where ε is a small positive parameter.

In the following part, we consider

$$\begin{cases} A^\varepsilon(u_\varepsilon) = f, & in \quad \Omega, \\ u_\varepsilon = 0 \quad on & \Gamma = \partial\Omega \end{cases} . \tag{1.1}$$

From homogenization theory (see [3]), we have an elliptic operator A^0, such that $u_\varepsilon \longrightarrow u$ where u is the solution of the following equation

$$\begin{cases} A^0 u = f, & \text{in } \Omega, \\ u = 0 \quad on \, \Gamma \end{cases} . \tag{1.2}$$

Then, A^0 is the homogenization operator of A^ε, and (1.2) is the homogenized equation.

Let V be a closed set in $H^1(\Omega)$,

$$H_0^1(\Omega) \subseteq V \subseteq H^1(\Omega).$$

For any $u, v \in H^1(\Omega)$, define

$$a^\varepsilon(u,v) = \int_\Omega a_{ij}^\varepsilon(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx + \int_\Omega a_0^\varepsilon uv dx,$$

where

$$a_{ij}^\varepsilon(x) = a_{ij}(x/\varepsilon), \quad a_0^\varepsilon(x) = a_0(x/\varepsilon).$$

Then it can be proved that

$$a^\varepsilon(v,v) \geq min(a, a_0)\|v\|_{H^1(\Omega)}^2, \quad \forall v \in H^1(\Omega),$$

with bilinear variational form

$$\begin{cases} a^\varepsilon(u_\varepsilon, v) = (f, v) & \forall v \in H^1(\Omega), \\ u_\varepsilon \in V, \end{cases} \tag{1.3}$$

where

$$(f,v) = \int_\Omega f v dx, \quad f \in L^2(\Omega).$$

1.4 Asymptotic Expansion

For any function $\phi(x,y), x \in \Omega, y \in R^n$ that is Y-periodic in the Y-direction, we denote $\phi(x, x/\varepsilon)$ by $\phi(x,y)$, i.e.: $y = x/\varepsilon$.

In order to find $u_\varepsilon(x)$, we expand $u_\varepsilon(x)$ as follows (see [3])

$$u_\varepsilon(x) = u_0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + ..., \tag{1.4}$$

where $u_j(x,y)$ is Y-periodic in the Y-direction.

The main idea of this method is to substitute (1.4) into the original equation to determine the coefficient of ε. Equating to the order of ϵ on the both sides of the equation, then we can obtain differential equations to be satisfied by the u_i.

In this method, we look on x, y as independent parameter. Then, the operator $\frac{\partial}{\partial x_j}$ becomes $\frac{\partial}{\partial x_j} + \frac{1}{\epsilon}\frac{\partial}{\partial y_j}$. Based on this idea, A^ε can be denoted as

$$A^\varepsilon = \varepsilon^{-2} A_1 + \varepsilon^{-1} A_2 + \varepsilon A_3 + \text{higher order terms}, \tag{1.5}$$

where

$$A_1 = -\frac{\partial}{\partial y_i}[a_{ij}(y)\frac{\partial}{\partial y_j}],$$
$$A_2 = -\frac{\partial}{\partial y_i}[a_{ij}(y)\frac{\partial}{\partial x_j}] - \frac{\partial}{\partial x_i}[a_{ij}(y)\frac{\partial}{\partial y_j}],$$
$$A_3 = -\frac{\partial}{\partial x_i}[a_{ij}(y)\frac{\partial}{\partial x_j}] + a_0.$$

By (1.4) and (1.5), equation (1.1) become

$$A_1 u_0 = 0, \tag{1.6}$$

$$A_1 u_1 + A_2 u_0 = 0, \tag{1.7}$$

$$A_1 u_2 + A_2 u_1 + A_3 u_0 = f, \tag{1.8}$$

and

$$A_1 u_3 + A_2 u_2 + A_3 u_1 = 0, ... \tag{1.9}$$

In the following part, we will derive the homogenized operator ϱ from (1.6), (1.7) and (1.8).

If

$$\int_Y F(y)dy = 0, \tag{1.10}$$

we see that

$$\begin{cases} A_1\phi = F, & \text{in } Y, \\ \phi : & Y - periordic, \end{cases} \tag{1.11}$$

has a unique solution.

Set

$$W(Y) = \{\phi | \phi \in H^1(Y), \quad \phi : Y - periodic\}.$$

For ϕ, ψ, let

$$a_1(\phi, \varphi) = \int_Y a_{ij}(y) \frac{\partial \phi}{\partial y_j} \frac{\partial \psi}{\partial y_i} dy,$$

$$(F, \psi)_Y = \int_Y F(y)\psi(y)dy,$$

then (1.11) becomes

$$\begin{cases} \phi \in W(y), \\ a_1(\phi, \psi) = (F, \psi)_Y, \quad \forall \psi \in W(y). \end{cases} \tag{1.12}$$

From (1.10), we have

$$W^*(Y) = W(Y)/R, \tag{1.13}$$

Moreover $\psi \to (F, \psi)_Y = (F, \psi + c)_Y, \forall c \in R$ is also linear form in W^*.

The process is as follows:

a) Solve (1.6)

From (1.10), (1.11), if x is a parameter , then the unique solution of (1.6) is $u_0 = const$. That is

$$u_0(x, y) = u(x), \tag{1.14}$$

b) Solve (1.7)

From (1.14), (1.7) can be reduced to

$$A_1 u_1 = [\frac{\partial}{\partial y_i} a_{ij}(y)] \frac{\partial u}{\partial x_j}, \tag{1.15}$$

Let $N^j = N^j(y)$ be the solution of the following equation

$$\begin{cases} A_1 N^j = A_1 y_j = -\frac{\partial}{\partial y_i} a_{ij}(y), \\ N^j : \quad Y - periodic. \end{cases} \tag{1.16}$$

Since $\int_Y A_1 y_j dy = 0$, from (1.10) and (1.11), we know that x^j exists. Then the solution of equation (1.15) is

$$u_1(x,y) = -N^j(y)\frac{\partial u}{\partial x_j}(x) + \widetilde{u}_1(x), \tag{1.17}$$

c) Solve (1.8)

Consider equation (1.8), where u_2 is unknown, x is a parameter, from (1.10) and (1.11) we have that if

$$\int_Y (A_2 u_1 + A_3 u_0) dy = \int_Y f dy = |Y| f, \tag{1.18}$$

then u_2 exists. Equation (1.18) is the homogenized equation we are looking for.

Since

$$\int_Y A_2 u_1 dy = -\frac{\partial}{\partial x_i}\int_Y a_{ik}(y)\frac{\partial u_1}{\partial y_k} dy,$$

from(1.17), we can derive that

$$\int_Y A_2 u_1 dy = \frac{\partial}{\partial x_i}\int_Y a_{ik}(y)\frac{\partial x^j}{\partial y_k} dy \frac{\partial u}{\partial x_j},$$

then (1.18) turns into

$$-\frac{1}{|Y|}[\int_Y (a_{ij} - a_{ik}\frac{\partial x^j}{\partial y_k}) dy]\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{1}{|Y|}[\int_Y a_0(y) dy] u = f. \tag{1.19}$$

Then we can draw the conclusion that the principle of calculating the homogenized operator ϱ is as follows

(1) Solve (1.16) in unit cell.

(2) Obtain A^0 by (1.19).

Bibliography

[1] R. A. Adams, *Sobolev Spaces,* Academic press, 1975.

[2] I. Babuška and A. K. Aziz, *Survey Lectures on Mathematical Foundations of the Finite Element Method,* in The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, ed. by A.Aziz, Academic Press New York and London, 1972, PP. 5-359.

[3] A. Bensoussan, J. L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures,* North-Holland, Amsterdam, New-York, Oxford, 1978.

[4] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems,* North-Holland, Amsterdam, 1978.

Chapter 2

Superconvergent Techniques in Multiscale Methods

2.1 Introduction

Early papers (such as [1], [4], [6]) concentrated on multi-scale methods mainly based on the theory of asymptotic expansion and homogenization. Later, various different but related multi-scale methods were proposed, including multigrid numerical homogenization method ([18], [19], [27], [28]), the multiscale finite element method (MsFEM) ([21], [22], [16]), the heterogeneous multiscale method (HMM) ([11], [12], [13], [14]), finite element method based on the *Residual-Free Bubble* method ([7], [17], [20], [23]), wavelet homogenization method ([9]) and so on. Each of these methods has its own advantage in some special fields. As we know, the multi-grid method as a classical multi-scale technique achieves optimal efficiency by relaxing the errors at different scales on different grids. It can give an accurate approximation to the detailed solution of fine scale problems. HMM is a specific strategy to compute the macro-scale behavior of the system with a standard macro-scale scheme in which the missing micro-scale data can be evaluated concurrently by using the micro-scale model. It can deal with many multi-scale problems efficiently even for problems whose period is unknown. MsFEM can obtain large scale solutions accurately and efficiently without resolving the small scale details. The main idea of it is to, in each element, construct finite element base functions which can capture the small scale information. Such small-scale information is then brought to the large scales through the coupling of the global stiffness matrix.

Although the methods above are very efficient to deal with some practical problems, sometimes the cost is still huge. For example, in order to simulate elliptic problems with non-uniformly oscillating coefficients by HMM, at least one unit cell in each element will be calculated to obtain the homogenized equation and grasp the information of microstructure, which involves many calculations if the number of elements is large. For some cases in which the domain and the solution are smooth enough, it is necessary for us to find a more efficient method or technique to reduce the needed calculation. It is known that, see [8], a fast technique, post-processing algorithm, has been used to analyze a multiscale method, which is based on asymptotic expansion. But in [8], the authors just analyzed elliptic problems with uniformly highly oscillatory coefficients. In practice, there are many multiscale problems with non-uniformly oscillating coefficients, and by using the post processing technique directly, it is impossible to improve the order of the error estimate of the whole domain if one just uses the linear interpolation in the unit cells, which have been simulated. For instance, under the conditions above, the error estimate of HMM for the H^1-broken norm is just $O(H)$. If we use a high order interpolation technique, then the number of the unit cells involved in the calculation will increase greatly in HMM. So, it is very important to reduce the number of unit cells needed. In this chapter, we show that it is not necessary to choose at least one unit cell in each element for the calculation. We just simulate unit cells on a new mesh, which is different from the partition of the whole domain. The size of the former is much bigger than that of the latter. This idea is different from that used in HMM and some other multiscale methods. Then, by using high order interpolation technique for the solved unit cells, we successfully reduce the computational effort. Moreover, we can use the superconvergence technique to deal with the numerical solution of homogenized equation to improve its accuracy. Based on these ideas, some improved error estimates are given. In this chapter, we just investigate the superconvergent techniques in the homogenized equations presented in [4] and [13]. In fact, superconvergent technique can also be applied to some other multiscale methods. In addition, in this chapter, we just discuss elliptic problems. For parabolic multiscale problems with suitable conditions, the superconvergent technique is also valid.

In the past forty years, superconvergence for finite element methods has been an active research field in numerical analysis. Early papers concentrated on superconvergence at isolated points (see [10] *et al*). Later various type of superconvergent recoveries were established, either in the strict sense or in an approximate way (see [2], [3], [29], [30], [31], [32], [33], [24], [26] *et al*). In this chapter, we just want to give a framework to demonstrate that superconvergent technique suited to multi-scale methods and can improve the accuracy efficiently. Thus, we only employ certain postprocessing techniques proposed in [24], [26], to improve the existing approximation accuracy. In fact, some other superconvergent techniques, such as the *Zienkiewicz-Zhu superconvergent Patch Recovery*(ZZ-SPR), can also be used to improve the order of error estimates of multi-scale methods . In future work, for some special cases in multi-scale methods, we plan to investigate ZZ-SPR or some other superconvergent techniques.

The outline of this chapter is as follows. In the next section, we introduce the model problem and provide two similar homogenized equations. Moreover, the error estimate between the exact solution of the original problem and the asymptotic expansion of order one is presented, and the estimates

$$\|u^\epsilon - u_1^\epsilon\|_{1,D} \le C\sqrt{\epsilon}\|U_0\|_{3,\infty,D}$$

$$\|u^\epsilon - \widetilde{u}_1^\epsilon\|_{1,D} \le (Ch^k\|u_0\|_{1,D} + \sqrt{\epsilon}\|u_0\|_{3,\infty,D}),$$

are obtained.

Based on this result, we present the principal results of this chapter in section 3. The error estimate between the exact solution and the numerical solution of the first order multiscale solution corrected by postprocessing, is shown to be

$$\|u^\epsilon - \overline{u}^\epsilon\|_{1,D} \le C(\sqrt{\epsilon}\|u_0\|_{3,\infty,D} + h^k\|u_0\|_{1,D} + H^m\|\widetilde{u}_0\|_{m+1,D}),$$

and

$$\begin{aligned}\|u^\epsilon - \overline{u}^\epsilon\|_{1,D_0} \;&\le\; C(\sqrt{\epsilon}\|u_0\|_{3,\infty,D_1} + h^k\|u_0\|_{1,D_1} \\ &\quad + H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}).\end{aligned}$$

In section 4 the superconvergent technique is extended to HMM and some useful error estimates are given. Moreover, from the analysis of the orders of the error estimates, we observe that the accuracy of the approximation is reasonably improved. In the last section, we discuss briefly some possible future work.

2.2 A model problem and its homogenized equations

Once again, we adopt the standard notation $W^{m,p}(D)$ for Sobolev spaces on D with norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$; $W_0^{m,p} \equiv \{\omega \in W^{m,p}(D) : \omega|_{\partial D} = 0\}$ and denote $W^{m,2}(D)$ $(W_0^{m,2}(D))$ by $H^m(D)$ $(H_0^m(D))$ with norm $\|\cdot\|_{m,D}$ and semi-norm $|\cdot|_{m,D}$. In addition, c or C denotes a positive constant independent of the sizes of the finite elements and micro-structure size ϵ.

Consider the model problem:

$$\begin{cases} -\nabla\cdot(A(x,\frac{x}{\epsilon})\nabla u^\epsilon) = f(x) \quad in\ D \\ u^\epsilon|_{\partial D} = 0 \end{cases}, \tag{2.1}$$

where D is a bounded convex domain in R^2 with a Lipschitz boundary ∂D (for simplicity, we only discuss the model problem in R^2, in fact, the conclusions can be extended to R^d $(d > 2)$, ϵ is a small positive number,

$$A(x,Y) = \begin{pmatrix} a_{11}(x,Y) & a_{12}(x,Y) \\ a_{21}(x,Y) & a_{22}(x,Y) \end{pmatrix}$$

such that A is symmetric and

$$c\xi_i\xi_i \le |a_{ij}(x,Y)\xi_i\xi_j| \le C\xi_i\xi_i, \quad \forall\, \xi_i, \xi_j \in R^2, \quad i,j = 1,2. \tag{2.2}$$

Moreover, $a_{ij}(x,Y), f \in L^\infty(D)$ are all Q-periodic in Y, where $Y = x/\epsilon$, $Q = (0,1)\times(0,1)$.

We first introduce more notation. Let $m_Y(v)$ be the integral average of v on Q:

$$m_Y(v) = \frac{1}{|Q|}\int_Q v dY = \int_Q v dY \quad \forall v \in L^2(Q),$$

where $Q = (0,1)\times(0,1)$ is a unit cell which is the referred domain of the micro-structure Q_ϵ in D, and $|Q|$ is the area of Q.

Then, the homogenized bilinear equation of (2.1) reduces to finding $U_0(x) \in H_0^1(D)$ such that (see [4])

$$A_0(U_0, v) = (f, v) \qquad \forall\ v \in H_0^1(D), \tag{2.3}$$

where A_0 is defined by

$$A_0(v,\omega) = (\widetilde{A}\nabla v, \nabla\omega), \quad \forall\, v,\omega \in H_0^1(D), \tag{2.4}$$

with

$$\widetilde{A} = (\widetilde{A}_{ij})_{2\times 2}, \quad \widetilde{A}_{ij} = m_Y(a_{ij} + a_{ik}\frac{\partial N^j}{\partial Y_k}), \tag{2.5}$$

and N^j is the periodic solution of the equation:

$$\frac{\partial}{\partial Y_i}(a_{ik}(x,Y)\frac{\partial N^j(x,Y)}{\partial Y_k}) = -\frac{\partial}{\partial Y_i}a_{ij}(x,Y) \quad in\ Q, \quad \int_Q N^j dY = 0. \tag{2.6}$$

For (2.6), we just want to obtain the solution $N^j(x,Y)$ in the Y-direction. But, unfortunately, there are two variables x, Y in this equation. So, it is very difficult to directly simulate the solution by any numerical method since the coefficient matrix of any numerical scheme is not a constant matrix, but a matrix with parameter x. In order to solve this difficulty, many papers firstly gave a partition of the whole domain, then calculated (2.6) on some fixed points of the given mesh and finally derived a homogenized equation in the same partition. For instance, in ([11]), ([15]), ([13]), cell problems are solved at each quadrature point of every element. Similarly, in some other papers, the vertexes of each element are chosen as centers of unit cells in order to solve (2.6). For these examples, we note that the number of unit cells calculated in the whole domain is $O(n^2)$ if the number of elements in one direction of the partition is n. So, it is obvious that the total calculation expended on the unit cells is huge if n is very big. In order to reduce the calculation on unit cells, in this chapter, we use P_k-interpolation technique for the obtained unit cells in a new mesh, which is not necessary the same as the partition of homogenized equation. That is: perhaps we use two different meshes to simulate the multi-scale problems. The bigger one is for unit cells and the other one is for the homogenized equation. This idea is different from those we have mentioned above. Under the same accuracy as the method in ([11]), from the following Theorem 2.2.1, it is shown that the required number of the unit cells is just $O(n)$ if we use P_2-interpolation. So, it is obvious that we can reduce the calculation of unit cells.

Theorem 2.2.1 *Let $\rho(x)$ be a function satisfying $\rho \in W^{k+1,\infty}(D)$, and let the P_k-interpolation of $\rho(x)$ be denoted by $\Pi_k\rho(x)$, then it can be shown that (see [1]),*

$$\|\rho(x) - \Pi_k\rho(x)\|_{s,\infty} = \begin{cases} O(H^{k+1-s-\delta}) & \text{for } \delta > 0, \quad \text{if } k = 1, \\ O(H^{k+1-s}) & \text{if } k \geq 2. \ (s = 0, 1) \end{cases} \tag{2.7}$$

Let T_h be a regular partition of D with elements e with size h_e, and define $h := \max_{e\in T_h} h_e$. Let P_k be the space of polynomials with degree no more than k. Then, from Theorem 2.2.1, we have

$$\|N^j(x, Y) - \Pi_k N^j(x_n, Y)\|_{1,\infty} \leq Ch^k, \qquad j = 1, 2, \quad k \geq 2, \tag{2.8}$$

where x_n is chosen point of T_h.

In addition, set T_H to be another regular partition of D with elements K with size h_K, and define $H := \max_{K\in T_H} h_K$. We define the finite element space to be

$$X_H := \{v \in H_0^1(D) : \ \ v|_K \in P_1(K) \ \ \forall\, K \in T_H\}.$$

From (2.8), the homogenized bilinear equation (2.3) can be turned into

$$\int_D \widetilde{a}(x)\nabla U_0 \cdot \nabla v dx = \int_D f v dx$$

where $\widetilde{a}(x) = (\widetilde{a_{ij}}(x))$, and

$$\widetilde{a_{ij}}(x)|_K = m_Y\left(a_{ij}(x, Y) + a_{im}\frac{\partial}{\partial Y_m}\Pi_k N^j(x_n, Y)\right), \tag{2.9}$$

For any $v, \omega \in X_H$, define the bilinear form:

$$A_H(v, \omega) = \sum_{K\in T_H} \int_K \left(\widetilde{a_{ij}}(x)\frac{\partial v}{\partial x_j}\frac{\partial \omega}{\partial x_i}\right) dx \tag{2.10}$$

Then the homogenized numerical solution is to obtain $U_H \in X_H$ such that

$$A_H(U_H, v) = (f, v), \quad \forall\, v \in X_H. \tag{2.11}$$

Remark 2.2.1 *Our main interest is the numerical approximation to (2.1). Therefore, we assume that the theoretical solution is reasonably regular for the estimates that follow to apply. In particular, it is convenient for our presentation to assume $N^j(x, Y) \in W^{k+1,\infty}(D \times Q)$ (j=1,2), $(k \geq 2)$.*

In the following, we give the important error estimates of this part.

Theorem 2.2.2 *Let $u^\epsilon(x)$ be the solution of the equation (2.1), let U_0 be the solution of (2.3), and*

$$u_1^\epsilon = u_0 + \epsilon u_1 = U_0 + \epsilon N^k \frac{\partial U_0}{\partial x_k}.$$

Assume that D is a smooth domain, $a_{ij}(x, Y) \in W^{1,\infty}(D)$ and $f \in L^2(D)$. Then (See [10]),

$$\|u^\epsilon - u_1^\epsilon\|_{1,D} \leq C\sqrt{\epsilon}\|U_0\|_{3,\infty,D}. \tag{2.12}$$

Remark 2.2.2 *From Theorem 2.2.2, it is easy to see that*

$$\|u^\epsilon - U_0\|_{0,D} \leq \|u^\epsilon - u_1^\epsilon\|_{0,D} + \epsilon\|N^k \frac{\partial U_0}{\partial x_k}\|_{0,D} \leq C\sqrt{\epsilon}\|U_0\|_{3,\infty,D}. \tag{2.13}$$

Assume that $\widetilde{u}_0$ is the exact solution of the following equation:

$$\widetilde{a}(\widetilde{u}_0, v) := \sum_{K \in T_H} \int_K \left(\widetilde{a}(x)\nabla\widetilde{u}_0 \cdot \nabla v\right) dx = \int_D f v dx, \quad \forall v \in H_0^1(D). \tag{2.14}$$

Let $N_{h_0}^j$ be the numerical solution of $N^j(x_0, Y)$ in Y-direction, where x_0 is a fixed point. It is known that the contribution of the error estimate $\|N^j - N_{h_0}^j\|_{1,D}$ is very small relative to the error estimates of this part and can be neglected. So, the error $\|N^j - N_{h_0}^j\|_{1,D}$ is not considered in the following part.

Next, we will give the error estimation between u_0 and $\widetilde{u}_0$. First, we present some useful lemmas.

Lemma 2.2.1 *Assume that $N^j(x, Y)$ is the solution of (2.6), satisfying $N^j(x, Y) \in W^{k+1,\infty}(D)$ $(k \geq 2)$. Let $\widetilde{A}_{ij}(x)$ and $\widetilde{a_{ij}}(x)$ be as defined in (2.5), and (2.9), respectively. Then, we have*

$$\|\widetilde{A}_{ij}(x) - \widetilde{a_{ij}}(x)\|_{0,\infty,D} \leq Ch^k. \tag{2.15}$$

Proof: From Theorem 2.2.1, it follows that there exists a positive constant C, such that

$$\max_{K \in T_H} \|N^j(x, Y) - \Pi_k N^j(x_n, Y)\|_{0,\infty,K} \leq \|N^j(x_n, Y) - \Pi_k N^j(x, Y)\|_{0,\infty,D} \leq Ch^k.$$

Using Minkowski's Integral Inequality, a direct calculation gives,

$$\begin{aligned}
\|\widetilde{A}_{ij}(x) - \widetilde{a_{ij}}(x)\|_{0,\infty,D} &= \max_{K\in T_H} \|m_Y(a_{ik}\frac{\partial}{\partial Y_k}(N^j(x,Y) - \Pi_k N^j(x_n,Y)))\|_{0,\infty,K} \\
&\le \max_{K\in T_H} m_Y(\|a_{ik}\frac{\partial}{\partial Y_k}(N^j(x,Y) - \Pi_k N^j(x_n,Y))\|_{0,\infty,K}) \\
&\le \max_{K\in T_H} m_Y(\|a_{ik}\|_{0,\infty,K} \cdot \frac{\partial}{\partial Y_k}\|N^j(x,Y) - \Pi_k N^j(x_n,Y)\|_{0,\infty,K}) \\
&\le Ch^k.
\end{aligned}$$

So, (2.15) is shown.

□

Lemma 2.2.2 *Assume that $A_0(u,v)$ is defined as in (2.4) and satisfies the inf-sup condition. Then for sufficiently small h, we have: $\widetilde{a}(u,v)$ also satisfies the inf-sup condition, that is, there exists a positive constant c, such that*

$$\sup_{0\neq v\in H_0^1} \frac{|\widetilde{a}(u,v)|}{\|v\|_{1,D}} \ge c\|u\|_{1,D}. \tag{2.16}$$

Proof: By using lemma 2.2.1, it can be easily shown that for all $u, v \in H_0^1(D)$,

$$\begin{aligned}
&|A_0(u,v) - \widetilde{a}(u,v)| \\
&= \sum_{K\in T_H} |A_0(u,v) - \widetilde{a}(u,v)|_K \\
&= \sum_{K\in T_H} |\int_K \left((\widetilde{A}(x) - \widetilde{a}(x))\nabla u \cdot \nabla v\right) dx| \\
&\le \sum_{K\in T_H} Ch^k \|\nabla u\|_{0,K}\|\nabla v\|_{0,K} \\
&\le Ch^k\|u\|_{1,D}\|v\|_{1,D}
\end{aligned}$$

Then, for any $v \in H_0^1(D)$ we have

$$\sup_{0\neq v\in H_0^1} \frac{|\widetilde{a}(u,v)|}{\|v\|_{1,D}} \ge \sup_{0\neq v\in H_0^1} \frac{|A_0(u,v)|}{\|v\|_{1,D}} - Ch^k\|u\|_{1,D} \tag{2.17}$$

$A_0(u,v)$ satisfies inf-sup condition, that is, there exists a constant $\widehat{C} > 0$, such that

$$\sup_{0\neq v\in H_0^1} \frac{|A_0(u,v)|}{\|v\|_{1,D}} \ge \widehat{C}\|u\|_{1,D}. \tag{2.18}$$

Combining the above inequalities (2.17) and (2.18), we obtain that for sufficiently small h, there exists a positive constant c, such that

$$\sup_{0\neq v\in H_0^1}\frac{|\widetilde{a}(u,v)|}{\|v\|_{1,D}}\geq(\widehat{C}-Ch^k)\|u\|_{1,D}\geq c\|u\|_{1,D}.$$

Then, this lemma is proved.

□

Based on the Lemma 2.2.1 and Lemma 2.2.2, we give an error estimate as follows.

Theorem 2.2.3 *Assume that the conditions of Lemma 2.2.2 are satisfied, u_0 is the solution of the homogenized equation (2.3) and $\widetilde{u}_0$ is the exact solution of (2.14), then for sufficiently small h, we have*

$$\|u_0-\widetilde{u}_0\|_{1,D}\leq Ch^k\|u_0\|_{1,D} \tag{2.19}$$

Proof: From Lemma 2.2.1 and Lemma 2.2.2, for sufficiently small h, we have

$$\begin{aligned} c\|u_0-\widetilde{u}_0\|_{1,D} &\leq \sup_{0\neq v\in H_0^1}\frac{|\widetilde{a}(u_0-\widetilde{u}_0,v)|}{\|v\|_{1,D}} \\ &\leq \sum_{K\in T_H}\sup_{0\neq v\in H_0^1}\frac{|\int_K(\widetilde{A}(x)-\widetilde{a}(x))\nabla u_0\cdot\nabla v dx|}{\|v\|_{1,K}} \\ &\leq \sum_{K\in T_H}\widetilde{c}h^k\|u_0\|_{1,K} \\ &= \widetilde{c}h^k\|u_0\|_{1,D} \end{aligned}$$

From the inequality above, we obtain that there exists a positive constant C, such that

$$\|u_0-\widetilde{u}_0\|_{1,D}\leq Ch^k\|u_0\|_{1,D}$$

This theorem is proved.

□

Remark 2.2.3 *If all conditions in Theorem 2.2.3 are valid, then for sufficiently small h, from Remark 2.2.2 and Theorem 2.2.3, it is easy to see that*

$$\|u^\epsilon-\widetilde{u}_0\|_{0,D}\leq\|u^\epsilon-u_0\|_{0,D}+\|u_0-\widetilde{u}_0\|_{0,D}\leq(Ch^k\|u_0\|_{1,D}+\sqrt{\epsilon}\|u_0\|_{3,\infty,D}), \tag{2.20}$$

$$\|u^\epsilon - \widetilde{u}_1^\epsilon\|_{1,D} \le \|u^\epsilon - u_1^\epsilon\|_{1,D} + \|u_1^\epsilon - \widetilde{u}_1^\epsilon\|_{1,D} \le (Ch^k\|u_0\|_{1,D} + \sqrt{\epsilon}\|u_0\|_{3,\infty,D}), \quad (2.21)$$

where

$$\widetilde{u}_1^\epsilon = \widetilde{u}_0 + \epsilon \Pi_k N^j(x_n, Y)\frac{\partial \widetilde{u}_0}{\partial x_j}. \quad (2.22)$$

2.3 Superconvergent techniques in multi-scale method

By the standard theory of finite element method and the Nitsche technique, it is very easy to obtain the following theorem.

Theorem 2.3.1 *Let U_H be the numerical solution of problem (2.11), and $\widetilde{u}_0$ be the exact solution of the equation (2.14). Then,*

$$\|\widetilde{u}_0 - U_H\|_{1,D} \le CH\|\widetilde{u}_0\|_{2,D}, \quad (2.23)$$

$$\|\widetilde{u}_0 - U_H\|_{0,D} \le CH^2\|\widetilde{u}_0\|_{2,D}.$$

From Theorem 2.3.1, the error estimate between the exact solution of (2.14) and its numerical approximation has been obtained. In the following, postprocessing techniques in [24], [26] are used to improve the accuracy of multiscale method. In this part, for simplicity, we just give the superconvergent error estimate on a rectangular mesh. In fact, it can be extended successfully to triangular mesh.

Firstly, construct a postprocessing interpolation operator Π_{2H}^m, such that (see [24], [25], [26]):

1) Combining four neighboring elements into a big element, $\tilde{e} = \bigcup_{i=1}^4 e_i$, such that

$$\Pi_{2H}^m \omega \in Q_m(\tilde{e}), \quad \forall\, \omega \in C(\tilde{e}), \quad (2.24)$$

where Q_m is bi-P_m polynomial space.

2)

$$\|\Pi_{2H}^m \omega - \omega\|_l \le CH^{r+1-l}\|\omega\|_{r+1}, \quad 0 \le r \le m, \quad l = 0, 1; \quad (2.25)$$

3)

$$\|\Pi_{2H}^m v\|_l \le C\|v\|_l, \quad \forall\, v \in V^H(D), \quad l = 0, 1, \tag{2.26}$$

where $V^H(D)$ is finite element space.

4)

$$\Pi_{2H}^m \omega^I = \Pi_{2H}^m \omega, \tag{2.27}$$

where $\omega^I \in V^H$ is the finite element interpolation of ω.

In the following, the result of superconvergence in the whole domain is obtained based on the theory of high order interpolation operators.

Theorem 2.3.2 *(see [25]) Let $\widetilde{u}_0$ be the exact solution of the equation (2.14), U_H, u^I be the finite element solution and finite element interpolation of $\widetilde{u}_0$, respectively, and satisfy:*

$$\|U_H - u^I\|_l \le C H^{\alpha+1-l}\|\widetilde{u}_0\|_{m+1}, \quad \alpha > p, \quad m \ge \alpha, \quad l = 0, 1,$$

where p is the order of the finite element polynomial space. Then,

$$\|\Pi_{2H}^m U^H - \widetilde{u}_0\|_l \le C H^{\alpha+1-l}\|\widetilde{u}_0\|_{m+1},$$

where Π_{2H}^m satisfies (2.24), (2.25), (2.26) and(2.27).

It is often more useful and/or necessary to give some superconvergent error estimates in local subdomains.

Theorem 2.3.3 *(See [26]) Let $\widetilde{u}_0$ be the exact solution of the equation (2.14), U_p^H, u_p^I be the finite element solution and finite element interpolation of $\widetilde{u}_0$, respectively, $D_0 \subset\subset D_1 \subset\subset D$. If $\widetilde{u}_0$ is smooth enough and the mesh in D_1 is almost uniform, then,*

$$\|U_p^H - u_p^I\|_{1,D_0} \le C(H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1});$$

$$\|U_p^H - u_p^I\|_{1,\infty,D_0} \le C(H^{p+1}|\ln H|^{\lambda}\|\widetilde{u}_0\|_{p+2,\infty,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}),$$

where p is the order of the finite element polynomial space, s is any non-negtive integer, and

$$\lambda = \begin{cases} 1, & if\ p = 1 \\ 0, & if\ p \ge 2. \end{cases}$$

By using the postprocessing interpolation operator, we have

Theorem 2.3.4 *(See [26]) Under the conditions of Theorem 2.3.3, then*

$$\|\Pi_{2H}^{p+1}U_p^H - \widetilde{u}_0\|_{1,D_0} \le C(H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1});$$

$$\|\Pi_{2H}^{p+1}U_p^H - \widetilde{u}_0\|_{1,\infty,D_0} \le C(H^{p+1}|\ln H|^{\lambda}\|\widetilde{u}_0\|_{p+2,\infty,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}).$$

In the following, we retrieve the microscopic information in the whole domain from $\Pi_{2H}^m U_H$ and give the most important results of this part.

Assume that

$$R(v) = v + \epsilon \Pi_k N^j(x_n, Y)\frac{\partial v}{\partial x_j}, \tag{2.28}$$

Define

$$\overline{u}^\epsilon|_K = R(\Pi_{2H}^m U_H)|_K. \tag{2.29}$$

Theorem 2.3.5 *Let u^ϵ be the solution of (2.1), $\overline{u}^\epsilon$ be given by (2.29). Assume that all conditions of Theorem 2.3.1 are valid. Then,*

$$\|u^\epsilon - \overline{u}^\epsilon\|_{1,D} \le C(\sqrt{\epsilon}\|u_0\|_{3,\infty,D} + h^k\|u_0\|_{1,D} + H^m\|\widetilde{u}_0\|_{m+1,D}). \tag{2.30}$$

Proof: Note that on each element K,

$$\begin{aligned}\frac{\partial \overline{u}^\epsilon}{\partial x_i} &= \frac{\partial}{\partial x_i}\Pi_{2H}^m U_H + (\frac{\partial}{\partial Y_i} + \epsilon\frac{\partial}{\partial x_i})\Pi_k N^j(x_n, Y)\cdot\frac{\partial}{\partial x_j}\Pi_{2H}^m U_H \\ &+\epsilon\Pi_k N^j(x_n, Y)\cdot\frac{\partial^2}{\partial x_i\partial x_j}\Pi_{2H}^m U_H.\end{aligned} \tag{2.31}$$

Furthermore,

$$\frac{\partial \widetilde{u}_1^\epsilon}{\partial x_i} = \frac{\partial \widetilde{u}_0}{\partial x_i} + (\epsilon\frac{\partial}{\partial x_i} + \frac{\partial}{\partial Y_i})\Pi_k N^j(x_n, Y)\frac{\partial \widetilde{u}_0}{\partial x_j} + \epsilon\Pi_k N^j(x_n, Y)\frac{\partial^2 \widetilde{u}_0}{\partial x_i\partial x_j}. \tag{2.32}$$

It follows from (2.31) and (2.32) that

$$\begin{aligned}\frac{\partial}{\partial x_i}(\overline{u}^\epsilon - \widetilde{u}_1^\epsilon) &= \frac{\partial}{\partial x_i}(\Pi_{2H}^m U_H - \widetilde{u}_0) + (\frac{\partial}{\partial Y_i} + \epsilon\frac{\partial}{\partial x_i})\Pi_k N^j(x_n, Y)\cdot\frac{\partial}{\partial x_j}(\Pi_{2H}^m U_H - \widetilde{u}_0) \\ &+\epsilon\Pi_k N^j(x_n, Y)\cdot\frac{\partial^2}{\partial x_i\partial x_j}(\Pi_{2H}^m U_H - \widetilde{u}_0).\end{aligned}$$

From Theorem 2.3.2, we can obtain that

$$\begin{aligned}\|\nabla(\overline{u}^\epsilon - \widetilde{u}_1^\epsilon)\|_{0,D} &\leq C\|\nabla(\Pi_{2H}^m U_H - \widetilde{u}_0)\|_{0,D} + C\epsilon\|\widetilde{u}_0\|_{2,D} \\ &\leq CH^m\|\widetilde{u}_0\|_{m+1,D} + C\epsilon\|\widetilde{u}_0\|_{2,D}.\end{aligned}$$

Moreover,

$$\begin{aligned}\|\overline{u}^\epsilon - \widetilde{u}_1^\epsilon\|_{0,D} &\leq C\|\Pi_{2H}^m U_H - \widetilde{u}_0\|_{0,D} + C\epsilon\|\widetilde{u}_0\|_{1,D} \\ &\leq CH^{m+1}\|\widetilde{u}_0\|_{m+1,D} + C\epsilon\|\widetilde{u}_0\|_{1,D}.\end{aligned}$$

From the inequalities above, it follows that

$$\|\overline{u}^\epsilon - \widetilde{u}_1^\epsilon\|_{1,D} \leq CH^m\|\widetilde{u}_0\|_{m+1,D} + C\epsilon\|\widetilde{u}_0\|_{2,D}.$$

Combining with (2.21), it is easy to obtain (2.30). This proves Theorem 2.3.5.

□

Remark 2.3.1 *In applications, the superconvergent error estimate in a local domain is sometimes more important. By the same method used in proving Theorem 2.3.5 and from Theorem 2.3.4, it follows that*

$$\begin{aligned}\|u^\epsilon - \overline{u}^\epsilon\|_{1,D_0} &\leq C(\sqrt{\epsilon}\|u_0\|_{3,\infty,D_1} + h^k\|u_0\|_{1,D_1} \\ &\quad + H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}),\end{aligned} \tag{2.33}$$

where, $\overline{u}^\epsilon = \Pi_{2H}^{p+1} U_p^H$.

2.4 Superconvergent technique for HMM

In this section, the superconvergent technique will be successfully used in the HMM to reduce its calculations. First, let's recall that the HMM scheme is as follows (see [15]).

Consider the classical problem

$$\begin{cases} -\nabla\cdot(a^\epsilon(x)\nabla u^\epsilon(x)) = f(x) & in\ D \subset \mathcal{R}^d, \\ u^\epsilon(x) = 0 & on\ x \in \partial D. \end{cases} \tag{2.34}$$

In this part, a conventional P_k finite element method on a triangulation T_H of element size H is chosen and we just consider the case $d = 2$. Let A_H be defined as

$$A_H(V,V) = \sum_{K \in T_H} |K| \sum_{x_l \in K} \omega_l (\nabla V \cdot \mathcal{A}_H \nabla V)(x_l), \tag{2.35}$$

where x_l and w_l are the quadrature points and weights in K, $K \in T_H$. In the absence of explicit knowledge of $\mathcal{A}_H(x)$, let

$$(\nabla V \cdot \mathcal{A}_H \nabla V)(x_l) = \frac{1}{\delta^d} \int_{I_\delta(x_l)} \nabla v_l^\epsilon(x) \cdot a^\epsilon(x) \nabla v_l^\epsilon dx, \tag{2.36}$$

where $I_\delta(x_l) = x_l + \delta I$, $I = [0,1]^2$. Here δ is chosen such that a^ϵ restricted to $I_\delta(x_l)$ gives an accurate enough representation of the local variations of a^ϵ, while $v_l^\epsilon(x)$ is the solution of the problem:

$$\begin{cases} -\nabla \cdot (a^\epsilon(x) \nabla v_l^\epsilon(x)) = 0 & in\ I_\delta(x_l), \\ v_l^\epsilon(x) = V_l(x) & on\ \partial I_\delta(x_l), \end{cases} \tag{2.37}$$

where V_l is the linear approximation of V at x_l.

Then, the HMM solution $u_H \in X_H$ is defined by

$$A_H(u_H, V) = (f, V), \quad \forall\, V \in X_H. \tag{2.38}$$

For problem (2.37), set $w_l^\epsilon(x) = v_l^\epsilon(x) - V_l(x)$, then we have

$$\begin{cases} -\nabla \cdot (a^\epsilon(x) \nabla w_l^\epsilon(x)) = \nabla \cdot (a^\epsilon(x) \nabla V_l(x)) & in\ I_\delta(x_l), \\ w_l^\epsilon(x) = 0 & on\ \partial I_\delta(x_l), \end{cases} \tag{2.39}$$

Since $\nabla V_l(x)$ is constant, if $N_j^\epsilon(x)$ satisfy:

$$\begin{cases} -\nabla \cdot (a^\epsilon(x) \nabla N_j^\epsilon(x)) = \frac{\partial}{\partial x_i}(a_{ij}^\epsilon)(x), & in\ I_\delta(x), \\ N_j^\epsilon(x) = 0 & on\ \partial I_\delta(x), \end{cases} \tag{2.40}$$

where $I_\delta(x) = x + \delta I$, then

$$w_l^\epsilon(x) = N_j^\epsilon(x) \frac{\partial V_l(x)}{\partial x_j}.$$

It follows that

$$v_l^\epsilon(x) = V_l(x) + N_j^\epsilon(x) \frac{\partial V_l(x)}{\partial x_j} \tag{2.41}$$

Let T_h, h be defined as in section 2. Assume that $N_j^\epsilon(x) \in W^{k+1,\infty}(D)$, then from Theorem 2.2.1, we have

$$\|N_j^\epsilon(x) - \Pi_k N_j^\epsilon(x)\|_{1,\infty} \leq Ch^k, \quad j = 1, 2, \quad k \geq 2. \tag{2.42}$$

Set

$$\widetilde{\mathcal{A}}_H(V,V) = \sum_{K \in T_H} |K| \sum_{x_l \in K} \omega_l (\nabla V \cdot \widetilde{\mathcal{A}}_H \nabla V)(x_l), \tag{2.43}$$

where

$$(\nabla V \cdot \widetilde{\mathcal{A}}_H \nabla V)(x_l) = \frac{1}{\delta^d} \int_{I_\delta(x_l)} \nabla \widetilde{v}_l^\epsilon(x) \cdot a^\epsilon(x) \nabla \widetilde{v}_l^\epsilon(x) dx, \tag{2.44}$$

and

$$\widetilde{v}_l^\epsilon(x) = V_l(x) + \Pi_k N_j^\epsilon(x) \frac{\partial V_l(x)}{\partial x_j}. \tag{2.45}$$

Then, the revised HMM solution $U_H \in X_H$ is defined by

$$\widetilde{\mathcal{A}}_H(U_H, V) = (f, V), \quad \forall\, V \in X_H. \tag{2.46}$$

Theorem 2.4.1 *Let $\mathcal{A}_H$ and $\widetilde{\mathcal{A}}_H$ be defined by (2.36) and (2.44), respectively. Then, we have*

$$\max_{x_l \in K} \|\mathcal{A}_H - \widetilde{\mathcal{A}}_H\| \le Ch^k. \tag{2.47}$$

Proof: From inequalities (2.41), (2.42) and (2.45), it is easy to obtain that

$$\|\nabla v_l^\epsilon(x) - \nabla \widetilde{v}_l^\epsilon(x)\|_{0,I_\delta(x_l)} \le Ch^k \|\nabla V_l(x)\|_{0,I_\delta(x_l)}.$$

So, from (2.36) and (2.44), it follows that

$$\begin{aligned}
&|\nabla W(x_l)(A_H - \widetilde{\mathcal{A}}_H)\nabla V(x_l)| \\
&= |\frac{1}{\delta^d} \int_{I_\delta(x_l)} \Big(\nabla w_l^\epsilon(x) \cdot a^\epsilon(x) \nabla v_l^\epsilon(x) - \nabla \widetilde{w}_l^\epsilon(x) \cdot a^\epsilon(x) \nabla \widetilde{v}_l^\epsilon(x) \Big) dx| \\
&= |\frac{1}{\delta^d} \int_{I_\delta(x_l)} (\nabla w_l^\epsilon(x) - \nabla \widetilde{w}_l^\epsilon(x)) \cdot a^\epsilon(x) (\nabla v_l^\epsilon(x) - \nabla \widetilde{v}_l^\epsilon(x)) dx \\
&+ \frac{1}{\delta^d} \int_{I_\delta(x_l)} (\nabla w_l^\epsilon(x) - \nabla \widetilde{w}_l^\epsilon(x)) \cdot a^\epsilon(x) \nabla \widetilde{v}_l^\epsilon(x) dx \\
&+ \frac{1}{\delta^d} \int_{I_\delta(x_l)} \nabla \widetilde{w}_l^\epsilon(x) \cdot a^\epsilon(x) (\nabla v_l^\epsilon(x) - \nabla \widetilde{v}_l^\epsilon(x)) dx| \\
&\le C \Big(h^{2k} \|\nabla W(x_l)\|_{0,I_\delta(x_l)} \|\nabla V(x_l)\|_{0,I_\delta(x_l)} + h^k \|\nabla W(x_l)\|_{0,I_\delta(x_l)} \|\nabla \widetilde{v}_l^\epsilon\|_{0,I_\delta(x_l)} \\
&+ h^k \|\nabla \widetilde{w}_l^\epsilon(x_l)\|_{0,I_\delta(x_l)} \|\nabla V_l(x)\|_{0,I_\delta(x_l)} \Big) \\
&\le Ch^k \|\nabla W(x_l)\|_{0,I_\delta(x_l)} \|\nabla V(x_l)\|_{0,I_\delta(x_l)}.
\end{aligned}$$

The inequality above give the desired result (2.47).

□

The homogenized equation of (2.34) is (see [15]):

$$\begin{cases} -\nabla \cdot (\mathcal{A}(x)\nabla U(x)) = f(x) & in\ D \subset \mathcal{R}^d, \\ U(x) = 0 & on\ x \in \partial D. \end{cases} \tag{2.48}$$

where $\mathcal{A}(x)$ is the homogenized coefficient.

Lemma 2.4.1 *Let*

$$e(\text{HMM}) = \max_{x_l \in K} \|\mathcal{A}(x_l) - \mathcal{A}_H(x_l)\|,$$

then for the periodic homogenization problems (see [15]),

$$e(\text{HMM}) \le \begin{cases} C\delta, & if\ \delta\ is\ an\ integer\ multiple\ of\ \epsilon, \\ C(\epsilon/\delta + \delta), & if\ \delta\ is\ not\ an\ integer\ multiple\ of\ \epsilon. \end{cases} \tag{2.49}$$

Theorem 2.4.2 *Assume that u^ϵ is the exact solution of the problem (2.34), U_0 is the exact solution of equation (2.48), $\widetilde{U}_0$ is the exact solution of (2.46) with the space X_H replaced by H_0^1. Moreover, set $a^\epsilon(x) = a(x, x/\epsilon)$. Then we have*

$$\|u^\epsilon - \widetilde{U}_0\|_{0,D} \le C(\sqrt{\epsilon} + h^k + e(\text{HMM})), \tag{2.50}$$

$$\|u^\epsilon - \widetilde{u}_1^\epsilon\|_{1,D} \le C(\sqrt{\epsilon} + h^k + e(\text{HMM})), \tag{2.51}$$

where $\widetilde{u}_1^\epsilon = \widetilde{U}_0 + \Pi_k N_j^\epsilon \frac{\partial \widetilde{U}_0}{\partial x_j}$.

Proof: From (2.47) and (2.49), it follows that

$$\max_{x_l \in K} \|\mathcal{A}(x_l) - \widetilde{\mathcal{A}}_H(x_l)\| \le C(h^k + e(\text{HMM})). \tag{2.52}$$

In view of (2.44), (2.48) and (2.52), we have

$$\begin{aligned} c\|U_0 - \widetilde{U}_0\|_{1,D}\|W\|_{1,D} &\le |A(U_0 - \widetilde{U}_0, W)| \\ &= |A(U_0, W) - A(\widetilde{U}_0, W)| \\ &= |A(U_0, W) - (A - \widetilde{A}_H)(\widetilde{U}_0, W) - \widetilde{A}_H(\widetilde{U}_0, W)| \\ &= |(f, W) - (A - \widetilde{A}_H)(\widetilde{U}_0, W) - (f, W)| \\ &= |(A - \widetilde{A}_H)(\widetilde{U}_0, W)| \\ &\le C(h^k + e(\text{HMM}))\|\widetilde{U}_0\|_{1,D}\|W\|_{1,D}. \end{aligned}$$

So,

$$\|U_0 - \widetilde{U}_0\|_{1,D} \le C(h^k + e(\text{HMM}))\|\widetilde{U}_0\|_{1,D}$$

Hence,

$$\|u^\epsilon - \widetilde{U}_0\|_{0,D} \le \|u^\epsilon - U_0\|_{0,D} + \|U_0 - \widetilde{U}_0\|_{0,D} \le C(\sqrt{\epsilon} + h^k + e(\text{HMM})).$$

In addition, if $a^\epsilon(x) = a(x, x/\epsilon)$, then we have $N_j^\epsilon(x) = \epsilon N^j(x)$.

So, we can obtain,

$$\begin{aligned}
\|u_1^\epsilon - \widetilde{u}_1^\epsilon\|_{1,D} &\le \|U_0 - \widetilde{U}_0\|_{1,D} + \|N_j^\epsilon \frac{\partial U_0}{\partial x_j} - \Pi_k N_j^\epsilon \frac{\partial \widetilde{U}_0}{\partial x_j}\|_{1,D} \\
&\le \|U_0 - \widetilde{U}_0\|_{1,D} + \|(N_j^\epsilon - \Pi_k N_j^\epsilon)\frac{\partial U_0}{\partial x_j}\|_{1,D} + \|\Pi_k N_j^\epsilon \cdot \frac{\partial}{\partial x_j}(U_0 - \widetilde{U}_0)\|_{1,D} \\
&\le C(\sqrt{\epsilon} + h^k + e(\text{HMM})).
\end{aligned}$$

Then this theorem is proved.

□

As theorem 2.3.1, we can have

Theorem 2.4.3 *Let U_H be the numerical solution of problem (2.46), and $\widetilde{u}_0$ be the exact solution of the equation (2.46) with X_H replaced by $H_0^1(D)$. Then (see [1]),*

$$\|\widetilde{u}_0 - U_H\|_{1,D} \le CH\|\widetilde{u}_0\|_{2,D}, \tag{2.53}$$

$$\|\widetilde{u}_0 - U_H\|_{0,D} \le CH^2\|\widetilde{u}_0\|_{2,D}.$$

Next, superconvergent techniques are applied to HMM to improve its accuracy.

Firstly, define a postprocessing interpolation operator Π_{2H}^m that satisfies all the conditions (2.24), (2.25), (2.26) and (2.27). Then the result of superconvergence in the whole domain follows

Theorem 2.4.4 *(see [25]) Let $\widetilde{u}_0$ be the exact solution of equation (2.46) with X_H replaced by $H_0^1(D)$, and U_H, u^I be the finite element solution and finite element interpolation of $\widetilde{u}_0$, respectively, assumed to satisfy:*

$$\|U_H - u^I\|_l \le CH^{\alpha+1-l}\|\widetilde{u}_0\|_{m+1}, \quad \alpha > p, \quad m \ge \alpha, \quad l = 0, 1,$$

where p is the order of the finite element polynomial space. Then,

$$\|\Pi_{2H}^m U^H - \widetilde{u}_0\|_l \le CH^{\alpha+1-l}\|\widetilde{u}_0\|_{m+1}.$$

Concurrently, we have some superconvergent error estimates in local domains.

Theorem 2.4.5 *(See [26]) Let $\widetilde{u}_0$ be the exact solution of equation (2.46) with X_H replaced by $H_0^1(D)$, and U_p^H, u_p^I be the finite element solution and finite element interpolation of $\widetilde{u}_0$, respectively, $D_0 \subset\subset D_1 \subset\subset D$. If $\widetilde{u}_0$ is smooth enough and the mesh in D_1 is almost uniform, then,*

$$\|U_p^H - u_p^I\|_{1,D_0} \le C(H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1});$$

$$\|U_p^H - u_p^I\|_{1,\infty,D_0} \le C(H^{p+1}|\ln H|^\lambda\|\widetilde{u}_0\|_{p+2,\infty,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}),$$

where p is the order of finite element polynomial space, s is any non-negtive integer, and

$$\lambda = \begin{cases} 1, & if\ p=1 \\ 0, & if\ p \ge 2. \end{cases}$$

By using the postprocessing interpolation operator, we have

Theorem 2.4.6 *(See [26]) Under the condition of Theorem 2.4.5, then*

$$\|\Pi_{2H}^{p+1} U_p^H - \widetilde{u}_0\|_{1,D_0} \le C(H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1});$$

$$\|\Pi_{2H}^{p+1} U_p^H - \widetilde{u}_0\|_{1,\infty,D_0} \le C(H^{p+1}|\ln H|^\lambda\|\widetilde{u}_0\|_{p+2,\infty,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}).$$

In the following, we retrieve the microscopic information in the whole domain from $\Pi_{2H}^m U_H$ and give the most important results of this part.

Assume that

$$R(v) = v + \Pi_k N_j^\epsilon(x)\frac{\partial v}{\partial x_j}, \tag{2.54}$$

Define

$$\overline{u}^\epsilon|_K = R(\Pi_{2H}^m U_H)|_K. \tag{2.55}$$

Theorem 2.4.7 *Let u^ϵ be the solution of (2.34), $\overline{u}^\epsilon$ be given by (2.55). Assume that all conditions of Theorem 2.4.3 are valid. Then,*

$$\|u^\epsilon - \overline{u}^\epsilon\|_{1,D} \le C(\sqrt{\epsilon}\|u_0\|_{3,\infty,D} + h^k\|u_0\|_{1,D} + H^m\|\widetilde{u}_0\|_{m+1,D}). \tag{2.56}$$

Proof: Note that on each element K,

$$\begin{aligned}\frac{\partial \overline{u}^\epsilon}{\partial x_i} &= \frac{\partial}{\partial x_i}\Pi_{2H}^m U_H + \frac{\partial}{\partial x_i}\Pi_k N_j^\epsilon(x) \cdot \frac{\partial}{\partial x_j}\Pi_{2H}^m U_H \\ &+\Pi_k N_j^\epsilon(x) \cdot \frac{\partial^2}{\partial x_i \partial x_j}\Pi_{2H}^m U_H. \end{aligned} \tag{2.57}$$

Furthermore,

$$\frac{\partial \widetilde{u}_1^\epsilon}{\partial x_i} = \frac{\partial \widetilde{u}_0}{\partial x_i} + \frac{\partial}{\partial x_i}\Pi_k N_j^\epsilon(x)\frac{\partial \widetilde{u}_0}{\partial x_j} + \Pi_k N_j^\epsilon(x)\frac{\partial^2 \widetilde{u}_0}{\partial x_i \partial x_j}. \tag{2.58}$$

It follows from (2.57) and (2.58) that

$$\begin{aligned}\frac{\partial}{\partial x_i}(\overline{u}^\epsilon - \widetilde{u}_1^\epsilon) &= \frac{\partial}{\partial x_i}(\Pi_{2H}^m U_H - \widetilde{u}_0) + \frac{\partial}{\partial x_i}\Pi_k N_j^\epsilon(x) \cdot \frac{\partial}{\partial x_j}(\Pi_{2H}^m U_H - \widetilde{u}_0) \\ &+\Pi_k N_j^\epsilon(x) \cdot \frac{\partial^2}{\partial x_i \partial x_j}(\Pi_{2H}^m U_H - \widetilde{u}_0).\end{aligned}$$

From Theorem 2.4.4 and $N_{lj}^\epsilon(x) = O(\epsilon)$, it follows that

$$\begin{aligned}\|\nabla(\overline{u}^\epsilon - \widetilde{u}_1^\epsilon)\|_{0,D} &\leq C\|\nabla(\Pi_{2H}^m U_H - \widetilde{u}_0)\|_{0,D} + C\epsilon\|\widetilde{u}_0\|_{2,D} \\ &\leq CH^m\|\widetilde{u}_0\|_{m+1,D} + C\epsilon\|\widetilde{u}_0\|_{2,D}.\end{aligned}$$

Moreover,

$$\begin{aligned}\|\overline{u}^\epsilon - \widetilde{u}_1^\epsilon\|_{0,D} &\leq C\|\Pi_{2H}^m U_H - \widetilde{u}_0\|_{0,D} + C\epsilon\|\widetilde{u}_0\|_{1,D} \\ &\leq CH^{m+1}\|\widetilde{u}_0\|_{m+1,D} + C\epsilon\|\widetilde{u}_0\|_{1,D}.\end{aligned}$$

From the inequalities above, it follows that

$$\|\overline{u}^\epsilon - \widetilde{u}_1^\epsilon\|_{1,D} \leq CH^m\|\widetilde{u}_0\|_{m+1,D} + C\epsilon\|\widetilde{u}_0\|_{2,D}.$$

Combining with (2.51), it is easy to obtain (2.56). This proves Theorem 2.4.7.

□

Remark 2.4.1 *In some cases, the superconvergent error estimate in a local domain is more important. By the same method as in Theorem 2.4.7 and from Theorem 2.4.6, it follows that*

$$\begin{aligned}\|u^\epsilon - \overline{u}^\epsilon\|_{1,D_0} &\leq C(\sqrt{\epsilon}\|u_0\|_{3,\infty,D_1} + h^k\|u_0\|_{1,D_1} \\ &+H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}),\end{aligned} \tag{2.59}$$

where, $\overline{u}^\epsilon = \Pi_{2H}^{p+1} U_p^H$.

2.5 Discussion

In this chapter, we have discussed superconvergent techniques in multi-scale methods, especially in HMM. For simplicity, we assumed that the conditions of the model problems are smooth enough to derive the reasonable error estimates. In practical problems, some of these conditions can't be satisfied. But we can still use this method by some other retrieving techniques, such as error expansion and defect correction. In the future work, we plan to do some research on its application in practical problems.

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Bibliography

[1] I. Babuška, *Homogenization and its applications, mathematical and computational problems,* Numerical Solutions of Partial Differential Equations-III, (SYNSPADE 1975, College Park MD, May 1975), (B. Hubbard ed.), Academic Press, New York, 1976, pp. 89-116.

[2] I. Babuška, T. Strouboulis, S. K. Gangaraj, and C. S. Upadhyay $\eta\%$ *superconvergence in the interior of locally refined meshes of quadrilaterals: superconvergence of the gradient in finite element solutions of Laplace's and Poisson's equations*, Applied Numerical Mathematics, 122, 273-305(1994).

[3] I. Babuška, T. Strouboulis, and C. S. Upadhyay, $\eta\%$ *superconvergence of finite element approximations in the interior of meshes of triangles,* Computer Methods in Applied Mechanics and Engineering, 16, 3-49(1994).

[4] A. Bensoussan, J. L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam, New-York, Oxford, 1978.

[5] J. F. Bourgat, *Numerical experiments of the homogenization method for operators with periodic coefficients*, Lecture Notes in Mathematics, Vol. 707. 330-356.

[6] A. Brandt, *Multi-level adaptive solutions to boundary-value problems*, Math. Comp., 31(1977), 333-390.

[7] F. Brezzi, L. P. Franca, T. J. R. Hughes and A. Russo, $b = \int gdx$, Comput. Meth. Appl. Mech. Engrg., 145(1997), 329-339.

[8] L. Cao, J. Cui and J. Luo, *Multiscale asymptotic expansion and a post-processing algorithm for second-order elliptic problems with highly oscillatory coefficients*

over general convex domains, Journal of Computational and Applied Mathematics, 157(2003) 1-29.

[9] M. Dorobantu and B. Engquist, *Wavelet-based numerical homogenization*, SIAM J. Numer. Anal., 35(1998), 540-559.

[10] J. Douslas and T. Dupont, *Some superconvergence results for Galerkin methods for the approximate solution of two-point boundary value problems*, Topics in Numerical Analysis, Academic Press, pp. 89-92(1973).

[11] W. E, B. Engquist, X. Li, W. Ren, E. Vanden-Eijnden, *The Heterogeneous Multiscale Method: A Review*. http://www.math.princeton.edu/multiscale/

[12] W. E, *Analysis of the heterogeneous multiscale method for ordinary differential equations*, Comm. Math. Sci., **1(3)**, 423-436 (2003).

[13] W. E and B. Engquist, *The Heterogeneous Multi-scale Methods,* Comm. Math. Sci., 1, 87-133 (2003).

[14] W. E, B. Engquist, *Multiscale modeling and computation*, Notice Amer. Math. Soc., 50(2003), 1062-1070.

[15] Weinan E, Pingbing Ming, Pingwen Zhang *Analysis of the heterogeneous multiscale method for elliptic homogenization problems.* J. Amer. Math. Soc. 18 (2005), no. 1, 121–156.

[16] Y. R. Efendiev, T. Hou and X. Wu, *Convergence of a nonconforming multiscale finite element method*, SIAM J. Numer. Anal., 37(2000), 888-910.

[17] C. Farhat. I. Harari and L. P. Franca, *The discontinuous enrichment method*, Comput. Meth. Appl. Mech. Engrg., 190(2001), 6455-6479.

[18] J. Fish and V. Belsky, *Multigrid method for a periodic heterogeneous medium, Part I: Convergence studies for one-dimensional case*, Comput. Meth. Appl. Mech. Engrg., 126(1995), 1-16.

[19] J. Fish and V. Belsky, *Multigrid method for a periodic heterogeneous medium, Part I: Multi-scale modeling and quality in multi-dimensional case*, Comput. Meth. Appl. Mech. Engrg., 126(1995), 17-38.

[20] J. Fish and Z. Yuan, *Multiscale enrichment based on partition of unity*, Inter. J. Numer. Meth. Engrg., 62(2005), 1341-1359.

[21] T. Y. Hou and X. H. Wu, *A multiscale finite element method for elliptic problems in composite materials and porous media,* J. Comput. Phys., **134**(1997), 169-189.

[22] T. Y. Hou, X. H. Wu and Z. Cai, *Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating cofficients,* Math. Comput., **68**(1999), 913-943.

[23] T. J. R. Hughes, *Multiscale phenomena: Green's functions, the Dirichlet to Neumann formulation, subgrid scale models, bubbles and the origin of stabilized methods*, Comput. Meth. Appl. Mech. Engrg., 127(1995), 387-401.

[24] Q. Lin, *Global error expansion and superconvergence for higher order interpolation of finite elements*, J. Comp. Math., Supplementary Issue, 286-289(1992).

[25] Q. Lin, N. N. Yan, A. Zhou, *A rectangle test for interpolated finite elements* Proc. Sys. Sci. & Sys. Engrg., Great wall Culture Publ. Co., Hong Kong, 217-229(1991).

[26] Q. Lin, N. N. Yan, *Construction and Analysis for Efficient Finite Element Methods*, Publishing of Hebei University (in Chinese), 1996.

[27] J. D. Moulton, J. E. Dendy and J. M. Hyman, *The black box multigrid numerical homogenization algorithm*, J. Comput. Phys., 141(1998), 1-29.

[28] N. Neuss, W. Jäger and G. Wittum, *Homoginization and multigrid*, Computing., 66(2001), 1-26.

[29] A. H. Schatz and L. B. Wahlbin, *Superconvergence in finite element methods and meshes which are locally symmetric with respect to a point*, SIAM J. on Numer. Anal, 33, 505-521(1996).

[30] A. H. Schatz, *Pointwise error estimates and asymptotic expansion error inequalities for the finite element mehtod on irregualr grids, Part I: global estimates*, Math of Compt, 67, 877-899(1998).

[31] O. C. Zienkiewicz and J. Z. Zhu, *The superconvergent patch recovery and a posteriori error estimates. Part 1: The recovery technique*, Internation J. of Numer. Methods in Engineering, 33, 1331-1364(1992a).

[32] O. C. Zienkiewicz and J. Z. Zhu, *The superconvergent patch recovery and a posteriori error estimates. Part 2: Error estimates and adaptivity*, Internation J. of Numer. Methods in Engineering, 33, 1365-1382(1992b).

[33] O. C. Zienkiewicz and J. Z. Zhu, *The superconvergent patch recovery(SPR) and adaptive finite element refinement*, Computer Methods in Applied Methanics and Engineering, 101, 207-224(1992c).

Chapter 3

A Multiscale Method for Semi Linear Elliptic Equations

3.1 Introduction

It is well known that nonlinear problems with microstructure occur in many scientific and engineering applications. These include: material science, porous media, turbulent transport in high Reynolds number flows, etc. These problems are characterized by a great number of spatial and time scales, consequently it is difficult to simulate the solution numerically using standard finite element methods. In recent years, approaches for solving nonlinear equations by multiscale finite element methods or multiscale finite volume methods based on the homogenization method have been studied extensively, see [2], [3], [4], [5], [6], [7], etc. In these papers, the first step is to find an efficient homogenization of the original problem. It is know that it is difficult to find the homogenization equation for complex nonlinear systems. Furthermore, for different problems, the homogenization equations are different. In this paper, we present an efficient multiscale finite element method, based on the theory of upper and lower solutions, which reduces the solution of the original nonlinear problems to that of a finite calculable number of linear equations, thus bypassing the difficulties of dealing with the nonlinear case. Furthermore the method can be used to find, for example, positive solution for problems that also admit the zero solution. We point out that not all nonlinear problems can be treated in this manner. In particular, we implicitly deal with nonlinearities that are sublinear at infinity to ensure the existence of the upper/lower solution.

Finally, we explicitly consider, for simplicity of presentation, only the situation where the first order corrector is connected in the ϵ-approximation. This means that we only need to calculate the solution N^j once. We comment that not all constants can be estimated, but this is no worse that the situation when homogenization is applied to linear problems, since as mentioned above the number of nonlinear problems can be estimated.

The outline of this paper is as follows. In the next section, we introduce the model problem and recall the basic theory of upper and lower solutions. In section 3, a multiscale method based on the method of upper and lower solutions is provided, and the error estimate between the exact solution and the asymptotic expansion of order one is presented. Based on the results provided in section 3, we present the principal results of this paper in section 4: the error estimates between the exact solution and the approximation of multiscale method are provided. Some numerical examples demonstrating our theoretical results are shown in Section 5. In the last section, we discuss briefly some possible extensions of the completion presented.

3.2 A semi linear model and the method of upper and lower solutions

In this paper, assume that D is a convex bounded domain with Lipschitz continuous boundary. Moreover, we adopt the standard notation: $W^{m,p}(D)$ for Sobolev spaces on D with norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. Set $W_0^{m,p} \equiv \{\omega \in W^{m,p}(D) : \omega|_{\partial D} = 0\}$ and denote $W^{m,2}(D)$ $(W_0^{m,2}(D))$ by $H^m(D)$ $(H_0^m(D))$ with norm $\|\cdot\|_{m,D}$ and semi-norm $|\cdot|_{m,D}$. In addition, c or C denotes a positive constant independent of the sizes of the finite elements and micro-structure size ϵ.

Consider the nonlinear multiscale model:

$$\begin{cases} -L^\epsilon u^\epsilon + c(x/\epsilon)u^\epsilon = f(x, u^\epsilon) & in\ D \\ u^\epsilon|_{\partial D} = 0 \end{cases}, \tag{3.1}$$

where L^ϵ is a symmetric operator given by:

$$L^\epsilon u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(\frac{x}{\epsilon})\frac{\partial u}{\partial x_j}\right). \tag{3.2}$$

We assume that L^ϵ is uniformly elliptic in $\overline{D}$, that is: $a_{ij}(\frac{x}{\epsilon})$ satisfies:

$$C_\xi \xi_i \xi_i \le a_{ij}(\frac{x}{\epsilon})\xi_i \xi_j \le C\xi_i \xi_i, \quad \forall\, \xi_i, \xi_j \in \mathcal{R}^n, \quad i,j = 1,2,...n, \tag{3.3}$$

where the Einstein notation has been used.

To simplify the technical details, we assume that D, a_{ij}, c, f are smooth and $c \ge 0$. The results hold in more general situation with obvious changes, for example: the conditions on f need only be postulated in the order interval determined by the upper/lower solution, c could be negative (depending on the least eigenvalue of $-L^\epsilon$ with Dirichlet conditions), etc. We point out that all equations are to be understood in the usual work sense, and denote by $B(\cdot,\cdot)$ the quadratic form associated with the left hand side of (3.1).

We then recall, based on results of [9] and elsewhere:

Definition 3.2.1 *A function $\widetilde{u} \in C^\alpha(\overline{D}) \bigcap H^1_I(D)$ is called an upper solution of (3.1) if*

$$\begin{cases} -L^\epsilon \widetilde{u} + c(x/\epsilon)\widetilde{u} \ge f(x,\widetilde{u}) & in\ D \\ \widetilde{u}|_{\partial D} \ge 0 \end{cases}, \tag{3.4}$$

similarly, a function $\widehat{u} \in C^\alpha(\overline{D}) \bigcap H^1_I(D)$ is a lower solution if it satisfies the reverse inequalities in (3.4).

We also observe that if $\widehat{u} \le \widetilde{u}$, we can construct solutions $\underline{u}$, $\overline{u}$ of (3.1) (with possibly $\underline{u} = \overline{u}$) by considering the pointwise monotone limit of the process to the linear problems:

$$\begin{cases} -L^\epsilon u_\epsilon^{(k)} + c(x/\epsilon)u_\epsilon^{(k)} = f(x, u_\epsilon^{(k-1)}) & in\ D \\ u_\epsilon^{(k)}|_{\partial D} = 0 \end{cases}, \tag{3.5}$$

with $u_\epsilon^{(0)}$ chosen to be either $\widetilde{u}$ or $\widehat{u}$. In the former case, $u_\epsilon^{(k)} \downarrow \overline{u}$, in the latter $u_\epsilon^{(k)} \uparrow \underline{u}$.

Since we need only consider $f(x, \xi(x))$ with $\xi(x)$ in the order interval between $\widehat{u}$, $\widetilde{u}$, we assume without loss of generality that f satisfies a global Lipschitz condition in u with constant K_f. Henceforth, we only explicitly consider the case of $u_\epsilon^{(0)} = \widetilde{u}$ (The often possibilities is treated identically), and by the solution u^ϵ of (3.1) we shall choose $u^\epsilon = \overline{u}$ in the case of multiple solutions. It is important in the sequel to estimate $u^\epsilon - u_\epsilon^{(k)}$ in terms of k, $\widetilde{u}$, $\widehat{u}$, D and the coefficients of (3.1), but not ϵ. We then have

Theorem 3.2.1 *Let u^ϵ solve (3.1), and $u_\epsilon^{(k)}$ solve (3.5), we then have:*

$$\|u^\epsilon - u_\epsilon^{(k)}\|_{0,D} \le \left(\frac{K_f}{\lambda_1}\right)^k \|\widetilde{u} - \widehat{u}\|_{0,D}, \tag{3.6}$$

and

$$\|u^\epsilon - u_\epsilon^{(k)}\|_{1,D} \le \frac{K_f\sqrt{2}}{\sqrt{\min(C_0,\lambda_1)}}\left(\frac{K_f}{\lambda_1}\right)^{k-1} \frac{\|\widetilde{u} - \widehat{u}\|_{0,D}}{\sqrt{\lambda_1 C_0}}, \tag{3.7}$$

where C_0 is the ellipticity constant of a_{ij} and λ_1 is the least eigenvalue of the Dirichlet problem for $-L^\epsilon + c(x/\epsilon)$.

Proof: Observe that

$$B(u^\epsilon - u_\epsilon^{(k)}, u^\epsilon - u_\epsilon^{(k)}) = (f(x,u^\epsilon) - f(x,u_\epsilon^{(k-1)}), u^\epsilon - u_\epsilon^{(k)}). \tag{3.8}$$

From the Poincaré Min-Max Principle, we obtain

$$\lambda_1\|u^\epsilon - u_\epsilon^{(k)}\|_{0,D} \le \|f(x,u^\epsilon) - f(x,u_\epsilon^{(k-1)})\|_{0,D} \le K_f\|u^\epsilon - u_\epsilon^{(k-1)}\|_{0,D}. \tag{3.9}$$

Thus,

$$\|u^\epsilon - u_\epsilon^{(k)}\|_{0,D} \le \left(\frac{K_f}{\lambda_1}\right)^k \|u^\epsilon - \widetilde{u}\|_{0,D} \le \left(\frac{K_f}{\lambda_1}\right)^k \|\widetilde{u} - \widehat{u}\|_{0,D}. \tag{3.10}$$

In the some way,

$$B(u^\epsilon - u_\epsilon^{(k)}, u^\epsilon - u_\epsilon^{(k)}) \ge C_0|u^\epsilon - u_\epsilon^{(k)}|_{1,D}^2,$$

and,

$$B(u^\epsilon - u_\epsilon^{(k)}, u^\epsilon - u_\epsilon^{(k)}) \ge \frac{\min(C_0,\lambda_1)}{2}\|u^\epsilon - u_\epsilon^{(k)}\|_{1,D}^2.$$

Moreover,

$$\begin{aligned}
|B(u^\epsilon - u_\epsilon^{(k)}, u^\epsilon - u_\epsilon^{(k)})| &\le \frac{\|f\|}{\sqrt{\lambda_1 C_0}}\sqrt{\lambda_1 C_0}\|u^\epsilon - u_\epsilon^{(k-1)}\|_{0,D} \\
&\le \frac{\|f\|}{\sqrt{\lambda_1 C_0}}\sqrt{C_0}|u^\epsilon - u_\epsilon^{(k-1)}|_{1,D} \\
&\le \frac{\|f\|}{\sqrt{\lambda_1 C_0}}\sqrt{B(u^\epsilon - u_\epsilon^{(k-1)}, u^\epsilon - u_\epsilon^{(k-1)})}.
\end{aligned}$$

So,

$$\sqrt{\frac{\min(C_0,\lambda_1)}{2}}\|u^\epsilon - u_\epsilon^{(k)}\|_{1,D} \le \frac{K_f}{\sqrt{\lambda_1 C_0}}\|u^\epsilon - u_\epsilon^{(k-1)}\|_{0,D}.$$

Thus

$$\sqrt{\frac{\min(C_0,\lambda_1)}{2}}\|u^\epsilon - u_\epsilon^{(k)}\|_{1,D} \le \frac{K_f}{\sqrt{\lambda_1 C_0}}\|u^\epsilon - u_\epsilon^{(k-1)}\|_{0,D} \le \frac{K_f}{\sqrt{\lambda_1 C_0}}\left(\frac{K_f}{\lambda_1}\right)^{k-1}\|\widetilde{u}-\widehat{u}\|_{0,D}.$$

Then result follows.

□

We observe that all the terms on the right hand side of (3.6) and (3.7) can be explicitly estimated, and they are independent of ϵ. The shortcoming is that we require $K_f < \lambda_1$ for a meaning result. Note also that since u^ϵ, $u_\epsilon^{(k)}$ are uniformly bounded in k by the upper/lower solution, then $u_\epsilon^{(k)} \to u_\epsilon$ in $L^p(D)$ for any p and thus in $C^\alpha(\overline{D})$ for some small $\alpha > 0$, independent of ϵ. However, the value of α and the constants now appear difficult to estimate explicitly.

3.3 A Multiscale Method Based on Upper and Lower Solutions

By the usual linear homogenization approach, we set in (3.5):

$$u_\epsilon^{(k)} \cong \sum_{l=0}^{+\infty} \epsilon^l u_l^{(k)}(x,Y), \qquad k = 0,1,2,... \tag{3.11}$$

with $Y = \frac{x}{\epsilon}$, treated formally as an independent variable. We thus obtain, neglecting terms of order ϵ or higher,

$$-L^\epsilon + c(\frac{x}{\epsilon}) \cong \epsilon^{-2}A_1 + \epsilon^{-1}A_2 + \epsilon^0 A_3, \tag{3.12}$$

with

$$A_1 = -\frac{\partial}{\partial Y_i}[a_{ij}(Y)\frac{\partial}{\partial Y_j}],$$
$$A_2 = -\frac{\partial}{\partial Y_i}[a_{ij}(Y)\frac{\partial}{\partial x_j}] - \frac{\partial}{\partial x_i}[a_{ij}(Y)\frac{\partial}{\partial Y_j}],$$
$$A_3 = -\frac{\partial}{\partial x_i}[a_{ij}(Y)\frac{\partial}{\partial x_j}] + c(Y).$$

By classic results, the linear equation:

$$\begin{cases} -L^\epsilon \omega + c(x/\epsilon)\omega = g(x) & in\ D \\ \omega|_{\partial D} = 0 \end{cases}, \tag{3.13}$$

is then approximated by $\widetilde{\omega} = \omega_0 + \epsilon\omega_1$ with $\omega_1 = N^k \frac{\partial \omega_0}{\partial x_k}$ and N_k, ω_0 are respectively solutions of:

$$\frac{\partial}{\partial Y_i}(a_{ik}(Y)\frac{\partial N^j(Y)}{\partial Y_k}) = -\frac{\partial}{\partial Y_i}a_{ij}(Y) \tag{3.14}$$

with N^j periodic, $\int_Q N^j d_Y = 0$, and:

$$\begin{cases} -\frac{\partial}{\partial x_i}(\widetilde{a}_{ij}\frac{\partial \omega_0}{\partial x_k}) + m_Y(c)\omega_0 = g(x) & in\ D \\ \omega_0|_{\partial D} = 0 \end{cases}, \tag{3.15}$$

with

$$\widetilde{a}_{ij} = m_Y(a_{ij} + a_{ik}\frac{\partial N^j}{\partial Y_k}). \tag{3.16}$$

We recall that if $\frac{\partial N^j}{\partial Y_k} \in L^\infty(Q)$ (this can be ensured by suitable assumptions on a_{ik}, and in particular, is easy to obtain in one dimension), then

$$||\omega - \widetilde{\omega}||_{1,D} \leq C_2\sqrt{\epsilon}||\omega_0||_{2,2,D}.$$

Furthermore, note that both N^j and the coefficients on the left hand side of (3.15) are independent of $g(x)$. It follows from the assumed smoothness of the coefficients that

$$||\omega_0||_{2,2,D} \leq C_3||g||_{2,D}.$$

In summary, we have

$$||\omega - \widetilde{\omega}||_{1,D} \leq C_4\sqrt{\epsilon}||g||_{2,D}. \tag{3.17}$$

The constant C_4 appears difficult to estimate. It depends, in particular, on the shape of D.

Note that (3.17) implies:

$$||\omega - \omega_0||_{0,D} \leq C_5\sqrt{\epsilon}||g||_{2,D}. \tag{3.18}$$

by the assumption on N^j, ω_0.

Assume now that $\widetilde{u}$, the supersolution, can be expanded as $u_0^{(0)}(x) + \epsilon u_1^{(0)}(x,\epsilon)$ with error of order $\sqrt{\epsilon}$. Observe that this will be the case if f is sublinear since $\widetilde{u}$ can be chosen to be a large positive constant. Note also that the subsolution does not play a role (except for estimates (3.6), (3.7)) in this calculation. Consider the sequence of linear problems:

$$\begin{cases} \Big(-L^\epsilon + c(\frac{x}{\epsilon})\Big)v^{(k)} = f(x, v_0^{(k-1)}) & in\ D \\ v^{(k)}|_{\partial D} = 0 \end{cases}, \tag{3.19}$$

where $v_0^{(0)}(x) = u_0^{(0)}(x)$, $v_0^{(k-1)}$ denotes the solution of the homogenized equation for (3.19) if $k > 1$. We estimate the difference between $v^{(k)}$ and the solution $u^{(k)}$ of (3.5) as follows:

Theorem 3.3.1

$$\|u^{(k)} - v^{(k)}\|_{1,D} \leq CM^k\sqrt{\epsilon},$$

for some constants C and M independent of k, ϵ.

Proof: We have

$$\begin{aligned} \|u^{(k)} - v^{(k)}\|_{1,D} &\leq CK_f\|u^{(k-1)} - v_0^{(k-1)}\|_{0,D} \\ &\leq CK_f\{\|u^{(k-1)} - v^{(k-1)}\|_{0,D} + \|v^{(k-1)} - v_0^{(k-1)}\|_{0,D}\}. \quad (3.20) \end{aligned}$$

Now

$$\|u^{(1)} - v^{(1)}\|_{1,D} \leq CK_f\|u^{(0)} - v_0\|_{0,D} \leq MK_f\sqrt{\epsilon},$$

$$\|v^{(k-1)} - v_0^{(k-1)}\|_{0,D} \leq C\sqrt{\epsilon},$$

with C independent of k by (3.18) and the properties of f. Put $z_k = \|u^{(k)} - v^{(k)}\|_{1,D}$. Then (3.20) becomes

$$z_k \leq CK_f\{z_{k-1} + C\sqrt{\epsilon}\}$$

or

$$z_k \leq F\{z_{k-1} + \sqrt{\epsilon}\}$$

with F independent of k, ϵ and $z_1 \leq F\sqrt{\epsilon}$.

The result follows by induction.

□

3.4 Error estimate for the multi-scale finite element method

Let T_h be a regular partition of D with elements K with size h_K, and define $h := \max_{K\in T_h} h_K$. Let P_k be the space of polynomials with degree no more than k. We define the finite element space to be

$$X_h := \{v \in H_0^1(D) : \quad v|_K \in P_1(K) \;\; \forall\, K \in T_h\}.$$

Let $U_{0h}^{(k)} \in X_h$ be the linear finite element solution of the following equation,

$$A_0(U_{0h}^{(k)}, v) = (f(x, U_{0h}^{(k-1)}), v), \qquad \forall\ v \in X_h. \tag{3.21}$$

where $A_0(\cdot, \cdot)$ denotes the form associated with the homogenized equation (3.15) and with $U_{0,h}^{(0)} = (v_0^{(0)})_h$. We estimate the difference between $U_{0,h}^{(k)}$ and $v_0^{(k)}$, the solution of the homogenized equation for (3.19). We have

$$A_0(v_0^{(k)}) = f(x, v_0^{(k-1)})$$

and defining r to be the solution of

$$A_0(r) = f(x, v_0^{(k-1)}) - f(x, U_{0,h}^{(k-1)})$$

yields

$$A_0(v_0^{(k)} - r) = f(x, U_{0,h}^{(k-1)}),$$

where A_0 is the homogenized operator of (3.19).

Thus

$$\|v_0^{(k)} - r - U_{0,h}^{(k)}\|_{1,D} \leq Ch\|v_0^{(k)} - r\|_{2,D} \leq Ch.$$

Due, once again, to the uniform boundedness of f. We then obtain

$$\|v_0^{(k)} - U_{0,h}^{(k)}\|_{1,D} \leq Ch + K_f\|v_0^{(k-1)} - U_{0,h}^{(k-1)}\|_{0,D}.$$

This estimate is identical inform to (3.19) with h replacing $\sqrt{\epsilon}$, and we obtain

$$\|v_0^{(k)} - U_{0,h}^{(k)}\|_{1,D} \leq CM^k h.$$

Assume that

$$R(v) = v + \epsilon N^j(x/\epsilon)\frac{\partial v}{\partial x_j}, \tag{3.22}$$

where, for simplicity, we assume N^j has been calculated exactly.

Define

$$\overline{u}_\epsilon^{(k)}|_K = R(U_{0h}^{(k)})|_K. \tag{3.23}$$

We observe that $\overline{u}_\epsilon^{(k)}$ is discontinuous across the element face. Let a broken H^1-norm be given by:

$$|\overline{u}_\epsilon^{(k)}|_{H,D} = \left(\sum_{K\in T_h} \|\nabla \overline{u}_\epsilon^{(k)}\|_{0,K}^2\right)^{\frac{1}{2}}.$$

We then have the following theorem which is the main result of this section,

Theorem 3.4.1 *Assume that u^ϵ is the exact solution of (3.1), $\overline{u}_\epsilon^{(k)}$ is defined by (3.23), then*

$$\|u^\epsilon - \overline{u}_\epsilon^{(k)}\|_{H,D} \leq \left(\frac{K_f\sqrt{2}}{\min(C_0,\lambda_1)}\right)\left(\frac{K_f}{\lambda_1}\right)^{k-1}\frac{\|\widetilde{u}-\widehat{u}\|_{0,D}}{\sqrt{C_0\lambda_1}} + CM^k(\sqrt{\epsilon}+h). \quad (3.24)$$

with C, M independent of ϵ, h, k.

Proof:

$$\|u^\epsilon - \overline{u}_\epsilon^{(k)}\|_{H,D} \leq \|u^\epsilon - u^{(k)}\|_{1,D} + \|u^{(k)} - v^{(k)}\|_{1,D} + \|v^{(k)} - v_1^{(k)}\|_{1,D} + \|v_1^{(k)} - \overline{u}_\epsilon^{(k)}\|_{H,D}.$$

We need only estimate $\|v_1^{(k)} - \overline{u}_\epsilon^{(k)}\|_{H,D}$. We have

$$\begin{aligned}\|v_1^{(k)} - \overline{u}_\epsilon^{(k)}\|_{H,K} &\leq C\|v_0^{(k)} - U_{0,h}^{(k)}\|_{1,K} + C\epsilon(\|v_0^{(k)}\|_{2,K} + \|U_{0,h}^{(k)}\|_{2,K}) \\ &\leq CM^k h + C\epsilon.\end{aligned}$$

□

3.5 A Numerical example

In this section, we illustrate with a numerical example, the accuracy of the proposed multi-scale method for solving the semi linear problem (3.1). Let now $D = [0,1]$ and recall $Q = [0,1]$ is the unit cell. Consider the nonlinear problem given by:

$$\begin{cases} -(a(\frac{x}{\epsilon})u')' + u = 1 + \frac{u^2}{1+u^2} & in\ (0,1), \\ u = 0, \quad at\ x = 0,\ x = 1. \end{cases} \quad (3.25)$$

where

$$a(\frac{x}{\epsilon}) = \frac{1}{2 + sin(\frac{x}{\epsilon})}.$$

For this case, the solutions, obtained by the standard linear finite element method based on upper and lower solutions, are looked on as the exact solutions, which is used to compare with the numerical solutions obtained by the multi-scale method. That is, we use the finite element method with the mesh $h_0 \ll \epsilon$ to simulate the following problems:

$$\begin{cases} -(a(\frac{x}{\epsilon})(u^{(k)})')' + u^{(k)} = 1 + \frac{(u^{(k-1)})^2}{1+(u^{(k-1)})^2} & in\ (0,1), \\ u^{(k)} = 0, \quad at\ x = 0,\ x = 1. \end{cases} \quad (3.26)$$

It is easy to check that $f(u)$ is Lipschitz continuous and $a(\frac{x}{\epsilon})$ satisfies (3.3). So, we can use the multi-scale method provided in this paper to solve (3.25). Let N be the number of elements in the x-direction and M be the number of elements in ξ-direction in each unit cell, where $\xi = x/\epsilon$. Thus the mesh size in the whole domain is $H = 1/N$. For all of the following cases, let $M = 100$. Then the convergence of this method is as follows.

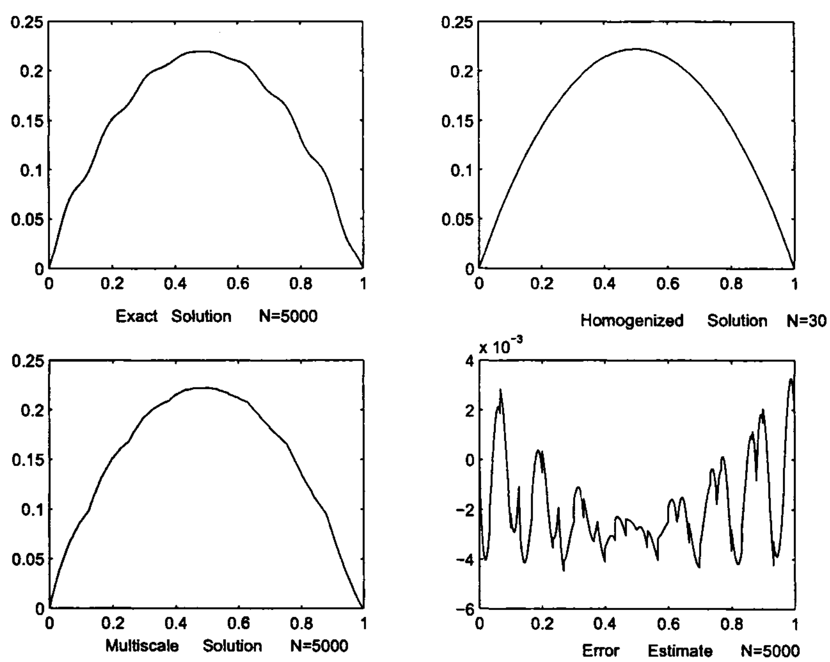
case	Ms-FEM			FEM		error	
	N	ϵ	k	N	k	$\|e\|_{0,D}$	$\|e\|_{1,D}$
case 1	30	0.004	7	5000	9	0.001580	0.147568
case 1	60	0.004	7	5000	9	0.001657	0.144240
case 1	120	0.004	7	5000	9	0.001677	0.143634

Table 1. Convergence for Ms-FEM under the condition $\epsilon \ll H \ll 1$.

case	Ms-FEM			FEM		error	
	N	ϵ	k	N	k	$\|e\|_{0,D}$	rate
case 2	30	0.02	7	5000	9	0.002591	
case 2	30	0.01	7	5000	9	0.001961	0.40
case 2	30	0.005	7	5000	9	0.001533	0.36

Table 2. Convergence for Ms-FEM under the condition $\epsilon \ll H \ll 1$.

In the pictures, case 2 for $\epsilon = 0.02$ is given.



In Tables 1-2 above, Ms-FEM denotes the multi-scale finite element method of this paper. Moreover, let $e = u^\epsilon - \overline{u}_\epsilon^{(k)}$, and $\overline{u}_\epsilon^{(k)}$ is as defined in Section 4. From the results of these two tables, we observe that in case 1 the difference of the error estimates under L^2-norm is not big as H become smaller, and so is it under H^1-norm. This phenomena demonstrates that the most important contributor in the error estimates of (3.24) is the term $\sqrt{\epsilon}$, not the term h^k. That is, the term of h^k goes to zero quickly under a suitable condition as k become bigger. In table 2, H is fixed, the order of the error estimate is around $\frac{1}{2}$ as ϵ become smaller. It proves that this rate is consistent with the order estimate of ϵ in (3.24).

In the following part, some pictures about the numerical results are given.

3.6 Discussion

In this chapter, we mainly discuss the multiscale method for the semi linear problems. In future work, we will try to apply this method to some nonlinear problems although it is difficult to give the error estimate for these problems.

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Bibliography

[1] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[2] Y. Efendiev and A. Pankov, *Numerical homogenization of monotone elliptic operators*, SIAM Multiscale Model. Simul. 1 (2003), 62-79.

[3] Y. Efendiev and A. Pankov, *Numerical homogenization of nonlinear random parabolic operators*, SIAM Multiscale Model. Simul. 2 (2004), 237-268.

[4] Y. Efendiev and A. Pankov, *Numerical homogenization and correctors for nonlinear elliptic equations*, SIAM J. Appl. Math. 65 (2004), 43-68.

[5] Y. Efendiev and A. Pankov, *Homogenization of nonlinear random parabolic operators*, Adv. Differ. Equat. (Accepted).

[6] T. Y. Hou, *Multiscale Modeling and Computation of Incompressible Flow* , Applied Mathematics Entering the 21st Century, Invited Talks from the ICIAM2003 Congress, ed. J. M. Hill and R. Moore, SIAM Publications, 177-210 (2004).

[7] T. Y. Hou, *Numerical Approximations to Multiscale Solutions in Partial Differential Equations* , Frontier in Numerical Analysis , ed. J. F. Blowey, A. W. Craig, and T. Shardlow, Springer Publications, 241-302 (2003).

[8] O. A. Oleinik, A. S. Shamaev and G. A. Yosifian, *Mathematical Problems in Elasticity and Homogenization*, North-Holland, Amsterdam, 1992.

[9] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York and London, 1992.

[10] V. V. Zhikov, S. M. Kozlov and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functions,* Springer-verlag Berlin Heidelberg, 1994.

Chapter 4

Conclusion

In this thesis, we mainly discuss superconvergent techniques in multiscale methods and a multiscale method for semi linear elliptic equations.

In Chapter 2, we successfully use superconvergent techniques to improve the efficiency of some multiscale methods. Moreover, we put forward a method that the points on which unit cells calculated are not related to the mesh of homogenized equations. So, we can greatly reduce the number of the unit cells needed to calculate in the practical problems. In addition, in this chapter, we just use a general superconvergent technique to multiscale method. In fact, for many practical problems, we can use different superconvergent techniques according to the conditions of the problems.

In Chapter 3, a multiscale method for semi linear elliptic equations is discussed. In this chapter, we try to find another efficient method to solve semi linear and nonlinear problems. That is, the first step of it is to find an iteration process to approximation the exact solution of the original problems based on the upper and lower solution. The second step is to retrieve the detailed information of unit cells. This method is same as that of linear problems after using iteration process. So, in this chapter, we don't need to find the nonlinear asymptotic series of the original problems. Moreover, for this method, it can be used to some nonlinear parabolic problems under some suitable conditions.

From above, we can draw a conclusion that by the technique in Chapter 2 or the method in Chapter 3, some practical problems can be solved more efficiently than in the past.