DESMOND PAUL HENRY. *The truncation of truth-functional calculation.* ***Notre Dame journal of formal logic,*** vol. 2 (1961), pp. 193–205.

The author presents an algorithm for enumerating the verifying truth-value assignments to the variables of a wff of two-valued propositional logic that purports to be significantly more economical than testing *seriatim* each possible value assignment. When 1 and 0 are truth and falsehood respectively and a fixed order of value assignments to variables is adopted, one can represent the full disjunctive normal form of a wff in n variables as a sequence of 2^n zeros and ones which is called a *selector.* Ingenious matrix-like *arrays* abbreviate selectors by telescoping repetitive patterns together and reducing patternless stretches of digits to single digits. In outline, the algorithm is: Beginning with a wff B in variables $p_1, \cdots, p_n$, one associates with each p_i a multi-tiered array for a wff in variables $p_1, \cdots, p_n$ that is equivalent to p_i. Rules are given for determining the array for B by operating successively on the tiers in the arrays for variables beginning with the bottom tiers. The result is an array, or a disjunction or conjunction of arrays, from which one can read off the selector for B. For wffs that fall exclusively under thesis (8), the algorithm is quite economical when large numbers of variables are involved. For other wffs, however, it is not evident that the algorithm is as economical as the method of reading off verifying value assignments from the branches of a semantic tableau.

The informality and brevity of the elegant exposition often frustrates comprehension. Algorithm rule S4, for example, can be read in several ways and none of the unintended readings is ruled out by the author's examples. (S4 should be read as stipulating that the ones that terminate calculation rows that are exactly alike with respect to occurrences of K, L, and M shall be grouped together to consitute, when gaps have been filled with zeros, a distinct outcome component.) The author's terminology is sometimes unexplained and misleading. For example, selectors are called *mutually exclusive* if they correspond to mutually exclusive wffs. But without further explanation the author speaks of *mutually exclusive arrays* which are not arrays that abbreviate mutually exclusive selectors but rather are arrays having mutually exclusive selectors on those tiers that cause the construction of an array to split into several arrays.

Justification of the eleven theses on which the algorithm depends is reduced to two fundamental theses (6) and (7) which are formulated too restrictedly. They must be understood so that, if a and b are selectors of degree m and n respectively, then $a(\alpha)$ is a function of the first m variables while $b(\beta)$ is a function of the next n variables. The unexplained function J on page 197 is NE. The variable u in the penultimate formula on page 203 should be v. None of the other misprints impedes comprehension. GERALD J. MASSEY and CARL J. POSY

IRVING M. COPI. ***The theory of logical types.*** Routledge & Kegan Paul, London 1971, x + 129 pp.

In the preface of this book, Copi explains that he has "tried to give an account of the Theory of Logical Types which shall not be so technical as to repel the non-specialist nor so informal as to disappoint the serious student who wants to see exactly what it is and how it works" and that he has "tried to provide a sufficiently trustworthy account of the theory to permit responsible philosophizing about it."

The book opens with a discussion of the "paradoxes," which are classified into the usual Ramseyan "logical" and "epistemological" ("semantic") categories. This section, in the reviewer's opinion, is particularly good: It not only clearly explains the "paradoxes" but also takes some pains to show to the philosopher who might not be overly interested in the foundations of mathematics that they are of the utmost importance for "ordinary arithmetic"—that our everyday intuitions about simple arithmetical and set-theoretic concepts are self-contradictory.

The simple theory of types (so-called because it was a simplification, made popular by Ramsey, of the ramified theory of types which had appeared in ***Principia mathematica*** twelve years earlier) unfolds into two parts: a semantical hierarchy of "types" and a syntactical rule for well-formedness (although this way of putting the point was not then available—the distinction between object language and meta-language had not yet become popular). In the set-theoretic case, we set up our hierarchy by letting type 0 be a set of individuals, type 1 be a set of all sets of individuals, type 2 a set of all sets of sets of individuals, etc. The rule for well-formedness would state that all variables and names are assigned a type and $x \in y$ is a wff iff the type of y is one more than the type of x. As Copi mentions, this is sufficient to avoid the set-theoretic paradoxes of Cantor, Burali-Forti, and Mirimanoff.

Such a system would be severely hampered in expressive power. For instance, the expression (1) $\{x, \{x, y\}\}$ must be ill-formed; otherwise both x and $\{x, y\}$ would be of the same type, and then it would be impossible that $x \in \{x, y\}$ is true (or even meaningful, by the well-formedness rule). So this set theory cannot define $\langle x, y\rangle$ by the formula (1). Instead it will define it as (2) $\{\{x\}, \{x, y\}\}$. Copi (p. 57) incorrectly "generalizes" this to $\langle x_1, x_2, \cdots, x_n\rangle = \{\{x_1\}, \{x_1, x_2\}, \cdots, \{x_1, x_2, \cdots, x_n\}\}$; it is incorrect because it follows from such a definition that $\langle a, a, b\rangle = \langle a, b, b\rangle = \langle a, b\rangle$, which is disastrous. This error has implications for the generality of his discussion of the inadequacies of the present set theory, but for simplicity the reviewer will consider only the problem Copi raises as it appears with ("simple") ordered pairs.

If binary relations are reduced to sets of ordered pairs, then the present theory would admit homogeneous relations (where the relata were of the same type) but not heterogeneous relations. This hampers the expressive power of the language, says Copi; and he suggests (pp. 59–60) that we might "inflate" the type of the lower-type expression (e.g. if x is one type "too low" for y, then consider the relation holding between $\{x\}$ and y). There seems to be something profoundly unsatisfying about this suggestion, for we are no longer "talking about" the same relation, but a different one holding between different relata. And in any case, this set theory could more easily be rejected simply by noting that it prohibits the well-formedness of (1) for *any* x and y.

Copi's own system embodying the simple theory of types does not attempt to have this simple hierarchy. Rather it has the following: (3) 0 is a type index. (4) If $a, b, c, \cdots$ are type indices, then $(a, b, c, \cdots)$ is a type index. And the key clause in the rules for well-formedness is (5) If $G_1, G_2, \cdots, G_n$ are respectively of types $a, b, \cdots, n$, then $F(G_1, G_2, \cdots, G_n)$ is a wff iff F is of type $(a, b, \cdots, n)$. In the intended interpretation, individuals are assigned type 0, monadic properties of individuals are type (0), monadic properties of these monadic properties are type ((0)), binary relations between individuals are of kind (0, 0), ternary relation between individuals are of type (0, 0, 0), monadic properties of binary relations between individuals are of type ((0, 0)), binary relations holding between monadic properties of individuals are of type ((0), (0)), binary relations holding between an individual and a binary relation whose arguments are monadic properties of binary relations between individuals are of type (0, (((0, 0)), ((0, 0)))), etc. Thus Copi's system admits of a kind of heterogeneous relation without "inflating" the type of one of the relata. Copi claims that this shows his system "stronger," without spelling out what this amounts to (p. 60). The reason is that (say) $F^{((0),(0))}(G^{(0)}, H^{(0)})$ and $F^{(0,(0))}(G^0, H^{(0)})$ can both be expressed in this language. In the former language only the first would have an analogue. (Copi apparently has in mind the procedure of "inflating" the type of one of the relata—here G^0—in the former language, in order to arrive at an "equivalent" expression. In his system this procedure is not required.)

There seem to be two drawbacks to Copi's system. First there is an intuitive dissatisfaction at the vast number of *different* hierarchies of types. Intuitively, what one wants from the simple theory of types is a "linear" arrangement of the types, one "above" the other, so that for any two distinct types one can say that one of them is "higher" than the other. This can be achieved by defining our types in the following way (sometimes called the "cumulative theory of types"): (6) $\{a, b, c, \cdots\}$ is type 0. (7) $[\mathscr{P}(\bigcup_{i \leq n} \text{type } i)] \sim [\bigcup_{i \leq n} \text{type } i]$ is type $n + 1$. Thus in the intended interpretation, type 0 is a set of individuals, and every other type is "constructed" by taking the union of all "lower" types, then taking the power set of this union (the set of all subsets), and finally subtracting from this the union of all the lower types. (This last insures that each "object" appears in exactly one type.) We can now treat an n-place relation R as a set of ordered n-tuples, and find the type of R by first determining the type of the n-tuple. We define a one-tuple $\langle x\rangle$ as x, and an n-tuple $\langle x_1, x_2, \cdots, x_n\rangle$ as the ordered pair $\langle x_1, \langle x_2, \cdots, x_n\rangle\rangle$ (where the second member is an $n - 1$ tuple). With this and the definition of ordered pair ((1) above) in mind, the type of an n-tuple $\langle x_1, x_2, \cdots, x_n\rangle$ will be the maximum type of x_1 and the $(n - 1)$-tuple $\langle x_2, \cdots, x_n\rangle$, plus 2. This can be stated more directly (where n is at least 2) as (8) $\max[\max_{1 \leq i < n}(\text{type } x_i + 2i), \text{type } x_n + 2(n - 1)]$. Since an n-ary relation R is a set of these n-tuples, we shall want to determine the largest type of any of them, and add one to that; R will be at that type. That there will be such a set at that type can be verified by constructing them in accordance with definitions (6) and (7).

Copi had said that his system, but not the other system, allowed of heterogeneous relations. This is only true in a restricted sense of 'heterogeneous.' Copi's system can indeed admit formulas like $F^{((0),0)}(G^{(0)}, H^{0})$ indicating a relation between a type (0) entity and a type 0 entity. But that *same* relation cannot hold between a type ((0)) entity and a type 0 entity (for example). This means that there can be no set such as (9) $\{\langle x, y\rangle, \langle z, y\rangle\}$, where x and z are of different types. This seems to be a much too restrictive condition: In any recognized formulation of set theory we can always construct from x, y, and z, (10) $\{\{\{x\}, \{x, y\}\}, \{\{z\}, \{z, y\}\}\}$, which is the definition of (9) in accordance with (2), or (11) $\{\{x, \{x, y\}\}, \{z, \{z, y\}\}\}$, which is the definition of (9) in accordance with (1). In the system the reviewer has sketched, letting y be at type 0, x be at type 1, and z be at type 2, we find that (11) is in type 5 and thus this system does allow truly heterogeneous relations which Copi's does not: viz., relations wherein the same argument-position can be occupied by expressions of different types. Furthermore, the system presented here allows certain other statements that are ill-formed in Copi's system. The formula (a) $F^{((0,(0)))}(G^{(0,(0))})$ is well-formed in Copi, but (b) $F^{((0,(0)))}(H^{((0),0)})$ is not. However, in the system given here, any binary relation between an individual and a monadic property of individuals (as both $G^{(0,(0))}$ and $H^{((0),0)}$ intuitively indicate) is always in type 4. And a monadic property which is assertable of such a relation is in type 5. Thus the analogues of both (a) and (b) are well-formed. Copi might argue that under the intended interpretation, a formula like (b) cannot possibly be true, since $F^{((0,(0)))}$ designates a monadic property of a binary relation between individuals and monadic properties of individuals (in that order), and $H^{((0),0)}$ is not such a relation. Quite so, but the proper response is that (b) is false, not ill-formed, and that the negation of (b) is true, not ill-formed. (The material in these last two paragraphs has benefited considerably from discussions with Charles Morgan.)

After presenting the formal structure of the simple theory of types, Copi considers three well-known objections to it and suggests various responses. The three objections are: 1. There is no unique universal or empty class, nor a unique $0, 1, 2, \cdots$, etc. Everything is reduplicated on each type. This is intuitively repugnant and technically awkward. 2. The simple theory requires the axiom of infinity, and this cannot be regarded as a logical truth in any reasonable sense of the phrase. 3. The theory itself is contradictory. The theory states that nothing can be said about *all* properties, *all* functions, *all* classes. But in so stating, it violated the law it states. In suggesting responses to these, he canvasses various historical figures and makes some suggestions of his own. Not all of these latter suggestions are both well taken and illuminating (e.g. his "rod and pipe language" explication of Wittgenstein on objection 3, p. 75).

The Ramified "Theory of Types . . . consists really of two distinct parts directed respectively against the two groups of contradictions. . . . These two parts were unified by being both deduced in a rather sloppy way from the 'vicious circle principle'" (Ramsey). Copi spends some time trying to explicate the "vicious circle principle," and demonstrating that the simple theory *is* a part of the ramified theory but that it cannot be deduced from the "vicious circle principle." Since the ramified theory contains the restrictions of simple theory, it too avoids the logical paradoxes; however, it is weaker in the sense that it contains restrictions to avoid the semantical ones also. The ramified theory divides each type above 0 into a further hierarchy of *orders*, according to the "vicious circle principle." (For whatever intuitive value it has: A new order on the same type is generated whenever we must refer to the totality of functions of the previous order of that type. Thus $F^{n}(G^{n-1})$ is a function at type n of the argument G^{n-1}; if we were to quantify over F^{n} as, e.g. $(\exists F^{n})F^{n}(G^{n-1})$, we would have a function at type n of the argument G^{n-1} which is one *order* higher than the original function.) Copi informally shows that the ramified theory avoids Richard's paradox, and more formally shows how it avoids Grelling's paradox. The theory is also consistent (Fitch, Lorenzen, Schütte).

The main problem with the ramified theory is that only a fraction of classical mathematics can be reconstructed. There is no longer a least upper bound for every bounded collection of real numbers, mathematical induction is restricted, Cantor's theorem about the cardinality of the power set of a class cannot be proved, and various definitions of identity no longer apply. These restrictions are eased by the axiom of reducibility: For any function of any type, any order, any number of arguments, there is a formally equivalent function of the same type and number of arguments, but of first order. Copi suggests (pp. 96–97) that this axiom does not reinstate the semantic paradoxes; but that, nonetheless, it is difficult to maintain it is "a truth of logic." Other objections of Fitch, Quine, W. and M. Kneale, Gödel, and Fraenkel and Bar-Hillel are canvassed, and some (all-too-short) comments on modern "streamlined" versions of the ramified theory are quoted from Fraenkel and Bar-Hillel. The final section of this chapter is a discussion of the familiar distinction between the object language and the meta-language as a method of avoiding the semantic paradoxes. The reviewer thinks it very helpful that this widely accepted resolution is contrasted with the ramified theory so that philosophers can understand just which presuppositions are, and which ones are not, being made when they accept this resolution.

The book ends with Russell's *The hierarchy of jokes*. Except for occurrences of bold-face type, and some difficulty under 'Quine' in the (quite extensive) bibliography, the book is remarkably free of typographical errors. The reviewer recommends the book to anyone, especially a non-logician, who wishes a clear introduction to the theory (and history of the theory) of logical types. FRANCIS JEFFRY PELLETIER

IRVING M. COPI. ***Symbolic logic.*** Third edition of XIX 282. The Macmillan Company, New York, and Collier-Macmillan Limited, London, 1967, xvi + 400 pp.

The contents of the new edition are the same as those of the preceding one (XXXII 252): No new topics are introduced and no sections of the second edition are deleted. The quality, however, is better: Minor blemishes and a major error that occur in the preceding edition are corrected.

First, the minor corrections. The notational convention used in formulating the rules of quantification is modified slightly to accommodate vacuous quantification (p. 106 and p. 108), a modification required if the claim (p. 302) that the postulates of RS_1 can be derived in the natural deduction system is to hold. The principle of identity, omitted in the earlier edition, is now stated (p. 161) along with Leibniz's law in formulating the logic of identity, thereby making that logic complete. Russell's distinction of primary and secondary occurrence of a descriptive phrase, ignored in the second edition, is introduced (pp. 167–168) into the discussion of definite descriptions, and with it available the author is able to comment more accurately on the truth conditions of sentences containing non-denoting descriptions. The characterization of the strengthened rule of conditional proof (**CP**), a basic component of the author's system of natural deduction, is supplemented in the new edition by several sentences which make explicit the restrictions, adhered to in practice throughout the earlier edition but never stated there, required to make the procedure a valid one: "When the scope of an assumption has been ended, the assumption is said to have been *discharged*, and no subsequent line can be justified by reference to it or to any line lying between it and the line inferred by the rule of Conditional Proof that discharges it.... Scopes of different assumptions may follow each other with no overlap, or one scope may be contained entirely within another. But two scopes may not partially overlap: that is strictly forbidden." (p. 86)

The major correction consists in new formulations of the quantification rules existential instantiation (**EI**) and universal generalization (**UG**). The formulations of these rules given in the second edition were not satisfactory (see Parry's XXXI 286). The substantial modification occurs in the statement of **EI**: The earlier Quine-type formulation is replaced by one that closely approximates Gentzen's original version of the rule.

$$
\begin{array}{ll}
\textbf{EI:} & (\exists\mu)\Phi\mu \\
 & \quad\cdots \\
 & \rightarrow \Phi\nu \\
 & \quad\cdots \\
 & \quad p \\
 & \therefore p
\end{array}
$$

"provided that ν is a variable that does not occur free either in p or in any line preceding $\Phi\nu$" (p. 113). This revised system of natural deduction, in contrast to that of the second edition, does