Essays on Esscher Pricing Measures and Densities

by

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Abstract

This thesis investigates the important statistical tool of Esscher transform with its applications in mathematical finance for various environments. This transformation was introduced by F. Esscher and it became very popular afterwards, in both actuarial sciences and finance due to its role in premium calculations and pricing. The thesis discusses the Esscher pricing measure (or equivalently density) for both the discrete and continuous time settings. In discrete time, we describe the Esscher martingale measure for the general case, and we illustrate the results on the two popular models for stock, namely the binomial and trinomial models. In the continuous time framework, we focus on the Black-Scholes model for stock (geometric Brownian motion) only. The innovation of this thesis lies principally in considering two level of informations: The "public" flow information denoted by $\mathbb{F}$ that represents the flow of information available to all agents through time, and a bigger flow of information denoted by $\mathbb{G}$. This latter flow of information incorporates both the flow $\mathbb{F}$ and the information about a death time of an agent τ as it occurs. Thus, for this larger flow of information, we describe the Esscher martingale measures, and the Esscher prices for death-linked contracts. The Greeks of these Esscher prices are derived as well, besides the comparison with the Black-Scholes pricing formula.

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Chapter 1

Introduction

Mathematical finance, also known as quantitative finance, was pioneered by Louis Bachelier, in his Ph.D thesis defended in 1900 at Paris Sorbonne. His stunning work on "Théorie de la Spéculation" (Theory of Speculation) was published in one of the famous French scientific journals called Annales Scientiques de l'école Normale Suprieure. He used the trajectories of Brownian motion for modeling the stock price movements and pricing the European option. In 1973, Fischer Black and Myron Scholes published the paper "The Pricing of Options and Corporate Liabilities" [3] and Robert Merton published the paper "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates" [15]. These two papers introduced the Black-Scholes-Merton formula for pricing European call and put options and it made a very big impact in the field of mathematical finance and the stock and financial derivatives market . In 1975, two years after the Black-Scholes model was introduced, a lot of traders started using it for hedging and valuing the financial derivatives. In 2008, Schachermayer and Teichmann [16] showed that the price comes from the

Black-Scholes-Merton formula and the option pricing formulas derived by Louis Bachelier are equal. Nowadays, Mathematical finance is one of the rapidly developing areas. It is a combination between mathematical analysis and financial economy and focuses on the mathematical techniques from the probability theory, stochastic process and its properties and stochastic differential equations.

Mathematical finance is very important for many financial institutions like banks because they need a very good mathematical skills along with a good financial decisions making ability. It provides the mathematical modeling to price the financial products like derivatives. In the area of mathematical finance, there are many major problems in the financial market such as hedging and pricing different financial products. There are two popular approaches in mathematical finance for option pricing. In the first approach, the price of any financial security equals to an expected value under the risk-neutral probability while in the second approach the price of the option can be calculated by the Black-Scholes formula. There are many important topics in mathematical finance such as risk measures and portfolio management, hedging, volatility, risk management, credit risk, insurance analysis and financial derivatives pricing under the risk-neutral probability measure. In this thesis, we consider the topic of pricing financial products under Esscher measure for models stopped at the death time.

Esscher Transform is one of the useful topics in mathematical finance. This transformation was introduced by the Swedish actuary F. Esscher in 1932 [7], it is named in honor of him. Gerber and Shiu [8] (1994, 1996) used the Esscher transform for pricing some derivatives namely options. In 2002, Kallsen and Shiryaev [11] introduced the Esscher martingale measure for exponential processes and the Esscher martingale transform for linear processes. Esscher transform plays a vital role in the financial market because it is a great tool for pricing many contingent claims (individual risks) and financial derivatives like options. It is mainly a transformation of distribution function $F(x)$. Esscher martingale transform is a popular approach for option pricing for incomplete models. There are two different Esscher martingale transforms for Levy processes depending on the choice of the parameter. The first one makes the ordinary exponential process a martingale while the other one makes the stochastic exponential a martingale. Esscher transform creates an equivalent martingale measure on stock price process. Here are some examples of the stochastic processes: Poisson process, Weiner process (Brownian motion) and the inverse Gaussian process. Brownian motion and Poisson process are the most common examples of the stochastic processes with stationary and independent increments.

A general definition of Esscher transform relies on the change of measure for certain class of stochastic processes for modelling financial security prices, in which the Esscher parameter is determined in such a way that the discounted price process of the primitive securities becomes a martingale under the Esscher measure . In mathematical finance and insurance mathematics areas, Esscher transform is a very popular way to price many financial products. It solves major problems in quantitative finance such as option pricing, asset allocation, and risk management. Esscher transform can be used to price the financial derivatives of asset pools or multiple risky assets. A very notable application of Esscher transform is inventing some hedging strategies to avoid the risk of losses. There are other applications such as the valuation of non risky assets.

This thesis is organized as follows. There are six chapters including the current chapter.

The second chapter recalls some stochastic elements, for both discrete and continuous time. In chapter three and four of this thesis, we address the Esscher pricing measure/density for the discrete time models with a particular illustration on the binomial and trinomial models. In chapter 3, we consider the model, without incorporating the mortality, denoted by (S,$\mathbb{F}$). Here S represents the price process of the underlying risky assets (stocks). In chapter 4, we incorporate the mortality into the model and work with the model (S^τ,$\mathbb{G}$). Here S^τ is the stocks' price process stopped at the death time τ, and $\mathbb{G}$ is the resulting flow of information from the expansion of the flow $\mathbb{F}$ with τ.

In chapter five and six, we discuss the Esscher pricing measure for the continuous time namely for the Brownian motion (Weiner process). In chapter 5, the model of the Black-Scholes for the stock without the mortality is considered. Chapter six incorporates the mortality via considering the stopped Black-Scholes model at the death time with its larger flow of information $\mathbb{G}$. Herein, we describe the Esscher pricing measure, we price some death-linked contracts, and we compare the obtained Esscher under mortality prices with the Black-Scholes prices under no mortality. The Greeks for the Esscher prices are also derived in this chapter.

Chapter 2

Mathematical Tools and Preliminaries

In this chapter, we introduce some of mathematical tools that we used in our studies. This chapter is divided into two sections. The first section recalls mathematical and statistical tools for discrete-time setting, while the second section addresses the continuous-time setting.

Throughout this thesis, we suppose given on a probability space $(\Omega, \mathcal{F}, P)$. Here, Ω is representing all the possible outcomes, $\mathcal{F}$ is σ-algebra of subset of Ω and P represents the probability measure. Below, we recall one of the most useful definitions in the field of mathematics which is the definition of σ-algebra (see [17]).

Definition 2.0.1. *A σ-algebra on Ω is any collection $\mathcal{B}$ of subsets of Ω satisfying the following:*
(i) (Empty set) $\emptyset \in \mathcal{B}$.

(ii) For any $E \in \mathcal{B}$, *then* $E^c := \Omega \setminus E \in \mathcal{B}$.

(iii) For any sequence $(E_n)_{n\geq 1}$ *of elements of* $\mathcal{B}$, *we have* $\cup_{n=1}^{\infty} E_n \in \mathcal{B}$.

Definition 2.0.2. *Let* A *and* B *be two events on the probability space* $(\Omega, \mathcal{F}, P)$ *such that* $P(B) > 0$. *Then*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

In chapter 4, we change the probability measure, and hence we need the following results that can be found in many probability books see for instance [13]

Lemma 2.0.1. *Let* $Q \sim P$ *be equivalent two probability measures on the probability space* $(\Omega, \mathcal{F})$ *with density* Z, X *be a nonnegative random variable or* XZ *is integrable, and* $\mathcal{G}$ *is a sub-*σ*-algebra of* $\mathcal{F}$. *Then*

$$E_Q[X|\mathcal{G}] = \frac{E_P[XZ|\mathcal{G}]}{E_P[Z|\mathcal{G}]}.$$

2.1 Elements from discrete-time stochastic

Throughout this section, we consider the setting of discrete-time. A filtration $\mathbb{F} := (\mathcal{F}_t)_{t=0,\ldots,T}$ is an increasing sequence of σ-algebras

$$\mathcal{F}_0 \subseteq \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}_T, \qquad 0 \leq s \leq t \leq T.$$

Definition 2.1.1. *A process* $X = (X_t)_{t=0,\ldots..,T}$ *is said to be* $\mathbb{F}$*-adapted if* X_t *is* $\mathcal{F}_t$*-measurable for all* $t = 0, \ldots., T$.

Definition 2.1.2. *A process* $X = (X_t)_{t=0,\ldots..,T}$ *is said to be* $\mathbb{F}$*-predictable if* X_t *is* $\mathcal{F}_{t-1}$*-measurable for all* $t = 0, \ldots., T$.

Definition 2.1.3. *Let* $M = (M_t)_{t=0,....,T}$ *be a stochastic process.* M *is a martingale if*

(1) M *is* $\mathbb{F}$*-adapted.*

(2) $E|M_t| < +\infty \quad \forall\ t = 0, ..., T.$

(3) $E[M_t|\mathcal{F}_{t-1}] = M_{t-1}, \quad \forall\ t = 1, ..., T.$

Then, M_t is sub-martingale if and only if (1) and (2) hold, and (3) replaced by

$$E[M_t|\mathcal{F}_{t-1}] \geq M_{t-1}, \quad \text{a.s.} \quad \forall\ t = 1, ..., T.$$

M_t is super-martingale if and only if (1) and (2) hold, and (3) replaced by

$$E[M_t|\mathcal{F}_{t-1}] \leq M_{t-1}, \quad \text{a.s.} \quad \forall\ t = 1, ..., T.$$

Stopping time is one of the important random times in the area of mathematical finance, we state below the definition of it [17]

Definition 2.1.4. *On the probability space* $(\Omega, \mathcal{F}, P)$*, any nonnegative random variable* τ *is a stopping time with respect to the filtration* $\mathbb{F} := (\mathcal{F}_t)_{t=0,...,T}$*, if and only if*

$$\{\tau = t\} \in \mathcal{F}_t, \quad t = 0, ..., T.$$

2.2 Elements from continuous-time stochastic

In this section, we introduce the Brownian motion and its properties. Brownian motion is a continuous-time stochastic process that is widely used in physics and finance for modelling random behaviour over time. It is used in mathematical finance to model stock prices. Below, we define it precisely

Definition 2.2.1. *Let $W = (W_t)_{t\geq 0}$ be a stochastic process. Then W is called a Brownian motion if the following hold [14]:*
1) The initial value of the process $W_0 = 0$,
2) W_t has stationary and independent increments,
3) The process W_t is continuous in t,
4) The increments $W_t - W_s$ are normally distributed with mean zero and variance $|t - s|$,

$$W_t - W_s \sim N(0, |t - s|) \tag{2.1}$$

Beside the definition of the Brownian motion itself, we need to define the filtration of it

Definition 2.2.2. *Let a Brownian motion W_t be defined on the probability $(\Omega, \mathcal{F}, P)$. A filtration for the Brownian motion is a collection of σ-algebras $(\mathcal{F}_t)_{t\geq 0}$, satisfies [17]:*
(i) Every set in $\mathcal{F}_s$ is also in $\mathcal{F}_t$, for every $0 \leq s < t$
(ii) At time t, W_t is $\mathcal{F}_t$-measurable,
(iii) The increment of $(W_u - W_t)$ is independent of $\mathcal{F}_t$, for $0 \leq t < u$

In order to introduce Ito's lemma, we have to define the stochastic integral [17]. We have to start constructing the Itô's integral. Let W_t be a one dimensional Brownian motion with the filtration $\mathcal{F}_t$. An adapted stochastic process X_t is called elementary process, if it is written in the form

$$X_t = \sum_{i=0}^{n} \xi_i I_{(t_i, t_{i+1}]} t, \tag{2.2}$$

where $0 = t_0 < t_1 < ..... < t_n < \infty$, and for each index i the random variable ξ_i is $\mathcal{F}_{t_i}$-measurable.

Definition 2.2.3. *For a simple process $(X_t)_{t\geq 0}$ satisfying (2.2), the Itô's integral is defined as:*

$$I_t(X) = \int_0^t X_s dW_s := \sum_{i=0}^{n-1} \xi_i(W(t_{i+1} \wedge t) - W(t_i \wedge t)).$$

Then, we need to define the Ito's process

Definition 2.2.4. *Let X_t be a stochastic process, Then X_t is called Itô's process if it is written in the following form:*

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dW_s$$

where a and b are two progressively measurable process such that

$$\int_0^t [|a_s| + |b_s|^2] ds < +\infty \quad \forall \quad t \geq 0.$$

After that, we introduce Itô's lemma [17] and it is a very important part of Itô's calculus

Lemma 2.2.1. *Let $(W_t)_{t\geq 0}$ be a Brownian motion. Then we have*

$$f(W_t) - f(W_0) = \int_0^t f'(W_s) dW_s + \int_0^t \frac{1}{2} f''(W_s) ds$$

Theorem 2.2.2. *Let W be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t\geq 0}$ be the filtration. Then, let M_t be a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$, M_t can* *be written as follows*

$$M_t = M_0 + \int_0^t \psi_s dWs,$$

where ψ is a progressively measurable process such that

$$\int_0^t |\psi_s|^2 ds < +\infty \quad \forall \quad t \geq 0.$$

Theorem 2.2.3. *Let* $W = (W_t)_{t\geq 0}$ *be an m-dimensional Brownian motion* $x \in R$*, and* A, a, S_j, σ_j *be progressively measurable,real-valued stochastic process such that*

$$P\left\{\forall t \geq 0 : \int_0^t (|A(s)| + |a(s)|)ds < \infty\right\} = 1,$$

$$P\left\{\forall t \geq 0 : \int_0^t (\left|S_j^2(s)\right| + \left|\sigma_j^2(s)\right|)ds < \infty\right\} = 1.$$

Then the stochastic differential equation

$$dX_t = [A(t)X(t) + a(t)]\,dt + \sum_{j=1}^{m} [S_j(t)X(t) + \Sigma_j(t)]\,dW_j(t),$$

$$X(0) = x,$$

has a unique solution given by

$$X(t) = Z(t)\left(x + \int_0^t \frac{1}{Z_u}\left[a(u) - \sum_{j=1}^{m} S_j(u)\Sigma_j(u)\right]du + \sum_{j=1}^{m}\int_0^t \frac{\Sigma_j(u)}{Z(u)}dW_j(u)\right), \quad t \geq 0.$$

Here

$$Z(t) := exp\left(\int_0^t (A(u) - \frac{1}{2}\left\|S(u)\right\|^2)du + \int_0^t S(u)dW(u)\right), \quad t \geq 0.$$

Definition 2.2.5. *On the probability space* $(\Omega, \mathcal{F}, P)$*, any nonnegative random variable* τ *is a stopping time with respect to the filtration* $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$*, if and only if*

$$\{\tau \leq t\} \in \mathcal{F}_t, \quad 0 \leq t \leq T.$$

Chapter 3

The Esscher pricing measure in discrete time

In this chapter, we will focus on the Esscher pricing measure in discrete time setting. We consider the stochastic process $S = (S_t)_{t=0,\ldots,T}$, which represents the stock price process, that is defined on the filtered probability space $(\Omega, (\mathcal{F}_t)_{t=0,\ldots,T}, P)$. Throughout this section, we also consider the following notations

$$S^* = \frac{S}{B} = \text{discounted stock price process}, \quad \Delta S_t^* = S_t^* - S_{t-1}^* \ \ \text{t=1,....,T.}$$

Since our framework is a stochastic dynamic setting, we consider our extension of Esscher transform, called conditional Esscher transform. Up to our knowledge, this extension was introduced first in [4], and we refer the reader to this paper for more details and related discussion. Below, we recall the definition of Esscher pricing density.

Definition 3.0.1. *Suppose that a stochastic process* $H = (H_n)$ *with* $\Delta H_n = h_n$, $Ee^{a_k h_k} < \infty$, $k \geq 1$ *and* $a_1, a_2, ...$ *are constant then the following process* $Z = (Z_n)_{n\geq 1}$ *and* $Z_0 = 1$

$$Z_n = \prod_{k=1}^{n} \frac{e^{a_k h_k}}{E[e^{a_k h_k}|\mathcal{F}_{k-1}]} \quad , \quad n \geq 1 \tag{3.1}$$

is a martingale called Esscher martingale density, or Esscher pricing density.

By constructing the family of measures $(\widetilde{P}_N)$ such that $d\widetilde{P}_N = Z_N dP_N$ and $\widetilde{P}_N = \widetilde{P}_{N+1}|\mathcal{F}_N$. The conditional distribution

$$\widetilde{P}_N(h_N \in A|\mathcal{F}_{N-1}) = E[I_{A(h_N)} \frac{e^{a_N h_N}}{E[e^{a_N h_N}|\mathcal{F}_{N-1}]}|\mathcal{F}_{N-1}]$$

is called the conditional Esscher transform.

3.1 The general framework

This section describes the Esscher pricing density by using an integral equation. More importantly, we extend (3.1) to allow the sequence of constants $(a_i)_{i\geq 1}$ to be a predictable process.

Theorem 3.1.1. *The Esscher pricing density, denoted by* Z*, is given by:*

$$Z_t := \prod_{i=1}^{t} \frac{e^{\theta_i \Delta S_i^*}}{E[e^{\theta_i \Delta S_i^*}|\mathcal{F}_{i-1}]} \quad , \quad t = 1, ..., T, \quad Z_0 = 1, \tag{3.2}$$

where θ_t is an $\mathcal{F}_{t-1}$-measurable random variable that is the unique root of

$$0 = E[\Delta S_t^* e^{\alpha \Delta S_t^*} | \mathcal{F}_{t-1}] =: \int (x - S_{t-1}^*) e^{\alpha(x - S_{t-1}^*)} F_t(dx), \text{ where } F_t(dx) := P\left(S_t^* \in dx \middle| \mathcal{F}_{t-1}\right) \quad (3.3)$$

Proof. This proof contains three steps. In the first step, we prove that the process Z defined in (3.2) is a positive martingale, while the second step proves that the process ZS^* also a martingale when θ satisfies (3.3). The last step proves that the equation (3.3) has in fact a unique solution.

Step 1. It is clear that Z_t is positive since it is the product of positive random variables. It is also clear that Z_t is $\mathcal{F}_t$-measurable, and satisfies

$$Z_t = Z_{t-1} \quad \frac{e^{\theta_t \Delta S_t^*}}{E[e^{\theta_t \Delta S_t^*} | \mathcal{F}_{t-1}]}, \quad t = 1, ..., T. \quad (3.4)$$

Therefore, by taking conditional expectation in the above equality, we get

$$E[Z_t | \mathcal{F}_{t-1}] = Z_{t-1}, \quad t = 1, ..., T.$$

This proves that Z a positive martingale.

Step 2. By using (3.2) and (3.3), for $t = 1, ..., T$, we derive

$$E[Z_t S_t^* | \mathcal{F}_{t-1}] = \frac{Z_{t-1} E[e^{\theta_t \Delta S_t^*} S_t^* | \mathcal{F}_{t-1}]}{E[e^{\theta_t \Delta S_t^*} | \mathcal{F}_{t-1}]} = \frac{Z_{t-1} S_{t-1}^* E[e^{\theta_t \Delta S_t^*} | \mathcal{F}_{t-1}]}{E[e^{\theta_t \Delta S_t^*} | \mathcal{F}_{t-1}]} = Z_{t-1} S_{t-1}^*.$$

This proves that ZS^* a martingale, and completes the second step.

Step 3. Here we prove that (3.3) has a unique solution. To this end, we put

$$f(\alpha) := \int e^{\alpha(x-S^*_{t-1})} F_t(dx), \qquad \alpha \in \mathbb{R}.$$

Then, we get

$$\begin{cases} g(\alpha) := f'(\alpha) = \int (x - S^*_{t-1}) e^{\alpha(x-S^*_{t-1})} F_t(dx) = 0, \\ g'(\alpha) := f''(\alpha) = \int (x - S^*_{t-1})^2 e^{\alpha(x-S^*_{t-1})} F_t(dx) > 0 \end{cases} \tag{3.5}$$

It is clear that g is strictly increasing and continuous. We need to calculate the limits when α goes to $(+\infty)$ and $(-\infty)$. Then

$$\lim_{\alpha\to+\infty} = \int_{x>S^*_{t-1}} (x-S^*_{t-1}) e^{\alpha(x-S^*_{t-1})} F_t(dx) + \int_{x<S^*_{t-1}} (x-S^*_{t-1}) e^{\alpha(x-S^*_{t-1})} F_t(dx) = \infty - 0 = +\infty$$

$$\lim_{\alpha\to-\infty} = \int_{x>S^*_{t-1}} (x-S^*_{t-1}) e^{\alpha(x-S^*_{t-1})} F_t(dx) + \int_{x<S^*_{t-1}} (x-S^*_{t-1}) e^{\alpha(x-S^*_{t-1})} F_t(dx) = 0 - \infty = -\infty$$

Therefore, we have that $g(+\infty) = +\infty$ and $g(-\infty) = -\infty$. Hence by using the intermediate value theorem, these exists a unique $\alpha \in \mathbb{R}$ particular $g(\alpha) = 0$. This proves that (3.2) has a unique solution, and the proof of the theorem is completed.

See [2] for the intermediate value theorem. □

3.2 Particular cases

In this section, we illustrate the results of the previous section on two important particular cases, namely the binomial and trinomial models.

3.2.1 The Binomial Model

Throughout this subsection, we assume that the stock price $S = (S_t)_{t=0,...,T}$ is given by

$$S_t = S_{t-1}Y_t = S_0 \prod_{i=1}^{t} Y_i, \quad t = 1, ..., T, \quad S_0 \text{ is a positive given number,} \tag{3.6}$$

and $(Y_t)_{t=1,....,T}$ are independent and identically distributed random variables satisfying the following

$$Y_t = \begin{cases} u & \text{with probability } p, \\ d & \text{with probability } (1-p). \end{cases} \tag{3.7}$$

Theorem 3.2.1. *The model described in (3.6)-(3.7), is arbitrage free if and only if*

$$0 < d < 1 + r < u,$$

and under this assumption there is one risk-neutral probability Q given by

$$Q(Stock\ goes\ up) = \frac{(1+r)-d}{u-d} =: p^*, \quad Q(Stock\ goes\ down) = \frac{u-(1+r)}{u-d} = 1 - p^*. \tag{3.8}$$

The proof of this theorem can be found in [18].

Theorem 3.2.2. *The Esscher pricing density process, denoted by Z, when S follows the binomial model of (3.6)-(3.7), is given by*

$$Z_t = \frac{\exp\left(\gamma \sum_{i=1}^{t} Y_i\right)}{\left(pe^{\gamma u} + (1-p)e^{\gamma d}\right)^t}, \quad where \quad \gamma = (u-d)^{-1} \ln\left(\frac{p^*(1-p)}{p(1-p^*)}\right). \tag{3.9}$$

Proof. The proof of the theorem follows from Theorem 3.1.1, as soon as, we solve (3.3).This is the aim of the rest of this proof. To this end, we calculate

$$\Delta S_t^* = S_t^* - S_{t-1}^* = S_{t-1}^* \frac{Y_t - (1+r)}{1+r}, \qquad t = 1, ..., T$$

Then, by inserting this in (3.3) and using the fact that Y_t is independent of $\mathcal{F}_{t-1}$ and has the same distribution as Y_1, we get

$$\begin{aligned} 0 \quad &= S_{t-1}^* E\left[\frac{Y_t - (1+r)}{1+r} \exp\left(\theta_t S_{t-1}^* (Y_t - (1+r))(1+r)^{-1}\right) | \mathcal{F}_{t-1}\right] \\ &= S_{t-1}^* E\left[\frac{Y_1 - 1 - r}{1+r} \exp\left(x(Y_1 - (1+r))(1+r)^{-1}\right)\right] I_{\{x=\theta_t S_{t-1}^*\}} \\ &= S_{t-1}^* \left[p\left(\frac{u-1-r}{1+r}\right) \exp\left(\theta_t S_{t-1}^* \frac{u-1-r}{1+r}\right) + (1-p)\left(\frac{d-1-r}{1+r}\right) \exp\left(\theta_t S_{t-1}^* \frac{d-1-r}{1+r}\right)\right] \end{aligned}$$

Hence, the solution to this equation is

$$\widetilde{\theta}_t = \frac{1+r}{(u-d)S_{t-1}^*} \ln\left[\frac{(1-p)(1+r-d)}{p(u-1-r)}\right] = \frac{1+r}{(u-d)S_{t-1}^*} \ln\left[\frac{p^*(1-p)}{(1-p^*)p}\right] \tag{3.10}$$

As a result, we get

$$\widetilde{\theta}_t \Delta S_t^* = \widetilde{\theta}_t S_{t-1}^* \frac{Y_t - (1+r)}{1+r} = \frac{Y_t - (1+r)}{u-d} \ln\left[\frac{p^*(1-p)}{(1-p^*)p}\right] = \gamma(Y_t - (1+r)), \tag{3.11}$$

where γ is given by (3.9). Then, by combining (3.10) and (3.11) we obtain for $t = 1, ..., T$

$$\begin{aligned}
\frac{e^{\widetilde{\theta}_t \Delta S_t^*}}{E[e^{\widetilde{\theta}_t \Delta S_t^*} | \mathcal{F}_{t-1}]} &= \frac{e^{\gamma(Y_t-(1+r))}}{E[e^{\gamma(Y_t-(1+r))} | \mathcal{F}_{t-1}]} \\
&= \frac{e^{\gamma Y_t}}{E[e^{\gamma Y_t} | \mathcal{F}_{t-1}]} = \frac{e^{\gamma Y_t}}{E[e^{\gamma Y_t}]} \\
&= \frac{e^{\gamma Y_1}}{pe^{\gamma u} + (1-p)e^{\gamma d}}.
\end{aligned}$$

Therefore, by inserting this in (3.2), we obtain

$$Z_t = \prod_{i=1}^{t} \frac{e^{\gamma Y_i}}{pe^{\gamma u} + (1-p)e^{\gamma d}} = \frac{\exp(\gamma \sum_{i=1}^{t} Y_i)}{(pe^{\gamma u} + (1-p)e^{\gamma d})^t}.$$

and hence (3.9) follows immediately. This ends the proof of the theorem. □

3.2.2 The Trinomial Model

Throughout this subsection, we assume that the stock price $S = (S_t)_{t=0,...,T}$ is given by

$$S_t = S_{t-1} Y_t = S_0 \prod_{i=1}^{t} Y_i, \quad t = 1, ..., T, \quad S_0 \text{ is a positive given number,} \tag{3.12}$$

and $(Y_t)_{t=1,....,T}$ are independent and identically distributed random variables satisfying the following

$$Y_t = \begin{cases} u & \text{with probability } p, \\ m & \text{with probability } q, \\ d & \text{with probability } (1-p-q) \end{cases} \tag{3.13}$$

where $p > 0$, $q > 0$ and $p + q < 1$.

Theorem 3.2.3. *The model described in (3.12)-(3.13), is arbitrage free if and only if*

$$0 < d < 1 + r < u,$$

and under this assumption the model is incomplete, or equivalently there are many risk-neutral probabilities.

The proof of this theorem can be found in [18].

Theorem 3.2.4. *The Esscher pricing density process, denoted by Z, when S follows the trinomial model of (3.12)-(3.13), is given by*

$$Z_t = \exp\left(\frac{\widetilde{\eta}}{1+r}\sum_{i=1}^{t} Y_i - t\left(f(\widetilde{\eta}) + \widetilde{\eta}\right)\right), \tag{3.14}$$

where f is a function given by

$$f(x) := \ln\left(pe^{\left(x\frac{u-1-r}{1+r}\right)} + qe^{\left(x\frac{m-1-r}{1+r}\right)} + (1-p-q)e^{\left(x\frac{d-1-r}{1+r}\right)}\right), \tag{3.15}$$

and $\widetilde{\eta}$ is the unique root of $f'(x) = 0$, or equivalently

$$[(u-r-1)p\exp[x(u-1-r)]+(m-r-1)q\exp[x(m-r-1)]+(1-p-q)(d-r-1)\exp[x(d-r-1)] = 0 \tag{3.16}$$

Proof. The proof of the theorem follows from Theorem 3.1.1, as soon as, we solve (3.3), and prove that (3.16) has a unique solution. This is the aim of the rest of this proof. To this end, we calculate

$$\Delta S_t^* = S_t^* - S_{t-1}^* = S_{t-1}^* \frac{Y_t - (1+r)}{1+r}, \qquad t = 1, ..., T$$

Then, by inserting this in (3.3), and using $S_{t-1}^* > 0$ and the fact that Y_t is independent of $\mathcal{F}_{t-1}$ and has the same distribution as Y_1, we get

$$\begin{aligned} 0 \quad &= E\left[\frac{Y_t - (1+r)}{1+r} \exp\left(\alpha S_{t-1}^* (Y_t - (1+r))(1+r)^{-1}\right) | \mathcal{F}_{t-1}\right] \\ &= E\left[\frac{Y_1 - (1+r)}{1+r} \exp\left(\alpha x (Y_1 - (1+r))(1+r)^{-1}\right)\right] I_{\{x = S_{t-1}^*\}}, \quad \text{(since } Y_t \text{ is independent of } \mathcal{F}_{t-1}\text{)} \\ &= p\frac{u-1-r}{1+r} \exp\left(\alpha S_{t-1}^* \frac{u-r-1}{1+r}\right) + q\frac{m-r-1}{1+r} \exp\left(\alpha S_{t-1}^* \frac{m-r-1}{1+r}\right) \\ &+ (1-p-q)\frac{d-r-1}{1+r} \exp\left(\alpha S_{t-1}^* \frac{d-r-1}{1+r}\right). \end{aligned}$$

Therefore, if θ_t is the root for

$$0 = p(u-r-1)e^{\alpha S_{t-1}^*(u-r-1)} + q(m-r-1)e^{\alpha S_{t-1}^*(m-r-1)} + (1-p-q)(d-r-1)e^{\alpha S_{t-1}^*(d-r-1)}.$$

then $\eta_t := \theta_t S_{t-1}^*$ is the root for (3.16). As a result, we get

$$\theta_t \Delta S_t^* = \theta_t S_{t-1}^* \frac{Y_t - (1+r)}{1+r} = \frac{Y_t - (1+r)}{1+r}\, \widetilde{\eta}, \qquad t = 1, ..., T, \tag{3.17}$$

where $\widetilde{\eta}$ is the unique root of (3.16), which is a real constant. Therefore, we get

$$\begin{aligned} E[e^{\theta_t \Delta S_t^*}|F_{t-1}] &= E[e^{\theta_t S_{t-1}^* \frac{Y_t-(1+r)}{1+r}}|\mathcal{F}_{t-1}] = E[e^{\widetilde{\eta}\frac{Y_1-(1+r)}{1+r}}] \\ &= pe^{\left(\widetilde{\eta}\frac{u-r-1}{1+r}\right)} + qe^{\left(\widetilde{\eta}\frac{m-r-1}{1+r}\right)} + (1-p-q)e^{\left(\widetilde{\eta}\frac{d-r-1}{1+r}\right)} \\ &= \exp(f(\widetilde{\eta})) \end{aligned} \tag{3.18}$$

where f is the function defined in (3.15). Thus, by inserting (3.17) and (3.18) in (3.2) afterwards, we obtain

$$Z_t = \exp\left(\widetilde{\eta}\sum_{i=1}^{t}\frac{Y_i-(1+r)}{1+r} - tf(\widetilde{\eta})\right) = \exp\left(\frac{\widetilde{\eta}}{1+r}\sum_{i=1}^{t}(Y_i-(1+r)) - tf(\widetilde{\eta})\right)$$

This implies (3.14). Then, the proof of the theorem will be achieved as soon as we prove that (3.16) admits a unique solution. To this end, we put

$$h(x) = p(u-r-1)\exp(x(u-r-1)) + q(m-r-1)\exp(x(m-r-1)) + (1-p-q)(d-r-1)\exp(x(d-r-1)).$$

Then we calculate the derivative of $h(x)$

$$h'(x) = p(u-r-1)^2 e^{x(u-r-1)} + q(m-r-1)^2 e^{x(m-r-1)} + (1-p-q)(d-r-1)^2 e^{x(d-r-1)} > 0$$

Hence h is strictly increasing, continuous and $h(+\infty) = +\infty$ and $h(-\infty) = -\infty$. This is due to the fact that

$$\lim_{x\to+\infty} \alpha e^{\alpha x} = \begin{cases} 0 & \text{if} \quad \alpha \leq 0, \\ +\infty & \text{if} \quad \alpha > 0 \end{cases} \tag{3.19}$$

$$\lim_{x \to -\infty} \alpha e^{+\alpha x} = \begin{cases} 0 \quad \text{if} \quad \alpha \geq 0, \\ \\ -\infty \quad \text{if} \quad \alpha < 0 \end{cases} \tag{3.20}$$

□

Chapter 4

The Esscher under mortality for discrete-time

In this chapter, we derive the Esscher pricing density in the discrete time framework, for the model S^τ. Since τ is not an $\mathbb{F}$-stopping time, the first obstacle is to find another filtration that incorporates the information in τ and that in $\mathbb{F}$. Thus, herein, we follow the footsteps of [6] regarding this issue, and we consider the filtration $\mathbb{G}$ given by

$$\mathbb{G} = (\mathcal{G}_t)_{t=0,..,T}, \qquad \mathcal{G}_t = \mathcal{F}_t \vee \sigma\{(\tau = k), k = 0, ..., t\}, \quad t \geq 1.$$

Therefore, for the filtration $\mathbb{G}$ (as stated in [6]) τ is a stopping time and our problem reduces to discuss Esscher pricing measure for the model $(S^\tau, \mathbb{G})$. Throughout this chapter, we consider the following lemma (See Lemma (2.4) and the equality (2.7) in Choulli and Deng. [6]), that we state below

Lemma 4.0.1. *The following assertions hold.*

(1) For any nonnegative (or integrable) random variable X, we have

$$E\left[X|\mathcal{G}_{t-1}\right] I_{\{t\leq\tau\}} = E\left[X I_{\{t\leq\tau\}}|\mathcal{F}_{t-1}\right] (G_{t-1})^{-1} I_{\{t\leq\tau\}}. \tag{4.1}$$

(2) For any $\mathcal{G}_{t-1}$-measurable random variable, $\theta_t^{\mathbb{G}}$, there exists $\mathcal{F}_{t-1}$-measurable random variable, $\theta_t^{\mathbb{F}}$, such that

$$\theta_t^{\mathbb{G}} = \theta_t^{\mathbb{F}} \quad on\ (t \leq \tau). \tag{4.2}$$

Lemma 4.0.2. *The following process*

$$L_t := \prod_{k=1}^{t} \frac{\tilde{G}_k}{G_{k-1}}, \qquad t = 1, ..., T, \tag{4.3}$$

is a positive martingale with respect to $(\mathcal{F}_t)_{t=0,..,T}$.

Proof. Remark that L_t is positive, $\mathcal{F}_t$-measurable, and satisfies

$$E[L_t|\mathcal{F}_{t-1}] = L_{t-1} E\left[\frac{\tilde{G}_t}{G_{t-1}}|\mathcal{F}_{t-1}\right].$$

Due to $E[\tilde{G}_t|\mathcal{F}_{t-1}] = G_{t-1}$, which follows from $\tilde{G}_t = P(t \leq \tau|\mathcal{F}_t) = P(\tau > t-1|\mathcal{F}_t)$, we deduce that

$$E[L_t|\mathcal{F}_{t-1}] = L_{t-1} \qquad \text{t=1,......,T.}$$

This proves that L is a positive martingale. □

The next theorem proves that the Esscher martingale measure for $(S^\tau,\mathbb{G})$ coincides with the Esscher martingale measure of (S,Q,$\mathbb{F}$) for some probability Q.

Theorem 4.0.3. *Consider the probability $Q := L_T \cdot P$, where L is given in (4.3), and let $\widetilde{Z}^Q$ be the Esscher pricing density for (S,Q,$\mathbb{F}$), and $\widetilde{Z}^{\mathbb{G}}$ be the Esscher pricing density for $(S^\tau, \mathbb{G})$. Then it holds that*

$$\widetilde{Z}^Q_{t\wedge\tau} = \widetilde{Z}^{\mathbb{G}}_t, \qquad t = 0, 1, ..., T.$$

Proof. First, we have the density $Z^{\mathbb{G}}$, is given by

$$Z^{\mathbb{G}}_t = \prod_{i=1}^{t} \quad \frac{e^{\theta^{\mathbb{G}}_i \Delta S^*_i}}{E[e^{\theta^{\mathbb{G}}_i \Delta S^*_i} | \mathcal{G}_{i-1}]} \quad , \quad t = 1, ..., T, \quad Z_0 = 1, \tag{4.4}$$

where $(\theta^{\mathbb{G}}_i)_{i\geq 1}$ is a $\mathbb{G}$-predictable process.

Thus, by applying (4.2), we replace $\theta^{\mathbb{G}}$ by an $\mathbb{F}$-predictable process $\theta^{\mathbb{F}}$.

Afterwards, we combine (4.1) and the Bayes' rule, and get on $(t \leq \tau)$,

$$E\left[e^{\theta^{\mathbb{F}}_t \Delta S^*_t} | \mathcal{G}_{t-1}\right] = \frac{E\left[e^{\theta^{\mathbb{F}}_t \Delta S^*_t} \widetilde{G}_t | \mathcal{F}_{t-1}\right]}{G_{t-1}} = E_Q[e^{\theta^{\mathbb{F}}_t \Delta S^*_t} | \mathcal{F}_{t-1}].$$

As a result, we derive for $t \leq \tau$:

$$Z^{\mathbb{G}}_t = \prod_{i=1}^{t} \frac{e^{\theta^{\mathbb{G}}_i \Delta S^*_i}}{E\left[e^{\theta^{\mathbb{G}}_i \Delta S^*_i} | \mathcal{G}_{i-1}\right]} = \prod_{i=1}^{t} \frac{e^{\theta^{\mathbb{F}}_i \Delta S^*_i}}{E_Q\left[e^{\theta^{\mathbb{F}}_i \Delta S^*_i} | \mathcal{F}_{i-1}\right]} = Z^Q_{t\wedge\tau}, \qquad Z_0 = 1$$

This ends the proof of the theorem. □

4.1 The general framework

The aim of this section is to describe the Esscher pricing density for (S^τ, $\mathbb{G}$). To this end, we start describing the Esscher pricing density for the model (S, $\mathbb{F}$, Q), where Q is given in (4.0.3). Then, we combine the obtained result with Theorem 4.0.3 to achieve our goal.

Theorem 4.1.1. *The Esscher pricing density for (S, $\mathbb{F} := (\mathcal{F}_t)_{t=0,..,T}$, Q),denoted by Z, is given by:*

$$Z_t := \prod_{i=1}^{t} \frac{e^{\theta_i \Delta S_i^*} G_{i-1}}{E[e^{\theta_i \Delta S_i^*} \widetilde{G}_i | \mathcal{F}_{i-1}]} \quad , \quad t = 1, ..., T, \qquad Z_0 = 1, \tag{4.5}$$

where θ_t is $\mathcal{F}_{t-1}$ measurable random variable that is a unique root of

$$E[\Delta S_t^* e^{\alpha \Delta S_t^*} \widetilde{G}_t | \mathcal{F}_{t-1}] = 0 \tag{4.6}$$

Proof. This proof is divided into two steps. In the first step, we prove that Z is a martingale under Q. In the second step, we prove that ZS^* is a martingale under Q.

Step 1. We need to change the measure by applying Bayes rule:

$$E_Q[Z_t | \mathcal{F}_{t-1}] = \frac{E[Z_t L_t | \mathcal{F}_{t-1}]}{L_{t-1}}.$$

Then we derive

$$\begin{aligned} E_Q[Z_t|\mathcal{F}_{t-1}] &= \frac{1}{L_{t-1}} E\left[\prod_{i=1}^{t} \frac{e^{\theta_i \Delta S_i^*} G_{i-1}}{E[e^{\theta_i \Delta S_i^*} \widetilde{G}_i|\mathcal{F}_{i-1}]} \prod_{i=1}^{t} \frac{\widetilde{G}_i}{G_{i-1}} |\mathcal{F}_{t-1}\right] \\ &= \frac{1}{L_{t-1}} E\left[Z_{t-1} L_{t-1} \frac{e^{\theta_t \Delta S_t^*} \widetilde{G}_t G_{t-1}}{E[e^{\theta_t \Delta S_t^*} \widetilde{G}_t|\mathcal{F}_{t-1}] G_{t-1}} |\mathcal{F}_{t-1}\right] \\ &= Z_{t-1}. \end{aligned}$$

This proves that Z a martingale under Q.

Step 2. Here we prove that ZS^* is a martingale under Q if and only if (4.6) holds. To this end, we remark that

$$E_Q[Z_t S_t^*|\mathcal{F}_{t-1}] = \frac{E[Z_t L_t S_t^*|\mathcal{F}_{t-1}]}{L_{t-1}} = \frac{Z_{t-1} E[S_t^* e^{\alpha \Delta S_t^*} \widetilde{G}_t|\mathcal{F}_{t-1}]}{E[e^{\alpha \Delta S_t^*} \widetilde{G}_t|\mathcal{F}_{t-1}]}.$$

Thus, we conclude that ZS^* is a martingale under Q if and only if $S_{t-1}^* = \frac{E[S_t^* e^{\alpha \Delta S_t^*} \widetilde{G}_t|\mathcal{F}_{t-1}]}{E[e^{\alpha \Delta S_t^*} \widetilde{G}_t|\mathcal{F}_{t-1}]}$, or equivalently θ_t is the root of $E[\Delta S_t e^{\alpha \Delta S_t^*} \widetilde{G}_t|\mathcal{F}_{t-1}] = 0$. This proves the theorem. □

Theorem 4.1.2. *The Esscher pricing density for* $(S^\tau, \mathbb{G})$*, denoted by* $Z^{\mathbb{G}}$*, is given by*

$$Z_t^{\mathbb{G}} := \prod_{i=1}^{t\wedge\tau} \frac{e^{\widetilde{\theta}_i \Delta S_i^*}}{E[e^{\widetilde{\theta}_i \Delta S_i^*}|\mathcal{G}_{i-1}]}, \quad t = 1, ..., T, \quad Z_0 = 1, \tag{4.7}$$

where $\widetilde{\theta}_t$ *is* $\mathcal{F}_{t-1}$*-measurable random variable and the root of*

$$E[\Delta S_t^* e^{\alpha \Delta S_t^*} \widetilde{G}_t|\mathcal{F}_{t-1}] = 0 \tag{4.8}$$

Proof. First, we need to define two conditional survival probabilities as follow:

$$G_t = P(t < \tau|\mathcal{F}_t) \qquad and \qquad \widetilde{G}_t = P(t \leq \tau|\mathcal{F}_t). \tag{4.9}$$

Second, we will change the measure by applying (4.2) and (4.1)

$$E\left[e^{\theta_t^{\mathbb{F}}\Delta S_t^*}|\mathcal{G}_{t-1}\right] I_{\{t\leq\tau\}} = E\left[e^{\theta_i^{\mathbb{F}}\Delta S_t^*} I_{\{t\leq\tau\}}|\mathcal{F}_{t-1}\right] (G_{t-1})^{-1} I_{\{t\leq\tau\}}$$

Rewrite Z as follow, on $(t \leq \tau)$

$$Z_t^{\mathbb{G}} \qquad = \prod_{i=1}^{t} \frac{e^{\theta_i^{\mathbb{F}}\Delta S_i^*} G_{i-1}}{E[e^{\theta_i^{\mathbb{F}}\Delta S_i^*}\widetilde{G}_i|\mathcal{F}_{i-1}]}$$

Then,

$$\begin{aligned}
E[Z_t|\mathcal{G}_{t-1}] \qquad &= E[\prod_{i=1}^{t} \frac{e^{\theta_i^{\mathbb{F}}\Delta S_i^*} G_{i-1}}{E[e^{\theta_i^{\mathbb{F}}\Delta S_i^*}\widetilde{G}_i|\mathcal{F}_{i-1}]}|\mathcal{G}_{t-1}] \\
&= E[Z_{t-1}\frac{e^{\theta_t^{\mathbb{F}}\Delta S_t^*} G_{t-1}}{E[e^{\theta_t^{\mathbb{F}}\Delta S_t^*}\widetilde{G}_t|\mathcal{F}_{t-1}]}|\mathcal{G}_{t-1}] \\
&= E[Z_{t-1}\frac{e^{\theta_t^{\mathbb{F}}\Delta S_t^*} G_{t-1}}{E[e^{\theta_t^{\mathbb{F}}\Delta S_t^*}\widetilde{G}_t|\mathcal{F}_{t-1}]}|\mathcal{F}_{t-1}] \\
&= Z_{t-1}.
\end{aligned}$$

This proves that $Z^{\mathbb{G}}$ a martingale. Then, we will prove that ZS^* is a martingale. Remark that on $(t \leq \tau)$:

$$E[Z_tS_t^*|\mathcal{G}_{t-1}] = E\left[Z_{t-1}S_t^*\frac{G_{t-1}e^{\theta_t\Delta S_t^*}}{E[\widetilde{G}_te^{\theta_t\Delta S_t^*}|\mathcal{F}_{t-1}]}|\mathcal{G}_{t-1}\right] = Z_{t-1}G_{t-1}\frac{E[S_t^*e^{\theta_t\Delta S_t^*}|\mathcal{G}_{t-1}]}{E[\widetilde{G}_te^{\theta_t\Delta S_t^*}|\mathcal{F}_{t-1}]}.$$

By using (4.1), we obtain

$$\begin{aligned} E[Z_t S_t^* | \mathcal{G}_{t-1}] &= \frac{Z_{t-1} G_{t-1}}{E[\widetilde{G}_t e^{\theta_t \Delta S_t^*} | \mathcal{F}_{t-1}]} E[S_t^* \widetilde{G}_t e^{\theta_t \Delta S_t^*} | \mathcal{F}_{t-1}] (G_{t-1})^{-1} \\ &= \frac{Z_{t-1}}{E[\widetilde{G}_t e^{\theta_t \Delta S_t^*} | \mathcal{F}_{t-1}]} E[S_t^* \widetilde{G}_t e^{\theta_t \Delta S_t^*} | \mathcal{F}_{t-1}]. \end{aligned}$$

Thus, from this equality, we deduce that ZS^* is a martingale if and only if θ is the root of (4.8). This ends the proof of the theorem. □

4.2 Particular cases

In this section, we illustrate the results of the previous section on two important particular cases, namely the binomial and trinomial models.

Throughout this section, we consider a sequence $(Y_i)_{i \geq 1}$ of independent and identically distributed random variables and another sequence $(\epsilon_i)_{i \geq 1}$ of independent and identically distributed random variables that are also independent of $(Y_i)_{i \geq 1}$. Then throughout the rest of this chapter, the filtration $\mathbb{F} := (\mathcal{F}_t)_{t=0,..,T}$ will be

$$\mathbb{F} := (\mathcal{F}_n)_{n=0,..,T}, \quad \mathcal{F}_n = \sigma(Y_1, ..., Y_n, \epsilon_n, ..., \epsilon_1), \; n \geq 1, \quad \mathcal{F}_0 := \{\emptyset, \Omega\}.$$

Lemma 4.2.1. *Let a, b, and c be real numbers. The process*

$$K_t := \exp\left[\sum_{i=1}^{t} aY_i + b\sum_{i=1}^{t} \epsilon_i - ct\right], \quad t \geq 0.$$

is a super-martingale if and only if $\varphi(a)\psi(b)e^{-c} \leq 1$, *where*

$$\varphi(x) := E[e^{xY_1}], \quad \text{and} \quad \psi(x) := E[e^{x\epsilon_1}]. \tag{4.10}$$

Proof. Due to the independence of (Y_t, ϵ_t) and $\mathcal{F}_{t-1}$, we derive

$$\begin{aligned} E[K_t|\mathcal{F}_{t-1}] \quad &= \exp\left[\sum_{i=1}^{t} aY_i + b\sum_{i=1}^{t}\epsilon_i - ct|\mathcal{F}_{t-1}\right] \\ &= K_{t-1}E[e^{(aY_t+b\epsilon_t-c)}] \\ &= K_{t-1}E[e^{aY_t}]E[e^{b\epsilon_t}]e^{-c} \end{aligned}$$

As a result, K is a super-martingale if $E[K_t|\mathcal{F}_{t-1}] \leq K_{t-1}$, which is equivalently to $E[e^{aY_t}]E[e^{b\epsilon_t}]e^{-c} \leq 1$, or equivalently to $\varphi(a)\psi(b)e^{-c} \leq 1$. This proves that K_t a super-martingale if and only if $\varphi(a)\psi(b)e^{-c} \leq 1$. □

Throughout the rest of this chapter, we assume that τ follows a model given by its Azéma-supermartingale

$$\widetilde{G}_t := \exp\left[\sum_{i=1}^{t} aY_i + b\sum_{i=1}^{t}\epsilon_i - ct\right], \quad \text{with} \quad \varphi(a)\psi(b)e^{-c} < 1, \tag{4.11}$$

where $\varphi(a)$ and $\psi(b)$ are defined in (4.10).

4.2.1 The Binomial model under mortality

In this subsection, we suppose that the sequence $(Y_i)_{i\geq 1}$ follows from Subsection 3.2.1 where Y_1 is a binomial random variable.

Theorem 4.2.2. *The Esscher pricing density, when S follows a binomial model, for $(S^\tau, \mathbb{G})$, is given by*

$$Z_t = \exp\left[\frac{\tilde{\gamma}}{1+r}\sum_{i=1}^{t\wedge\tau} Y_i - \Lambda(t\wedge\tau)\right], \tag{4.12}$$

where

$$\tilde{\gamma} = \frac{(1+r)\ln\left[\frac{p^*(1-p)}{p(1-p^*)}\right]}{u-d} - (1+r)a, \qquad \Lambda := \ln\left(\frac{\varphi\left(\frac{\tilde{\gamma}}{1+r}+a\right)}{\varphi(a)}\right), \tag{4.13}$$

and $\varphi(a)$ is the function given in (4.10).

Proof. The proof of the theorem follows from (4.5), as soon as, we solve (4.6). This is the aim of the rest of this proof. To this end, we calculate:

$$\Delta S_t^* = S_t^* - S_{t-1}^* = S_{t-1}^*\frac{Y_t - (1+r)}{1+r}, \qquad t = 1, \ldots.., T$$

Then, by inserting this in (4.6), on $(t \leq \tau)$ we get

$$\begin{aligned}
0 \quad &= S_{t-1}^* E\left[\frac{Y_t-(1+r)}{1+r}\exp\left(\theta_t S_{t-1}^*\frac{Y_t-(1+r)}{1+r}\right)\widetilde{G}_t|\mathcal{F}_{t-1}\right] \\
&= S_{t-1}^*\widetilde{G}_{t-1}E[e^{b\epsilon_t - c}]E\left[\frac{Y_1-(1+r)}{1+r}\exp\left(x\frac{Y_1-(1+r)}{1+r}+aY_1\right)\right]I_{\{x=\theta_t S_{t-1}^*\}} \\
&= S_{t-1}^*\widetilde{G}_{t-1}E[e^{b\epsilon_t - c}]\left\{p\frac{u-r-1}{1+r}\exp\left(\theta_t S_{t-1}^*\frac{u-r-1}{1+r}+au\right)+\right. \\
&\left.(1-p)\frac{d-r-1}{1+r}\exp\left(\theta_t S_{t-1}^*\frac{d-r-1}{1+r}+ad\right)\right\}.
\end{aligned}$$

Hence, the solution to this equation is given by

$$\widetilde{\theta}_t = \frac{1+r}{(u-d)S^*_{t-1}} \ln\left[\frac{(1-p)(1+r-d)}{p(u-1-r)}\right] - \frac{a(1+r)}{S^*_{t-1}} = \frac{1+r}{(u-d)S^*_{t-1}} \ln\left[\frac{p^*(1-p)}{(1-p^*)p}\right] - \frac{a(1+r)}{S^*_{t-1}} \tag{4.14}$$

As a result, we get

$$\widetilde{\theta}_t \Delta S^*_t = \widetilde{\theta}_t S^*_{t-1} \frac{Y_t - (1+r)}{1+r} = \tilde{\gamma} \frac{Y_t - (1+r)}{1+r} \tag{4.15}$$

where $\tilde{\gamma}$ is defined in (4.13). Remark that G_{t-1} and $\widetilde{G}_t$ are related by the following equation

$$\widetilde{G}_{t-1}\varphi(a)\psi(b)e^{-c} = E[\tilde{G}_t|\mathcal{F}_{t-1}] = G_{t-1} \tag{4.16}$$

This implies that

$$\widetilde{G}_{t-1}\psi(b)e^{-c} = \frac{G_{t-1}}{\varphi(a)},$$

and by combining this equality with (4.14), we get

$$\begin{aligned} E[e^{\widetilde{\theta}_t \Delta S^*_t}\widetilde{G}_t|\mathcal{F}_{t-1}] \quad &= \widetilde{G}_{t-1}\psi(b)e^{-c}E\left[\exp\left(\tilde{\gamma}\frac{Y_1-(1+r)}{1+r} + aY_1\right)\right] \\ &= \frac{G_{t-1}}{\varphi(a)}e^{-\tilde{\gamma}}E\left[\exp\left((\frac{\tilde{\gamma}}{1+r} + a)Y_1\right)\right] \\ &= \frac{G_{t-1}}{\varphi(a)}\varphi\left(\frac{\tilde{\gamma}}{1+r} + a\right)e^{-\tilde{\gamma}} \end{aligned}$$

Then, by inserting this equality (4.15) and (4.16) in (4.5), we obtain

$$\begin{aligned} Z_t \quad &= \prod_{i=1}^{t} \frac{e^{\tilde{\gamma}\frac{Y_i-(1+r)}{1+r}} G_{i-1}}{E\left[e^{\tilde{\gamma}\frac{Y_i-(1+r)}{1+r}} \widetilde{G}_i | \mathcal{F}_{i-1}\right]} \\ &= \prod_{i=1}^{t} \frac{e^{\frac{\tilde{\gamma}Y_i}{1+r}} e^{-\tilde{\gamma}} \varphi(a) G_{i-1}}{G_{i-1} e^{-\tilde{\gamma}} \varphi\left(\frac{\tilde{\gamma}}{1+r} + a\right)} \\ &= \exp\left(\frac{\tilde{\gamma}}{1+r} \sum_{i=1}^{t} Y_i - \Lambda t\right) \end{aligned}$$

where Λ is given in (4.13). This implies (4.12) and the proof of this theorem is completed.

□

4.2.2 The Trinomial Model Under Mortality

Hence in this subsection, we suppose that the sequence $(Y_i)_{i\geq 0}$ follows Subsection 3.2.2 where Y_1 is a trinomial random variable.

Theorem 4.2.3. *The Esscher pricing density, when S follows the trinomial model, for $(S^\tau, \mathbb{G})$, is given by*

$$Z_t = \exp\left(\widetilde{\eta} \sum_{i=1}^{t} Y_i - \Gamma(t \wedge \tau)\right), \tag{4.17}$$

where $\widetilde{\eta}$ is the unique root of

$$p(u-r-1)e^{(\alpha+a)u} + q(m-r-1)e^{(\alpha+a)m} + (1-p-q)(d-r-1)e^{(\alpha+a)d} = 0, \tag{4.18}$$

and

$$\Gamma := \ln\left(\frac{\varphi(\widetilde{\eta}+a)}{\varphi(a)}\right). \tag{4.19}$$

Proof. The proof of the theorem follows from (4.5), as soon as, we solve (4.6). This is the aim of the rest of this proof. To this end, we calculate

$$\Delta S_t^* = S_t^* - S_{t-1}^* = S_{t-1}^* \frac{Y_t - (1+r)}{1+r}, \qquad t = 1, ..., T$$

Then, by inserting this in (4.6) and using the fact that Y_t is independent of $\mathcal{F}_{t-1}$ and has the same distribution as Y_1, on $(t \leq \tau)$ we get

$$\begin{aligned} 0 \quad &= E\left[\frac{Y_t - (1+r)}{1+r} \exp\left(\alpha S_{t-1}^* \frac{Y_t - (1+r)}{1+r}\right) \widetilde{G}_t | \mathcal{F}_{t-1}\right] \\ &= \widetilde{G}_{t-1} E[e^{b\epsilon_t - c}] E\left[\frac{Y_1 - (1+r)}{1+r} \exp\left(x \frac{Y_1 - (1+r)}{1+r} + aY_1\right)\right] I_{\{x = \alpha S_{t-1}^*\}}, \\ &= \widetilde{G}_{t-1} E[e^{b\epsilon_t - c}] \left\{ p \frac{u-r-1}{1+r} \exp\left(\alpha S_{t-1}^* \frac{u-r-1}{1+r} + au\right)\right. \\ &\left. + q \frac{m-r-1}{1+r} \exp\left(\alpha S_{t-1}^* \frac{m-r-1}{1+r} + am\right) + (1-p-q) \frac{d-r-1}{1+r} \exp\left(\alpha S_{t-1}^* \frac{d-r-1}{1+r} + ad\right)\right\}. \end{aligned}$$

Therefore, if θ_t is the root for

$$0 = p(u-r-1)e^{\alpha S_{t-1}^* \frac{u-r-1}{1+r} + au} + q(m-r-1)e^{\alpha S_{t-1}^* \frac{m-r-1}{1+r} + am} + (1-p-q)(d-r-1)e^{\alpha S_{t-1}^* \frac{d-r-1}{1+r} + ad}.$$

then $\eta_t := \frac{\theta_t S_{t-1}^*}{1+r}$ is the root of

$$0 = p(u-r-1)e^{(x+a)u} + q(m-r-1)e^{(x+a)m} + (1-p-q)(d-r-1)e^{(x+a)d}.$$

As a result, we get

$$\theta_t \Delta S_t^* = \theta_t S_{t-1}^* \frac{Y_t - (1+r)}{1+r} = (Y_t - (1+r))\widetilde{\eta}, \qquad t = 1, ..., T, \tag{4.20}$$

where $\widetilde{\eta}$ is the unique root of (4.18). By combining this with (4.16),we get

$$\begin{aligned} E[e^{\theta_t \Delta S_t^*}\widetilde{G}_t|\mathcal{F}_{t-1}] &= E[e^{\theta_t S_{t-1}^* \frac{Y_t-(1+r)}{1+r}}\widetilde{G}_t|\mathcal{F}_{t-1}] \\ &= \widetilde{G}_{t-1}E[e^{b\epsilon_t - c}]E[e^{\widetilde{\eta}(Y_1-(1+r))+aY_1}] \\ &= \tfrac{G_{t-1}}{\varphi(a)}[pe^{\widetilde{\eta}(u-r-1)+au} + qe^{\widetilde{\eta}(m-r-1)+am} + (1-p-q)e^{\widetilde{\eta}(d-r-1)+ad}] \\ &= \tfrac{G_{t-1}}{\varphi(a)}e^{-\widetilde{\eta}(1+r)}\varphi(\widetilde{\eta}+a). \end{aligned} \tag{4.21}$$

Then, by inserting this equality (4.20) and (4.21) in (4.5), we obtain

$$\begin{aligned} Z_t &= \prod_{i=1}^{t\wedge\tau} \frac{e^{\widetilde{\eta}(Y_i-(1+r))}G_{i-1}}{E[e^{\widetilde{\eta}(Y_i-(1+r))}\widetilde{G}_i|\mathcal{F}_{t-1}]} \\ &= \prod_{i=1}^{t\wedge\tau} \frac{e^{\widetilde{\eta}Y_i}\varphi(a)}{\varphi(\widetilde{\eta}+a)} \\ &= \exp\left(\widetilde{\eta}\sum_{i=1}^{t\wedge\tau} Y_i - \Gamma t\right), \end{aligned}$$

where Λ is given in (4.19). This implies (3.14). Now, we prove that (4.18) has a unique solution. To this end, we put

$$f(\alpha) = p(u-r-1)e^{\alpha(u-r-1)+au} + q(m-r-1)e^{\alpha(m-r-1)+am} + (1-p-q)(d-r-1)e^{\alpha(d-r-1)+ad}, \qquad \alpha \in \mathbb{R}.$$

Then, we have

$$f'(\alpha) = p(u-r-1)^2e^{\alpha(u-r-1)+au} + q(m-r-1)^2e^{\alpha(m-r-1)+am} + (1-p-q)(d-r-1)^2e^{\alpha(d-r-1)+ad} > 0.$$

It is clear that f is strictly increasing and continuous, and satisfies $f(+\infty) = +\infty$ and $f(-\infty) = -\infty$. Hence, thanks again to the intermediate value theorem, these exists unique $\tilde{\eta} \in \mathbb{R}$ particular $f(\tilde{\eta}) = 0$. This ends the proof of the theorem. $\square$

Chapter 5

The Esscher for the continuous-time

In this chapter, we extend our studies to the continuous-time setting. Hence, throughout this chapter, we suppose that $W = (W_t)_{t\geq 0}$ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Then the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is the augmented natural filtration of W. For simplicity, we assume that the risk free rate $r = 0$. We suppose that the stock price process $S = (S_t)_{0\leq t\leq T}$ is given by

$$S_t = S_0 e^{X_t}, \qquad \text{where} \qquad X_t = (\mu - \frac{\sigma^2}{2})t + \sigma W_t, \qquad 0 \leq t \leq T, \tag{5.1}$$

where S_0 is a positive number, $\mu \in \mathbb{R}$ is the drift rate and σ is a positive number representing the volatility rate.

A direct application of *Itô*'s formula allows us to conclude that S is the unique solution to the following stochastic differential equation

$$dS_t = S_t d\widetilde{X}_t, \qquad \text{where} \qquad \widetilde{X}_t = \mu t + \sigma W_t. \tag{5.2}$$

5.1 Definitions of Esscher pricing measure/density

For continuous-time models, there are two definitions of Esscher transform in the literature, depending wether we consider process X or the process $\widetilde{X}$ that appeared in the dynamics of S in (5.1) and (5.2) respectively.

Definition 5.1.1. *(Esscher for exponential processes) We call exponential Esscher martingale density for the model (S, $\mathbb{F}$), the process $\widetilde{Z}$ is given by*

$$\widetilde{Z}_t = \exp(\widetilde{\theta} X_t - \gamma(\widetilde{\theta})t), \qquad 0 \leq t \leq T.$$

Here $\gamma(\theta)$ is a function of $\theta \in \mathbb{R}$, such that the process $\exp(\theta X_t - \gamma(\theta)t)$ is a martingale, and $\widetilde{\theta}$ is a real number such that $(\widetilde{Z}_t S_t)_{0\leq t\leq T}$ is a martingale.

Definition 5.1.2. *We call linear Esscher martingale density for (S, $\mathbb{F}$), the process*

$$\widehat{Z}_t = \exp(\widehat{\theta}\widetilde{X}_t - k(\widehat{\theta})t), \qquad 0 \leq t \leq T,$$

where $k(\theta)$ is a function of a real number θ such that $\exp(\theta\widetilde{X}_t - k(\theta)t)$ is a martingale, and $\widehat{\theta}$ is a real number such that $(\widehat{Z}_t S_t)_{0\leq t\leq T}$ is also a martingale.

5.2 Esscher Pricing Measures

Theorem 5.2.1. *The Esscher pricing density Z, for the model $(S, \mathbb{F})$ defined in (5.1), or equivalently (5.2), is given by*

$$Z_t = \exp\left(\frac{-\mu}{\sigma}W_t - \frac{\mu^2}{2\sigma^2}t\right), \qquad 0 \leq t \leq T. \tag{5.3}$$

Proof. This proof is divided into two parts. In the first part, we need to find the process of $V_t = \int_0^t v_s(\theta)ds$ that makes the process

$$Z^\theta := \exp(\int_0^t \theta_s dX_s + \int_0^t v_s(\theta)ds)$$

a martingale. In the second part, we have to find the value of $\widetilde{\theta}$ that makes the process $Z^{\widetilde{\theta}}S$ a martingale.

Part 1. Consider the process Z^θ

$$Z_t^\theta = e^{Y_t}, \quad \text{where} \quad Y_t = \int_0^t \theta_u dX_u + \int_0^t v_u(\theta)du \tag{5.4}$$

where S is given in (5.1).

By applying Itô's formula to Z^θ, and get

$$\begin{aligned} dZ_t^\theta &= Z_t dY_t + \frac{1}{2}Z_t d\left\langle Y\right\rangle_t \\ &= Z_t\left[\theta_t\mu - \frac{\theta_t\sigma^2}{2} + v_t + \frac{\theta^2\sigma^2}{2}\right]dt + \theta_t\sigma dW_t. \end{aligned}$$

As a result, Z^θ is a martingale if and only if

$$v_t(\theta) = -\theta_t\mu + \frac{\theta_t\sigma^2}{2} - \frac{\theta^2\sigma^2}{2}. \tag{5.5}$$

Part 2. In order to find the value of θ_t, we need to calculate the dynamics of $Z^\theta S$ using *Itô*'s formula. This leads to

$$d(Z^\theta S)_t = Z_t S_t[(\mu + \theta_t\sigma^2)dt + \sigma(1+\theta_t)dW_t].$$

Therefore, $Z^\theta S$ is a martingale if and only if

$$\theta_t = \frac{-\mu}{\sigma^2}. \tag{5.6}$$

Then, by inserting (5.5) and (5.6) in (5.4), we obtain (5.3). This ends the proof of the theorem. □

We didn't specify in the previous theorem wether we consider the linear Esscher pricing density or the exponential Esscher pricing density for the model $(\mathrm{S}, \mathbb{F})$. This is due to the fact that both densities coincide in this Brownian setting. In fact, the exponential Esscher pricing density, defined by

$$\widetilde{Z}_t = \exp(\widetilde{\theta}X_t - \gamma(\widetilde{\theta})t), \qquad 0 \le t \le T,$$

can be obtained as follows. A direct application of *Itô*'s formula, we get

$$dZ_t^\theta \qquad = Z_t^\theta(\theta\sigma dW_t + (\theta(\mu - \frac{\sigma^2}{2}) + \frac{\theta^2\sigma^2}{2} - \gamma(\theta))dt),$$

and hence Z^θ is a martingale if and only if

$$\gamma(\theta) = \theta(\mu - \frac{\sigma^2}{2}) + \frac{\theta^2\sigma^2}{2},$$

and in this case we have

$$dZ_t^\theta = Z_t^\theta \sigma\theta dW_t, \quad Z_0^\theta = 1 \tag{5.7}$$

Similarly, we recall that the linear Esscher density

$$L_t^\theta = \exp(\theta \widetilde{X}_t - k(\theta)t).$$

Due to a direct application of *Itô*'s formula, we get

$$dL_t^\theta \qquad = L_t^\theta(\theta\sigma dW_t + (\frac{\theta^2\sigma^2}{2} - k(\theta) + \theta\mu)dt)$$

L^θ is a martingale if and only if

$$k(\theta) = \frac{\theta^2\sigma^2}{2} + \theta\mu,$$

and in this case we have

$$dL_t^\theta = L_t^\theta \sigma\theta dW_t, \quad L_0^\theta = 1 \tag{5.8}$$

Therefore, in this Brownian setting, Z^θ and L^θ are solutions of the following equation

$$dZ_t = \theta\sigma Z_t dW_t, \quad Z_0 = 1.$$

Thus, Z^θ and L^θ conicide with the solution to this SDE given by (see Theorem 2.2.3)

$$Z_t^\theta = L_t^\theta = \exp\left(\theta\sigma W_t - \frac{\sigma^2\theta^2}{2}t\right).$$

This ends the proof of the theorem.

Chapter 6

Esscher under Mortality for Continuous-time

In this chapter, we address the Esscher pricing measure under mortality for the continuous time setting. Precisely, we suppose that the stock's price process follows the Brownian motion model of chapter 5 (see (5.1)-(5.2)).

Lemma 6.0.1. *Let α, β be two real numbers.The following process*

$$K_t = \exp(\alpha W_t + \beta t),$$

is a super-martingale if and only if $\beta \leq \frac{-\alpha^2}{2}$.

Proof. By applying Itô's formula, we get

$$dK_t = K_t \left[\alpha dW_t + \beta dt + \frac{1}{2}\alpha^2 dt \right].$$

Then, K_t is a super-martingale if and only if

$$\beta + \frac{\alpha^2}{2} \leq 0.$$

This ends the proof of the lemma. $\square$

Throughout this chapter, we suppose that the death time τ follows the model given by its survival conditional probability $\widetilde{G}$ as follows,

$$\widetilde{G}_t := P(\tau \geq t | \mathcal{F}_t) = \exp(\alpha W_t + \beta t), \quad \text{with } \beta < \frac{-\alpha^2}{2}. \tag{6.1}$$

6.1 Esscher under Mortality

In this section, we consider the following process

$$\widehat{W}_t := W_{t\wedge\tau} - \alpha(t \wedge \tau), \tag{6.2}$$

which is a martingale under $\mathbb{G}$, see [1] for details about this and related discussion. The stock's price process stopped at τ, S^τ, is given by

$$S_t^\tau = S_0 e^{X_{t\wedge\tau}}, \qquad \text{where} \qquad X_{t\wedge\tau} = (\mu - \frac{\sigma^2}{2} + \alpha\sigma)(t \wedge \tau) + \sigma\widehat{W}_t, \tag{6.3}$$

where S_0 is a positive number, $\mu \in \mathbb{R}$ is the drift rate, and σ is a positive number representing the volatility rate. The dynamics of S^τ are given by

$$dS_t^\tau = S_t^\tau d\widetilde{X}_{t\wedge\tau}, \qquad \text{where} \qquad \widetilde{X}_{t\wedge\tau} := (\mu + \alpha\sigma)(t \wedge \tau) + \sigma\widehat{W}_t. \tag{6.4}$$

Proposition 6.1.1. *Let θ be a real number, and consider the two processes*

$$Z_t^\theta := \exp(\theta\widetilde{X}_{t\wedge\tau} - k(\theta)(t \wedge \tau)), \quad L_t^\theta := \exp(\theta X_{t\wedge\tau} - \gamma(\theta)(t \wedge \tau)), \qquad 0 \leq t \leq T.$$

where $k(\theta)$ and $\gamma(\theta)$ are functions of θ.Then the following properties hold:
(1) Z^θ is a $\mathbb{G}$-martingale if and only if

$$k(\theta) = \theta\mu + \frac{\theta^2\sigma^2}{2} + \alpha\theta\sigma. \tag{6.5}$$

(2) L^θ is a $\mathbb{G}$-martingale if and only if

$$\gamma(\theta) = \theta\mu + \frac{\theta^2\sigma^2}{2} + \alpha\theta\sigma - \frac{\theta\sigma^2}{2}. \tag{6.6}$$

(3) The two processes Z^θ and L^θ coincide and

$$Z^\theta = L^\theta = \exp\left(\theta\sigma\widehat{W}_t - \frac{\theta^2\sigma^2}{2}(t \wedge \tau)\right), \qquad 0 \leq t \leq T.$$

Proof. (1) To prove property (1), we directly apply Itô's lemma and get

$$\begin{aligned} dZ_t^\theta &= Z_t^\theta \theta d\widetilde{X}_t + \frac{1}{2} Z_t^\theta \theta^2 d\left\langle \widetilde{X} \right\rangle_t \\ &= Z_t^\theta \left(\theta\sigma d\widehat{W}_t + \left(\theta(\mu + \sigma\alpha) + \frac{\theta^2\sigma^2}{2} - k(\theta) \right) I_{]0,\tau]} dt \right). \end{aligned}$$

Hence, Z^θ is a martingale if and only if

$$k(\theta) = \theta(\mu + \sigma\alpha) + \frac{\theta^2\sigma^2}{2}.$$

This proves the property(1).

(2) Similarly, a direct application of Itô's lemma, we get

$$\begin{aligned} dL_t^\theta &= L_t^\theta \theta dX_t + \frac{1}{2} L_t^\theta \theta^2 d\left\langle X \right\rangle_t \\ &= L_t^\theta \left(\theta\sigma d\widehat{W}_t + \left(\theta(\mu - \frac{\sigma^2}{2} + \sigma\alpha) + \frac{\theta^2\sigma^2}{2} - \gamma(\theta) \right) I_{]0,\tau]} dt \right), \end{aligned}$$

and L^θ is a martingale if and only if

$$\gamma(\theta) = \theta(\mu + \sigma\alpha) + \frac{\theta^2\sigma^2}{2} - \frac{\theta\sigma^2}{2}.$$

This proves the property (2).

(3) Finally, we prove the third property. By inserting (6.5) in the quantity

$$Z_t^\theta = \exp(\theta\widetilde{X}_{t\wedge\tau} - k(\theta)(t \wedge \tau)),$$

we obtain

$$\begin{aligned} Z_t^\theta &= \exp(\theta \widetilde{X}_{t\wedge\tau} - k(\theta)(t\wedge\tau)) \\ &= \exp(\theta(\mu+\alpha\sigma)t\wedge\tau + \theta\sigma\widehat{W}_t - (\theta(\mu+\sigma\alpha) + \frac{\theta^2\sigma^2}{2})(t\wedge\tau)) \\ &= \exp(\theta\sigma\widehat{W}_t - \frac{\theta^2\sigma^2}{2}(t\wedge\tau)). \end{aligned}$$

Similarly, by inserting (6.6) in the quantity

$$L_t^\theta = \exp(\theta X_{t\wedge\tau} - \gamma(\theta)(t\wedge\tau)),$$

we obtain

$$\begin{aligned} L_t^\theta &= \exp(\theta X_{t\wedge\tau} - \gamma(\theta)(t\wedge\tau)) \\ &= \exp(\theta(\mu - \frac{\sigma^2}{2} + \alpha\sigma)t\wedge\tau + \theta\sigma\widehat{W}_t - (\theta(\mu - \frac{\sigma^2}{2} + \sigma\alpha) + \frac{\theta^2\sigma^2}{2})(t\wedge\tau)) \\ &= \exp(\theta\sigma\widehat{W}_t - \frac{\theta^2\sigma^2}{2}(t\wedge\tau)) \end{aligned}$$

These calculations prove the property (3), and the proof of the proposition is complete. □

Proposition 6.1.1.tells us that the linear and exponential Esscher pricing densities coincide also for (S^τ, $\mathbb{G}$). Then, in the following, we determine this Esscher pricing density.

Theorem 6.1.1. *Suppose that r=0, then the Esscher pricing density for (S^τ, $\mathbb{G}$), is given* *by:*

$$\begin{aligned} Z_t^{\mathbb{G}} &= \exp\left[-\left(\frac{\mu+\sigma\alpha}{\sigma}\right)\widehat{W}_t - \frac{1}{2}\frac{(\mu+\sigma\alpha)^2}{\sigma^2}(t\wedge\tau)\right] \\ &= \exp\left[-\left(\frac{\mu}{\sigma}+\alpha\right)W_{t\wedge\tau} - \frac{1}{2}\left(\left(\frac{\mu}{\sigma}\right)^2 - \alpha^2\right)(t\wedge\tau)\right]. \end{aligned} \tag{6.7}$$

Proof. This proof contains two steps. In the first step, we need to find the value of y_t that makes

$$Z_t^\theta = \exp(\int_0^{t\wedge\tau} \theta_s d\widetilde{X}_s + \int_0^{t\wedge\tau} y_s ds)$$

a martingale. In the second step, we need to find the value of θ that makes $Z^\theta S^\tau$ a martingale.

Step 1. Remark that on $(t \leq \tau)$:

$$Z_t^\theta = e^{Y_t}, \quad \text{where } Y_t = \int_0^t \theta_s d\widetilde{X}_s^\tau + \int_0^t y_s ds. \tag{6.8}$$

Then, in order to find the value of y_t, we have to derive the dynamics of Z_t^θ using *Itô*'s formula, and get

$$\begin{aligned} dZ_t &= Z_t dY_t + \frac{1}{2} Z_t d\left\langle Y \right\rangle_t \\ &= Z_t\left[(\theta_t\mu + \theta_t\sigma\alpha + y_t + \frac{\theta_t^2\sigma^2}{2})dt + \sigma\theta_t d\widehat{W}_t\right]. \end{aligned}$$

As a result, Z^θ is a martingale if and only if

$$y_t = -\theta_t\mu - \theta_t\sigma\alpha - \frac{\theta_t^2\sigma^2}{2}. \tag{6.9}$$

Step 2. A direct application of *Itô*'s formula, we get

$$d(Z^\theta S^\tau)_t = Z_t S_t^\tau \left[(\mu + \sigma\alpha + \theta_t \sigma^2) I_{]0,\tau]} dt + (\sigma + \theta_t \sigma) d\widehat{W}_t \right],$$

Therefore, $Z^\theta S^\tau$ is a martingale if and only if

$$\theta_t = \frac{-(\mu + \sigma\alpha)}{\sigma^2}. \tag{6.10}$$

Then, by inserting (6.9) and (6.10) in (6.8), we obtain (6.7). This ends the proof of the theorem. □

6.2 Esscher pricing for some contracts

Throughout this chapter, the survival process $\widetilde{G}$ defined in (6.1) satisfies

$$G_t = P(\tau > t | \mathcal{F}_t) = \widetilde{G}_t =: 1 + m_t - V_t,$$

where m_t is a martingale and V is an increasing process given by

$$m_t := \alpha \int_0^t G_s dW_s \quad \text{and} \quad V_t := -(\beta + \frac{\alpha^2}{2}) \int_0^t G_s ds.$$

Lemma 6.2.1. *For any process $\mathbb{F}$-adapted process L, we have*

$$E[L_{\tau\wedge T}] = E\left[\int_0^T L_t dV_t + G_T L_T \right] = -(\beta + \frac{\alpha^2}{2}) E\left[\int_0^T L_t G_t dt \right] + E[G_T L_T].$$

For the proof of this lemma we refer the reader to [5].

Throughout the rest of this chapter, we consider the following notations

$$\begin{aligned} d(t) &:= \frac{\ln\left(\frac{S_0}{K}\right) + (\mu - \frac{\sigma^2}{2})t}{\sigma}, \qquad 0 \le t \le T, \\ d_1(t) &:= \frac{\ln\left(\frac{S_0}{K}\right) - \frac{\sigma^2}{2}t}{\sigma\sqrt{t}}, \quad d_2(t) := \frac{\ln\left(\frac{S_0}{K}\right) + \frac{\sigma^2}{2}t}{\sigma\sqrt{t}}. \end{aligned} \tag{6.11}$$

Definition 6.2.1. *We call variable annuity contract a contract that has the payoff*

$$D(T) = \max(K, S(T \wedge \tau)).$$

Theorem 6.2.2. *Consider the notations in (6.11) and let $Z^{\mathbb{G}}$ and Φ be the martingale defined in (6.7), and the cumulative distribution function of the standard normal distribution. Then the following hold.*

(a) *The Esscher price of the call option, based on the model (S^τ,$\mathbb{G}$), is given by:*

$$\begin{aligned} C^E(0) &= E_Q[(S(\tau \wedge T) - K)^+ e^{-rT\wedge\tau}] = E(Z_T^{\mathbb{G}}(S(\tau \wedge T) - K)^+ e^{-rT\wedge\tau}) \\ &= -(\frac{\alpha^2}{2} + \beta)\left(S_0 \int_0^T e^{(\beta - r + \frac{\alpha^2}{2})t}\Phi(d_2(t))dt - K\int_0^T e^{(\beta - r + \frac{\alpha^2}{2})t}\Phi(d_1(t))dt\right) + \\ & e^{(\beta - r + \frac{\alpha^2}{2})T}\left(S_0\Phi(d_2(T)) - K\Phi(d_1(T))\right). \end{aligned} \tag{6.12}$$

(b) *The Esscher price of the put, for the model (S^τ, $\mathbb{G}$), is given by:*

$$\begin{aligned} P^E(0) &= E_Q[(S(\tau \wedge T) - K)^- e^{-rT\wedge\tau}] = E(Z_T^{\mathbb{G}}(S(\tau \wedge T) - K)^- e^{-rT\wedge\tau}) \\ &= C^E(0) - S(0) - K(\frac{\alpha^2}{2} + \beta)\int_0^T e^{(\beta - r + \frac{\alpha^2}{2})t}dt + Ke^{(\beta - r + \frac{\alpha^2}{2})T}. \end{aligned} \tag{6.13}$$

(c) *The Esscher price* $D(0)$ *of the variable annuity contract, for the model* (S^τ, $\mathbb{G}$), *is given by:*

$$\begin{aligned} D(0) &= E[Z_T^{\mathbb{G}} D(T) e^{-r(\tau\wedge T)}] \\ &= C^E(0) - K(\frac{\alpha^2}{2}+\beta)\int_0^T e^{(\beta-r+\frac{\alpha^2}{2})t}dt + Ke^{(\beta-r+\frac{\alpha^2}{2})T}. \end{aligned} \tag{6.14}$$

Proof. This proof is divided into three parts. In the first part, we prove the call option price under Esscher measure. In the second part, we show that by using the call put parity, we get the put option price under the Esscher measure. The third part calculates the Esscher price of the variable annuity contract.

Part 1. Remark that:

$$(S_{\tau\wedge T} - K)^+ = (S_{\tau\wedge T} - K)I_{\{S_{\tau\wedge T}>K\}} = S_{\tau\wedge T} I_{\{S_{\tau\wedge T}>K\}} - KI_{\{S_{\tau\wedge T}>K\}} \tag{6.15}$$

We will calculate the quantity of $E[Z_T^{\mathbb{G}} e^{-r(\tau\wedge T)} S_{\tau\wedge T} I_{\{S_{\tau\wedge T}>K\}}]$ as follows

$$\begin{aligned} E[Z_T^{\mathbb{G}} e^{-r(T\wedge\tau)} S_{T\wedge\tau} I_{\{S_{T\wedge\tau}>K\}}] &= -E\int_0^T e^{-rt} Z_t S_t I_{\{S_t>K\}} G_t(\frac{\alpha^2}{2}+\beta)dt + E[e^{-rT} Z_T S_T G_T I_{\{S_T>K\}}] \\ &= -(\frac{\alpha^2}{2}+\beta)E\int_0^T e^{-rt} Z_t S_t I_{\{S_t>K\}} G_t dt + E[e^{-rT} Z_T S_T G_T I_{\{S_T>K\}}] \\ &= -(\frac{\alpha^2}{2}+\beta)\int_0^T E[e^{-rt} Z_t S_t I_{\{S_t>K\}} G_t]dt + E[e^{-rT} Z_T S_T G_T I_{\{S_T>K\}}] \end{aligned}$$

Thus, we need to calculate $E[e^{-rt}Z_tS_tI_{\{S_t>K\}}G_t]$ for $0 \leq t \leq T$,

$$\begin{aligned} E[e^{-rt}Z_tS_tI_{\{S_t>K\}}G_t] &= E[e^{-rt}S_0e^{X_t}e^{-(\frac{\mu}{\sigma}+\alpha)W_t-\frac{1}{2}((\frac{\mu}{\sigma})^2-\alpha^2)t}e^{\alpha W_t+\beta t}I_{\{S_0e^{X_t}>K\}}] \\ &= E[e^{-rt}S_0e^{(\mu-\frac{\sigma^2}{2}+\beta-\frac{1}{2}((\frac{\mu}{\sigma})^2-\alpha^2))t+(\sigma+\alpha-\frac{\mu}{\sigma}-\alpha)W_t}I_{\{S_0e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t}>K\}}] \\ &= E[e^{-rt}S_0e^{(\sigma-\frac{\mu}{\sigma})W_t+(\mu-\frac{\sigma^2}{2}+\beta-\frac{1}{2}((\frac{\mu}{\sigma})^2-\alpha^2))t}I_{\{S_0e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t}>K\}}]. \end{aligned}$$

Since $(S_0e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t} > K) = (W_t > -d(t))$, we obtain

$$\begin{aligned} E[e^{-rt}Z_tS_tI_{\{S_t>K\}}G_t] &= e^{-rt}S_0\exp\left((\mu-\frac{\sigma^2}{2}+\beta-\frac{1}{2}((\frac{\mu}{\sigma})^2-\alpha^2))t\right)\int_{-d(t)}^{\infty}\frac{e^{\Gamma x-\frac{x^2}{2t}}}{\sqrt{2\pi t}}dx \\ &= e^{-rt}S_0\exp\left((\mu-\frac{\sigma^2}{2}+\beta-\frac{1}{2}((\frac{\mu}{\sigma})^2-\alpha^2))t\right)\int_{-d(t)}^{\infty}\frac{e^{\frac{-1}{2t}[(x-\Gamma t)^2-\Gamma^2t^2]}}{\sqrt{2\pi t}}dx \\ &= e^{-rt}S_0\exp\left((\mu-\frac{\sigma^2}{2}+\beta-\frac{1}{2}((\frac{\mu}{\sigma})^2-\alpha^2)+\frac{\Gamma^2}{2})t\right)P\left[Z>\left(\frac{-d(t)}{\sqrt{t}}-\Gamma\sqrt{t}\right)\right] \\ &= e^{-rt}S_0\exp\left((\beta+\frac{\alpha^2}{2})t\right)\left[1-\Phi\left(\frac{-d(t)}{\sqrt{t}}-\Gamma\sqrt{t}\right)\right] \\ &= e^{-rt}S_0\exp\left((\beta+\frac{\alpha^2}{2})t\right)\Phi\left(d_2(t)\right), \end{aligned}$$

where $\Gamma = \sigma - \frac{\mu}{\sigma}$ and $d_2(t)$ is given by (6.11).

In particular, when $t = T$, we get

$$E[e^{-rT}Z_TS_TG_TI_{\{S_T>K\}}] = S_0e^{(\beta-r+\frac{\alpha^2}{2})T}\Phi\left(d_2(T)\right).$$

By combining the calculations of the two parts, we get

$$
\begin{aligned}
E[e^{-r(T\wedge\tau)}Z_{T\wedge\tau}S_{T\wedge\tau}I_{\{S_{T\wedge\tau}>K\}}] \quad &= -(\frac{\alpha^2}{2}+\beta)\int_0^T S_0 e^{(\beta-r+\frac{\alpha^2}{2})t}\Phi(d_2(t))dt \\
&+S_0^{(\beta-r+\frac{\alpha^2}{2})T}\Phi(d_2(T)). \qquad (6.16)
\end{aligned}
$$

Next, we calculate the strike price part i.e $E[KZ_T^{\mathbb{G}}e^{-r(\tau\wedge T)}I_{\{S_{\tau\wedge T}>K\}}]$

$$
\begin{aligned}
KE[Z_T^{\mathbb{G}}e^{-r(T\wedge\tau)}I_{\{S_{T\wedge\tau}>K\}}] \quad &= -KE\int_0^T e^{-rt}Z_tI_{\{S_t>K\}}G_t(\frac{\alpha^2}{2}+\beta)dt + KE[e^{-rT}Z_TI_{\{S_T>K\}}G_T] \\
&= -K(\frac{\alpha^2}{2}+\beta)E\int_0^T e^{-rt}Z_tI_{\{S_t>K\}}G_tdt + KE[e^{-rT}Z_TI_{\{S_T>K\}}G_T] \\
&= -K(\frac{\alpha^2}{2}+\beta)\int_0^T E[e^{-rt}Z_tI_{\{S_t>K\}}G_t]dt + KE[e^{-rT}Z_TI_{\{S_T>K\}}G_T]
\end{aligned}
$$

Similarly as before, we need to calculate $E[e^{-rt}Z_tI_{\{S_t>K\}}G_t]$ for $0 \leq t \leq T$.

$$
\begin{aligned}
KE[e^{-rt}Z_tI_{\{S_t>K\}}G_t] \quad &= KE[e^{-rt}e^{-(\frac{\mu}{\sigma}+\alpha)W_t-\frac{1}{2}((\frac{\mu}{\sigma})^2-\alpha^2))t}e^{\alpha W_t+\beta t}I_{\{S_0e^{X_t}>K\}}] \\
&= KE[e^{-rt}e^{(\alpha-(\frac{\mu}{\sigma}+\alpha))W_t+(\beta-\frac{1}{2}((\frac{\mu}{\sigma})^2-\alpha^2))t}I_{\{S_0e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t}>K\}}] \\
&= KE[e^{-rt}e^{-\frac{\mu}{\sigma}W_t+(\beta-\frac{1}{2}((\frac{\mu}{\sigma})^2-\alpha^2))t}I_{\{S_0e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t}>K\}}]
\end{aligned}
$$

Since $(S_0 e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t} > K) = (W_t > -d(t))$, we get

$$\begin{aligned}
KE[e^{-rt}Z_t I_{\{S_t>K\}}G_t] &= Ke^{-rt}\exp\left((\beta-\frac{1}{2}((\frac{\mu}{\sigma})^2-\alpha^2))t\right)\int_{-d(t)}^{\infty}\frac{e^{-(\mu/\sigma)x-\frac{x^2}{2t}}}{\sqrt{2\pi t}}dx \\
&= Ke^{-rt}\exp\left((\beta-\frac{1}{2}((\frac{\mu}{\sigma})^2-\alpha^2))t\right)\int_{-d(t)}^{\infty}\frac{\exp(\frac{-(x+t\mu/\sigma)^2}{2t}+(\mu^2/2\sigma^2)t)}{\sqrt{2\pi t}}dx \\
&= Ke^{-rt}\exp\left((\beta-\frac{1}{2}((\frac{\mu}{\sigma})^2-\alpha^2)+\frac{\mu^2}{2\sigma^2})t\right)P\left[Z>\left(\frac{-d(t)}{\sqrt{t}}+(\mu/\sigma)\sqrt{t}\right)\right] \\
&= Ke^{-rt}\exp\left((\beta+\frac{\alpha^2}{2})t\right)\left[1-\Phi\left(\frac{-d(t)}{\sqrt{t}}+(\mu/\sigma)\sqrt{t}\right)\right] \\
&= Ke^{-rt}\exp\left((\beta+\frac{\alpha^2}{2})t\right)\Phi(d_1(t)),
\end{aligned}$$

where $d_1(t)$ is given by (6.11).

In particular, when $t = T$, we get

$$KE[e^{-rT}Z_T I_{\{S_T>K\}}G_T] = Ke^{-rT}e^{(\beta+\frac{\alpha^2}{2})T}\Phi(d_1(T)),$$

By putting together the calculations of the two parts of the strike price calculation, we get

$$\begin{aligned}
KE[e^{-r(T\wedge\tau)}Z_{T\wedge\tau}I_{\{S_{T\wedge\tau}>K\}}] &= -K(\frac{\alpha^2}{2}+\beta-r)\int_0^T e^{(\beta-r+\frac{\alpha^2}{2})t}\Phi(d_1(t))dt \\
&+Ke^{-rT}e^{(\beta+\frac{\alpha^2}{2})T}\Phi(d_1(T)). \qquad (6.17)
\end{aligned}$$

As a result, by combining (6.15), (6.16) and (6.17), (6.12) follows

$$\begin{aligned}
C^E(0) &= -(\frac{\alpha^2}{2}+\beta)\int_0^T S_0 e^{(\beta-r+\frac{\alpha^2}{2})t}\Phi(d_2(t))dt + S_0 e^{(\beta-r+\frac{\alpha^2}{2})T}\Phi(d_2(T)) + K(\frac{\alpha^2}{2}+\beta) \\
&\int_0^T e^{(\beta-r+\frac{\alpha^2}{2})t}\Phi(d_1(t))dt - Ke^{(\beta-r+\frac{\alpha^2}{2})T}\Phi(d_1(T)).
\end{aligned}$$

Part 2. Here, we prove (6.13). To this end, we remark that

$$S_{\tau\wedge T} - K = (S_{\tau\wedge T} - K)^+ - (S_{\tau\wedge T} - K)^-,$$

Then calculate

$$\begin{aligned} E[Z_T^{\mathbb{G}}(S_{T\wedge\tau} - K)e^{-r(\tau\wedge T)}] &= S_0 - KE[Z_T^{\mathbb{G}} e^{-r(\tau\wedge T)}] \\ &= S_0 + K(\frac{\alpha^2}{2} + \beta)\int_0^T E[e^{-rt} Z_t G_t]dt - KE[e^{-rT} Z_T G_T] \\ &= S_0 + K(\frac{\alpha^2}{2} + \beta)\int_0^T e^{(\beta - r + \frac{\alpha^2}{2})t}dt - Ke^{(\beta - r + \frac{\alpha^2}{2})T} \quad (6.18) \end{aligned}$$

Therefore, we derive

$$P^E(0) = E[Z_T^{\mathbb{G}}(S_{\tau\wedge T} - K)^- e^{-r(\tau\wedge T)}] = C^E(0) - S_0 - K(\frac{\alpha^2}{2} + \beta)\int_0^T e^{(\beta - r + \frac{\alpha^2}{2})t}dt + Ke^{(\beta - r + \frac{\alpha^2}{2})T}.$$

This proves (6.13).

Part 3. Here, we prove assertion (3). Remark that the payoff $D(T)$ satisfies

$$\begin{aligned} D(T) &= \max(S(T), K) \\ &= \max(S(T), K) - K + K \\ &= (S(T) - K)^+ + K \end{aligned}$$

Therefore,

$$\begin{aligned} D(0) &= E[Z_T^{\mathbb{G}} D(T) e^{-r(\tau\wedge T)}] \\ &= E[Z_T^{\mathbb{G}} e^{-r(\tau\wedge T)}(S(T)-K)^+] + E[e^{-r(\tau\wedge T)} Z_T^{\mathbb{G}} K] \\ &= C^E(0) + E[e^{-r(\tau\wedge T)} Z_T^{\mathbb{G}} K] \\ &= C^E(0) - K(\frac{\alpha^2}{2}+\beta)\int_0^T e^{(\beta-r+\frac{\alpha^2}{2})t}dt + Ke^{(\beta-r+\frac{\alpha^2}{2})T} \end{aligned}$$

This proves (6.14), the last equality follows from the calculation in part 2 and the proof of the theorem is complete. □

Corollary 6.2.1. *Suppose that $\alpha = 0$. Then the prices $C^E(0)$, $P^E(0)$ and $D(0)$ become*

$$\begin{aligned} C^E(0) &= -\beta\int_0^T S_0 e^{(\beta-r)t}\Phi(d_2(t))dt + S_0 e^{(\beta-r)T}\Phi(d_2(T)) + K\beta\int_0^T e^{(\beta-r)t}\Phi(d_1(t))dt - \\ &\quad Ke^{(\beta-r)T}\Phi(d_1(T)) \\ P^E(0) &= C^E(0) - S(0) - K\beta\int_0^T e^{(\beta-r)t}dt + Ke^{(\beta-r)T} \\ D(0) &= C^E(0) - K\beta\int_0^T e^{(\beta-r)t}dt + Ke^{(\beta-r)T}. \end{aligned}$$

This corollary follows immediately from Theorem 6.2.2. by putting $\alpha = 0$.

Corollary 6.2.2. *For $t \in [0,T]$*

$$\begin{aligned} C_{ESR}(t) &= -(\frac{\alpha^2}{2}+\beta)\left(S_t\int_0^{T-t} e^{(\beta-r+\frac{\alpha^2}{2})s}\Phi(d_2(s))ds + K\int_0^{T-t} e^{(\beta-r+\frac{\alpha^2}{2})s}\Phi(d_1(s))ds\right) + \\ &\quad e^{(\beta-r+\frac{\alpha^2}{2})(T-t)}\left(S_t\Phi(d_2(T-t)) - K\Phi(d_1(T-t))\right) \end{aligned}$$

$$P_{ESR}(t) \quad = C_{ESR}(t) - S(t) - K(\frac{\alpha^2}{2} + \beta)\int_0^{T-t} e^{(\beta - r + \frac{\alpha^2}{2})s} ds + Ke^{(\beta - r + \frac{\alpha^2}{2})(T-t)}$$

$$D_{ESR}(t) \quad = C_{ESR}(t) - K(\frac{\alpha^2}{2} + \beta)\int_0^{T-t} e^{(\beta - r + \frac{\alpha^2}{2})s} ds + Ke^{(\beta - r + \frac{\alpha^2}{2})(T-t)}.$$

This corollary follows directly from Theorem 6.2.2. by replacing T and S_0 with $T - t$ and $S(t)$ respectively.

6.3 Relationship to Black-Scholes Prices

Definition 6.3.1. *The Esscher price for an European call (respectively put) with strike K and maturity T for the model (S,$\mathbb{F}$), will be denoted by*

$$C^{\mathbb{F}}_{ESR}(S_0, K, r, \sigma, T) \qquad (\textit{ respectively } P^{\mathbb{F}}_{ESR}(S_0, K, r, \sigma, T))$$

Definition 6.3.2. *The Esscher price for an European call (respectively put) with strike K and maturity T for the model (S^{τ},$\mathbb{G}$), will be denoted by*

$$C^{\mathbb{G}}_{ESR}(S_0, K, r, \sigma, T) \qquad (\textit{ respectively } P^{\mathbb{G}}_{ESR}(S_0, K, r, \sigma, T))$$

Definition 6.3.3. *The Black-Scholes price for an European call (respectively put) with strike K and maturity T for the model (S,$\mathbb{F}$), will be denoted by*

$$C_{BS}(S_0, K, r, \sigma, T) \qquad (\textit{ respectively } P_{BS}(S_0, K, r, \sigma, T))$$

Theorem 6.3.1. *Consider a call and put option on the stock given by (5.1), that expire at time T with a strike price K. Then*

$$C_{BS}(S_0, K, r, \sigma, T) = S_0\Phi(\delta_1) - Ke^{-rT}\Phi(\delta_2),$$

$$P_{BS}(S_0, K, r, \sigma, T) = -S_0\Phi(\delta_1) + Ke^{-rT}\Phi(\delta_2).$$

where $\delta_1 = \frac{\ln(\frac{S_0}{K})+T(r+\sigma^2/2)}{\sigma\sqrt{(T)}}$ *and* $\delta_2 = \frac{\ln(\frac{S_0}{K})+T(r-\sigma^2/2)}{\sigma\sqrt{(T)}}$

The proof of the theorem can be found in various of graduate textbook such as in corollary 3.9 of [12].

Theorem 6.3.2. *Put*

$$\widehat{\Gamma} = \beta + \frac{\alpha^2}{2}.$$

Then the following hold.

$$C^{\mathbb{G}}_{ESR}(S_0, K, r, \sigma, T) = -(\frac{\alpha^2}{2} + \beta)\int_0^T e^{\widehat{\Gamma}t}C_{BS}(S_0e^{-rt}, K, r, \sigma, t)dt + e^{\widehat{\Gamma}T}C_{BS}(S_0e^{-rT}, K, r, \sigma, T)$$

$$P^{\mathbb{G}}_{ESR}(S_0, K, r, \sigma, T) = -(\frac{\alpha^2}{2} + \beta)\int_0^T e^{\widehat{\Gamma}t}P_{BS}(S_0e^{-rt}, K, r, \sigma, t)dt + e^{\widehat{\Gamma}T}P_{BS}(S_0e^{-rT}, K, r, \sigma, T)$$

Proof. The proof of the theorem follows from combining Theorem 6.2.2. with Theorem 6.3.1. as follows. Notice that for $0 \leq t \leq T$, we have

$$\begin{aligned} C_{BS}(S_0e^{-rt}, K, r, \sigma, t) &= S_0e^{-rt}\Phi\left(\frac{\ln\left(\frac{S_0e^{-rt}}{K}\right) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right) - Ke^{-rt}\Phi\left(\frac{\ln\left(\frac{S_0e^{-rt}}{K}\right) + (r - \sigma^2/2)t}{\sigma\sqrt{t}}\right) \\ &= e^{-rt}(S_0\Phi(d_2(t)) - K\Phi(d_1(t))) \end{aligned}$$

This ends the proof of the theorem.

□

6.4 The Greeks for Esscher prices

Here in this subsection, we calculate the Greeks for Esscher prices of Section (6.2). To this end, we recall the following notations

$$d_1(t) := \frac{\ln\left(\frac{S_0}{K}\right) - \frac{\sigma^2}{2}t}{\sigma\sqrt{t}} = d_2(t) - \sigma\sqrt{t}, \quad d_2(t) := \frac{\ln\left(\frac{S_0}{K}\right) + \frac{\sigma^2}{2}t}{\sigma\sqrt{t}}.$$

Proposition 6.4.1.

$$\Delta = \frac{\partial C^E(0)}{\partial S_0} = e^{(\beta - r + \frac{\alpha^2}{2})T}\Phi(d_2(T)) - (\frac{\alpha^2}{2} + \beta)\int_0^T e^{(\beta - r + \frac{\alpha^2}{2})t}\Phi(d_2(t))dt$$

$$\Gamma = \frac{\partial^2 C^E(0)}{\partial S_0^2} = e^{(\beta - r + \frac{\alpha^2}{2})T}\frac{\Phi'(d_2(T))}{S_0\sigma\sqrt{T}} - (\frac{\alpha^2}{2} + \beta)\int_0^T e^{(\beta - r + \frac{\alpha^2}{2})t}\frac{\Phi'(d_2(t))}{S_0\sigma\sqrt{t}}dt$$

$$\frac{\partial C^E(0)}{\partial \sigma} = S_0 e^{(\beta - r + \frac{\alpha^2}{2})T}\Phi'(d_2(T))\sqrt{T} - (\frac{\alpha^2}{2} + \beta)S_0\int_0^T e^{(\beta - r + \frac{\alpha^2}{2})t}\Phi'(d_2(t))\sqrt{t}dt$$

To calculate Rho, we assume that $r \neq 0$

$$\begin{aligned} Rho &= \frac{\partial C^E(0)}{\partial r} = (\frac{\alpha^2}{2} + \beta)S_0\int_0^T te^{(\beta - r + \frac{\alpha^2}{2})t}\Phi(d_2(t))dt - K(\frac{\alpha^2}{2} + \beta)\int_0^T te^{(\beta - r + \frac{\alpha^2}{2})t}\Phi(d_1(t))dt \\ &-TS_0 e^{(\beta - r + \frac{\alpha^2}{2})T}\Phi(d_2(T)) + TKe^{(\beta - r + \frac{\alpha^2}{2})T}\Phi(d_1(T)) \end{aligned}$$

Proof. The proof of this proposition follows directly from Theorem 6.2.2. by differentiating and using the following fact.

$$S_0\Phi'(d_2(t)) - K\Phi'(d_1(t)) = 0 \qquad \forall \quad 0 \leq t \leq T \tag{6.19}$$

Then the rest of this proof we prove this fact. To this end, remark that
$d_1(t) = d_2(t) - \sigma\sqrt{t}$, and get

$$\begin{aligned} d_1(t)^2 \qquad &= (d_2(t) - \sigma\sqrt{t})^2 = d_2(t)^2 - 2d_2(t)\sigma\sqrt{t} + \sigma^2 t \\ &= d_2(t)^2 - 2\ln\left(\frac{S_0}{K}\right). \end{aligned}$$

Then, we get

$$\Phi'(d_1(t)) = \frac{e^{-d_1(t)^2/2}}{\sqrt{2\pi}} = \frac{e^{-d_2(t)^2/2}}{\sqrt{2\pi}}\frac{S_0}{K} = \Phi'(d_2(t))\frac{S_0}{K}.$$

Therefore (6.19) follows immediately, and the proof of the proposition is complete.

□

Chapter 7

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