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THE UNIVERSITY OF ALBERTA

ON THE RANGE OF A VECTOR MEASURE

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ON THE RANGE OF A VECTOR MEASURE submitted by RAJAN ANANTHARAMAN in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Pure Mathematics.

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Date: March 21, [illegible]

This thesis is respectfully dedicated to the memories of my grandfather, S.M. Narayanaswami Iyer, and my friend, T.V. Srinivasan.

"Full many a gem of purest ray serene
Do the dark unfathomed caves of ocean bear,
Full many a flower is born to blush unseen
And waste its sweetness in the desert air."

(T. Gray, Elegy)

ABSTRACT

We investigate in this thesis the properties and the extremal structure of the range of a vector measure. Let ν be a measure defined on a σ - algebra A of sets and with its values in a real Hausdorff quasi - complete locally convex space X, and let λ be a finite positive measure on A that controls ν, i.e. $\nu \ll \lambda$.

Let ν have any property hereditarily if it is possessed by the restriction ν_A of ν to every $A \in A$. Extending Liapounoff's theorem to infinite dimensions, we obtain some necessary and sufficient conditions for ν to have hereditarily convex range, e.g. when ν is semi-convex, i.e. when, for every $A \in A$, there exists $B \in A$, $B \subset A$ such that $\nu(B) = (1/2)\ \nu(A)$, and ν is proved to have hereditarily weakly compact range if and only if the non-atomic part of ν is semi-convex. These may be compared with a theorem of Kingman and Robertson. When X is Banach, the range of ν is proved to be relatively compact if and only if there exists a sequence $\{x_i\}$ in X and a sequence $\{A_i\}$ of sets in A such that the series $\Sigma\, x_i\, \lambda_{A_i}$ converges to ν uniformly on A. This extends the necessity part of the following result of Bolker to Banach spaces: When X is finite-dimensional, a compact convex set in X is the closed convex hull of the range of a measure if and only if it is the limit in the Hausdorff metric of finite sums of segments.

Let K be the closed convex hull of the range of ν. Every extreme point of K is proved to be strongly extreme, and that it is a support-point of K when X is Banach. When X has a weak*-separable dual and ν is non-atomic, we obtain the following extension of another theorem

of Liapounoff, who proved it in finite dimensions: A point of X is an extreme point of K if and only if it is assumed by ν once and only once. When X is a Fréchet space, we find a sufficient condition (that always holds in finite dimensions) under which the set of extreme points of K is closed.

The exposed points of the range of ν are proved to be strongly exposed. An element x' of the dual X' of X is proved to expose the range of ν if and only if the signed measure $x' \circ \nu \equiv \nu$, and when X is Banach, this yields a theorem of Rybakov. If K is a weakly compact convex set in a Banach space X of which every exposed point is strongly exposed, we prove that the functionals $x' \in X'$ that expose K form a residual G_δ subset of X' whenever it is dense in X'. This yields the following theorem due to B.J. Walsh: When X is Banach, the functionals $x' \in X'$ for which $x' \circ \nu \equiv \nu$ form a residual G_δ subset of X'.

We further study properties of ν as a map relative to the Fréchet-Nikodym topology on A induced by any control measure of ν, and prove that every finite dimensional non-atomic measure is open and monotone, whereas an arbitrary finite dimensional measure is biquotient.

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CHAPTER I

INTRODUCTION

1. Introduction. The present thesis is devoted to the properties of the range of a vector measure together with its extremal structure. We also obtain here some properties of the measure as a map relative to a suitable topology on its domain.

Let ν be a measure defined on a σ - algebra $\mathcal{A}$ of subsets of some set S and with values in a real Hausdorff locally convex space X. We assume ν to be controlled by some finite positive measure λ on $\mathcal{A}$ (i.e. $\nu \equiv \lambda$) which, according to the theorem of Bartle - Dunford - Schwartz [3], is always the case when X is normed. Let $\mathcal{A}$ itself denote the quotient σ - algebra of $\mathcal{A}$ modulo the σ - ideal of ν-null sets.

We first study in section 3 the properties of ν as a map. Let ρ be the Fréchet - Nikodym metric on $\mathcal{A}$ induced by λ, viz. $\rho(A,B) = \lambda(A \Delta B)$ for every $A, B \in \mathcal{A}$. The topology induced by this metric on $\mathcal{A}$ is independent of the choice of λ and $\nu: \mathcal{A} \to X$ is continuous relative to this topology.

When ν is finite-dimensional and (purely) atomic, as proved by Halmos [14] and Marczewski and Sikorski [31], $\mathcal{A}$ is compact, so ν becomes closed. In case ν is finite-dimensional and non-atomic, we prove in Theorem 3.6 that ν is always open. An arbitrary finite-dimensional measure is proved in Theorem 3.7 to be always biquotient.

The measure ν has been called by Halmos [15, p.417] semi-convex if for every $A \in \mathcal{A}$ there exists $B \in \mathcal{A}$, $B \subset A$ such that

$\nu(B) = (1/2)\,\nu(A)$. In Proposition 3.8 we prove that a semi-convex measure is always weakly monotone. A finite-dimensional non-atomic measure is shown, on the other hand, to be monotone.

We next investigate in section 4 properties of the range of ν. The first result in this direction is due to Liapounoff [27], who proved that the range of every finite-dimensional measure ν is compact, and it is further convex when ν is non-atomic. Several simpler proofs of this theorem have since appeared, see e.g. Halmos [15] and Lindenstrauss [30]. Later, Liapounoff [28] gave an example in ℓ_1 to show that the above results do not hold in infinite dimensions in general.

Let ν have any property P <u>hereditarily</u> if it holds for the restriction ν_A of ν to every set $A \in \mathcal{A}$. For every x' in the dual X' of X, let $f_{x'}$ denote the Radon-Nikodym derivative of $x' \cdot \nu$ relative to λ. Extending Liapounoff's theorem to infinite dimensions, Kingman and Robertson [21] proved that ν has hereditarily convex and hereditarily weakly compact range if and only if the subspace $\{f_{x'} : x' \in X'\}$ of $L_1(\lambda)$ is thin [21]. In Theorem 4.2 we give some necessary and sufficient conditions for ν to have hereditarily convex range, e.g. the semi-convexity of ν, and the range of ν is hereditarily weakly compact according to Theorem 4.3 if and only if the non-atomic part of ν is semi-convex. When X is [illegible]ach, we further obtain in Proposition 4.4 a sufficient condition for the range of ν to be closed.

In connection with Liapounoff's theorem, Bartle, Dunford and Schwartz [3] proved, on the other hand, the range of ν to be relatively weakly compact whenever X is Banach. This remains valid for a quasi-complete X as proved by Tweddle [44]. For some further results in this direction see Lew [26] and Hoffman-Jørgensen [17]. When X is Banach and ν is a Bochner integral with respect to a finite positive measure, Uhl [46] proved that ν has relatively compact range.

Let X be Banach and $||\nu||_p$ be the Pettis norm of ν, viz, $\sup\{||\nu(E)|| : E \in A\}$. In Theorem 4.5, we prove that the range of $\nu : A \to X$ is relatively compact if and only if there exists a sequence $\{x_i\}$ in X and a sequence $\{A_i\}$ in A such that the series $\sum_{i=1}^{\infty} x_i \chi_{A_i}$ converges to ν in the Pettis norm. It follows that the range of ν is hereditarily compact if and only if its non-atomic part is semi-convex and has a representation as above.

In case of finite dimensions, Bolker [5] has established the following characterization of the range of a measure: A compact convex set K is the closed convex hull of the range of a measure if and only if it is the limit in the Hausdorff metric of finite sums of segments through the origin. According to Theorem 4.5, the necessity part of Bolker's theorem remains valid in infinite dimensions.

Chapter III is devoted to the extremal structure of the range of a measure. In this direction, again it was Liapounoff [27, th. 3] who proved for finite-dimensional non-atomic measures that a point of the range of ν is its extreme point if and only if it is assumed by ν only once.

Let X'^* denote the algebraic dual of X' and let T be the weak integral map [6, p. 32] $\phi \mapsto \int \phi \, d\nu$ of $L_\infty(\lambda)$ to X'^*. Let further T_0 be the restriction of T to the set P of functions ϕ in $L_\infty(\lambda)$ for which $0 \le \phi \le 1$ λ - almost everywhere. Then $T_0(P)$ is the $\sigma(X'^*, X')$ - closed convex hull of $\nu(A)$ and we call the extreme points of $T_o(P)$ the extreme points of the range of ν.

In Proposition 5.1 we prove that a point in X'^* is an extreme point of the range of ν if and only if it is assumed by T_o once and only once on characteristic functions of sets in Λ. The following result of Kluvánek [24, th. 2] follows as a corollary: The range of a controlled measure ν contains all of its extreme points. In Proposition 5.2 we obtain the following extension of Liapounoff's result: When X' is weak* - separable, a point in X is an extreme point of the range of a non-atomic measure ν if and only if it is assumed by ν once and only once.

When X is quasi-complete, every extreme point of the range of ν is proved in Proposition 5.3 to be strongly extreme, and it is further found in Proposition 5.5 to be a support point of the closed convex hull of $\nu(\Lambda)$ when X is Banach. These properties do not hold for arbitrary weakly compact convex sets even when X is ℓ_1 or ℓ_2 [37 p. 75; 7, p. 160]. When X is Fréchet, the set of extreme points of the range is further proved in Proposition 5.7 to be closed whenever T_o is open relative to the $L_1(\lambda)$ - norm topology on its domain, as is always the case when X is finite-dimensional.

Coming now to exposed points, we prove in Proposition 6.3 that when X is quasi-complete, the range of ν and its closed convex hull have the same exposed points, which in turn are found in Theorem 6.4 to be strongly exposed. This again does not hold for arbitrary weakly compact convex sets even in ℓ_2 [29, p. 145]. In Theorem 6.7 we extend a result of Husain and Tweddle [18] on extreme points of the range of restrictions of ν to their exposed points.

Finally, in section 7 we investigate the set of functionals $x' \in X'$ that expose the range of ν. In connection with the Bartle – Dunford – Schwartz theorem on the existence of a control measure, Rybakov [39] has proved the following interesting result: When X is Banach, there always exists an $x' \in X'$ such that the signed measure $x' \circ \nu \equiv \nu$. This result has in turn been strengthened by Walsh [47], who showed that the functionals $x' \in X'$ for which the above holds form a residual G_δ set in X' relative to the norm topology.

We show in Lemma 6.1 that a functional $x' \in X'$ exposes the range of ν if and only if $x' \circ \nu \equiv \nu$, and the Rybakov theorem is deduced from this lemma. When a weakly compact convex set K in a Banach space X has each of its exposed points strongly exposed, we prove in Theorem 7.2 that the set of exposing functionals of K is residual in X' whenever they are dense in X'. This yields, in particular, the above theorem of Walsh. When X is quasi-complete, we obtain, in Proposition 7.4, a sufficient condition for the range of ν to have exposing functionals in terms of the Mackey – topology $m(X_o', X_o)$, where X_o is the linear span of the closed convex hull of $\nu(A)$.

2. Terminology and Notations

2.1. When X is a topological space, the set of all non-void closed subsets of X is denoted by 2^X. If X and Y are topological spaces, a map $f : Y \to 2^X$ is said to be lower semi-continuous [25, p. 173] if for every $y \in Y$ and for every open subset U of X for which $f(y) \cap U \neq \phi$ there exists a neighborhood V of y such that $f(z) \cap U \neq \phi$ for every $z \in V$.

The Vietoris (or finite) topology [32] on 2^X is generated by basic open sets of the form

$$\{F \in 2^X : F \subset \bigcup_{i=1}^{n} U_i \text{ , and } F \cap U_i \neq \phi \text{ for } 1 \leq i \leq n\},$$

where U_i is an open subset of X for $1 \leq i \leq n$.

When (A_n) is a net of subsets of a topological space X, the limit superior (or limit inferior) [25, pp. 335, 337] of (A_n) is defined to be the set of all $x \in X$ such that every neighborhood of x intersects the net frequently (respectively eventually), and it is denoted by LsA_n (respectively LiA_n).

When (X,ρ) is a metric space, the distance of a point $x \in X$ from a non-void set $A \subset X$ is denoted by $\rho(x,A)$. The Hausdorff distance [25, p. 214] $d(A,B)$ between two bounded elements A, B of 2^X is defined to be the maximum of the following two numbers:

$$\max\{\rho(a,B) : a \in A\} \quad \text{and} \quad \max\{\rho(b,A) : b \in B\}.$$

We shall call $A \in 2^X$ the Hausdorff limit of a sequence $\{A_n\}$ in 2^X if $d(A_n,A) \to 0$.

When X and Y are topological spaces and $f : X \to Y$ is continuous and onto, f is said to be open if $f(U)$ is open whenever U is open in X. The map f is further called <u>biquotient</u> [34] if for every $y \in Y$ and for every open cover $\mathcal{U}$ of $f^{-1}\{y\}$ there exists a neighborhood of y that is covered by finitely many f(U), $U \in \mathcal{U}$. When f is not onto, we still call f open or biquotient if it is so with respect to the relative topology on f(X).

The map f is further called <u>monotone</u> [49] if f^{-1} preserves connected sets, and <u>weakly monotone</u> [49] if $f^{-1}\{y\}$ is connected for every $y \in Y$.

2.2. We denote by ν, throughout this thesis, a measure (i.e. a countably additive set function) defined on a σ - algebra A of subsets of a set S, and with values in a Hausdorff locally convex space (L.C.S.) X over the field $\mathbb{R}$ of reals.

The continuous dual of X will be denoted by X'. For every $x' \in X'$ the finite signed measure $x' \circ \nu$ (denoted by $\nu_{x'}$) has a bounded range, and so the range of $\nu(A)$ is weakly bounded. By Mackey's theorem [7, p. 70], it is thus bounded. When X is Banach, the <u>Pettis-norm</u> of ν is defined to be $\sup \{||\nu(E)|| : E \in A\}$ and denoted by $||\nu||_p$. Further, the variation [12, p. 97] and the semi-variation [12, p. 320] of ν are denoted by $|\nu|$ and $||\nu||$ respectively.

The closed convex hull of the range of ν will be denoted by K. According to Mazur's theorem [7, p. 67], this set is the same for all topologies on X compatible with the duality (X,X'). When X is quasi-complete [7, p.9], the set K is weakly compact as proved by Tweddle [44].

The measure $\nu : A \to X$ is said to be <u>controlled</u> [11] if there exists a finite positive measure λ on A (called a <u>control measure</u> of

ν) such that $\nu \equiv \lambda$. For X Banach, ν was originally defined to be controlled [8] by λ if $\lim_{\lambda(E) \to 0} ||\nu||(E) = 0$ and $\lim_{||\nu||E \to 0} \lambda(E) = 0$. This, however, is equivalent to the above definition when X is Banach as is clear from the Pettis theorem [12, p. 318] and [12, p. 321, lem. 5]. It may be further observed that ν is controlled even if there exists a finite positive measure λ for which $\nu \ll \lambda$. For, if N is a maximal disjoint class of ν - null sets in A that are not λ - null, then N is countable, so that its union N belongs to A, and the restriction of λ to the complement of N clearly controls ν. The measure ν is assumed to be controlled (by λ) throughout this thesis.

We shall denote by A itself the quotient σ - algebra of A modulo ν - null sets. On this new A the Fréchet-Nikodym metric induced by λ is defined to be $\rho(A,B) = \lambda(A \Delta B)$, $A, B \in A$. The topology induced on A by ρ will be denoted by $\mathcal{T}_\nu$, for it is independent of the choice of the control measure.

The restriction of ν to $A \in A$ will be denoted by ν_A: i.e. $\nu_A(E) = \nu(A \cap E)$ for every $E \in A$. A set $A \in A$ is said to be an atom of ν if $\nu(A) \neq 0$ and for every $B \in A$, $B \subset A$, either $\nu(B) = 0$ or $\nu(A \setminus B) = 0$. The measure ν is said to be atomic if S is the union of atoms of ν, and non-atomic if it has no atoms. When ν is controlled, it has the same atoms as its control measure λ, and so ν can have at most countably many atoms, say $(A_i)_{i \in N}$. If $S_a = \bigcup_{i \in N} A_i$ and $S_n = S \setminus S_a$, then $\nu_a = \nu_{S_a}$ and $\nu_n = \nu_{S_n}$ are called the atomic and non-atomic parts of ν respectively [15, p. 417]. The σ - algebras $A_a = \{E \in A : E \subset S_a\}$ and $A_n = \{E \in A : E \subset S_n\}$ will be called the atomic, and non-atomic parts of $\mathcal{A}$ respectively. The measure ν is said to be semi-convex [15, p. 417] if for every $A \in \mathcal{A}$ there exists $B \in \mathcal{A}$, $B \subset A$ for which $\nu(B) = (1/2)\,\nu(A)$.

We call a member $A \in \mathcal{A}$ an element of ν if the range of ν_A is contained in a segment with zero as one of its end points. The measure ν is called elementary if S is the union of elements of ν, and ν is non-elementary if it has no elements. If A and B are two elements of ν such that the ranges of ν_A and ν_B are contained in two distinct rays through zero, then $A \cap B$ is clearly a ν - null set. If ν is controlled, the family of all maximal elements of ν is at most countable, say $\{A_n : n \in N\}$. If $E = \cup\{A_n : n \in N\}$ and E^c denotes the complement of E, then ν_E and ν_{E^c} are called the elementary and non-elementary parts of ν respectively.

2.3. The spaces $L_1(\lambda)$ and $L_\infty(\lambda)$ will be denoted by L_1 and L_∞ respectively.

We shall denote by P the set of all extended real-valued $\mathcal{A}$ - measurable functions f defined on S for which $f(s) \in [0,1]$ for λ - almost every $s \in S$. Unless otherwise stated, P is assumed to have the topology of L_1 - norm. The set P is convex and it is compact relative to the weak* - topology $\sigma(L_\infty, L_1)$ [21]. On P the induced $\sigma(L_1, L_\infty)$ - topology is Hausdorff and coarser than the weak* - topology, and so the two coincide. We further employ P_0 to denote the set of all characteristic functions χ_E of sets $E \in \mathcal{A}$. Then P_0 is the set of extreme points of P [21].

The algebraic dual of X' is denoted by X'^*. Since $\lambda \equiv \nu$, the signed measure $\nu_{x'} \ll \lambda$ for every $x' \in X'$, and so

$L_\infty \subset L_\infty(|\nu_{x'}|) \subset L_1(|\nu_{x'}|)$. Thus, for each $f \in L_\infty$ we may define the <u>weak integral</u> $\int f\, d\nu$, as in [6, p. 32], to be the linear functional on X' such that $(x', \int f\, d\nu) = \int f\, d\nu_{x'}$ for every $x' \in X'$. Following Lindenstrauss [30], we denote by T_ν the map from L_∞ to X'^* for which

$$T_\nu(f) = \int f\, d\nu \quad , f \in L_\infty .$$

The range of T is not always contained in X [6, p. 33]. It follows from the Radon Nikodym Theorem that T is continuous relative to the weak* - topologies $\sigma(L_\infty, L_1)$ and $\sigma(X'^*, X')$ [21].

The set T (P) is thus clearly the $\sigma(X'^*,X')$ - closed convex hull of $\nu(A)$. It may be observed that T (P) = K if and only if K is weakly compact. For if T (P) = K, this set is $\sigma(X'^*,X')$ - compact. As is well known, $(X,\sigma(X,X'))$ is linearly and homeomorphically embedded in $(X'^*,\sigma(X'^*,X'))$, and so K is compact relative to the former topology. The converse may be verified without difficulty. When X is quasi-complete, K is weakly compact [44], and so T (P) = K.

The restriction of T to P will be denoted by $T_\circ$. Since $T_\circ$ is continuous relative to the weak* - topologies $\sigma(L_\infty,L_1)$ and $\sigma(X'^*,X')$ and P is weak* - compact and convex, for every $a \in T_\circ(P)$ the set $T_\circ^{-1}\{a\}$ has extreme points by Krein - Milman theorem. For every a in the range of ν, it is easy to see that every characteristic function in $T_\circ^{-1}\{a\}$ is an extreme point of this level. We shall say that possesses <u>property (*)</u> if the extreme points of $T_\circ^{-1}\{a\}$ are only characteristic functions for every $a \in \nu(A)$.

2.4. A set in X is called a <u>hyperplane</u> if it is the level of some non-zero $x' \in X'$, and it is said to <u>support</u> a set $A \subset X$ at $x_0 \in A$ if

x' is not identically zero on A and $(x,x') \leq (x_o,x')$ for every $x \in A$. The point x_o is then called a support point [23] of A.

When $\nu : A \to X$ is a measure, we call the extreme points of the $\sigma(X'^*,X')$ - closed convex hull of $\nu(A)$ the extreme points of the range of ν. When $A \subset X$ is convex and closed, $x_o \in A$ is called a strongly extreme point [9, p. 97], or a denting point [37] of A, if x_o is not in the closed convex hull of $(A \setminus V)$ for any neighborhood V of x_o relative to A.

For every set $A \subset X$, a point $x_o \in A$ is called an exposed point [29] of A if there is an $x' \in X'$ such that $(x,x') < (x_o,x')$ for every $x \in A$, $x \neq x_o$, or, equivalently, if there exists an $x' \in X'$ and a real number α for which the hyperplane $H = \{x \in X : (x,x') = \alpha\}$ supports A at x_o and $A \cap H = \{x_o\}$. Moreover, $x_o \in A$ is called a strongly exposed point [29] of A if there exists an $x' \in X'$ such that (i) $(x,x') < (x_o,x')$ whenever $x \in A$ and $x \neq x_o$, and (ii) for every net (x_n) in A, $(x_n,x') \to (x_o,x')$ implies that $x_n \to x_o$.

We shall denote by ext A, st ext A, exp A and st exp A the sets of extreme, strongly extreme, exposed and strongly exposed points of A respectively. It is clear that st exp $A \subset$ exp A for any set $A \subset X$, whereas it is easy to verify that whenever A is convex, exp $A \subset$ ext A. When A is further closed, it may be verified without difficulty that st ext $A \subset$ ext A and st exp $A \subset$ st ext A.

CHAPTER II

MEASURE AS A MAP AND ITS RANGE

3. Measure as a map. In this section we assume, except in Proposition 3.1 and Lemma 3.5, X to be a quasi-complete L.C.S. and $\nu : \mathcal{A} \to X$ to be a measure, controlled by λ.

Let $S_{\infty,1}$ denote the unit ball of L_∞ with the topology of L_1.

PROPOSITION 3.1. If K is weakly compact, then the restriction of T to $S_{\infty,1}$ is continuous and has its range contained in X.

Proof: Since K is weakly compact, we have $T(P) = K$ (see §2) and so $T(S_{\infty,1}) = T(P - P) = T(P) - T(P) = K - K \subset X$. We shall first assume X to be normed. If $T_1 = T|_{S_{\infty,1}}$ is not continuous at some element f of $S_{\infty,1}$, then there exists $\varepsilon > 0$ and a sequence $\{f_n\}$ in $S_{\infty,1}$ converging to f in L_1 - norm, such that $||T_1 f_n - T_1 f|| \geq \varepsilon$ for every n. According to Lemma 5 in [12, p. 321] there exists a $\delta > 0$ such that $||\nu||(E) < \varepsilon/8$ whenever $E \in \mathcal{A}$ and $\lambda(E) < \delta$, where $||\nu||$ denotes the semi-variation of ν. Since $\{f_n\} \to f$ in measure, there exists, by Egoroff's theorem, a subsequence $\{f_m\}$ of $\{f_n\}$ converging to f almost uniformly. Hence there exists a set $E \in \mathcal{A}$ such that $\lambda(E) < \delta$ and $\{f_m\} \to f$ uniformly on the complement E^c of E. As the weak integral of every $f \in L_\infty$ is in X and it coincides with its Dunford-Schwartz integral [12, p. 323], we have

$$||T_1 f_m - T_1 f|| \le || \int_E (f_m - f)\, d\nu|| + || \int_{E^c} (f_m - f)\, d\nu|| .$$

$$\le \operatorname*{ess\,sup}_{s \in E} |f_m(s) - f(s)| \; ||\nu||(E) +$$

$$\operatorname*{ess\,sup}_{s \in E^c} |f_m(s) - f(s)| \; ||\nu|| \; (E^c) ,$$

$$\le 2 \cdot \varepsilon/8 + \operatorname*{ess\,sup}_{s \in E^c} |f_m(s) - f(s)| \cdot ||\nu|| \; (S).$$

Since $\operatorname*{ess\,sup}_{s \in E^c} |f_m(s) - f(s)| \to 0$ as $m \to \infty$, we have

$||T_1 f_m - T_1 f|| \le \varepsilon/2$ eventually, contrary to the hypothesis.

Thus, $T_1 : S_{\infty,1} \to X$ is continuous when X is normed.

In the general case, let p be a continuous semi-norm on X and X_p be the quotient normed space $X / p^{-1}\{0\}$. Let the measure $\nu_p : A \to X_p$ be defined by $\nu_p(A)$ = the equivalence class of $\nu(A)$ in X_p for $A \in A$. Now, if $f_n, f \in S_{\infty,1}$ and $||f_n - f||_1 \to 0$, then $p(T_1 f_n - T_1 f) \to 0$ by the above. Since this is true for every p, $T_1 f_n \to T_1 f$ in the topology of X, completing the proof.

Remark: Even when X is Banach, the map T need not be continuous on all of L_∞. For example, let X be a Hilbert space with a complete orthonormal basis $\{e_n : n \ge 1\}$, $S = [0,1]$, A the σ - algebra of Borel subsets of S, λ the Lebesgue measure on A and $A_n = (1/2^n , 1/2^{n-1}]$ for $n \ge 1$. Modifying the example of [6, p. 81, ex. 11], let $\nu(E) = \sum_{n=1}^{\infty} \frac{2^n}{n} \lambda(E \cap A_n)\, e_n$, $E \in A$. Then $\nu \equiv \lambda$, and it is easy to see that the sequence $f_n = n\chi_{A_n}$ $(n = 1,2,\ldots)$ converges to zero in L_1, while $||Tf_n|| = 1$ for every n.

We recall that T_o denotes the restriction of the map T to P.

LEMMA 3.2. If ν is a finite signed measure, then, for every sequence $\{x_n\}$ of elements of the range of T_o converging to the real number x, the sequence $\{T_o^{-1}\{x_n\}\}$ converges to $T_o^{-1}\{x\}$ in the Hausdorff metric on 2^P.

Proof: Let S^+, S^-, denote a Hahn - decomposition of S relative to ν and let $\beta = \nu(S^+)$, $\alpha = \nu(S^-)$. Then we have $T_o(P) = \overline{co}\ \nu\ (A) = [\alpha,\beta]$, and so $\alpha \le x \le \beta$. It will clearly suffice to prove that $d(T_o^{-1}\{x\}, T_o^{-1}\{y\}) \le |x - y|$ for every $y \in [\alpha,\beta]$.

We shall first prove that

(1) $\sup\{d(\phi,T_o^{-1}\{y\}) : \phi \in T_o^{-1}\{x\}\} \le |x - y|$.

Let $\phi \in T_o^{-1}\{x\}$. We shall show that there exists

$\psi \in T_o^{-1}\{y\}$ such that $||\psi - \phi||_1 = |x - y|$.

If $y = x$, we may take $\psi = \phi$. Suppose that $x > y$. Then $x > \alpha$, and we put

$$\psi = \phi + \frac{x - y}{x - \alpha}(\chi_{S^-} - \phi).$$

Then we have $\psi = \frac{y - \alpha}{x - \alpha}\phi + \frac{x - y}{x - \alpha}\chi_{S^-}$, and since $\alpha \le y < x$, both $\frac{y - \alpha}{x - \alpha}$ and $\frac{x - y}{x - \alpha}$ are between 0 and 1. Thus ψ is a convex combination of ϕ and χ_{S^-}, whence $\psi \in P$.

Moreover,

$$T_o(\psi) = x + \frac{(x - y)}{(x - \alpha)}(\alpha - x) = y,$$

and

$$||\psi - \phi||_1 = \frac{x - y}{x - \alpha}\int|\chi_{S^-} - \phi|d|\nu|$$

$$= \frac{x - y}{x - \alpha} \left\{ \int_{S^+} \phi \, d\nu - \int_{S^-} (1 - \phi) \, d\nu \right\}$$

$$= \frac{x - y}{x - \alpha} \left\{ \int \phi d\nu - \alpha \right\} = \frac{x - y}{x - \alpha} (x - \alpha) = x - y.$$

Suppose next that $x < y$. Then $x < \beta$, and we put

$$\psi = \phi + \frac{y - x}{\beta - x} (\chi_{S^+} - \phi)$$

instead. It follows as above that $\psi \in P$, $T_o(\psi) = y$ and $||\psi - \phi||_1 = y - x$. Thus (1) holds, and by interchanging x and y we obtain

(2) $$\sup \{ d(\psi, T_o^{-1}\{x\}) : \psi \in T_o^{-1}\{y\}\} \leq |x - y|.$$

From (1) and (2) it follows that $d(T_o^{-1}\{x\}, T_o^{-1}\{y\}) \leq |x - y|$. This completes the proof of the lemma.

<u>PROPOSITION</u> 3.3. If the range of ν is finite-dimensional, then T_o is open.

<u>Proof</u>: Let $\nu : A \to R^k$, and suppose that $\nu(A) = (\nu_1(A), \nu_2(A), \ldots, \nu_k(A))$ for every $A \in A$. Further let $T\phi = (T_1\phi, T_2\phi, \ldots, T_k\phi)$ for each $\phi \in P$, so that $T_i\phi = \int \phi \, d\nu_i$ for every $\phi \in P$ and $1 \leq i \leq k$. Suppose that some sequence $\{y_n\}$ of elements of $T_o(P)$ converges to $y \in T_o(P)$. We claim that the sequence $\{T_o^{-1}\{y_n\}\}$ converges to $T_o^{-1}\{y\}$ in the Hausdorff metric on 2^P.

Let, for every n, $y_n = (y_{n,i})_{i=1}^k$ and $y = (y_i)_{i=1}^k$. Since $y_n \to y$, we have $\{y_{n,i}\} \to y_i$ for each i, $1 \leq i \leq k$. As ν_i is a finite signed measure and $T_i(\phi) = \int \phi \, d\nu_i$ for every $\phi \in P$, we have $d(T_i^{-1}\{y_{n,i}\}, T_i^{-1}\{y_i\}) \to 0$ by Lemma 3.2. Since the operation $(A,B) \to A \cap B$ is continuous relative to the Hausdorff metric on 2^P,

[25], we have

$$d\left(\bigcap_{i=1}^{k} T_i^{-1}\{y_{n,i}\},\ \bigcap_{i=1}^{k} T_i^{-1}\{y_i\}\right) \to 0,$$

$$\text{i.e., } d(T_\circ^{-1}\{y_n\}, T_\circ^{-1}\{y\}) \to 0.$$

Now, if $T_\circ$ is not open, there exists an open subset U of P such that $T_\circ(U)$ is not open in K, i.e. there exists a sequence $\{x_n\}$ of elements in K \ $T_\circ(U)$ converging to an element x of $T_\circ(U)$. There thus exists $\phi \in T_\circ^{-1}\{x\} \cap U$, and since U is open, there further exists $\varepsilon > 0$ such that the closed ball $S(\phi,\varepsilon) \subset U$. For every natural number n, we then have $d(\phi, T_\circ^{-1}\{x_n\}) \geq \varepsilon$, whence

$$d(T_\circ^{-1}\{x\}, T_\circ^{-1}\{x_n\}) \geq d(\phi, T_\circ^{-1}\{x_n\}) \geq \varepsilon,$$

which contradicts the above. Hence the proposition.

PROPOSITION 3.4. If X is quasi-complete, ν possesses property (*) and $T_\circ$ is open relative to the L_1 – norm [weak*] topology on P and the given [weak] topology on K, so is ν relative to $\mathcal{T}_\nu$ on $\mathcal{A}$ and the above topology on its range.

Proof: When (A_n) is a net of subsets of a topological space E, the limit superior [25] of (A_n) is defined to be the set of all x in E such that every neighborhood of x intersects the net frequently, and denoted by LsA_n. It follows from Hajek [13, Prop. 1] that a continuous map f on E to a topological space F is open if, and only if, for every net (y_n) of elements of F converging to $y \in F$ we have $f^{-1}\{y\} = Lsf^{-1}\{y_n\}$. For any net (A_n) of subsets of P, we shall denote the limit superior of (A_n) relative to the L_1 – norm and the weak* topologies on P by LsA_n and Ls^*A_n respectively.

As X is quasi-complete, $T_\circ(P) = K$. Let first $T_\circ$ be open relative to the weak* - topology on P and the weak topology on K. Let (x_n) be a net of elements in $\nu(A)$ which converges weakly to an element x of $\nu(A)$. Since $T_\circ$ is open, we have $T_\circ^{-1}\{x\} = Ls^*T_\circ^{-1}\{x_n\}$. Since $(T_\circ^{-1}\{x_n\})$ is a net of weak* - compact convex sets converging to the weak* - compact convex set $T_\circ^{-1}\{x\}$, according to Jerison [19], we have $Ls^*T_\circ^{-1}\{x_n\} = \overline{co}\,\{Ls^*\ ext\ T_\circ^{-1}\{x_n\}\}$. As ν possesses property (*), we have $ext\ T_\circ^{-1}\{x_n\} = \{\chi_E : E \in A, \nu(E) = x_n\}$ for every n, and so $T_\circ^{-1}\{x\} = \overline{co}\,(Ls^*\{\chi_E : E \in A, \nu(E) = x_n\})$. According to Milman's theorem [20, p. 132] we have $ext\ T_\circ^{-1}\{x\} \subset Ls^*\{\chi_E : E \in A, \nu(E) = x_n\}$,

i.e. $\{\chi_E : E \in A, \nu(E) = x\} \subset Ls^*\{\chi_E : E \in A, \nu(E) = x_n\}$.

As the weak* - topology coincides with the L_1 - topology on $P_\circ$, we obtain $\{\chi_E : E \in \nu^{-1}\{x\}\} \subset Ls\{\chi_E : E \in \nu^{-1}\{x_n\}\}$,

and since $(P_\circ, ||\cdot||_1)$ can be identified with (A, T_ν), we get $\nu^{-1}\{x\} \subset Ls\nu^{-1}\{x_n\}$. Further, since ν is continuous, we have $Ls\nu^{-1}\{x_n\} \subset \nu^{-1}\{x\}$, and so the required equality holds.

Let now $T_\circ$ be open relative to the L_1 - topology on P and the given topology on K, and let the net (x_n) of elements of $\nu(A)$ converge to the element x of $\nu(A)$ relative to the given topology of K. Since $T_\circ$ is open, we have $T_\circ^{-1}\{x\} = LsT^{-1}\{x_n\}$. Since the weak* - topology is coarser than the L_1 - norm topology, and $T_\circ$ is continuous, we have

$$T_\circ^{-1}\{x\} = \mathrm{Ls}\,T_\circ^{-1}\{x_n\} \subseteq \mathrm{Ls}^* T_\circ^{-1}\{x_n\} \subset T_\circ^{-1}\{x\},$$

and so we still have $T_\circ^{-1}\{x\} = \mathrm{Ls}^* T_\circ^{-1}\{x_n\}$. Hence we get, as proved above, $\nu^{-1}\{x\} = \mathrm{Ls}\,\nu^{-1}\{x_n\}$, and so ν is open by Hájek's theorem. This completes the proof of the proposition.

LEMMA 3.5. If ν is semi-convex, then the extreme points of every level of $T_\circ$ are characteristic functions.

Proof: Let $a \in T_\circ(P)$, $\phi \in T_\circ^{-1}\{a\}$, $\phi \notin P_\circ$. We need to show that ϕ is not an extreme point of $T_\circ^{-1}\{a\}$.

As ϕ is not a characteristic function, there exists a non-null set $A \in \mathcal{A}$ and a real number ε such that $0 < \varepsilon < \frac{1}{2}$, and $\varepsilon \le \phi \le 1 - \varepsilon$ on A. Since ν is semi-convex, there exists $B \in \mathcal{A}$, $B \subset A$ such that $\nu(B) = (1/2)\,\nu(A)$. Let

$$\phi_1 = \phi - \varepsilon\chi_B + \varepsilon\,\chi_{A\setminus B}, \qquad \phi_2 = \phi + \varepsilon\,\chi_B - \varepsilon\,\chi_{A\setminus B}.$$

Since ϕ is bounded by ε and $1 - \varepsilon$ on A, ϕ_1 and $\phi_2 \in P$. Moreover,

$$\begin{aligned} T_\circ(\phi_1) &= \int \phi_1\, d\nu = \int \phi\, d\nu - \varepsilon\,\nu(B) + \varepsilon\,\nu(A \setminus B) \\ &= \int \phi\, d\nu - \varepsilon\cdot(1/2)\nu(A) + \varepsilon\cdot(1/2)\nu(A) \\ &= \int \phi\, d\nu = a. \end{aligned}$$

Similarly, $T_\circ(\phi_2) = a$. Clearly, $\phi = (1/2)(\phi_1 + \phi_2)$, where $\phi_1 \neq \phi_2$ since they differ on the non-null set A. Hence ϕ cannot be an extreme point of $T_\circ^{-1}\{a\}$.

According to Halmos [15], every non-atomic finite-dimensional measure is semi-convex, and so it clearly possesses property (*). We thus get the following theorem from Propositions 3.3 and 3.4:

THEOREM 3.6. Every finite-dimensional non-atomic measure is open.

Remark: The above theorem need not hold in general when X is infinite dimensional. The following example was suggested to me by Professor J.L.B. Gamlen. Let $S = [-1,1]$, $\mathcal{A}$ be the σ - algebra of Borel subsets of S, λ the Lebesgue measure on $\mathcal{A}$ and $X = L_1[0,1]$. Define the measure $\nu : \mathcal{A} \to X$ by

$$\nu(A) = \chi_{A\cap[0,1]} + \lambda(A \cap [-1,0])\, \chi_{[0,1]},\quad A \in \mathcal{A}.$$

Then ν is clearly controlled by λ, hence non-atomic. Let $A_n = [0, 1 - 1/n]$ for every $n \geq 2$, and $A = [-1,0]$. Then $\|\nu(A_n) - \nu(A)\|_1 \to 0$, and it is clear that A cannot be the limit of any subsequence of $\{A_n\}_{n\geq 2}$, i.e. $\mathrm{Ls}\,\nu^{-1}\{\nu(A_n)\} \neq \nu^{-1}\{\nu(A)\}$, and so ν does not satisfy Hájek's criterion for openness of a map.

THEOREM 3.7. Every finite-dimensional measure is biquotient.

Proof: Let ν be a finite-dimensional measure and λ be a control measure for ν. Clearly, ν and λ have the same atoms. We denote the atomic and non-atomic parts of $\mathcal{A}$ (or ν) by $\mathcal{A}_a$, $\mathcal{A}_n$ (respectively ν_a, ν_n). For every $A \in \mathcal{A}$, let A_a and A_n denote the atomic and non-atomic parts of A respectively. Let further R_a and R_n denote the ranges of ν_a and ν_n respectively.

Now the product topology on $\mathcal{A}_a \times \mathcal{A}_n$ is clearly induced by the following metric:

$$\sigma((A,B),(C,D)) = \lambda(A \,\Delta\, C) + \lambda(B \,\Delta\, D),$$

for every $(A,B), (C,D) \in \mathcal{A}_a \times \mathcal{A}_n$.

Define $h : \mathcal{A} \to \mathcal{A}_a \times \mathcal{A}_n$ by $h(A) = (A_a, A_n)$ for every $A \in \mathcal{A}$. Further define $g : \mathcal{A}_a \times \mathcal{A}_n \to R_a \times R_n$ and $f : R_a \times R_n \to X$ by $g = \nu_a \times \nu_n$, $f(x,y) = x + y$ for $x \in R_a$, $y \in R_n$. Then $\nu = f \circ g \circ h$, and so according to Michael [34], it would suffice to prove that the maps f, g and h are biquotient.

First, h is an isometry. Since each $A \in \mathcal{A}$ has the unique decomposition $A = A_a \cup A_n$, $A_a \in \mathcal{A}_a$, $A_n \in \mathcal{A}_n$, h is one to one. Moreover, if A and B are in $\mathcal{A}$, then

$$\begin{aligned}\sigma(h(A),h(B)) &= \sigma((A_a,A_n),(B_a,B_n)) \\ &= \lambda(A_a \Delta B_a) + \lambda(A_n \Delta B_n) \\ &= \lambda((A_a \Delta B_a) \cup (A_n \Delta B_n)) = \lambda(A \Delta B).\end{aligned}$$

According to Halmos [14, lem. 3] $\mathcal{A}_a$ is compact. Since ν_a is continuous on $\mathcal{A}_a$, it is biquotient. On the other hand, ν_n is open by Theorem 3.6. Thus ν_a and ν_n are both biquotient, and it follows from Theorem 1.2 of Michael [34] that g is biquotient.

Finally, the set R_a is clearly compact, and since ν is finite-dimensional, according to Liapounoff's theorem R_n is also compact. Thus $R_a \times R_n$ is compact, and since f is continuous on it, it is biquotient. This completes the proof of the theorem.

Let, for every pair of elements A and B of $\mathcal{A}$, $A \leq_\nu B$ if $\lambda(A \setminus B) = 0$. Then $\leq_\nu$ is a partial order on $\mathcal{A}$ and $(\mathcal{A}, \leq_\nu)$ is a complete lattice [16, p. 169]. It can also be verified that the order $\leq_\nu$ is independent of the choice of λ. If $\mathcal{C}$ is a chain for $\leq_\nu$, then the order topology [4] of $\mathcal{C}$ coincides with the one induced by λ on $\mathcal{C}$. Indeed, if $C \in \mathcal{C}$ and (C_1, C_2) is an open interval containing C, then on putting $r = 1/2 \ \min\{\lambda(C_2 \setminus C), \lambda(C \setminus C_1)\}$, we have

$$B(C,r) \equiv \{D \in \mathcal{C} : \lambda(D \Delta C) < r\} \subset (C_1, C_2).$$

On the other hand, for every $C \in \mathcal{C}$ and $r > 0$, if $\delta = 1/2 \ \min\{\lambda(C), r\}$, and $C_1, C_2 \in \mathcal{C}$ are such that

$$C_1 <_\lambda C <_\lambda C_2, \qquad \lambda(C \setminus C_1) < \delta \text{ and } \lambda(C_2 \setminus C) < \delta,$$

we have $(C_1, C_2) \subset B(C,r)$.

<u>PROPOSITION</u> 3.8. Every semi-convex measure is weakly monotone.

<u>Proof:</u> Let ν be a semi-convex measure with λ as its control measure. Let A and B be any two distinct elements in $\mathcal{A}$ such that $\nu(A) = \nu(B)$. Hence $\nu(A \setminus B) = \nu(B \setminus A) = x$ say. Since ν is semi-convex, one can find, as in Schmets [41, p. 185] chains $\mathcal{C} = \{C_\theta : \theta \in [0,1]\}$ and $\mathcal{D} = \{D_\theta : \theta \in [0,1]\}$ of subsets of $A \setminus B$ and $B \setminus A$ respectively with the following properties: $C_o = D_o = \phi$, $C_1 = A \setminus B$, $D_1 = B \setminus A$,

$C_{\theta_1} <_\nu C_{\theta_2}$ if and only if $\theta_1 < \theta_2$,

$$D_{\theta_1} <_\nu D_{\theta_2} \text{ if and only if } \theta_1 < \theta_2, \text{ and}$$

$\nu(C_\theta) = \theta x = \nu(D_\theta)$ for every $\theta \in [0,1]$.

The maps $\theta \to C_\theta$ and $\theta \to D_\theta$ are clearly isomorphisms of the chain [0,1] onto the chains $\mathcal{C}$ and $\mathcal{D}$ respectively, and so they are also homeomorphisms relative to the chain topologies. As the operations of union and difference are continuous on $\mathcal{A} \times \mathcal{A}$ relative to the product topology $T_\lambda \times T_\lambda$ [16, p. 168], the map $\theta \to E_\theta = (A \setminus C_\theta) \cup D_\theta$, $\theta \in [0,1]$ is continuous. Hence $\mathcal{E} = \{E_\theta : \theta \in [0,1]\}$ is a connected subset of $\mathcal{A}$. Further, for every $\theta \in [0,1]$ we have

$$\nu(E_\theta) = \nu(A) - \nu(C_\theta) + \nu(D_\theta) = \nu(A), \text{ while}$$

$E_0 = A$, $E_1 = (A \setminus (A \setminus B)) \cup (B \setminus A) = B$. Thus any two distinct elements of $\nu^{-1}\{\nu(A)\}$ are contained in a connected subset of this level, and so $\nu^{-1}\{\nu(A)\}$ is connected for every $A \in \mathcal{A}$. This completes the proof of the proposition.

Every non-atomic finite dimensional measure is semi-convex, according to Halmos [15], whence weakly monotone by Proposition 3.8. On the other hand it is open (Theorem 3.6) and so with the help of Whyburn [49], we have

COROLLARY 3.8.1. Every non-atomic finite dimensional measure is monotone.

Remark: The converse of Proposition 3.8. does not hold in general, as is evident from any non-zero measure on $\mathcal{A} = \{\phi, S\}$. For an example of a measure in infinite dimensions, which is weakly monotone without being semi-convex, let $\mathcal{A}$ be the σ - algebra of Borel subsets of $S = [0,1]$, $X = L_1[0,1]$, and define $\nu : \mathcal{A} \to X$ by $\nu(E) = \chi_E$ for every $E \in \mathcal{A}$. It is easy to see that ν is non-atomic and one - to - one, whence weakly monotone. However, it is not semi-convex, for we have $\nu(S) = \chi_S$ whereas ν does not assume $(1/2)\chi_S$ anywhere.

In infinite dimensions, a non-atomic measure is not always weakly monotone. In the example following Theorem 3.6 it may be easily verified that $\nu^{-1}\{\chi_{[0,1]}\} = \{[-1,0],[0,1]\}$ which is not connected, i.e. ν is not weakly monotone.

4. Range of a measure. In this section, X is assumed to be Banach unless otherwise stated. Also $\nu : \mathcal{A} \to X$ continues to be controlled. If P is a property of measures, ν is said to possess P hereditarily if ν_A possesses P for every $A \in \mathcal{A}$.

LEMMA 4.1. Let X be any L.C.S. and $\nu : \mathcal{A} \to X$ a controlled measure. If ν possesses property (*), then the range of ν is convex, closed or weakly compact if and only if it is hereditarily so.

Proof: Let $A \in \mathcal{A}$. We shall denote the set $\{\phi \in P : 0 \le \phi \le \chi_A\}$ by P_A, and the restriction of $T_\circ$ to P_A by T_A. Then $T_A : P_A \to X'^*$ is also continuous relative to the weak* - topology on its domain and the $\sigma(X'^*,X')$ - topology on its range. As P_A is convex and T_A is linear, $T_A(P_A)$ is convex and it clearly contains $\nu_A(\mathcal{A})$.

Suppose first that $\nu(\mathcal{A})$ is convex, $x,y \in \nu_A(\mathcal{A})$, $0 < t < 1$ and $z = tx + (1-t)y$. Since $x,y \in T_A(P_A)$, we have $z \in T_A(P_A)$. We claim that $z \in \nu_A(\mathcal{A})$. Clearly, it will suffice to prove that there exists $E \in \mathcal{A}$, $E \subset A$ such that $\chi_E \in T_A^{-1}\{z\}$. Now the set P_A is weak*-compact and convex and T_A is linear, so that $T_A^{-1}\{z\}$ is also weak*-compact and convex, whence it has some extreme point ϕ by Krein-Milman theorem. Suppose that $\phi \notin P_\circ$. As $x,y \in \nu(\mathcal{A})$ also, we have $z \in \nu(\mathcal{A})$. Since $\phi = 0$ a.e. on A^c we have $T_\circ(\phi) = T_A(\phi) = z$. As ν has the property (*), $\phi \notin \text{ext } T_\circ^{-1}\{z\}$ and so there exist distinct members

ϕ_1, ϕ_2 of $T_\circ^{-1}\{z\}$ such that $\phi = 1/2 \ (\phi_1 + \phi_2)$. Then we have $\phi_1 = \phi_2 = 0$ on A^c, so that $\phi_1 \neq \phi_2$ on A, otherwise they will be equal on S. Moreover, $T_A(\phi_1) = T_\circ(\phi_1) = z$, i.e. $\phi_1 \in T_A^{-1}\{z\}$. Similarly $\phi_2 \in T_A^{-1}\{z\}$, so that $\phi \notin$ ext $T_A^{-1}\{z\}$. Thus ext $T_A^{-1}\{z\} \subset P_\circ$, proving the claim.

Let, now, $\nu(A)$ be closed in the given topology of X, and let (x_n) be a net in $\nu_A(A)$ converging to $x \in X$. Then (x_n) converges weakly, and since $(X,\sigma(X,X'))$ is embedded in $(X'^*,\sigma(X'^*,X'))$ (see §2) we have $(x_n) \to x$ relative to the latter topology. With P_A and T_A as above, the set $T_A(P_A)$ is $\sigma(X'^*,X')$ - compact, and since $x_n \in \nu_A(A) \subset T_A(P_A)$ for each n, $x \in T_A(P_A)$. As the net (x_n) is contained in $\nu(A)$, we have $x \in \nu(A)$. It follows again, as above, that $x \in \nu_A(A)$.

Finally, suppose that $\nu(A)$ is weakly compact and (x_n) is a net of elements of $\nu_A(A)$. As $\nu_A(A) \subset \nu(A)$, there exists a subnet (x_m) of (x_n) which converges weakly to some $x \in \nu(A)$. As in the previous proof, we have $x \in T_A(P_A)$ and it follows once again that $x \in \nu_A(A)$. Since the converses are obvious, this completes the proof of the lemma.

<u>THEOREM</u> 4.2. A controlled measure ν with values in a L.C.S. X has hereditarily convex range if and only if any one of the following holds:

(a) ν is semi - convex.

(b) For every $a \in T_\circ(P)$, ext $T_\circ^{-1}\{a\} \subset P_\circ$.

(c) ν possesses property (*) and has convex range.

Proof: It is obvious that if ν has hereditarily convex range, ν is then semi-convex. The implication (a) implies (b) was proved in Lemma 3.5. It would suffice to prove that (b) implies (c), for when (c) holds then it follows from Lemma 4.1 that ν has hereditarily convex range.

Suppose ν satisfies (b). It then clearly has property (*). Let $a \in T_\circ(P)$. Since ext $T_\circ^{-1}\{a\} \subset P_\circ$, it follows from the Krein-Milman theorem that $a \in T_\circ(P_\circ)$, i.e. $T_\circ(P) \subset T_\circ(P_\circ) = \nu(A)$. As the reverse inclusion is obvious, we have $\nu(A) = T_\circ(P)$. Since P is convex and $T_\circ$ is affine, the set $\nu(A)$ is convex. This completes the proof of the theorem.

For any controlled measure ν, we recall that ν_a and ν_n denote its atomic and non-atomic parts respectively.

THEOREM 4.3. A controlled measure ν with values in a L.C.S. has hereditarily weakly compact range if and only if one of the following holds:

(a) ν_n is semi-convex.

(b) ν_n possesses property (*) and has weakly compact range.

Proof: Since ν_a has hereditarily compact range [14], it follows that ν has hereditarily weakly compact range if and only if ν_n has hereditarily weakly compact range. Thus it suffices to prove the theorem for ν_n and we may as well assume $\nu = \nu_n$.

Let ν have hereditarily weakly compact range, and let $A \in \mathcal{A}$. As $\nu_A(A)$ is weakly compact and $(X,\sigma(X,X'))$ is embedded in $(X'^*,\sigma(X'^*,X))$

(see §2) the set $\nu_A(\mathcal{A})$ is equally compact in the latter topology. As observed in the proof of Lemma 4.1, the map $T_A : P_A \to X'^*$ is continuous in the weak* - topology on its domain and the $\sigma(X'^*,X')$ - topology on its range, and so $Q = T_A^{-1}\{\nu_A(\mathcal{A})\}$ is weak* - closed. As ν is non-atomic, so are λ and λ_A and the set $(P_\circ)_A = \{\chi_E : E \in \mathcal{A},\ E \subset A\}$ is weak* - dense in P_A [21, lem. 3]. Since Q clearly contains $(P_\circ)_A$, we have $Q = P_A$, whence $\nu_A(\mathcal{A}) = T_A(Q) = T_A(P_A)$, i.e. $\nu_A(\mathcal{A})$ is convex. Hence ν has hereditarily convex range, and so is semi-convex.

Now suppose that (a) holds. Then ext $T_\circ^{-1}\{a\} \subset P_\circ$ for each $a \in T_\circ(P)$ by Theorem 4.2, and so ν clearly has property (*). As in the proof of (b) implies (c) of Theorem 4.2, we have $\nu(\mathcal{A}) = T_\circ(P)$ and since $T_\circ(P)$ is $\sigma(X'^*,X')$ - compact and $(X,\sigma(X,X'))$ is embedded in $(X'^*,\sigma(X'^*,X'))$, it follows that $\nu(\mathcal{A})$ is weakly compact.

If (b) holds, then ν has hereditarily weakly compact range by Lemma 4.1. This completes the proof of the theorem.

From the above theorem we obtain

<u>COROLLARY</u> 4.3.1. A controlled measure ν with values in a L.C.S. is semi-convex if and only if any of the following holds:

(a) ν is non-atomic and has a hereditarily weakly compact range.

(b) ν is non-atomic, possesses property (*) and has weakly compact range.

<u>Remark</u>: A subspace N of $L_1(\lambda)$ is called <u>thin</u> if for every non-null set $A \in \mathcal{A}$ there exists a non-zero function $\phi \in L_\infty(\lambda)$ such that $\phi = 0$ a.e. on the complement of A and $\int \phi f \, d\lambda = 0$ for each $f \in N$. Given a

measure $\nu : \mathcal{A} \to X$ controlled by λ, the subspace $N_\nu = \{\frac{dx'\cdot\nu}{d\lambda} : x' \in X'\}$ of $L_1(\lambda)$ has been proved by Kingman and Robertson [21] (see also Wegmann [48]) to be thin if and only if the range of ν is hereditarily convex and hereditarily weakly compact. The subspace N_ν is thus thin if and only if ν is semi-convex, and so, according to Theorems 4.2 and 4.3, N_ν is thin if and only if the range of ν is hereditarily convex, or equivalently, if and only if ν is non-atomic and its range is hereditarily weakly compact.

PROPOSITION 4.4. If X is Banach, ν possesses property (*), and $T_\circ$ is open, then the range of ν is norm closed.

Proof: Since $T_\circ$ is open and ν possesses property (*), ν is open by Proposition 3.4. On identifying $\mathcal{A}$ with $P_\circ$, the maps $T^{-1} : K \to 2^P$ and $\nu^{-1} : \nu(\mathcal{A}) \to 2^{P_\circ}$ are then lower semi-continuous (l.s.c.) [25, p. 174]. Since K and $\nu(\mathcal{A})$ have the induced norm topology of X, they are metrisable, and so are paracompact. Since ν and $T_\circ$ are continuous and their domains are complete metric spaces, $\nu^{-1}\{\nu(A)\}$ and $T_\circ^{-1}\{x\}$ are complete for every $A \in \mathcal{A}$ and $x \in K$. As proved by Michael [33] there exist l.s.c. maps $f : K \to 2^P$ and $g : \nu(\mathcal{A}) \to 2^{P_\circ}$ such that $f(x)$ is a compact subset of $T_\circ^{-1}\{x\}$ for every $x \in K$, while $g(\nu(A))$ is a compact subset of $\nu^{-1}\{\nu(A)\}$ for every $A \in \mathcal{A}$. Define $h : K \to 2^P$ by

$$\begin{aligned} h(x) &= f(x) \cup g(x) \quad \text{if } x \in \nu(\mathcal{A}). \\ &= f(x) \qquad\qquad \text{if } x \in K \setminus \nu(\mathcal{A}). \end{aligned}$$

Then $h(x)$ is compact for every $x \in K$.

Now let $\{x_n\}$ be a sequence of elements of $\nu(\mathcal{A})$ converging to $x_\circ$, and assume that $x_\circ \notin \nu(\mathcal{A})$. Then $h(x_\circ) = f(x_\circ)$ and since f is l.s.c., we have, according to Sikorski [42] $f(x_\circ) = Lsf(x_n) = Lif(x_n)$. Thus we have $h(x_\circ) = Lif(x_n) \subset Lih(x_n)$. On the other hand, $Lsg(x_n) = \phi$. For, if not, then there exists a sequence $\{\chi_{E_{n_k}}\}$ with $\chi_{E_{n_k}} \in g(x_{n_k})$ for every k, and $||\chi_{E_{n_k}} - \phi||_P \to 0$ for some $\phi \in P$. Since $P_\circ$ is closed in P, $\phi \in P_\circ$, say $\phi = \chi_E$ $(E \in A)$, and so $x_\circ = \lim x_{n_k} = \lim T_\circ(\chi_{E_{n_k}})$ $= T_\circ(\phi) = \nu(E)$, contradicting $x_\circ \notin \nu(A)$. According to [25, p.337] we thus have

$$Lsh(x_n) = Ls\{f(x_n) \cup g(x_n)\} = Lsf(x_n) \cup Lsg(x_n)$$
$$= Lsf(x_n) = f(x_\circ) = h(x_\circ).$$

Hence we have $h(x_\circ) \subset Lih(x_n) \subset Lsh(x_n) = h(x_\circ)$, so that

$$h(x_\circ) = Lih(x_n) = Lsh(x_n).$$

Thus $\{h(x_n)\}$ is a sequence in 2^P converging to $h(x_\circ)$ in the Vietoris topology [32], and so $\{h(x_n) : n \geq 0\}$ is a compact subset of 2^P. Since $h(x_n)$ is compact for every n, according to Michael [32] the set $C = \bigcup_{n \geq 0} h(x_n)$ is a compact subset of P. Choosing $\chi_{E_n} \in g(x_n)$ $(\subset h(x_n))$ for every $n \geq 1$, the sequence $\{\chi_{E_n}\}$ is contained in C, and so it has a subsequence $\{\chi_{E_{n_k}}\}$ which converges to some element ϕ of C. But then, $P_\circ$ being a closed subset of P, $\phi \in P_\circ$, and so $T_\circ\phi \in \nu(A)$. Thus, we have $x_\circ = \lim x_{n_k} = \lim T_\circ(\chi_{E_{n_k}}) = T_\circ(\phi)$ a contradiction. This completes the proof of the proposition.

When X is finite-dimensional and ν is non-atomic, then it possesses property (*) by Lemma 3.5, and $T_\circ$ is open by Proposition 3.3.

This gives

COROLLARY 4.4.1. (Liapounoff) The range of every finite dimensional non-atomic measure is a closed set.

Remark: When X is infinite dimensional and $T_\circ$ is open relative to the weak* and weak topologies, $\nu(A)$ need not be weakly closed even if ν posses property (*). In the second example following Corollary 3.8.1, both ν and $T_\circ$ are clearly one - to - one. Further, since (P,w^*) is compact, $T_\circ : (P,w^*) \to (K,w)$ is a homeomorphism. As $\nu(A) = P_\circ$, we have $K = \overline{co}\, P_\circ = P$. Since λ is non-atomic, according to [21, lem. 3], $P_\circ$ is weak* - dense in P, hence $\sigma(L_1,L_\infty)$ - dense in P (see §2). As $P \neq P_\circ$, the latter set cannot be weakly closed.

THEOREM 4.5. If X is Banach, then the range of a measure $\nu : A \to X$ is relatively compact if, and only if, there exists a sequence $\{x_n\}$ in X and a sequence $\{A_n\}$ in A such that the series $\sum_{n=1}^{\infty} x_n \lambda_{A_n}$ converges to ν in the Pettis norm.

Proof: Let us first prove the sufficiency part. Suppose that ν has the above representation. Let, for every n, $\nu_n = \sum_{i=1}^{n} x_i \lambda_{A_i}$.

It is then clear that $\nu_n(A)$ is compact. By hypothesis, $||\nu_n - \nu||_P = \sup\{||\nu_n(E) - \nu(E)||: E \in A\} \to 0$ as $n \to \infty$, and so $d(\nu_n(A),\nu(A)) \to 0$. Hence, it follows that $d(\nu_n(A),\overline{\nu(A)}) \to 0$. Since $\nu_n(A)$ is compact for every n, so is $\overline{\nu(A)}$, i.e. ν has relatively compact range.

Now let $\overline{\nu(A)}$ be compact. Then the set $K = \overline{co}\,\nu(A)$ is also compact. We shall first prove that ν is the limit in the Pettis norm of a sequence of elementary measures each of which has a finite number of elements.

Let us first assume that X has a Schauder basis, say $\{e_i : i \geq 1\}$. According to [40, p. 115], there exists a sequence of elements $\{x_i'\}$ of X' such that for every x in X we have

$x = \sum_{i=1}^{\infty} x_i'(x)e_i$, where the series converges uniformly on every compact subset of X. Thus

$$\sup_{x \in K} \left\| x - \sum_{i=1}^{n} x_i'(x)e_i \right\| \to 0 \quad \text{as } n \to \infty.$$

Let, for each natural number n,

$$\mu_n(E) = \sum_{i=1}^{n} x_i' . \nu(E)\, e_i \quad , \quad E \in A.$$

Clearly, μ_n is a finite dimensional measure $<<\lambda$, and as $\nu(A) \subseteq K$, we obtain

$$\|\nu - \mu_n\|_p = \sup \{\|\nu(E) - \mu_n(E)\| : E \in A\} \to 0 \text{ as } n \to \infty.$$

Now, for every n, there exists, by Radon-Nikodým theorem, a function $g_n : S \to X$ such that $\mu_n(E) = \int_E g_n \, d\lambda$ for each $E \in A$. Also, there exists a simple function $f_n : S \to X$ such that

$$\int \|f_n(s) - g_n(s)\| \, d\lambda \leq 1/n.$$

Let, for every n, $\nu_n(E) = \int_E f_n \, d\lambda$ for every $E \in A$. Then ν_n is an elementary measure with finitely many elements. Moreover, for every n,

$$\|\nu_n - \mu_n\|_p = \sup \left\{ \left\| \int_E f_n \, d\lambda - \int_E g_n \, d\lambda \right\| : E \in A \right\}$$

$$= \sup \left\{ \left\| \int_E (f_n - g_n) \, d\lambda \right\| : E \in A \right\}$$

$$\leq \int ||f_n - g_n|| \, d\lambda \leq 1/n.$$

We thus have

$$||\nu - \nu_n||_p \leq ||\nu - \mu_n||_p + ||\mu_n - \nu_n||_p \leq ||\nu - \mu_n||_p + 1/n \to 0 \text{ as } n \to \infty.$$

In case X does not have a basis, since K is compact, there exists [10, p. 99] a one-to-one continuous affine map g of K into a Hilbert space H. We may further assume that $g(0) = 0$. For, if not, then it may be replaced by the map $x \mapsto g(x) - g(0)$.

Now $K_1 = g(K)$ is also compact, and its closed linear span is a separable Hilbert space which may be denoted by H itself. Define $\mu : A \to H$ by $\mu(E) = g\cdot\nu(E)$ for $E \in A$. Clearly, μ is a measure, $\mu(A) \subset K_1$, and as H has a basis, there exists, by above, a sequence of elementary measures $\mu_n : A \to H$ $(n \geq 1)$ with finitely many elements such that $||\mu_n - \mu||_p \to 0$ as $n \to \infty$. As $d(\mu_n(A), \overline{\mu(A)}) \to 0$, the set $C = \overline{\mu(A)} \cup \bigcup_{n\geq 1} \mu_n(A)$ is compact, see [32]. Denote the extension of $h = g^{-1}$ to C by h itself.

Define, for $n \geq 1$, $\nu_n : A \to X$ by $\nu_n(E) = h\cdot\mu_n(E)$ for $E \in A$. Then ν_n is a measure. Moreover, for any $x' \in X'$, $x'\cdot h$ is continuous on the compact set C, hence uniformly continuous. As $||\mu_n - \mu||_p \to 0$, we thus have $||x'\cdot h\cdot(\mu_n - \mu)||_p \to 0$, i.e. $||x'\cdot(\nu_n - \nu)||_p \to 0$. Since the norm and weak-topologies coincide on the compact set $h(C)$, we then have $||\nu_n - \nu||_p \to 0$. As the measures ν_n are clearly elementary, this proves the assertion for general X.

To prove that ν has the required representation, there exists, by above, an elementary measure ν_1 with finitely many elements such that $||\nu - \nu_1||_p < 1/2$. Since ν_1 has compact range, $\nu - \nu_1$ is a measure with relatively compact range, and so there again exists an elementary measure ν_2 with finitely many elements such that $||(\nu - \nu_1) - \nu_2||_p < 1/2^2$. Continuing indefinitely, we find a sequence $\{\nu_n\}$ of elementary measures, each with finitely many elements, such that

$$||\nu - \sum_{i=1}^{n+1} \nu_i||_p < 1/2^{n+1} \text{ for every } n.$$

But then the series $\sum_{n=1}^{\infty} \nu_n$ converges to ν in the Pettis norm. By definition, for each n we have $\nu_n = \sum_{i=1}^{k_n} x_{n,i} \lambda_{A_{n,i}}$, where $\{x_{n,i} : 1 \le i \le n\} \subset X$ and $\{A_{n,i} : 1 \le i \le k_n\}$ is a partition of S. It is clear that the series $\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} x_{n,i} \lambda_{A_{n,i}}$ converges to ν in the Pettis norm. This completes the proof of the theorem.

Although we first proved the following result, it equally follows from the above theorem.

<u>COROLLARY</u> 4.5.1. The range of ν is relatively compact if and only if ν can be approximated arbitrarily in the Pettis norm by an elementary measure with finitely many elements.

If ν is a Bochner integral relative to some finite positive measure λ, then it clearly has a representation as in Theorem 4.5, and so we have

<u>COROLLARY</u> 4.5.2. (Uhl, [46]). A measure has relatively compact range whenever it is a Bochner integral relative to some finite positive

measure.

Since the atomic part of ν always has a hereditarily compact range (see [14]), the above theorem gives, with the help of Theorem 4.3,

COROLLARY 4.5.3. A measure ν has hereditarily compact range if and only if its non-atomic part ν_n is representable as in Theorem 4.5, and ν_n is either semi-convex, or has property (*) and a closed range.

Bolker [5] proved that a compact convex set in finite dimensions is the closed convex hull of the range of a measure if and only if it is the Hausdorff limit of finite sums of segments through the origin. According to Theorem 4.5, the necessity part of this theorem remains valid in infinite dimensions:

COROLLARY 4.5.4. If a compact convex set in X is the closed convex hull of the range of a measure, then it is the Hausdorff limit of finite sums of segments through the origin.

Problem: It would be interesting to investigate the hypotheses under which the sufficiency part of Bolker's theorem remains valid in infinite dimensions.

Remark: There exist measures, even with finite variation, whose ranges are relatively compact, but are not Bochner integrals; see Rieffel [38, p. 486] for an example in ℓ_∞.

There further exist measures with compact range, even in finite dimensions, that are non-elementary, and so in their representation (of Theorem 4.5) the sets (A_i) cannot be always made disjoint. As proved by Rickert [36, th. 1], there exists a non-atomic measure ν whose range is the closed unit disk K in R^2. Suppose ν is elementary. Then there clearly exist a non-trivial segment J with zero as one of its end points, and a set K_1 such that $K = J + K_1$. Without loss of generality, assume that J is along the positive x-axis. Since the point $(0,1) = b \in K$, there exist $\alpha \geq 0$ and $x_1 \in K_1$ such that $b = \alpha(1,0) + x_1$. If $\alpha > 0$, then $x_1 = (-\alpha,1)$, so that $x_1 \notin K$. On the other hand, as $0 \in J$, we have $0 + x_1 \in K$, a contradiction. In case $\alpha = 0$, $x_1 = b \in K$. Since J is non-trivial, there exists $\beta > 0$ such that $(\beta,1) = \beta(1,0) + b = \beta(1,0) + x_1 \in K$, which is not possible.

We recall that a measure ν with values in a Banach space is said to be σ-finite if its variation $|\nu|$ is so.

PROPOSITION 4.6. If X is an infinite dimensional Banach space, then the closed convex hull of the range of every σ-finite measure with values in X has empty interior.

Proof: Let $\nu : A \to X$ be a measure with its variation μ σ-finite. There then exists an increasing sequence $\{S_n\}$ of sets whose union is S such that the restriction μ_n of μ to S_n is finite for every n. Define $T_n : P \to X$ by $T_n(\phi) = \int_{S_n} \phi \, d\nu$, $\phi \in P$, for every n. Since X is Banach, $T_0(P) = K$, and $T_n(P) = \overline{co}\, \nu_{S_n}(A) = K_n$ (say) for every n.

Let us first consider the case when X is reflexive. According to the well-known theorem of Phillips [35, p. 130], each ν_{S_n} is a Bochner integral with respect to μ_n. Hence, by corollary 4.5.4, K_n is norm compact for every n. Since $S_n^c \downarrow \phi$, we have

$\lambda(S_n^c) \to 0$, where λ is a control measure of ν. Thus, $||\nu|| (S_n^c) \to 0$. For every n we have $K_n \subset K$. Moreover, for every $\phi \in P$, we have

$$||T\phi - T_n\phi|| = ||\int_S \phi \, d\nu - \int_{S_n} \phi \, d\nu|| = ||\int_{S_n^c} \phi \, d\nu|| \leq ||\nu|| (S_n^c) \to 0,$$

whence the sequence $\{K_n\}$ converges to K in the Hausdorff metric. Since K_n is compact for each n, so is K. As X is infinite dimensional, according to Riesz's theorem, the interior of K is empty.

Now let X be non-reflexive and, if possible, K have an interior point x. Then there exists $\varepsilon > 0$ such that the closed ball $S(x,\varepsilon) \subset K$, and since it is convex, it is weakly closed by Mazur's theorem [20, p. 154]. On the other hand, X being Banach, K is weakly compact, and so $S(x,\varepsilon)$ is also weakly compact. This is, however, not possible, since the closed unit ball of a non-reflexive space of infinite dimensions cannot be weakly compact.

CHAPTER III

EXTREMAL STRUCTURE OF THE RANGE OF A MEASURE

5. Extreme points of the range. We assume in this section X to be quasi-complete, except for Proposition 5.1. The measure $\nu : A \to X$ is assumed to be controlled as before. We recall that the extreme points of the range are defined to be those of its $\sigma(X'^*,X')$ - closed convex hull, i.e. of $T_\circ(P)$.

PROPOSITION 5.1. A point $a \in X'^*$ is an extreme point of $\nu(A)$ if and only if $T_\circ^{-1}\{a\}$ is a singleton which belongs to $P_\circ$.

Proof: Let a be an extreme point of $\nu(A)$. We shall first prove that $T_\circ^{-1}\{a\} \subset P_\circ$. Let, if possible, $\phi \in P \setminus P_\circ$ and $T_\circ(\phi) = a$. Then there exist an $\varepsilon > 0$ and a set $E \in A$ such that $\varepsilon < \frac{1}{2}$, $\lambda(E) > 0$ and

(1) $$\varepsilon \le \phi \le 1 - \varepsilon \qquad \lambda \text{ a.e. on } E.$$

We may further assume that $\tilde{\nu}(E) \neq 0$. For, as $\nu \equiv \lambda$, the set E is not ν - null and so contains a set $E_1 \in A$ for which $\nu(E_1) \neq 0$, and since $\lambda(E_1) > 0$, the set E may be replaced by E_1. Now define ϕ_1 and ϕ_2 by $\phi_1(s) = \phi_2(s) = \phi(s)$ on the complement E^c of E, while $\phi_1(s) = \phi(s) - \varepsilon$ and $\phi_2(s) = \phi(s) + \varepsilon$ on E. Then ϕ_1 and ϕ_2 are in P due to (1) and they differ on the non-null set E. Moreover,

$$T_\circ(\phi_1) = \int \phi_1 \, d\nu = \int_{E^c} \phi \, d\nu + \int_E (\phi - \varepsilon) \, d\nu = \int \phi \, d\nu - \varepsilon \, \nu(E) = a - \varepsilon\nu(E),$$

and similarly $T_\circ(\phi_2) = a + \varepsilon \, \nu(E)$. But then, $a = (1/2)(T_\circ(\phi_1) + T_\circ(\phi_2))$, and since $\nu(E) \neq 0$, we have $T_\circ(\phi_1) \neq T_\circ(\phi_2)$, so that $a \notin \text{ext}\, T_\circ(P)$, a contradiction.

Further, if $T_\circ^{-1}\{a\}$ contains two distinct elements, say χ_A and χ_B, we have $\lambda(A \Delta B) > 0$, so that $\psi = (1/2)(\chi_A + \chi_B) \notin P_\circ$, but since $\psi \in T_\circ^{-1}\{a\}$ this is again a contradiction. Hence $T_\circ^{-1}\{a\}$ is a singleton.

Conversely, suppose $T_\circ^{-1}\{a\} = \{\chi_A\}$ for some $A \in \mathcal{A}$. Now, if $a = (1/2)(T_\circ(\phi) + T_\circ(\psi))$ for ϕ, ψ in P, then $T_\circ(\chi_A) = T_\circ((1/2)(\phi + \psi))$, and so, according to the hypothesis, $\chi_A = (1/2)(\phi + \psi)$ λ a.e. But then it is easy to see that $\phi = \psi = \chi_A$ a.e., and so $T_\circ(\phi) = T_\circ(\psi) = a$, so that $a \in \text{ext}\, T_\circ(P)$. This completes the proof of the Proposition.

<u>COROLLARY</u> 5.1.1. The range of every controlled measure contains all of its extreme points.

This corollary has been recently established by Kluvánek [24, th. 2] for a wider class of measures called "closed measures". We also obtain from Proposition 5.1 the following extension of Theorem 3 of Liapounoff [27] to infinite dimensions, which in turn has been proved recently by Tweddle [45] for semi-convex measures.

<u>COROLLARY</u> 5.1.2. If ν possesses property (*), then a point $a \in X'^*$ is an extreme point of the range of ν if and only if $\nu^{-1}\{a\}$ is a singleton.

For, given $A \in \mathcal{A}$, since $\nu(A) = a$ if and only if $T_\circ(\chi_A) = a$, the necessity of the condition follows trivially; and if $\nu^{-1}\{a\} = \{A\}$, since $\text{ext}\, T_\circ^{-1}\{a\} \subset P_\circ$, we have $\text{ext}\, T_\circ^{-1}\{a\} = \{\chi_A\}$, so that $T_\circ^{-1}\{a\} = \overline{\text{co}}\, \{\chi_A\} = \{\chi_A\}$, and so $a \in \text{ext}\, \nu(A)$.

In the absence of property (*) we have, on the other hand, the following

PROPOSITION 5.2. If the dual of X is weak* - separable and ν is non-atomic, then a point $a \in X$ is an extreme point of the range of ν if and only if $\nu^{-1}\{a\}$ is a singleton.

Proof: The necessity of the condition follows from Proposition 5.1 as before.

Let $\{x_n'\}$ be a sequence of elements of X' that is weak* - dense in X'. As usual, we denote by R^ω the product of countably many copies of R, with the product topology. Define the map $\eta : X \to R^\omega$ by

$$\eta(x) = (x_1'(x),\ldots,x_n'(x),\ldots), \qquad x \in X.$$

Then η is linear. If $\eta(x) = 0$ for some $x \in X$, then $x_n'(x) = 0$ for every n, and since $\{x_n'\}$ is weak* - dense in X' we have $x'(x) = 0$ for every $x' \in X'$, i.e. $x = 0$. It may be easily verified that η is continuous relative to the weak-topology on X and the product topology on R^ω.

Define $\mu : A \to R^\omega$ by $\mu(E) = \eta\circ\nu(E)$, $E \in A$. Further, for each n, define $\mu_n : A \to R^\omega$ by

$$\mu_n(E) = (x_1'\circ\nu(E),\ldots\ldots,x_n'\circ\nu(E),0,0,\ldots).$$

Clearly, μ and μ_n are measures. Moreover, for every $E \in A$ we have $\mu_n(E) \to \mu(E)$. Define $T_n : P \to R^\omega$ by $T_n(\phi) = \int \phi \, d\mu_n$ for every $\phi \in P$. Then

$$T_n(\phi) = (\int \phi \, d\nu_{x_1'},\ldots\ldots,\int \phi \, d\nu_{x_n'}, 0,0,\ldots).$$

Now suppose that $\nu^{-1}\{a\} = \{A\}$ for some $A \in A$. As η is one-to-one, we have $\mu^{-1}\{\eta(a)\} = \{A\}$. For each n let

$$a_n = (x_1'(a),\ldots\ldots,x_n'(a),0,0,\ldots),$$

and $P_n = T_n^{-1}\{a_n\}$. We claim that $\{P_n\} \downarrow T_\circ^{-1}\{a\}$. For clearly $T_\circ^{-1}\{a\} \subset P_n$ for all n, while if $\phi \in P_n$ for every n, then $\int \phi \, d\nu_{x'_k} = x'_k(a)$ for all k, and since $\{x'_k\}$ is weak* - dense in X', we have $(\int \phi \, d\nu, x') = \int \phi \, d\nu_{x'} = x'(a)$ for every $x' \in X'$, so that $T_\circ(\phi) = \int \phi \, d\nu = a$. If $C_n = P_n \cap P_\circ$ for each n, then $C_n \downarrow T_\circ^{-1}\{a\} \cap P_\circ$ by above. By hypothesis, $\{E \in A : \nu(E) = a\} = \{\chi_A\}$, and so $C_n \downarrow \{\chi_A\}$.

As (P,w^*) is compact, the sequence $\{C_n\}$ converges to $\{\chi_A\}$ in the Vietoris topology on the power set of P [32]. If W is any closed weak* - neighborhood of χ_A relative to P, then $C_n \subset W$ eventually. Since ν is non-atomic, μ_n is non-atomic. As μ_n is further finite-dimensional, it possesses property (*) by Lemma 3.5, and so $C_n = \text{ext } P_n$. Since W is closed, convex and contains C_n eventually, W equally contains P_n, i.e. $\{P_n\} \to \{\chi_A\}$ relative to the Vietoris topology, whence $LsP_n = LiP_n = \{\chi_A\}$. Since $\{P_n\} \downarrow T_\circ^{-1}\{a\}$, we have $LsP_n = T_\circ^{-1}\{a\}$, and so $T_\circ^{-1}\{a\} = \{\chi_A\}$. But then $a \in \text{ext } \nu(A)$ by Proposition 5.1. This completes the proof of the proposition.

<u>PROPOSITION</u> 5.3. If X is quasi-complete and $\nu : A \to X$ is a measure, then every extreme point of its range is strongly extreme.

<u>Proof:</u> We first show that the sets of exposed and strongly exposed points of P relative to the L_1 - norm topology coincide with $P_\circ$.

Let $A \in A$. Then $f = \chi_A - \chi_{A^c}$ is in L_∞, and so

$$H = \{\phi \in L_1 : (\phi, f) \equiv \int \phi \, f \, d\lambda = \lambda(A)\}$$

is a hyperplane of L_1. Now, if $\phi \in P$, we have

$$(\phi, f) = \int_A \phi \, d\lambda - \int_{A^c} \phi \, d\lambda \le \lambda(A), \quad \text{while } (\chi_A, f) = \lambda(A),$$

i.e., H supports P at χ_A. Moreover, when $\phi \in H \cap P$, we have

$$-\int_{A^c} \phi \, d\lambda = \int_A (1 - \phi) \, d\lambda \quad , \quad \text{so that}$$

$\phi = \chi_A$ λ- a.e., and we have $H \cap P = \{\chi_A\}$. Hence χ_A is an exposed point of P. To show further that χ_A is strongly exposed, let $\{\phi_n\}$ be a sequence of elements in P such that $(\phi_n, f) \to (\chi_A, f)$. Since the latter is equal to $\lambda(A)$, we have

$$||\chi_A - \phi_n||_1 = \int_A (1 - \phi_n) \, d\lambda + \int_{A^c} \phi_n \, d\lambda = \lambda(A) - (\phi_n, f) \to 0 \quad ,$$

so that $\chi_A \in$ st exp P. Hence

$$P_\circ \subset \text{st exp } P \subset \text{exp } P \subset \text{ext } P \subset P_\circ \; ,$$

the last inclusion being observed in §2. This proves the assertion.

Now let $a \in$ ext K, and assume that $a \notin$ st ext K. Then there exists a neighborhood V of a relative to K such that $a \in \overline{\text{co}}\,(K \setminus V)$. Hence, there exists a net (x_n) of elements of co $(K \setminus V)$ converging to a. Let $x_n = \sum_{j=1}^{k_n} \alpha_{n,j} \, a_{n,j}$, $\alpha_{n,j} \geq 0$, $\sum_{j=1}^{k_n} \alpha_{n,j} = 1$ and $a_{n,j} \in (K \setminus V)$ for all j and n. Since X is quasi-complete, $T_\circ(P) = K$ (see §2) and so for every j and n there exists $\phi_{n,j} \in P$ such that $T_\circ(\phi_{n,j}) = a_{n,j}$. Let, for every n,

$$\phi_n = \sum_{j=1}^{k_n} \alpha_{n,j} \, \phi_{n,j} .$$

Since P is convex, it contains each ϕ_n, and since P is weak* - compact, there exists a subnet (ϕ_m) of (ϕ_n) that converges to some ϕ in P. As $T_\circ : (P, \omega^*) \to (K, \omega)$ is continuous, $T_\circ(\phi_m) \to T_\circ(\phi)$ weakly, while $T_\circ(\phi_n) = x_n \to a$ by assumption, so that $a = T_\circ(\phi)$. Since $a \in$ ext K, there exists, by Proposition 5.1, a set $A \in \mathcal{A}$ such that $\phi = \chi_A$ λ -a.e. However, the net (ϕ_n) is in co $T_\circ^{-1}$ $(K \setminus V)$ by construction, and so χ_A is in the weak* - closure of co $T_\circ^{-1}$ $(K \setminus V)$. As noted in §2, the weak* and $\sigma(L_1, L_\infty)$ - topologies coincide on P, and so χ_A is also in the $\sigma(L_1, L_\infty)$ - closed convex hull of $T_\circ^{-1}$ $(K \setminus V)$. But the latter is the same as the L_1 - closed convex hull of $T_\circ^{-1}$ $(K \setminus V)$ by Mazur's theorem [20, p. 154], so $\chi_A \in \overline{co}\, T_\circ^{-1}$ $(K \setminus V) = \overline{co}\, (P \setminus T_\circ^{-1}(V))$, as $V \subset K$. Since $T_\circ$ is continuous (see Proposition 3.1), $T_\circ^{-1}$ (V) is a L_1 - neighborhood of χ_A relative to P. This means that $\chi_A \notin$ st ext P, while, by the assertion made at the beginning of the proof, $\chi_A \in$ st exp P $\subset$ st ext P, a contradiction. Hence the proposition.

<u>LEMMA</u> 5.4. If X is quasi-complete and $\nu : \mathcal{A} \to X$ is a measure, then the weak topology coincides with the given one on the set of extreme points of the range.

<u>Proof:</u> What we need to show is that the weak topology on ext K is finer than the given topology. Let (x_i) be a net in ext K converging weakly to an element x in ext K. There then exist, by Proposition 5.1, E_i, $E \in \mathcal{A}$ such that $T_\circ^{-1}\{x_i\} = \{\chi_{E_i}\}$ for every i and $T_\circ^{-1}\{x\} = \{\chi_E\}$. Since X is quasi-complete, $T_\circ(P) = K$ (see §2) and as (P, w^*) is weak*-compact and $T_\circ : (P, w^*) \to (K, w)$ is continuous, it is biquotient. According to Michael [34, prop. 2.2] there exists a subnet (x_j) of (x_i) such that $\chi_{E_j} \xrightarrow{w^*} \chi_E$. Since the weak* - topology coincides with the L_1 - norm topology on $P_\circ$, $||\chi_{E_j} - \chi_E||_1 \to 0$. But then, according to Proposition 3.1, $T_\circ$ is equally continuous relative to the L_1 - norm

topology on P and the given topology on K, and so $T_o(\chi_{E_j}) \to T_o(\chi_E)$ in the given topology of K, i.e. $(x_j) \to x$ in the given topology. Thus every weakly convergent net in ext K has a subnet converging in the given topology. Hence the lemma.

PROPOSITION 5.5. If X is Banach, every extreme point of the range of ν is a support point of its closed convex hull.

Proof: The closed linear span of the range of ν is a Banach space, which may be denoted by X itself. Now K is weakly compact and convex, so according to Amir and Lindenstrauss [1, th. 4] we have $K = \overline{co}$ (exp K), whence it follows from Milman's theorem [20, p. 132] that the set exp K is weakly dense in the set ext K. Hence exp K is norm dense in ext K by Lemma 5.4.

Let $x \in$ ext K. Then there exists a sequence $\{x_n\}$ of exposed points of K such that $\|x_n - x\| \to 0$. Let S' denote the closed unit ball of X'. For every n there exists $x_n' \in S'$ exposing K at x_n. As X is the closed linear span of the weakly compact set K, according to Amir and Lindenstrauss [1, th. 2], the sequence $\{x_n'\}$ contains a subsequence $\{x_m'\}$ which is weak* - convergent to some $x' \in S'$. We claim that x' supports K at x. For let, for every m, $\beta_m = \sup x_m'(K)$ $(= x_m'(x_m))$ and $\gamma = x'(x)$, $\beta = \sup x'(K)$. It clearly suffices to prove that $\gamma = \beta$.

Since $\{x_m'\} \overset{w^*}{\to} x'$, we have $x_m' \circ \nu(A) \to x' \circ \nu(A)$ for every $A \in \mathcal{A}$. It follows from Vitali-Hahn-Saks theorem [12, p. 158] that $\lim_{\lambda(E) \to 0} |x_m' \circ \nu|(E) = 0$ uniformly in m. Since x_m, $x \in$ ext K, there exist unique sets E_m, E in $\mathcal{A}$ such that $x_m = \nu(E_m)$ for each m, and

$x = \nu(E)$. As the sequence $\{\nu(E_m)\}$ of extreme points of K converges to $\nu(E) \in \text{ext } K$, there exists, as in the proof of Lemma 5.4, a subsequence $\{\nu(E_i)\}$ of $\{\nu(E_m)\}$ such that $||\chi_{E_i} - \chi_E||_1 \to 0$, i.e. $\lambda(E_i \Delta E) \to 0$. Thus $\lim_{\lambda(E) \to 0} |x_i' \circ \nu|(E) = 0$ uniformly in i, so that

$$|x_i' \circ \nu(E_i) - x' \circ \nu(E)| \leq |x_i' \circ \nu(E_i) - x_i' \circ \nu(E)| + |x_i' \circ \nu(E) - x' \circ \nu(E)|$$

$$\leq |x_i' \circ \nu| \ (E_i \Delta E) + |x_i' \circ \nu(E) - x' \circ \nu(E)| \to 0 \ ,$$

whence $\{\beta_i\} \to x' \cdot \nu(E) = \gamma$. As the finite signed measure $x' \cdot \nu$ has compact range, it assumes its supremum and so there exists an $F \in A$ such that $x' \cdot \nu(F) = \max x' \cdot \nu(A) = \max x'(K) = \beta$. Now assume $\gamma < \beta$. Then there exists a $\delta > 0$ such that $\gamma + \delta < \beta$. Since $\{x_i'\} \overset{w^*}{\to} x'$, we have $\lim_i x_i' \circ \nu(F) = x' \circ \nu(F) = \beta$, and so $|x_i' \circ \nu(F) - \beta| < (1/2)\ \delta$ eventually, whereas $|\beta_i - \gamma| < (1/2)\ \delta$ eventually. Thus, we have, eventually,

$$\beta_i < \gamma + (1/2)\ \delta < \beta - (1/2)\ \delta < x_i' \circ \nu(F) \quad ,$$

which contradicts the definition of β_i. Thus $\gamma = \beta$, and so x' supports K at x. Hence the proposition.

<u>PROPOSITION</u> 5.7. Let X be Fréchet, $\nu : A \to X$ a measure, and T_o be open. Then the extreme points of the range of ν form a closed set.

<u>Proof</u>: Let $\{x_n\}$ be a sequence in the set ext K, converging to some $x \in X$. Since X is clearly quasi-complete, $K = T_o(P)$ is weakly compact (see §2), hence closed in the given topology, and so $x \in K$. According to Proposition 5.1, for each n there exists a unique $E_n \in A$ such that $x_n = \nu(E_n)$. As P and K are metrisable, and $T_o : P \to K$ is open,

according to Sikorski [42, v] we have $T_o^{-1}\{x\} = \text{Li}T_o^{-1}\{x_n\} = \text{Li}\{\chi_{E_n}\}$ and so the sequence $\{\chi_{E_n}\}$ is convergent, say to $\phi \in P$. Since $\chi_{E_n} \in P_o$ for every n and P_o is closed in P, $\phi \in P_o$, and so there exists $E \in A$ such that $\phi = \chi_E$. We then have $T_o^{-1}\{x\} = \{\chi_E\}$ and so $x \in \text{ext } K$ by Proposition 5.1, completing the proof.

The above proposition yields, with the help of Proposition 3.3,

COROLLARY 5.7.1. The extreme points of the range of every finite-dimensional measure form a closed set.

Remark: When X is infinite dimensional, the extreme points of the range of ν do not form a closed set in general. In the example considered after Theorem 3.6 it may be easily verified that

$$K\ (= \overline{co}\ \nu(A)) = \{\phi + \alpha\,\chi_{[0,1]} : \phi \in P,\ 0 \le \alpha \le 1\},$$

and

$$\text{ext } K = \{\chi_A : A \subset [0,1],\ 0 \le \lambda(A) < 1\} \cup \{\chi_A + \chi_{[0,1]} : A \subset [0,1],\ \lambda(A) > 0\}.$$

Thus, ext K contains the convergent sequence $\{\chi_{[0,1-(1/n)]}\}$ whose limit $\chi_{[0,1]}$ is not in ext K.

6. Exposed points of the range. In lemmas 6.1 and 6.2 of this section, X is any L.C.S., in lemma 6.3 and Theorem 6.4 it is quasi-complete and in the rest of the section, X is assumed to be a Banach space.

We recall that for every $x' \in X'$, $\nu_{x'}$ denotes the real measure $x' \circ \nu$.

LEMMA 6.1. If $\nu : A \to X$ is a measure, where X is a L.C.S., an $x' \in X'$ exposes the range of ν if and only if any of the following conditions holds:

a) The real measure $\nu_{x'} \equiv \nu$.

b) The real measure $\nu_{x'}$ assumes its maximum only once.

Proof: Suppose that $x' \in X'$ exposes $\nu(A)$ at $x_\circ$. Let $\beta = \max x' \circ \nu(A)$ and $H_\beta = \{x \in X : (x,x') = \beta\}$. Then $H_\beta \cap \nu(A) = \{x_\circ\}$. Since $\nu_{x'} << \nu$, we need to verify only that $\nu << \nu_{x'}$. Assume that $|\nu_{x'}|(N) = 0$ for some $N \in A$, but N is not ν - null. Let S^+, S^- be a Hahn-decomposition for $\nu_{x'}$, and $N^+ = N \cap S^+$, $N^- = N \cap S^+$. Then N^+ and N^- cannot be both ν - null. If N^+ is not ν - null, there exists $N_1 \in A$, $N_1 \subset N^+$ such that $\nu(N_1) \neq 0$, and since N_1 ($\subset N$) is $\nu_{x'}$ - null, we have $\beta = \max \nu_{x'}(A) = (\nu(S^+),x') = (\nu(S^+ \setminus N_1),x')$, i.e. $H_\beta \cap \nu(A)$ contains the distinct points $\nu(S^+)$ and $\nu(S^+ \setminus N_1)$, which is not possible. In case N^- is not ν - null, there similarly exists a $\nu_{x'}$ - null set $N_1 \in A$, $N_1 \subset N^-$ such that $\nu(N_1) \neq 0$, and then $\beta = \max \nu_{x'}(A) = (\nu(S^+),x') = (\nu(S^+ \cup N_1), x')$, which again leads to a contradiction. Hence N is ν - null, so that (a) holds.

Now let (a) hold for x'. With β, S^+ and S^- as before, $\beta = \nu_{x'}(S^+)$. Let E be any set in A for which $\nu_{x'}(E) = \beta$. Then $\nu_{x'}(S^+) = \nu_{x'}(E)$ and so $\nu_{x'}(S^+ \setminus E) = \nu_{x'}(E \setminus S^+)$. These must be zero since $S^+ \setminus E$ is purely $\nu_{x'}$ - positive and $E \setminus S^+$ is purely $\nu_{x'}$ - negative. But then these sets are $\nu_{x'}$ - null, and since $\nu_{x'} \equiv \nu$, they are also ν - null, whence $E \,\Delta\, S^+$ is ν - null, proving that (b) holds.

In case x' satisfies (b), we have $\nu(A) \cap H_\beta = \{\nu(S^+)\}$, and since H_β clearly supports $\nu(A)$, x' exposes $\nu(A)$ at $\nu(S^+)$. This completes the proof of the lemma.

<u>COROLLARY</u> 6.1.1. (Rybakov [39]). If X is Banach and $\nu : \mathcal{A} \to X$ is a measure, then there exists an $x' \in X'$ such that $\nu_{x'} \equiv \nu$.

<u>Proof</u>: We firstly observe that $\exp K \subset \exp \nu(A)$. For if $x \in \exp K$, then $x \in \text{ext } K$, so that $x \in \nu(A)$ by Corollary 5.1.1. Let H be a hyperplane of support of K such that $H \cap K = \{x\}$. Since $\nu(A) \subset K$ and $x \in \nu(A) \cap H$ it follows that H equally supports $\nu(A)$ at x, so $\{x\} \subset \nu(A) \cap H \subset K \cap H = \{x\}$, i.e. $\nu(A) \cap H = \{x\}$, whence $x \in \exp \nu(A)$. Now according to the Theorem 4 of Amir and Lindenstrauss [1], K has at least one exposed point. Hence $\exp \nu(A) \neq \phi$, so there exists an $x' \in X'$ exposing $\nu(A)$, and the result follows from Lemma 6.1.

<u>LEMMA</u> 6.2. If B is a closed convex set in a L.C.S., $A \subset B$ and A, B have the same supporting hyperplanes, then $\exp B \subset \exp A$ and $\text{st exp } B \subset \text{st exp } A$.

Moreover, if B is weakly compact and $\text{ext } B \subset A$, then A and B have the same exposed points.

<u>Proof</u>: Let $b \in \exp B$. Then there exists a hyperplane H of support of B such that $H \cap B = \{b\}$. By hypothesis, H equally supports A, so we have $\phi \neq H \cap A \subset H \cap B = \{b\}$, i.e. $H \cap A = \{b\}$, and $b \in \exp A$.

If $b \in \text{st exp } B$, then $b \in \exp B$, hence $b \in \exp A$ by above. Let $H = \{x \in X \; ; \; (x,x') = \alpha\}$ be a supporting hyperplane of A such that $H \cap A = \{b\}$, and let (a_n) be a net in A such that $(a_n,x') \to (b,x')$. Since $a_n \in B$ for every n, and b is a strongly exposed point of B, we have $(a_n) \to b$, so that $b \in \text{st exp } A$.

As for the second part, we need to show that exp A ⊂ exp B. Let a ∈ exp A and H be a supporting hyperplane of A such that H ∩ A = {a}. Then B is by hypothesis equally supported by H, and B ∩ H is a non-void weakly compact convex set. Now ext (B ∩ H) ⊂ ext B (see, e.g. [18, lemma 2]), whereas ext B ⊂ A by hypothesis, so that ext B ∩ H ⊂ A. Thus ext (B ∩ H) ⊂ A ∩ H = {a}, whence B ∩ H = $\overline{\text{co}}$ {a} = {a}. Hence a ∈ exp B. This completes the proof of the lemma.

<u>PROPOSITION</u> 6.3. If X is quasi-complete and ν : A → X is a measure, then ν(A) and K have the same exposed points.

<u>Proof:</u> Even if X is a L.C.S. the sets ν(A) and K have the same supporting hyperplanes. For if x' ∈ X' and $\beta_{x'}$, $\alpha_{x'}$ denote the sup x'∘ν(A) and inf x'∘ν(A) respectively, we have

$$x'(K) = x'(\overline{\text{co}}\ \nu(A)) = \overline{\text{co}}\ x'\circ\nu(A) = [\alpha_{x'},\beta_{x'}],$$

whence $\beta_{x'}$ = sup x'(K). Thus, every x' ∈ X' has the same supremum on ν(A) and K, and the assertion follows.

Further, since X is quasi-complete, the set K is weakly compact. Since ext K ⊂ ν(A) by Corollary 5.1.1, it follows from Lemma 6.2 that K and ν(A) have the same exposed points.

<u>THEOREM</u> 6.4. If X is quasi-complete and ν : A → X is a measure, then the exposed points of ν(A) and K are strongly exposed.

<u>Proof:</u> Let us first prove the inclusion exp ν(A) ⊂ st exp K. Let ν(A) ∈ exp ν(A). Then there exists an x' ∈ X' which exposes ν(A) at ν(A). According to Lemma , we have $\nu_{x'} \equiv \nu$. Let $\lambda = |\nu_{x'}|$. Suppose that (x_n) is a net in K such that $(x_n,x') \to (\nu(A),x') = \beta_{x'}$.

Let S^+, S^- be a Hahn-decomposition of S relative to $\nu_{x'}$. Then $A \Delta S^+$ is ν - null by Lemma 6.1. Since X is quasi-complete, K is weakly compact and $T_\circ(P) = K$ (see §2), so for every n there exists $\phi_n \in P$ for which $T_\circ(\phi_n) = x_n$. Then $(x_n, x') = (\int \phi_n \, d\nu, x') = \int \phi_n \, d\nu_{x'}$. Let $\nu^+_{x'}$, $\nu^-_{x'}$ be the positive and negative variations of $\nu_{x'}$. Then

$$\int |\chi_A - \phi_n| \, d\lambda = \int |\chi_{S^+} - \phi_n| \, d\lambda$$

$$= \int_{S^+} (1 - \phi_n) \, d\lambda + \int_{S^-} \phi_n \, d\lambda$$

$$= \lambda(S^+) - \int_{S^+} \phi_n \, d\nu^+_{x'} + \int_{S^-} \phi_n \, d\nu^-_{x'}$$

$$= \beta_{x'} - \int \phi_n \, d\nu_{x'}$$

$$= \beta_{x'} - (x_n, x') \to 0.$$

Since $T_\circ : P \to X$ is continuous relative to the L_1 - norm by Proposition 3.1, $T_\circ(\phi_n) \to T_\circ(\chi_A)$, i.e. $(x_n) \to \nu(A)$, so that $\nu(A) \in$ st exp K, whence exp $\nu(A) \subset$ st exp K.

As observed in the proof of Proposition 6.3, the sets K and $\nu(A)$ have the same supporting hyperplanes, and so by Lemma 6.2, we have st exp K $\subset$ st exp $\nu(A)$. We thus have

exp $\nu(A) \subset$ st exp K $\subset$ st exp $\nu(A) \subset$ exp $\nu(A)$, i.e.

exp $\nu(A)$ = st exp K = st exp $\nu(A)$, and since exp $\nu(A)$ = exp K by Proposition 6.3, this proves the theorem.

In the rest of this section, we extend some of the results of Husain and Tweddle [18] on extreme points to exposed points. The proof of the following proposition is a modification of theirs. From now on, X is assumed to be Banach.

If A and B are compact convex subsets of a L.C.S., $ext_B A$ is defined [18] to be the set of extreme points x of A for which there exists some extreme point y of B such that $x + y \in ext (A + B)$. The sets $exp_B A$, $st\ exp_B A$ and $st\ ext_B A$ may be defined analogously.

PROPOSITION 6.5. If A and B are two weakly compact convex sets in a Banach space X, then $st\ exp_B A$ is a weakly dense subset of ext A.

Proof: Let $C = \overline{co}\ st\ exp_B A$. We shall first prove that $A + B = C + B$.

Suppose that $x \in st\ exp (A + B)$. Then there exist $a \in A$ and $b \in B$ such that $x = a + b$. There further exists $x' \in X'$ strongly exposing $A + B$ at x. It then follows easily that x' strongly exposes A at a and B at b respectively, whence $a \in st\ exp_B A$, and so $x \in st\ exp_B A + B$. Thus, $st\ exp (A + B) \subset st\ exp_B A + B \subset C + B$. As C and B are weakly compact and convex, so is $C + B$, and since $C \subset A$ we have $\overline{co}\ st\ exp (A + B) \subset C + B \subset A + B$. Since $A + B$ is weakly compact and convex, according to Troyanski [43, Cor. 7], we have $\overline{co}\ st\ exp (A + B) = A + B$ and so $A + B = C + B$. Thus, $A = C$ by the cancellation law in [18, lem. 1] whence $A = \overline{co}\ st\ exp_B A$. The result now follows from Milman's theorem.

Since ext A contains the sets st ext A, exp A and st exp A, and $st\ exp_B A$ is contained in each of $ext_B A$, $st\ ext_B A$ and $exp_B A$, we have

COROLLARY 6.5.1. If A and B are two weakly compact convex sets in X, then $ext_B A$ is a weakly dense subset of ext A, and the same holds on replacing ext_B by exp, $st\ exp$ and $st\ ext$.

LEMMA 6.6. If $\nu : \mathcal{A} \to X$ is a measure and $A \in \mathcal{A}$, then we have
$\{\nu(E \cap A) : \nu(E) \in \exp \nu(\mathcal{A})\} \subset \exp \nu_A(\mathcal{A}) \subset \{\nu(E \cap A) : \nu(E) \in \exp\nu(\mathcal{A})\}^-$.

Proof: Let $K = \overline{co}\ \nu(\mathcal{A})$, $K_1 = \overline{co}\ \nu_A(\mathcal{A})$ and $K_2 = \overline{co}\ \nu_{A^c}(\mathcal{A})$. Then by Proposition 6.3 we have $\exp \nu(\mathcal{A}) = \exp K$ and $\exp \nu_A(\mathcal{A}) = \exp K_1$. To prove the first inclusion, let $E \in \mathcal{A}$ be such that $\nu(E) \in \exp K$. It is easy to see that $K = K_1 + K_2$ and that an $x' \in X'$ exposes K at $\nu(E)$ if and only if it exposes K_1 at $\nu(E \cap A)$ and K_2 at $\nu(E \cap A^c)$ respectively. Thus $\nu(E \cap A) \in \exp K_1$.

To prove the second inclusion, let $x_1 \in \exp K_1$. Then there exists an $x' \in X'$ which exposes K_1 at x_1. Let

$$K_{x'} = \{x \in K : (x,x') = \max x'(K)\},$$

and

$$K_{2,x'} = \{x \in K_2 : (x,x') = \max x'(K_2)\}.$$

It is easy to see that $\max x'(K) = \max x'(K_1) + \max x'(K_2)$, and since x' exposes K_1 at x_1, it follows that $K_{x'} = x_1 + K_{2,x'}$. Since $K_{x'}$ is weakly compact and convex, it has some extreme point y, and it is equally an extreme point of K (see, e.g. [18, lem. 2]). Then there exists $x_2 \in \text{ext } K_{2,x'}$ such that $y = x_1 + x_2$. According to Corollary 5.1.1, there exist $F, G \in \mathcal{A}$, $F \subset A$, $G \subset A^c$ such that $x_1 = \nu(F)$ and $x_2 = \nu(G)$.

As noted in the proof of Proposition 5.5, $\exp K$ is norm-dense in $\text{ext } K$, and so there exists a sequence $\{y_n\}$ in $\exp K$ converging to y. According to Corollary 5.1.1. there exists $H_n \in \mathcal{A}$ such that $y_n = \nu(H_n)$ for each n. As $T_\circ : (P,w^*) \to (K,w)$ is biquotient, there exists, as in the proof of Lemma 5.4, a subsequence $\{\nu(H_m)\}$ of

$\{\nu(H_n)\}$ such that $||\chi_{H_m} - \chi_{F \cup G}||_1 \to 0$, i.e. $H_m \to F \cup G$ in T_λ.
Since the operation $(A,B) \to A \cap B$ is continuous on $A \times A$ relative to the product topology [16, p. 168] we have $H_m \cap A \to (F \cup G) \cap A = F$. Due to the continuity of ν, we get $||\nu(H_m \cap A) - \nu(F)|| \to 0$. Since $\nu(H_m) \in \exp K$ for every m, it follows that $x_1 = \nu(F) \in \{\nu(E \cap A) : \nu(E) \in \exp K\}^-$, completing the proof.

<u>THEOREM</u> 6.7. If $\exp \nu(A)$ is weakly closed, then for every $A \in A$, we have

$$\exp \nu_A(A) = \{\nu(E \cap A) : \nu(E) \in \exp \nu(A)\}.$$

<u>Proof</u>: According to Lemma 6.6 it suffices to prove that the set on the right hand side is norm closed. Let $\nu(E_n) \in \exp \nu(A)$ for every n and suppose that $||\nu(E_n \cap A) - x|| \to 0$. The set $\exp \nu(A)$ is, by hypothesis, a weakly closed subset of the weakly compact set K, and so it is weakly compact. Since the norm and weak topologies coincide on $\exp \nu(A)$ by Lemma 5.4, this set is further norm compact. Hence there exists a subsequence $\{\nu(E_i)\}$ of $\{\nu(E_n)\}$ converging in the norm to some $\nu(E) \in \exp \nu(A)$. As in the previous proof, there exists a subsequence $\{\nu(E_j)\}$ of $\{\nu(E_i)\}$ such that $||\chi_{E_j} - \chi_E||_1 \to 0$, i.e. $E_j \to E$, and so, as before, $E_j \cap A \to E \cap A$. As ν is continuous, we have $||\nu(E_j \cap A) - \nu(E \cap A)|| \to 0$. But then, by hypothesis, $||\nu(E_j \cap A) - x|| \to 0$, and so we have $x = \nu(E \cap A)$. Since $\nu(E) \in \exp \nu(A)$, this completes the proof of the theorem.

<u>Remark</u>: When $\exp \nu(A)$ is not weakly closed, the above theorem need not hold even for a finite dimensional measure. Let ν_1 be a non-atomic

measure defined on the σ - algebra of Borel subsets of $A = [0,1]$, whose range is the closed unit disk in R^2 (see Rickert [36]), and let ν_2 be another non-atomic measure defined on the σ - algebra of Borel subsets of $B = [1,2]$ whose range is the segment from (0,0) to (1,0). Let $\mathcal{A}$ be the σ - algebra of Borel subsets of [0,2], and define the measure $\nu : \mathcal{A} \to R^2$ by $\nu(E) = \nu_1(E \cap A) + \nu_2(E \cap B)$, $E \in \mathcal{A}$. The range of ν is the convex hull of $\{x : |x| \leq 1\} \cup \{x : |x - 1| \leq 1\}$. Now, (0,1) is an exposed point of $\nu_A(\mathcal{A})$ and it is easy to see that (0,1) is not of the form $\nu(E \cap A)$ for any exposed point $\nu(E)$ of the range of ν.

7. Residuality of exposing functionals. In this section X is assumed to be Banach up to Theorem 7.2 and in Propositions 7.3 and 7.4 it is quasi-complete. If K is a weakly compact convex set in X, we let

$$K_{x'} = \{x \in K : (x,x') = \max x'(K)\},$$

and we define $\rho_K : X' \to R$ to be the map

$$\rho_K(x') = \operatorname{diam} K_{x'} \quad , \quad x' \in X'.$$

LEMMA 7.1. For any weakly compact convex set K, the map ρ_K is continuous at every x' in X' which strongly exposes K.

Proof: Let $x' \in X'$ be a strongly exposing functional of K. Then $K_{x'}$ is a singleton and so $\rho_K(x') = 0$. Suppose that ρ_K is not continuous at x'. Then there exists $\varepsilon > 0$ and a sequence $\{x_n'\}$ converging to x' for which $\rho_K(x_n') \geq \varepsilon$ for every n. Thus, for each n, there exist a_n and b_n in $K_{x_n'}$, such that $||a_n - b_n|| \geq (1/2)\ \varepsilon$.

Since K is weakly compact, $\{a_n\}$ contains, by Eberlein's theorem [40, p. 187], a subsequence $\{a_{n_i}\}$ which converges weakly to some point a in K. Similarly, $\{b_{n_i}\}$ contains a subsequence $\{b_{n_{i_j}}\}$ which converges weakly to some point b in K. Let us denote the sequences $\{a_{n_{i_j}}\}$ and $\{b_{n_{i_j}}\}$ simply by $\{a_j\}$ and $\{b_j\}$. Since $||x_j' - x'|| \to 0$, and K is bounded, we have $\sup_{x\in K} |x_j'(x) - x'(x)| \to 0$. Hence the sequence $\{x_j'(K)\}$ of compact intervals converges to the compact interval $x'(K)$ in the Hausdorff metric. If $\beta_j = \sup x_j'(K)$ and $\beta = \sup x'(K)$, then we have $\beta_j \to \beta$. Moreover,

$$|x_j'(a_j) - x'(a_j)| \leq \sup \{|x_j'(x) - x'(x)| : x \in K\} \to 0,$$

and since $\{a_j\} \to a$ weakly, we have $x'(a_j) \to x'(a)$, whence, $x_j'(a_j) \to x'(a)$. But then, for every j, we have $a_j \in K_{x_j'}$, so that $x_j'(a_j) = \beta_j$, and so $x'(a) = \lim \beta_j = \beta$. Further, since $x'(a_j) \to x'(a)$ and x' strongly exposes K at a, we have $||a_j - a|| \to 0$.

By a similar argument, one obtains $x'(b) = \beta$ and $||b_j - b|| \to 0$. Since x' exposes K, we have a = b, so that $||a_j - b_j|| \to 0$, contrary to the choice of a_n's and b_n's. Hence ρ_K is continuous at x'.

THEOREM 7.2. If K is a weakly compact convex set in X such that the set of strongly exposing functionals of K is dense in X', then the set of exposing functionals of K is residual in X'.

Further, if every exposing functional of K is strongly exposing, then they form a G_δ set residual in X'.

Proof: Let C denote the set of points of continuity of ρ_K, and let

X_e' and X_s' be the sets of exposing and strongly exposing functionals of K in X'. Since X_s' is, by hypothesis, dense in X' and ρ_K vanishes at every point of X_s', ρ_K is zero at every point where it is continuous. But then x' exposes K whenever $\rho_K(x') = 0$. We thus have, with the help of Lemma 7.1,

$$X_s' \subset C \subset X_e'.$$

Since the points of continuity of any real valued function form a G_δ set, X_e' contains the dense G_δ set C, and so X_e' is residual in X'. In case $X_e' \subset X_s'$, we have $X_e' = C$, whence the second part of the theorem.

In case of the range of a measure ν, according to Corollary 6.1.1 there exists at least one $x_o' \in X'$ which exposes $\nu(\Lambda)$. For any other functional $x' \in X'$ it is easy to verify (see [39]) that all but countably many elements in the segment from x_o' to x' expose $\nu(\Lambda)$, and so X_e' is dense in X'. It was proved in Theorem 6.4. that $X_e' \subset X_s'$. Hence, the above theorem gives

COROLLARY 7.2.1. For the range of a measure, its exposing functionals form a residual G_δ set in X'.

With the help of Lemma 6.1 this corollary yields in turn

COROLLARY 7.2.2. (Walsh [47]) For every measure ν, those $x' \in X'$ for which $x'\nu \equiv \nu$ form a residual G_δ set in X'.

Since the intersection of countably many residual subsets of X' is again residual in X', we further obtain

COROLLARY 7.2.3. (Drewnowski [11, th. 3.4]) If $\{\nu_n\}$ is a sequence of measures with values in X, then those $x' \in X'$ for which $x' \circ \nu_n \equiv \nu_n$ for every n form a residual G_δ set in X'.

Remark: The theorem of Walsh, or even Rybakov's theorem, does not hold in general when X is not Banach. In fact there exist even controlled measures with values in a separable Fréchet space for which Rybakov's theorem does not hold. Let, for example, X be the product of countably many copies of the real line with the product topology, $S = [0,2]$, $\mathcal{A}$ be the σ - algebra of Borel subsets of S and $A_n = (\frac{1}{2^n}, \frac{1}{2^{n-1}}]$ for $n = 0,1,2,\ldots$. For every $E \in \mathcal{A}$, let $\nu(E) = (\lambda(E \cap A_n))_{n=0}^{\infty}$, where λ is the Lebesgue measure on $\mathcal{A}$. Then $\nu : \mathcal{A} \to X$ is a measure controlled by λ and, clearly, $\nu(\mathcal{A}) = K = \prod_{n=1}^{\infty} [0, \frac{1}{2^n}]$. As observed by Klee [22, p. 96], K has no exposed points. It follows from Lemma 6.1 that Rybakov's theorem cannot hold for this ν.

From now on we assume X to be quasi-complete. Let X_o be the linear span of K and X_o' be the dual of X_o. The Mackey topology $m(X_o', X_o)$ for the duality (X_o, X_o') is the topology of uniform convergence on convex, circled and $\sigma(X_o, X_o')$ - compact subsets of X_o [20, p. 173]. The polar of a set $A \subset X_o$ is defined to be the following subset of X_o':

$$A^0 = \{x' \in X_o' : \sup_{x \in A} |(x,x')| \le 1\}.$$

We prove the following proposition for the existence of a bounded Mackey-neighborhood of zero in X_o' needed in the next one.

PROPOSITION 7.3. If $\nu : \mathcal{A} \to X$ is a controlled measure, then the space $(X_o', m(X_o', X_o))$ is normable.

<u>Proof</u>: Let ν have a control measure λ. As X is quasi-complete, K is $\sigma(X,X')$ - compact, and so according to the Hahn-Banach theorem, K is equally $\sigma(X_o,X_o')$ - compact. Thus $K_1 = K - K$ is convex, circled and $\sigma(X_o,X_o')$ - compact. The set $V = K_1^{\theta}$ is therefore a zero neighborhood for the Mackey topology on X_o'. It will suffice to prove that V is Mackey-bounded, for then $(X_o', m(X_o',X_o))$ will be normable [20, p. 44].

Since $\nu \equiv \lambda$, we have $\nu_{x'} << \lambda$ for every $x' \in X'$, and we identify $\nu_{x'}$ with its Radon-Nikodým derivative with respect to λ. Define $f : X_o' \to L_1(\lambda)$ by $f(x') = \nu_{x'}$ for every $x' \in X_o'$, and let the range of f be denoted by L_o. We claim that $f(V) = S_o$, where S_o is the closed unit ball of L_o, and that f is one-to-one.

For every $x' \in X_o'$ we have

(1) $$||f(x')||_1 = |\nu_{x'}|(S) = \beta_{x'} - \alpha_{x'} = \sup_{x \in K_1} |(x,x')|,$$

whence $x' \in V$ if and only if $||f(x')||_1 \leq 1$, and so $f(V) = S_o$. Further, if $f(x') = 0$, then x' vanishes on the range of ν. Since the linear span of $\nu(A)$ is all of X_o, $x' \equiv 0$, and so f is one-to-one. Thus, $g = f^{-1}$ is a map from L_o onto X_o'.

Next, we claim that g is continuous relative to the $\sigma(L_1,L_\infty)$ - topology on L_o and the $\sigma(X_o',X_o)$ - topology on X_o'. Let $W = \bigcap_{j=1}^{m} \{x' \in X_o' : |(x_j,x')| \leq 1\}$, where $x_j \in X_o$ for each j. Since K is weakly compact, $T(P) = K$ (see §2), $T(L_\infty)$ is then the linear span of K, and so is equal to X_o. Thus, for every j, there exists an element ϕ_j of L_∞ such that $x_j = \int \phi_j \, d\nu$. Then

$$U = \bigcap_{j=1}^{m} \{\mu \in L_o : |\int \phi_j \, d\mu| \leq 1\}$$

is clearly a $\sigma(L_1, L_\infty)$ - neighborhood of zero in L_o. If $\nu_{x'} \in U$, since $g(\nu_{x'}) = x'$, for each j we have $(x_j, x') = (\int \phi_j \, d\nu, x') = \int \phi_j \, d\nu_{x'}$, so that $x' \in W$, i.e. $g(U) \subset W$. Hence g is continuous.

Since $S_o = f(V)$ is bounded and g is one-to-one, linear and continuous, $V = g(S_o)$ is $\sigma(X_o', X_o)$ - bounded, and so Mackey-bounded. This completes the proof of the proposition.

PROPOSITION 7.4. If X is quasi-complete, those x' in X_o' that expose the range of ν are residual in every bounded $m(X_o', X_o)$ - neighborhood of zero that constitutes a Baire space.

Proof: Let f, V, S_o and L_o be as in the previous proof. As we saw there, we have $f(V) = S_o$, and so f is continuous relative to the Mackey-topology on X_o' and the L_1 - norm topology on its range L_o. Let W be a bounded Mackey-neighborhood of zero (the existence of which follows from Proposition 7.3) that constitutes a Baire space, and let $K' = f(W)$. Then according to Walsh [47] the set of all $x_o' \in W$ for which $\nu_{x'} \ll \nu_{x_o'}$ for every $x' \in W$ is a residual G_δ in W, and since W is absorbing, we have $\nu_{x'} \ll \nu_{x_o'}$ for every $x' \in X'$, i.e. $\nu \ll \nu_{x_o'}$ for such x_o'. The conclusion now follows from Lemma 6.1.

REFERENCES

1. Amir, D. and Lindenstrauss, J. The structure of weakly compact sets in Banach spaces. Ann. of Math. 88: 35-46, 1968.

2. Anantharaman, R. On exposed points of the range of a vector measure. Proc. of a conference held at the Univ. of Utah, Salt Lake City, Academic Press, New York. In press, 1974.

3. Bartle, R.G., Dunford, N. and Schwartz, J.T. Weak compactness and vector measures. Canad. J. Math. 7: 289-305, 1955.

4. Birkhoff, G. Lattice Theory. (Third Ed.) A.M.S. Providence, R.I. 1967.

5. Bolker, E.D. On a class of convex bodies, Trans. Amer. Math. Soc. 143: 323-345, 1969.

6. Bourbaki, N. Intégration, Ch. 6. Hermann (Paris) 1959.

7. Bourbaki, N. Espaces Vectoriels topologiques. Ch. 3-5. Hermann, (Paris) 1964.

8. Brooks, J.K. On the existence of a control measure for strongly bounded vector measures. Bull. Amer. Math. Soc. 77: 999-1001, 1971.

9. Choquet, G. Lectures in Analysis II Benjamin (New York) 1969.

10. Choquet, G. Lectures in Analysis III Benjamin (New York) 1969.

11. Drewnowski, L. On control submeasures and measures. Studia Math. 47. In press, 1973.

12. Dunford, N. and Schwartz, J. Linear Operators I Interscience (New York) 1958.

13. Hájek, O. Notes on quotient maps. Comm. Math. Univ. Carolinae, 7: 319-323, 1966.

14. Halmos, P.R. On the set of values of a finite measure. Bull. Amer. Math. Soc. 53: 138-141, 1947.

15. Halmos, P.R. The range of a vector measure. Bull. Amer. Math. Soc. 54: 416-421, 1948.

16. Halmos, P.R. Measure Theory. Van Nostrand (New York) 1950.

17. Hoffman-Jørgensen, J. Vector Measures. Math. Scand. 28: 5-32, 1971.

18. Husain, T. and Tweddle, I. On the extreme points of the sum of two compact convex sets, Math. Ann. 188: 113-122, 1970.

19. Jerison, M. A property of extreme points of compact convex sets. Proc. Amer. Math. Soc. 5: 782-783, 1954.

20. Kelley, J.L., Namioka, I. et. al. Linear Topological Spaces Van Nostrand (New York) 1961.

21. Kingman, J.F.C. and Robertson, A.P. On a theorem of Lyapounov. J. Lond. Math. Soc. 43: 347-351, 1968.

22. Klee, V., Jr. Extremal structure of convex sets. Math. Z. 69: 90-104, 1958.

23. Klee, V., Jr. Some new results on smoothness and rotundity in normed linear spaces. Math. Ann. 139: 51-63, 1959.

24. Kluvánek, I. The range of a vector-valued measure. Math. Systems theory. 7: 44-54, 1973.

25. Kuratowski, K. Topology, I Academic Press (New York) 1966.

26. Lew, J. The range of a vector measure with values in a Montel space. Math. Systems theory. 5: 145-147, 1971.

27. Liapounoff, A. Sur les fonctions-vecteurs complètement additives. Bull. Acad. Sci. U.R.S.S. Ser. Math. 4: 465-478, 1940. (Russian).

28. Liapounoff, A. Sur les fonctions-vecteurs complètement additives. Bull. Acad. Sci. U.R.S.S. Ser. Math. 10: 277-279, 1946. (Russian).

29. Lindenstrauss, J. On operators which attain their norm. Israel J. Math. 1: 139-148, 1963.

30. Lindenstrauss, J. A short proof of Liapounoff's convexity theorem. Jour. Math. Mech. 15: 971-972, 1966.

31. Marczewski, E. and Sikorski, R. On isomorphism types of measure algebras. Fund. Math. 38: 92-98, 1951.

32. Michael, E.A. Topologies on spaces of subsets. Trans. Amer. Math. Soc. 71: 152-182, 1951.

33. Michael, E.A. A Theorem on semi-continuous set-valued functions. Duke Math J. 26: 647-651, 1959.

34. Michael, E.A. Bi-quotient maps and cartesian products of quotient maps. Ann. Inst. Fourier, Grenoble 18: 287-302, 1968.

18. Husain, T. and Tweddle, I. On the extreme points of the sum of two compact convex sets, Math. Ann. 188: 113-122, 1970.

19. Jerison, M. A property of extreme points of compact convex sets. Proc. Amer. Math. Soc. 5: 782-783, 1954.

20. Kelley, J.L., Namioka, I. et. al. Linear Topological Spaces Van Nostrand (New York) 1961.

21. Kingman, J.F.C. and Robertson, A.P. On a theorem of Lyapounov. J. Lond. Math. Soc. 43: 347-351, 1968.

22. Klee, V., Jr. Extremal structure of convex sets. Math. Z. 69: 90-104, 1958.

23. Klee, V., Jr. Some new results on smoothness and rotundity in normed linear spaces. Math. Ann. 139: 51-63, 1959.

24. Kluvánek, I. The range of a vector-valued measure. Math. Systems theory. 7: 44-54, 1973.

25. Kuratowski, K. Topology, I Academic Press (New York) 1966.

26. Lew, J. The range of a vector measure with values in a Montel space. Math. Systems theory. 5: 145-147, 1971.

27. Liapounoff, A. Sur les fonctions-vecteurs complètement additives. Bull. Acad. Sci. U.R.S.S. Ser. Math. 4: 465-478, 1940. (Russian).

28. Liapounoff, A. Sur les fonctions-vecteurs complètement additives. Bull. Acad. Sci. U.R.S.S. Ser. Math. 10: 277-279, 1946. (Russian).

29. Lindenstrauss, J. On operators which attain their norm. Israel J. Math. 1: 139-148, 1963.

30. Lindenstrauss, J. A short proof of Liapounoff's convexity theorem. Jour. Math. Mech. 15: 971-972, 1966.

31. Marczewski, E. and Sikorski, R. On isomorphism types of measure algebras. Fund. Math. 38: 92-98, 1951.

32. Michael, E.A. Topologies on spaces of subsets. Trans. Amer. Math. Soc. 71: 152-182, 1951.

33. Michael, E.A. A Theorem on semi-continuous set-valued functions. Duke Math J. 26: 647-651, 1959.

34. Michael, E.A. Bi-quotient maps and cartesian products of quotient maps. Ann. Inst. Fourier, Grenoble 18: 287-302, 1968.