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University of Alberta

Mathematical Analysis on Some Models in Ecology and Epidemiology

BY

Xiaodong Lin ©

A Thesis
Submitted to the Faculty of Graduate Studies and Research
in Partial Fulfillment of the Requirements for the Degree
of Doctor of Philosophy

Department of Mathematics
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THE UNDERSIGNED CERTIFY THAT THEY HAVE READ, AND RECOMMEND TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH FOR ACCEPTANCE, A THESIS ENTITLED **Mathematical Analysis on Some Models in Ecology and Epidemiology** SUBMITTED BY **Xiaodong Lin** IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF **Doctor of Philosophy** IN **Applied Mathematics**.

P.L. Antonelli

N.D. Kazarinoff

W. Krawcewicz

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Y.S. Wong

Date: Sept. 27, 1990

To my parents

to whom one always owes more

than is said

ABSTRACT

A coral-starfish model which simulates the population dynamic of coral and starfish on the Great Barrier Reefs in Australia is considered. By employing the Hopf bifurcation theory, the Bautin's theory and Li's algorithm, it is shown that a unique Hopf bifurcation appears for this model and it is supercritical. These results support the recent finding of a wave pattern in the Great Barrier Reefs.

Next, an epidemic model with a time delay in the recovered class is considered. A conjecture of Hethcote et al. is solved. Furthermore, the global asymptotic behavior of this model is analyzed when the parameters are chosen so that a threshold is reached.

For an epidemic model with subpopulations, it is shown that the existence of an endemic equilibrium implies the persistence of the disease. A sufficient condition is given, on which the endemic equilibrium is globally asymptotically stable.

Finally, an HIV/AIDS transmission model is studied. A threshold theorem is obtained by applying techniques from the theory of monotone flows. It is also shown that AIDS will be epidemic if the threshold is exceeded. Several sufficient conditions are given to ensure the uniqueness, local stability and global stability of the endemic equilibrium.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Analyzing mathematical models has always been an important activity in theoretical ecology and epidemiology. After the formulation of questions about an observed phenomenon in terms of mathematical models, mathematical analysis of them then become essential. The qualitative conclusions based on the model analysis will not only increase our understanding but help us to make better predictions in the future.

The use of the qualitative theory of differential equations in analyzing ecological models has a rich and long history dating back over half a century and including the fundamental work of Volterra, Lotka and Gause. Further progress was made by Kolmogorov who systematically studied a class of ordinary differential equations arising from ecological models. With many discoveries in the qualitative theory of differential equations, such as the theory of Liapunov functions, the Poincare-Bendixon theory and the newly developed bifurcation theory, the mathematical theory of ecology has had a rapid growth in this decade. Many new mathematical methods and concepts for model analysis have been introduced. A lot of work has been done to investigate a wide range of ecological problems by many mathematician and biologists like May, Levin, Maynard Smith, Hirsch, Hoppensteadt, Antonelli and Kazarinoff, to name a few.

The ecological problem on the Great Barrier Reef(GBR) of Australia has attracted a lot of attention in the recent years. Between the 1960's and 1970's, there were two population outbreaks of a coral-eating starfish, called *Acanthaster planci.* Many coral reefs in the GBR were devastated by the huge numbers of this large starfish preying on coral. In 1989, Reichelt et al. revealed that there exists a wave pattern of the starfish population by analyzing the distribution of this coral-eating starfish over the GBR. However, there has been no satisfying theoretical analysis of this phenomenon so far. Hence, mathematical models are needed in order to give this phenomenon a satisfactory explanation and to predict the growth trendency of the populations of the starfish and its coral prey. Antonelli, Kazarinoff and their Australian marine scientist co-workers [2,3,4] have recently designed a series of mathematical models for this problem by taking the aggregation behavior of *A. planci* into account and using the adult-aggregation hypothesis of Dana et al.[13].

In the field of mathematical epidemiology, the modern research began with the study of Kermack and McKendrick. These authors use a simple ODE model(an SIR model-see the explanation later for the terminology) to investigate the problem of infectious diseases and found the celebrated Threshold Theorem, which showed that introducing infectious cases into susceptibles would not trigger an epidemic provided the density of susceptibles were below a certain value and there would be an epidemic if this value were exceeded. Since then, hundreds of mathematical models have been formulated to study the spreading of infectious diseases. In these studies, a population is often separated into disjoint classes: susceptibles class S, exposed(latent) class E, infectious class I and removed(recovered) class R by immunity, isolation or death, respectively(some models may have less classes depending on the characteristic of the diseases studied in these models). Models are usually referred to by a sequence of letters which indicate the flow of people between classes. For instance, in an SIRS model, people are initially suseptibles, then become infected, then recovered, and finally become susceptibles again after a period of immunity. An SIS model with two subpopulations were considered by Hethcote[19]. Lajmanovich and Yorke[31] studied an SIS model with n subpopulations by using a Liapunov function. The generalization of the Lajmanovich-Yorke model was studied by Nold[38]. SIS models with a time delay in the infectious class were analyzed by Cooke and Yorke[12], Greenberg and Hoppensteadt[16] and Brauer[8]. Hethcote and Waltman[24] analyzed an SIR model. Hethcote[19] also considered an SIR model with n subpopulations. SIRS models have been considered by many authors. Liu et al.[33,34] studied an ODE SIRS model. Hethcote, Lewis and van den Driessche[20], and Hethecote, Stech and van den Driessche[21] considered SIRS models with a time delay in the recovered class. It was shown that a Hopf bifurcation may occur if proper parameters are chosen in their SIRS model with a time delay. Since the discovery of the human immunodeficiency virus(HIV), the etiological agent of AIDS in 1983, research on modeling the dynamics of AIDS epidemic in human communities has become a major research topic in mathematical epidemiology. HIV/AIDS models have been developed by Anderson et al.[1], Castillo-Chavez et al.[11,27], Busenburg et al.[10], Jacquez et al.[28] and others.

The objective of this thesis is to analyze some mathematical models in the above areas by using various mathematical tools. In Chapter 2, we consider a coral-starfish model proposed by Antonelli and Kazarinoff. The model was formulated to describe the dynamics of the coral and starfish populations in the GBR. In this model, it is assumed that there are one coral and one starfish(actually, there are many kinds of coral in the GBR, but their behavior is very similar and the starfish , *A. planci* preys on them indiscriminately). We show that there exists a Hopf bifurcation for the model, which supports the recent finding of a wave pattern in the GBR. Then, using the Bautin's theory and Li's algorithm, we determine the direction of the Hopf bifurcation. The numerical results obtained by running the two computer codes, the Hassard's code BIFOR2 and Doedel's code AUTO also confirms our assertions. Chapter 3 is concerned with an SIRS model with a constant time delay in the recovered class. A conjecture of Hethcote et al. is solved. Furthermore, we analyze the asymptotic behavior of the model when the parameters are chosen so that the threshold is reached. The method we use in this chapter can also be adopted to more general model. In Chapter 4, we deal with an SIRS model with n subpopulations. We first give a new proof of the threshold theorem and the stability of the endemic equilibrium. Applying the persistence theory, we show that the disease will be epidemic when the threshold is exceeded. Finally, we present a sufficient condition on which the endemic equilibrium is globally asymptotically stable. A new method is developed in this proof. In Chapter 5, we discuss an HIV/AIDS transmission model with n subpopulations, which was proposed by Jacquez et al. The contact pattern between subpopulations is the widely used preferred mixing. We obtain the threshold theorem by applying techniques from the theory of monotone flow. We then use some results in the pesistence theory to show that the disease will be epidemic if the threshold is exceeded. Moreover, sufficient conditions are given to ensure the local and global stability of the endemic equilibrium. In the final chapter, we study the generalization of the model in Chapter 5. By dropping the restriction on the transmission rates, we find that there may exist multiple endemic equilibria. A sufficient condition is obtained on the existence and uniqueness of the endemic equilibrium.

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CHAPTER 2

BIFURCATION ANALYSIS ON A CORAL STARFISH MODEL

2.1 Introduction.

Australian marine scientists became aware of the devastation of the Great Barrier Reef (GBR) by coral eating crown-of-thorns starfish about 25 years ago. This starfish, which preys on hard corals, has outbroken twice between the 60's and 70's which caused large changes in coral reef communities. But still, there has been no completely satisfying explanation of why massive outbreaks of the coral eating starfish occur. The consensus opinion in 1981 and now was that adult starfish aggregate via chemical cues in the relatively shallow confines of the GBR. Thus any factor like fresh water run-off, pollution from land sources, cyclones, etc. could trigger a massive outbreak.

Based on optimal growth of the reef skeleton and aggregation by starfish, Antonelli and Kazarinoff formulated a mathematical model of ordinary differential equations

$$\frac{dN}{dt} = N(-2\alpha N - \delta F + \lambda)$$

$$\frac{dF}{dt} = F(2\beta N + rF^p - \varepsilon). \tag{2.1.1}$$

In this model, N and F represent the populations of coral and starfish respectively; λ is the intrinsic growth rate of coral and ε the intrinsic death rate of starfish. The coefficients α, δ, β and r have precisely defined chemical interpretation based on the Antonelli-Rhoades allometric plant response mechanism (Antonelli, 1985, Chapter 2D). The parameters α, δ, β and r are positive constants. The constant p in (2.1.1) is interpreted as density-dependent aggregation index and is taken to be close to one. It is pointed out that this model is different from traditional population model by the term rF^{p+1} called the cooperative term. Here, r is explained as an environmental control parameter depending only on the environmental factors which cause the aggregation of starfish. On the other hand, with power $p \neq 1$, the aggregation coefficient of F^2 is no longer constant. If $p < 1$, the coefficient is relatively large with small population F, which increases cooperation. If $p > 1$, the starfish decreases in cooperation while in small population. Using the theory of Hopf bifurcation and Hassard's computer code BIFOR 2, Antonelli and Kazarinoff conjecture that there exists a family of small-amplitude stable periodic solutions of this system for $p < 1$ (Antonelli and Kazarinoff, 1988).

Our purpose in this chapter is to present a mathematical proof of existence of such periodic solutions. In order to do this, the theory of Hopf bifurcation is employed. The environmental control parameter r is chosen as our bifurcation parameter. Hence, any environmental change could trigger the populations to cycle from their steady states according to the theory of Hopf bifurcation. The rest of this chapter is organized as follows. In Section 2.2, we study the model (2.1.1) in the case $p = 1$. Qualitatively, we give global phase portraits of solutions of (2.1.1) for various values of r; from which we will see later on that the model (2.1.1) is essentially different for $p \neq 1$ and $p = 1$. The existence and uniqueness of a Hopf bifurcation is shown in Section 2.3. The direction of the bifurcation is also determined: it is supercritical when $p < 1$ and subcritical when p is slightly greater than one (the definitions are in Section 2.3). Section 2.4 contains some numerical results: tables showing a numerical link between the coefficients and the period of the bifurcated solutions of the model; a figure providing evidence that periodic solutions with larger amplitude could also exist as the environmental control parameter r passes the bifurcation point.

2.2 Qualitative Analysis for the Case $p = 1$.

In this section, we consider the system

$$\frac{dN}{dt} = N(-2\alpha N - \delta F + \lambda)$$

$$\frac{dF}{dt} = F(2\beta N + rF - \varepsilon) \tag{2.2.1}$$

with $N \geq 0$, $F \geq 0$.

First, we make the following technical hypothesis:

HYPOTHESIS H

$$\beta\lambda - \alpha\varepsilon > 0, \quad \delta\varepsilon - \lambda r > 0.$$

Remark. If Hypothesis H fails, then either there is no equilibrium in the positive quadrant or the equilibrium in the positive quadrant never changes its stability. Both cases do not happen in the coral-starfish population dynamics of the GBR. So the hypothesis above is pretty reasonable.

We now turn to the equations (2.2.1). It is easy to see that under Hypothesis H, there is only one equilibrium, denoted by (N_0, F_0), in the positive orthant. In fact,

$$F_0 = \frac{\beta\lambda - \alpha\varepsilon}{\beta\delta - \alpha r}, \qquad N_0 = \frac{\lambda - \delta F_0}{2\alpha},$$

and Hypothesis H implies $\beta\delta - \alpha r > 0$.

Let us look at the Jacobian matrix A of (2.2.1) at (N_0, F_0). Evidently,

$$A = \begin{pmatrix} -2\alpha N_0 & -\delta N_0 \\ 2\beta F_0 & r F_0 \end{pmatrix}.$$

By simple calculation, we find that the real part of the eigenvalues of A are zero if and only if

$$r = \frac{2\alpha N_0}{F_0},$$

and the imaginary part of the eigenvalues of A do not vanish at this point. If we choose r as the bifurcation parameter there exists an unique Hopf bifurcation at $r = r_c$, where $r_c = \frac{2\alpha N_0}{F_0}$ (see Theorem 2.3.1.) Moreover, we have

THEOREM 2.2.1. *For the equations (2.2.1), the Hopf bifurcation at* $r = r_c$ *is neutral. In fact, if* $r \neq r_c$ *there is no periodic solution and if* $r = r_c$ *the equilibrium is a center (i.e. the solutions starting in a neighborhood of* (N_0, F_0) *all are periodic).*

PROOF. First, we suppose $r \neq r_c$. Let

$$B(N,F) = N^{k-1}F^{h-1},$$

where

$$k = \frac{r(\alpha+\beta)}{\beta\delta - \alpha r}, \qquad h = \frac{\alpha(r+\delta)}{\beta\delta - \alpha r}.$$

Then

$$\frac{\partial}{\partial N}[N(-2\alpha N - \delta F + \lambda)B(N,F)] + \frac{\partial}{\partial F}[F(2\beta N + rF - \varepsilon)B(N,F)]$$

$$= \frac{\lambda r(\alpha+\beta) - \alpha\varepsilon(\delta + r)}{2(\beta\delta - \alpha r)} B(N,F)$$

$$= (rF_0 - 2\alpha N_0)B(N,F) \neq 0.$$

By the Dulac criterion, there is no periodic nor homoclinic solution in the positive orthant.

Next, suppose $r = r_c$.

In this case, we have a first integral of (2.2.1)

$$G(N,F) = N^k F^h (2\alpha\varepsilon N + r_c\lambda F - \lambda\varepsilon) \equiv \text{ const.}$$

Therefore, every solution lying in the triangle

$$N = 0, \quad F = 0 \quad \text{and} \quad 2\alpha\varepsilon N + r_c\lambda F - \lambda\varepsilon = 0$$

is a periodic solution. ☐

REMARK. From the first integral of (2.2.1) at $r = r_c$, it is not difficult to draw the following global phase portrait.

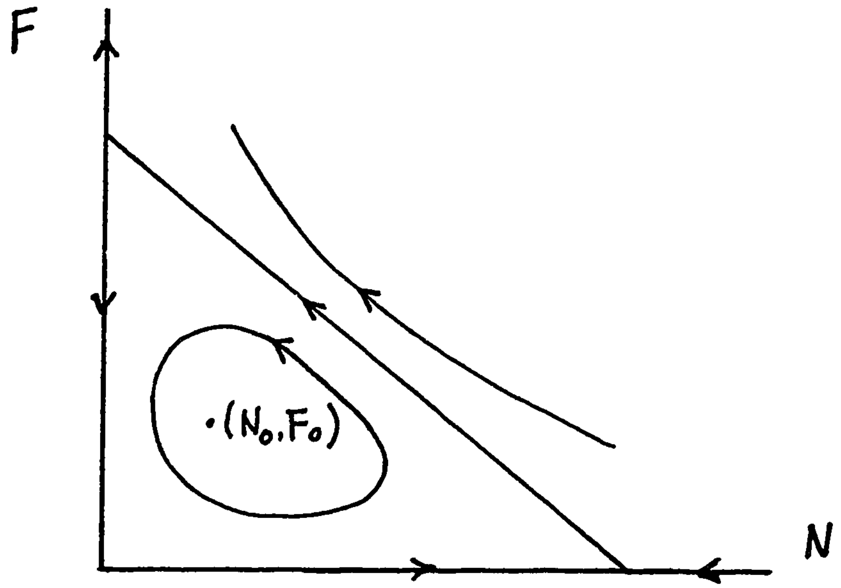


FIGURE 1 $r = r_c$

We now investigate the global structure of the system (2.2.1) when $r \neq r_c$. Obviously, there are four equilibria in the positive quadrant and its boundary:

$$E_0(0,0), \quad E_1(\frac{\lambda}{2\alpha}, 0), \quad E_2(0, \frac{\varepsilon}{r})$$

and $E_3(N_0, F_0)$.

The equilibria E_0, E_1 and E_2 are saddle points; E_3 is a stable spiral as $r < r_c$ and unstable spiral as $r > r_c$. Next, we study the singularity of (2.2.1) at infinity by using the Poincare transforms.

Under the Poincare transform with respect to the N-axis:

$$N = \frac{1}{z}, \quad F = \frac{u}{z}, \quad zd\tau = dt,$$

we obtain

$$\frac{du}{d\tau} = u[2(\alpha+\beta) + (r+\delta)u - (\lambda+\varepsilon)z]$$
$$\tag{2.2.2}$$
$$\frac{dz}{d\tau} = z(2\alpha + \delta u - \lambda z).$$

Under the Poincare transform with respect to the F-axis:

$$N = \frac{v}{z}, \quad F = \frac{1}{z}, \quad z d\tau = dt,$$

$$\frac{dv}{d\tau} = v[-(\delta + r) - 2(\alpha+\beta)v + (\lambda+\varepsilon)z] \tag{2.2.3}$$

$$\frac{dz}{d\tau} = z(-r - 2\beta v + \varepsilon z).$$

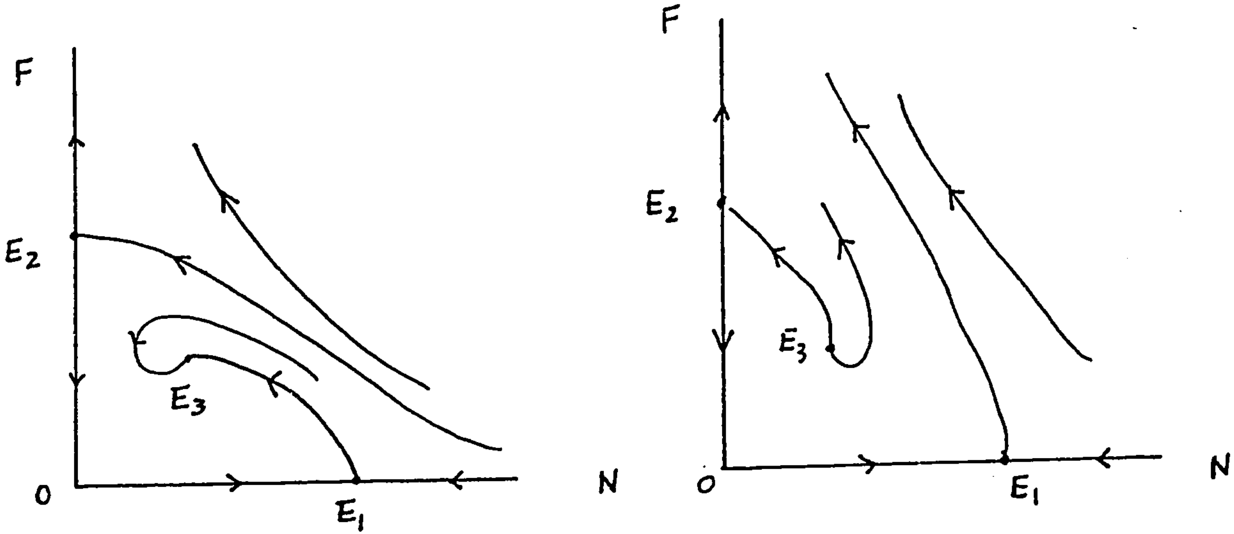
So there are only two critical points at infinity in the extended positive orthant:

$E_4(\infty, 0)$, which corresponds to $z = 0,\ u = 0$;

$E_5(0, \infty)$, which corresponds to $z = 0, v = 0$.

Calculating the Jacobi matrix of (2.2.2) at E_4, we find that E_4 is an unstable star node. The same argument shows E_5 is a stable star node. Combining with the fact that no periodic nor homoclinic trajectories exist in the positive orthant and the Poincare-Bendixson theory, we can conclude

THEROEM 2.2.2. *If* $r < r_c$, *then the global phase portrait is as shown in Figure 2a. If* $r_c < r < \delta\varepsilon/\lambda$, *the global phase portrait is as shown in Figure 2b.*



(a) $0 < r < r_c$

(b) $r_c < r < \delta\varepsilon/\lambda$

FIGURE 2

2.3 Bifurcation Analysis for the Case $p \neq 1$.

As we see in Section 2.2, the bifurcation is neutral for $p = 1$. However, the situation is quite different for $p \neq 1$.

Let us first consider the equations

$$\frac{dN}{dt} = N(-2\alpha N - \delta F + \lambda)$$

$$(2.3.1)$$

$$\frac{dF}{dt} = F(2\beta N + rF^p - \varepsilon)$$

with $0 < p < 1$.

We now assume $\beta\lambda - \varepsilon\alpha > 0$, then the two functions

$$z = F^p \quad \text{and} \quad z = \frac{\delta\beta}{\alpha r} F + \frac{\varepsilon\alpha - \beta\lambda}{\alpha r}$$

intersect at only one point in the positive orthant. Hence, there is an unique positive solution F_0 for

$$F^p = \frac{\delta\beta}{\alpha r} F + \frac{\varepsilon\alpha - \beta\lambda}{\alpha r},$$

Setting

$$N_0 = \frac{\lambda - \delta F_0}{2\alpha},$$

then (N_0, F_0) is an equilibrium of (2.3.1). Again, choosing r as the bifurcation parameter and letting $B(p)$ be the maximal value such that $N_0 > 0$ for $0 < r < B(p)$, we have

THEOREM 2.3.1. *There is a Hopf bifurcation for (2.3.1) at the value*

$$r_c = \frac{2\alpha N_0}{pF_0^p}, \qquad 0 < r_c < B(p),$$

provided $0 < p \leq 1$.

REMARK. $B(1) = \frac{\delta\varepsilon}{\lambda}$.

PROOF. From the last four equalities, it is easy to verify that $0 < r_c < B(p)$.

The Jacobi matrix of (2.3.1) at (N_0, F_0) is

$$A(r) = \begin{pmatrix} -2\alpha N_0 & -\delta N_0 \\ 2\beta F_0 & rpF_0^p \end{pmatrix}.$$

Thus, the corresponding characteristic equation is

$$\mu^2 + (2\alpha N_0 - rpF_0^p)\mu - 2\alpha rpN_0F_0^p + 2\beta\delta N_0F_0 = 0.$$

Hence, the real part $\operatorname{Re}\mu$ of the eigenvalues is zero only if

$$r = \frac{2\alpha N_0}{pF_0^p}.$$

On the other hand, since

$$2\beta\delta N_0F_0 - 2\alpha r_c pN_0F_0^p = 2pN_0(\beta\lambda - \varepsilon\alpha) + 2(1-p)\beta\delta N_0F_0 > 0$$

the real part $\operatorname{Re}\mu$ of the eigenvalues is zero if and only if $r = r_c$.

Finally, we show the transversality condition, i.e.

$$\frac{d}{dr}(\operatorname{Re}\mu)|_{r=r_c} > 0.$$

From the identity

$$r\alpha F_0^p - \beta\delta F_0 + \beta\lambda - \varepsilon\alpha = 0$$

by taking derivative of it at $r = r_c$, we get

$$\frac{dF_0}{dr}|_{r=r_c} = \frac{2\alpha N_0F_0^{p+1}}{2\beta\delta N_0F_0 - 2\alpha r_c pN_0F_0^p} > 0.$$

Hence

$$\frac{d}{dr}(\mathrm{Re}\,\mu) = \frac{d}{dr}(rpF_0^p - 2\alpha N_0)|_{r=r_c}$$
$$= pF_0^p + (\delta + rp^2F_0^{p-1})\frac{dF_0}{dr}|_{r=r_C} > 0.$$

So all conditions in the Hopf bifurcation theorem are satisfied. Therefore, the theorem is proved. □

REMARK. It can be seen from the above proof that if the parameter p is slightly greater than one, Theorem 2.3.1 still holds.

Next, we are going to decide the direction of the bifurcation obtained from Theorem 2.3.1 and the stability of the bifurcated periodic solutions.

Before stating our result, we give two lemmas. Interested readers may find the proofs in [3,7].

LEMMA 2.3.2. [3]. *Given a two dimensional analytic ordinary differential equation in polar coordinates*

$$\frac{d\rho}{d\theta} = R_2(\theta)\rho^2 + R_3(\theta)\rho^3 + \ldots.$$

If

$$\int_0^{2\pi} R_2(\theta)d\theta = 0$$

and

$$g_3 = \int_0^{2\pi}[R_3(\theta) + 2R_2(\theta)r_2(\theta)]d\theta \neq 0,$$

where

$$r_2(\theta) = \int_0^\theta R_2(\theta)d\theta,$$

then, the origin is a spiral. Moreover, if $g_3 < 0$(*resp.* > 0), *the origin is stable(resp. unstable).*

LEMMA 2.3.3. [7]. *For a quadratic system*

$$\frac{dx}{dt} = -y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \qquad (2.3.2)$$

$$\frac{dy}{dt} = x + b_{20}x^2 + b_{11}xy + b_{02}y^2.$$

Let

$$A = a_{20} + a_{02}, \qquad B = b_{20} + b_{02}$$

$$\bar{\alpha} = a_{11} + 2b_{02}, \qquad \bar{\beta} = b_{11} + 2a_{20}$$

$$\text{and} \qquad W_1 = A\bar{\alpha} - B\bar{\beta}.$$

If we rewrite (2.3.2) in the polar coordinates

$$\frac{d\rho}{d\theta} = R_2(\theta)\rho^2 + R_3(\theta)\rho^3 + \dots$$

then $g_3 = \frac{\pi}{4} W_1$, *where* g_3 *is given in Lemma 2.3.2.*

We now state and prove our result.

THEOREM 2.3.4. *The Hopf bifurcation obtained in Theorem 2.3.1 is supercritical when* $0 < p < 1$. *That is, as the bifurcation parameter* r *passes* r_c *a family of small amplitude periodic solution appears and each of them is stable.*

Remark. We will see in the proof below that for the case p is slightly greater than one the Hopf bifurcation is subcritical, that is a family of small amplitude unstable periodic solutions appears when r approaches but is less than r_c.

Proof. We first bring the equilibrium (N_0, F_0) to the origin by

$$x = N - N_0, \qquad y = F - F_0.$$

The equations (2.3.1) become

$$\begin{aligned} \frac{dx}{dt} &= -2\alpha N_0 x - \delta N_0 y - 2\alpha x^2 - \delta xy, \\ \frac{dy}{dt} &= 2\beta F_0 x + 2\beta xy + (r_c F_0 + y)[(F_0 + y)^p - F_0^p]. \end{aligned} \tag{2.3.3}$$

Observing that g_3 in Lemma 2.3.2 only depends on $R_2(\theta)$ and $R_3(\theta)$, it suffices to consider the third-order terms of (2.3.3)

$$\frac{dx}{dt} = -2\alpha N_0 x - \delta N_0 y - 2\alpha x^2 - \delta xy,$$

$$\begin{aligned} \frac{dy}{dt} = {} & 2\beta F_0 x + 2\alpha N_0 y + 2\beta xy \\ & + r_c p(\frac{p+1}{2})F_0^{p-1} y^2 + r_c p(\frac{p-1}{2})(\frac{p+1}{3})F_0^{p-2} y^3. \end{aligned}$$

Set

$$\omega_0 = [2\beta\delta N_0 F_0 - (2\alpha N_0)^2]^{1/2}$$

and

$$T = \begin{pmatrix} -2\alpha N_0 & -\omega_0 \\ 2\beta F_0 & 0 \end{pmatrix}.$$

Then under the transform

$$\begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \tau = \omega_0 t,$$

have

$$\begin{aligned} \frac{du}{d\tau} &= -v + \frac{2\alpha\beta(p-1)N_0}{\omega_0} u^2 - 2\beta uv \\ &\quad + \frac{2\beta^2 r_c (p-1)p(p+1)F_0^p}{3\omega_0} u^3, \end{aligned} \tag{2.3.4}$$

$$\begin{aligned} \frac{dv}{d\tau} &= u - 2\alpha\left[1 + \frac{2\alpha\beta(p-1)N_0^2}{\omega_0^2}\right]u^2 \\ &\quad + \frac{2}{\omega_0}(4\alpha^2 N_0 + 2\alpha\beta N_0 - \beta\delta F_0)uv + 2\alpha v^2 \\ &\quad - \frac{4\alpha\beta^2 r_c (p-1)p(p+1)N_0 F_0^p}{3\omega_0} u^3. \end{aligned}$$

Hence, for the truncated system without the third-order terms, we have

$$W_1 = \frac{4\alpha\beta(p-1)N_0}{\omega_0^3}\,[2(\alpha-\beta)\beta\delta N_0 N_0 F_0 + 4\alpha^2\beta(p+1)N_0^2]$$

where W_1 is defined in Lemma 2.3.3.

We now consider the system (2.3.4) and its truncated system in the polar coordinates. Suppose they are of forms

$$\begin{aligned} \frac{d\rho}{d\tau} &= a_2(\theta)\rho^2 + a_3(\theta)\rho^3 \\ \frac{d\theta}{d\tau} &= 1 + b_1(\theta)\rho + b_2(\theta)\rho^2 \end{aligned} \tag{2.3.5}$$

and

$$\frac{d\rho}{d\tau} = a_2(\theta)\rho^2$$

$$\frac{d\theta}{d\tau} = 1 + b_1(\theta)\rho,$$

respectively, where $a_2(\theta) = c_1 \cos^2\theta \sin\theta + c_2 \sin^2\theta \cos\theta$ for some constants c_1 and c_2.

Then by Lemma 2.3.3,

$$\frac{\pi}{4} W_1 = \int_0^{2\pi} [-b_1(\theta)a_2(\theta) + a_2(\theta)r_2(\theta)]d\theta,$$

here

$$r_2(\theta) = \int_0^{\theta} a_2(s)ds.$$

Also, using Lemma 2.3.2, to compute g_3 for system (2.3.5), we get

$$\begin{aligned} g_3 &= \int_0^{2\pi} [a_3(\theta) - b_1(\theta)a_2(\theta) + a_2(\theta)r_2(\theta)]d\theta \\ &= \int_0^{2\pi} a_3(\theta) + \frac{\pi}{4} W_1 \end{aligned}$$

where

$$a_3(\theta) = \frac{2\beta^2 r_c(p-1)p(p+1)F_0^p}{3\omega_0} \cos^4\theta - \frac{4\alpha\beta^2 r_c(p-1)p(p+1)N_0 F_0^p}{3\omega_0} \sin\theta\cos\theta.$$

So,

$$g_3 = \frac{\pi\beta^2 r_c(p-1)p(p+1)F_0^p}{2\omega_0} + \frac{\pi}{4}W_1$$

$$= \frac{2\pi\alpha\beta^2\delta(\alpha+p\beta)(p-1)N_0^2F_0}{\omega_0^3}.$$

Hence, when $0 < p < 1$, $g_3 < 0$. That implies the equilibrium (N_0, F_0) is a stable spiral. Therefore, by the continuity of solutions with respect to parameters and the Poincare-Bendixson theory, there exists a limit cycle near (N_0, F_0) when r just passes r_c. □

2.4 Numerical Results.

In this section, we present numerical results obtained by applying B.D. Hassard's Computer Code BIFOR2 and E. Doedel's computer code AUTO to the model (2.1.1). All the computations are performed on a Sun Workstation granted by NSERC.

Using the code BIFOR2, we computed the equilibrium (N_0, F_0), the basic frequency ω_0, the Hopf bifurcation point r_c, the direction of bifurcation μ_2, the correction coefficient τ_2 to the basic period $2\pi/\omega_0$, and the stability exponent β_2 to the model (1.1) for the following ranges of parameters: α is from 0.05 to 5.0; β is from 0.5 to 3.5; p is from 0.5 to 1.0; $\delta = 0.5$, $\varepsilon = 1.0$, and $\lambda = 2.0$.

The result are shown in Table 1-3. In each "cell", the numbers in the first column are, in order, values of N_0, F_0 and r_c. Those in the second column are, in order, values of $\omega_0, 10^2\mu_2$, τ_2, and β_2. Those results show that as p decreases, the stability of the periodic, bifurcating solutions strengthens. For fixed p, their period lengthens as either α decreases or β increases.

We next run the code AUTO on the Sun Workstation to trace the periodic, bifurcating solutions for $p = 0.5$, $\alpha = 0.05$ and $\beta = 0.5$. The plots in Figure 3 shows a relation between the norm of the bifurcating solutions and the bifurcation parameter r. The norm chosen here is the usual L_2-norm of the periodic solutions over one period. From it, we see that the norm of the bifurcating solutions increases rapidly as r passes the bifurcation point r_c. Thus the bifurcating, periodic solutions are very sensitive to changes in r, which supports the observation in the G.B.R. in recent years.

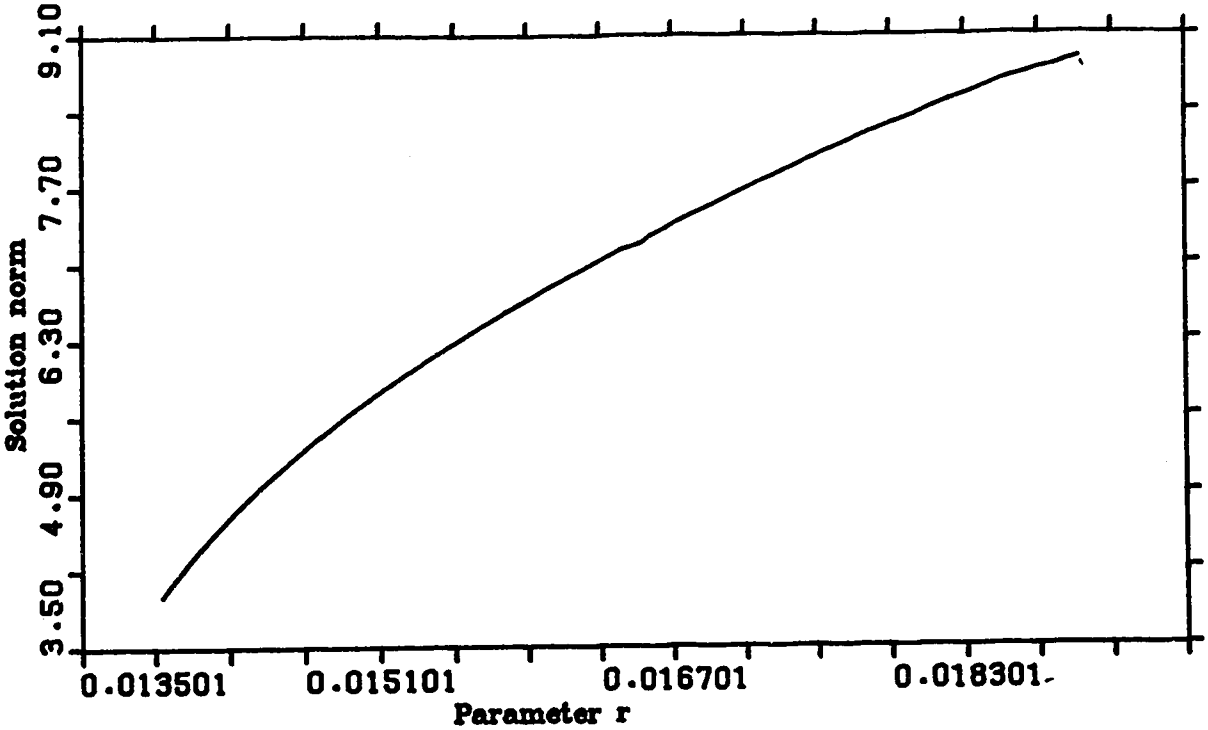

Solution norm
9.10
7.70
6.30
4.90
3.50
0.013501
0.015101
0.016701
0.018301
Parameter r

FIGURE 3

p =0.5

	3.5		3.0		2.5		2.0		1.5		1.0		0.5	
0.05	0.139	1.390	0.161	1.386	0.192	1.380	0.238	1.372	0.312	1.358	0.455	1.332	0.833	1.261
	3.972	1.105	3.968	0.949	3.962	0.793	3.952	0.636	3.938	0.480	3.909	0.323	3.833	0.167
	0.014	3.240	0.016	2.403	0.019	1.690	0.024	1.103	0.032	0.640	0.046	0.303	0.085	0.090
		-1.133		-0.977		-0.821		-0.664		-0.508		-0.352		-0.197
0.10	0.135	1.366	0.156	1.358	0.185	1.348	0.227	1.332	0.294	1.307	0.417	1.261	0.714	1.143
	3.946	2.233	3.938	1.920	3.926	1.607	3.909	1.294	3.882	0.980	3.833	0.666	3.714	0.349
	0.027	3.424	0.032	2.561	0.037	1.824	0.046	1.212	0.060	0.724	0.085	0.362	0.148	0.125
		-2.345		-2.033		-1.721		-1.409		-1.097		-0.786		-0.479
0.20	0.128	1.321	0.147	1.307	0.172	1.288	0.208	1.261	0.263	1.219	0.357	1.143	0.556	0.969
	3.897	4.549	3.882	3.922	3.862	3.294	3.833	2.665	3.789	2.033	3.714	1.397	3.556	0.746
	0.052	3.809	0.060	2.896	0.070	2.109	0.085	1.447	0.108	0.910	0.148	0.498	0.236	0.214
		-5.011		-4.388		-3.765		-3.144		-2.526		-1.914		-1.332
0.30	0.122	1.280	0.139	1.261	0.161	1.235	0.192	1.198	0.238	1.143	0.312	1.048	0.455	0.843
	3.854	6.940	3.833	5.996	3.806	5.050	3.769	4.100	3.714	3.143	3.625	2.173	3.455	1.168
	0.075	4.217	0.085	3.255	0.099	2.418	0.119	1.707	0.148	1.121	0.197	0.662	0.293	0.336
		-8.005		-7.074		-6.146		-5.221		-4.307		-3.416		-2.638
0.40	0.116	1.242	0.132	1.219	0.152	1.187	0.179	1.143	0.217	1.077	0.278	0.969	0.385	0.746
	3.814	9.398	3.789	8.134	3.758	6.864	3.714	5.588	3.652	4.298	3.556	2.983	3.385	1.606
	0.095	4.650	0.108	3.639	0.125	2.753	0.148	1.993	0.182	1.359	0.236	0.855	0.334	0.498
		-11.338		-10.103		-8.874		-7.656		-6.462		-5.329		-4.495
3.50	0.111	1.207	0.125	1.179	0.143	1.143	0.167	1.093	0.200	1.020	0.250	0.901	0.333	0.667
	3.778	11.917	3.750	10.328	3.714	8.731	3.667	7.121	3.600	5.490	3.500	3.818	3.333	2.054
	0.114	5.108	0.129	4.048	0.148	3.113	0.174	2.306	0.211	1.626	0.267	1.081	0.365	0.709
		-15.016		-13.483		-11.963		-10.465		-9.014		-7.692		-7.031

p =0.7

	3.5		3.0		2.5		2.0		1.5		1.0		0.5	
0.05	0.140	1.395	0.163	1.392	0.194	1.388	0.241	1.381	0.318	1.370	0.467	1.349	0.875	1.291
	3.972	0.503	3.967	0.432	3.961	0.361	3.952	0.290	3.936	0.219	3.907	0.148	3.825	0.077
	0.008	3.201	0.009	2.369	0.011	1.662	0.013	1.080	0.017	0.623	0.026	0.291	0.049	0.084
		-0.944		-0.813		-0.682		-0.551		-0.419		-0.288		-0.158
0.10	0.137	1.376	0.159	1.370	0.189	1.362	0.233	1.349	0.304	1.329	0.438	1.291	0.778	1.188
	3.945	1.018	3.936	0.876	3.924	0.734	3.907	0.592	3.878	0.449	3.825	0.307	3.689	0.163
	0.015	3.343	0.017	2.491	0.021	1.765	0.026	1.163	0.034	0.687	0.049	0.336	0.089	0.110
		-1.940		-1.678		-1.415		-1.153		-0.891		-0.630		-0.372
0.20	0.132	1.341	0.152	1.329	0.179	1.314	0.219	1.291	0.280	1.254	0.389	1.188	0.636	1.023
	3.894	2.082	3.878	1.797	3.856	1.512	3.825	1.227	3.776	0.940	3.689	0.652	3.491	0.358
	0.029	3.639	0.034	2.748	0.040	1.983	0.049	1.343	0.063	0.828	0.089	0.439	0.152	0.176
		-4.089		-3.566		-3.043		-2.521		-2.002		-1.489		-1.004
0.30	0.127	1.307	0.146	1.291	0.171	1.269	0.206	1.237	0.259	1.188	0.350	1.099	0.538	0.894
	3.847	3.188	3.825	2.760	3.795	2.331	3.753	1.900	3.689	1.468	3.580	1.030	3.354	0.575
	0.043	3.951	0.049	3.022	0.058	2.218	0.070	1.540	0.089	0.987	0.123	0.561	0.198	0.269
		-6.455		-5.673		-4.893		-4.117		-3.350		-2.606		-1.972
0.40	0.123	1.275	0.140	1.254	0.163	1.227	0.194	1.188	0.241	1.127	0.318	1.023	0.467	0.787
	3.804	4.334	3.776	3.761	3.740	3.187	3.689	2.609	3.614	2.026	3.491	1.433	3.253	0.808
	0.055	4.280	0.063	3.313	0.074	2.471	0.089	1.755	0.112	1.166	0.152	0.706	0.234	0.394
		-9.045		-8.008		-6.977		-5.956		-4.956		-4.015		-3.389
3.50	0.119	1.244	0.135	1.220	0.156	1.188	0.184	1.142	0.226	1.073	0.292	0.955	0.412	0.696
	3.763	5.518	3.731	4.799	3.689	4.077	3.632	3.349	3.548	2.613	3.417	1.858	3.176	1.051
	0.067	4.627	0.077	3.622	0.089	2.742	0.107	1.990	0.133	1.365	0.176	0.875	0.262	0.567
		-11.867		-10.581		-9.306		-8.052		-6.843		-5.760		-5.419

p =1.0

	3.5		3.0		2.5		2.0		1.5		1.0		0.5	
0.05	0.141	1.399	0.164	1.397	0.196	1.393	0.244	1.388	0.323	1.380	0.476	1.363	0.909	1.314
	3.972	0.000	3.967	0.000	3.961	0.000	3.951	0.000	3.935	0.000	3.905	0.000	3.818	0.000
	0.004	3.173	0.004	2.345	0.005	1.642	0.006	1.064	0.008	0.611	0.012	0.283	0.024	0.080
		0.000		0.000		0.000		0.000		0.000		0.000		0.000
0.10	0.139	1.384	0.161	1.380	0.192	1.373	0.238	1.363	0.313	1.346	0.455	1.314	0.833	1.225
	3.944	0.000	3.935	0.000	3.923	0.000	3.905	0.000	3.875	0.000	3.818	0.000	3.667	0.000
	0.007	3.287	0.008	2.443	0.010	1.724	0.012	1.131	0.016	0.662	0.024	0.318	0.046	0.100
		0.000		0.000		0.000		0.000		0.000		0.000		0.000
0.20	0.135	1.356	0.156	1.346	0.185	1.333	0.227	1.314	0.294	1.283	0.417	1.225	0.714	1.069
	3.892	0.000	3.875	0.000	3.852	0.000	3.818	0.000	3.765	0.000	3.667	0.000	3.429	0.000
	0.014	3.523	0.016	2.648	0.019	1.698	0.024	1.274	0.031	0.774	0.046	0.400	0.083	0.153
		0.000		0.000		0.000		0.000		0.000		0.000		0.000
0.30	0.132	1.328	0.152	1.314	0.179	1.296	0.217	1.268	0.278	1.225	0.385	1.144	0.625	0.935
	3.842	0.000	3.818	0.000	3.786	0.000	3.739	0.000	3.667	0.000	3.538	0.000	3.250	0.000
	0.021	3.772	0.024	2.866	0.028	2.085	0.035	1.430	0.046	0.900	0.065	0.497	0.115	0.229
		0.000		0.000		0.000		0.000		0.000		0.000		0.000
0.40	0.128	1.301	0.147	1.283	0.172	1.259	0.208	1.225	0.263	1.170	0.357	1.069	0.556	0.816
	3.795	0.000	3.765	0.000	3.724	0.000	3.667	0.000	3.579	0.000	3.429	0.000	3.111	0.000
	0.027	4.033	0.031	3.096	0.037	2.285	0.046	1.600	0.059	1.041	0.083	0.613	0.143	0.338
		0.000		0.000		0.000		0.000		0.000		0.000		0.000
3.50	0.125	1.275	0.143	1.254	0.167	1.225	0.200	1.183	0.250	1.118	0.333	1.000	0.500	0.707
	3.750	0.000	3.714	0.000	3.667	0.000	3.600	0.000	3.500	0.000	3.333	0.000	3.000	0.000
	0.033	4.308	0.039	3.341	0.046	2.500	0.056	1.786	0.071	1.200	0.100	0.750	0.167	0.500
		0.000		0.000		0.000		0.000		0.000		0.000		0.000

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CHAPTER 3

ASYMPTOTIC BEHAVIOR OF AN EPIDEMIC MODEL WITH A TIME DELAY AND A NONLINEAR INCIDENCE RATE

3.1 Introduction.

To study the spread of an infectious disease in a constant population, one usually divides the total population into three disjoint classes: susceptible, infectious and recovered (removed). In the susceptible class, individuals can incur the disease but are not yet infected. The infectious class consists of those who are transmitting the disease to others. The recovered class consists of individuals who have removed from the disease and have temporary immunity. We denote the fraction of the population in each class at time t by $x(t), y(t)$ and $z(t)$, respectively.

In recent years, various epidemiological models (so called SIRS models) have been formulated to investigate such interactions between these three classes. Hethcote [2] studied a model with a bilinear incidence rate βxy, where β is the contact rate. Hethcote, Stech and van den Driessche [3] studied a model with a bilinear incidence rate and a time delay in the recovered class, which means individuals gain a period of temporary immunity right after recovering. Liu, Hethcote and Lewis [6] considered a model with a nonlinear incidence rate of form $\beta x^q y^p$, where p and q are positive numbers, but without time delay. The model we will study in this chapter was proposed by Hethcote, Lewis and van den Driessche [5]. A constant time delay and a nonlinear incidence rate are taken into account in this model. It is assumed that the incidence rate is nonlinear and is of the form $\beta x y^p$. The recovery rate of individuals in the infectious class is proportional to the infectious fraction at rate γ and then the average infectious period is $1/\gamma$. Furthermore, a constant period ω of temporary immunity is introduced so that the probability, $P(t)$, of individuals remaining immune t unit time after recovering is 1 for $0 \le t \le \omega$, and 0 for $t > \omega$. According to the above assumptions, Hethcote et al. [5] developed the following epidemiological model

$$y'(t) = -\gamma y(t) + \beta y^p(t) x(t) \tag{3.1.1}$$

$$z(t) = z_0(t) + \gamma \int_0^t y(s) P(t-s) ds \tag{3.1.2}$$

$$x'(t) = \begin{cases} -z_0'(t) - \beta x(t) y^p(t), & \text{for} \quad t \le \omega, \\ \gamma y(t-\omega) - \beta x(t) y^p(t), & \text{for} \quad t > \omega, \end{cases} \tag{3.1.3}$$

where $z_0(t)$ is the fraction of the initial population which is initially in the recovered class and is still in it at time t. It is reasonable to assume $z_0(t)$ is a differentiable, nonincreasing function with $z_0(t) = 0$ for $t > \omega$.

In [5], Hethcote et al. analyzed the model (3.1.1)-(3.1.3) to determine the equilibria and examine their stability by varying the index p and the contact number $\sigma = \beta/\gamma$. For the case $p > 1$ in particular, they proved that there exists a threshold value σ^* such that the number of positive endemic equilibria is zero(resp. one, two) , when the contact number σ is below(resp. equal to, above) the threshold value σ^*. They then examined the local stability of the disease-free equilibrium in each case and the positive endemic equilibria in the case of $\sigma > \sigma^*$. Moreover, they gave a sufficient condition for the global stability of the disease-free equilibrium. They showed that there is a positive $\sigma_1 < \sigma^*$ such that if $\sigma < \sigma_1$, the disease-free equilibrium is globally asymptotically stable. They further conjectured that the disease-free equilibrium is globally asymptotically stable for all $\sigma < \sigma^*$.

In the present chapter, we will solve this conjecture. Also, we will analyze the asymptotic behavior of solutions of the model (3.1.1)-(3.1.3) as the contact number σ takes the threshold value σ^*. The results we will prove show that there is no periodic solution when $\sigma = \sigma^*$. Every solution tends to either the disease-free equilibrium or the unique positive endemic equilibrium.

The organization of the rest of this chapter is as follows. In the next section, we present some known results obtained by Hethcote et al.[5]. for the model (3.1.1)-(3.1.3) and their conjecture. Section 3.3 and Section 3.4 are the statements and proofs of our results. In the final section, we discuss our results and compare them with the results of Hethcote et al.[5].

3.2 Summary of the Previous Results and the Conjecture.

We are going to recall some results obtained by Hethcote et al. in [5]. Before doing that, we reduce the model (3.1.1)-(3.1.3) to an equivalent scalar integrodifferential equation.

Since $x(t) + y(t) + z(t) = 1$ for all $t \geq 0$,

$$y'(t) = -\gamma y(t) + \beta y^p(t)[1 - y(t) - z(t)].$$

Replacing $z(t)$ by $z_0(t) + \gamma \int_0^t y(s)P(t-s)ds$ and noting $P(t) = 1$ for $t \leq \omega$, and $P(t) = 0$ for $t > \omega$, we can write the following equivalent integrodifferential equation: for $t \leq \omega$,

$$y'(t) = -\gamma y(t) + \beta y^p(t)[1 - y(t) - z_0(t) - \gamma \int_0^t y(s)ds], \tag{3.2.1}$$

and for $t > \omega$,

$$y'(t) = -\gamma y(t) + \beta y^p(t)[1 - y(t) - \gamma \int_{t-\omega}^t y(s)ds]. \tag{3.2.2}$$

Obviously, $y(t) \equiv 0$ is always an equilibrium of (3.2.1)-(3.2.2) This corresponds to the equilibrium $(x, y, z) = (1, 0, 0)$ for (3.1.1)-(3.1.3). We call it the disease-free equilibrium. It is easy to see that a solution of (3.2.1)-(3.2.2) tends to an equilibrium if and only if the corresponding solution for (3.1.1)-(3.1.3) tends to the corresponding equilibrium. Hence, studying the stability of equilibria of the model (3.1.1)-(3.1.3) can be done by studying the stability of equilibria of the equation (3.2.1)-(3.2.2).

It can be shown that the initial value problem of (3.2.1)-(3.2.2) is well-posed and the interval $[0, 1]$ is a positive invariant set and an attractive region for all nonnegative solutions. Moreover, the maximal interval for every nonnegative solution is $[0, \infty)$ (cf. [5]). Besides that, by analyzing the equation

(3.2.2), Hethcote and his collaborators established the following theorems. We refer the interested readers to [3] and [5] for the proofs and more details.

THEOREM 3.2.1. *If* $p < 1$, *then the disease-free equilibrium is unstable. If* $p = 1$, *then the disease-free equilibrium is globally asymptotically stable for* $\sigma = \beta/\gamma \leq 1$ and *unstable for* $\sigma > 1$. *If* $p > 1$, *the disease-free equilibrium is locally asymptotically stable.*

THEOREM 3.2.2. *Let* $\sigma^* = p^p(1+r)^{p-1}/(p-1)^{p-1}$ *where* $r = \gamma\omega$. *For the equation (2.1)-(2.2), if*

(i) $p < 1$, *then it has one positive equilibrium,*

(ii) $p = 1$ *and* $\sigma \leq 1$, *then it has no positive equilibrium,*

(iii) $p = 1$ *and* $\sigma > 1$, *then it has one positive equilibrium,*

(iv) $p > 1$ *and* $\sigma < \sigma^*$, *then it has no positive equilibrium,*

(v) $p > 1$ *and* $\sigma = \sigma^*$, *then it has one positive equilibrium,*

(vi) $p > 1$ *and* $\sigma > \sigma^*$, *then it has two positive equilibria and the smaller positive equilibrium is locally asymptotically stable.*

REMARK. In the cases (i), (iii) and (vi), the stability of the positive equilibrium (the larger positive equilibrium in (vi)) depends on the value of

σ. Hethcote et al. have proved there is a Hopf bifurcation near it when one properly chooses the value of σ. (See [3] and [5].)

THEOREM 3.2.3. *Let* $\sigma_1 = p^p/(p-1)^{p-1}$. *If* $p > 1$ *and* $\sigma < \sigma_1$, *then all solutions of (3.2.1)-(3.2.2) which start in* $[0,1]$ *approach to the disease-free equilibrium.*

Based on this theorem, Hethcote et al. further made the following conjecture.

CONJECTURE: If $p > 1$ and $\sigma < \sigma^*$, then the conclusion of Theorem 3.2.3 is still true, i.e. all solutions of (3.2.1)-(3.2.2), starting in $[0,1]$, approach to the disease-free equilibrium.

3.3 Global Stability of the Disease-free Equilibrium when $p > 1$ and $\sigma < \sigma^*$.

In this section, we will study the asymptotic behavior of solutions of (3.2.1)-(3.2.2) when $p > 1$ and $\sigma < \sigma^*$. We will prove that if $p > 1$ and $\sigma < \sigma^*$, all nonnegative solutions will approach to the disease-free equilibrium. As a consequence, the conjecture of Hethcote et al. is established.

First, we introduce a definition, which will be used in our proofs.

DEFINITION: Let $f(t)$, $t \in [0, \infty)$, be a continuous function. $f(t)$ is said to be eventually (strictly) monotonic if there is a positive $T > 0$ such that $f(t)$ is (strictly) monotonic for $t \geq T$.

We now state and prove our results.

THEOREM 3.3.1. *Suppose* $p > 1$ *and* $\sigma < \sigma^*$. *Then, every nonnegative solution of (3.2.1)-(3.2.2) is an eventually decreasing function and it tends to the disease-free equilibrium.*

To prove Theorem 3.3.1, we need the following lemmas.

Let $y(t)$ be a nonnegative solution of (3.2.1)-(3.2.2). Since $[0,1]$ is the attractive region, without loss of generality we may assume $0 \leq y(t) \leq 1$. Define

$$\delta_n = \min\{y(t);\ t \in [(n-1)\omega, n\omega]\}, \quad n = 1, 2, \dots .$$

Then, for each n, there is a $t_n \in [(n-1)\omega, n\omega]$ such that $y(t_n) = \delta_n$.

LEMMA 3.3.2. *Suppose* $p > 1$ *and* $\sigma \leq \sigma^*$. *Then,* $y(t)$ *is an eventually decreasing function or* $t_n \neq n\omega$ *for all* $n = 1, 2, \cdots$.

PROOF. If $t_n = n\omega$ for some n, then

$$y(t_n) \leq y(t), \quad t \in [(n-1)\omega, n\omega].$$

Thus

$$\begin{aligned} y'(t_n) &= -\gamma y(t_n) + \beta y^p(t_n)[1 - y(t_n) - \gamma \int_{t_n-\omega}^{t_n} y(s)ds] \\ &\leq -\gamma y(t_n) + \beta y^p(t_n)[1-(1+r)y(t_n)] \quad (\text{since} t_n = n\omega) \\ &\leq \gamma y(t_n)(-1 + \frac{\sigma}{\sigma^*}). \end{aligned}$$

The last two inequalities become equalities if and only if $y(t) \equiv 0$ or $y(t) \equiv (p-1)/p(1+r)$. In these cases, $y(t)$ is already an eventually decreasing function. Hence, we assume that $y(t)$ is not either one. So, we have $y'(t_n) < 0$. This implies that there is a $t' > t_n$ such that $y(t)$ is a strictly decreasing function on $[t_n, t')$.

Let

$$T_n = \sup\{t'; y(t) \text{ strictly decreases on } [t_n, t')\}.$$

If $T_n < \infty$, then T_n is a minimal point of $y(t)$. Thus, $y'(T_n) = 0$, and $y(T_n) < y(t)$ on $[t_n - \omega, T_n)$. In particular, $y(T_n) \leq y(t)$ and $y(T_n) \neq y(t)$, for $t \in [T_n - \omega, T_n]$. Therefore, we have

$$\begin{aligned} y'(T_n) &= -\gamma y(T_n) + \beta y^p(T_n)[1 - y(T_n) - \gamma \int_{T_n-\omega}^{T_n} y(s)ds] \\ &< -\gamma y(T_n) + \beta y^p(T_n)[1-(1+r)y(T_n)] \\ &\leq \gamma y(T_n)(-1 + \frac{\sigma}{\sigma^*}) \leq 0, \end{aligned}$$

which contradicts to $y'(T_n) = 0$.

Hence, $T_n = \infty$. Consequently, $y(t)$ is an eventually strictly decreasing function. We complete the proof. □

Lemma 3.3.3. *Suppose* $p > 1$ *and* $\sigma \le \sigma^*$. *If* $t_n \ne n\omega$, *for all* $n = 1, 2, \cdots$, *then* $\delta_{n+1} > \delta_n$ *for all* $n = 1, 2, \cdots$.

Proof. If $\delta_{n+1} \le \delta_n$ for some n, then δ_{n+1} is the minimal value of $y(t)$ in $[(n-1)\omega, (n+1)\omega]$. From $t_{n+1} \ne (n+1)\omega$, $t_n \in ((n-1)\omega, (n+1)\omega)$. Thus $y'(t_{n+1}) = 0$. Again, we have

$$\begin{aligned} y'(t_{n+1}) &= -\gamma y(t_{n+1}) + \beta y^p(t_{n+1})[1 - y(t_{n+1}) - \gamma \int_{t_{n+1}-\omega}^{t_{n+1}} y(s)ds] \\ &< \gamma y(t_{n+1})\Big(-1 + \frac{\sigma}{\sigma^*}\Big) \le 0, \end{aligned}$$

a contradiction to $y'(t_{n+1}) = 0$. Hence the lemma is proved. □

Lemma 3.3.4. *Under the condition of Lemma 3.3.3, we have*

$$\delta_{n+1} - \delta_n > \frac{1}{\sigma r}\Big(1 - \frac{\sigma}{\sigma^*}\Big).$$

Proof. Since t_{n+1} is a minimal point in $[n\omega, (n+1)\omega]$ and $t_{n+1} \ne (n+1)\omega$,

$y'(t_{n+1}) \ge 0$. On the other hand,

$$y'(t_{n+1}) = -\gamma y(t_{n+1}) + \beta y^p(t_{n+1})[1 - y(t_{n+1}) - \gamma \int_{t_{n+1}-\omega}^{t_{n+1}} y(s)ds].$$

By Lemma 3.3.3, δ_n is the minimal value in $[(n-1)\omega, (n+1)\omega]$. Hence,

$$\begin{aligned} y'(t_{n+1}) &< -\gamma\delta_{n+1} + \beta\delta_{n+1}^p[1 - \delta_{n+1} - r\delta_n] \\ &= \gamma\delta_{n+1}\{-1 + \sigma\delta_{n+1}^{p-1}[1 - (1+r)\delta_{n+1} + r(\delta_{n+1} - \delta_n)]\}. \end{aligned}$$

From $y'(t_{n+1}) \geq 0$,

$$\begin{aligned}\sigma r(\delta_{n+1} - \delta_n) &\geq \sigma r \delta_{n+1}^{p-1}(\delta_{n+1} - \delta_n) \\ &> 1 - \sigma \delta_{n+1}^{p-1}[1 - (1+r)\delta_{n+1}] \\ &\geq 1 - \frac{\sigma}{\sigma^*},\end{aligned}$$

i.e.

$$\delta_{n+1} - \delta_n > \frac{1}{\sigma r}(1 - \frac{\sigma}{\sigma^*}).$$

□

LEMMA 3.3.5. *If* $p > 1$ *and* $\sigma < \sigma^*$, *then* $y(t)$ *is an eventually decreasing function.*

PROOF. If not, by Lemmas 3.3.2 and 3.3.4,

$$\delta_{n+1} - \delta_n > \frac{1}{\sigma r}(1 - \frac{\sigma}{\sigma^*}) > 0.$$

Thus, $\lim_{n \to \infty} \delta_n = \infty$, which is impossible because $y(t)$ is bounded. Therefore we prove the lemma. □

PROOF OF THEOREM 3.3.1. From Lemma 3.3.5, we know that every nonnegative solution $y(t)$ of (3.2.1)-(3.2.2) is eventually decreasing. Let y_0 denote the limit of $y(t)$, as

$t \to \infty$. Since $y(t)$ and $y'(t)$ are bounded, we have $\lim_{t\to\infty} y'(t) = 0$. Hence, let $t \to \infty$ in the equation (3.2.2), we obtain

$$-\gamma y_0 + \beta y_0^p[1-(1+r)y_0] = 0.$$

But, for $p > 1$ and $\sigma < \sigma^*$, only $y \equiv 0$ satisfy this equality. Hence $y_0 = 0$. This implies

$$\lim_{t\to\infty} y(t) = 0.$$

Therefore, the disease-free equilibrium is globally asymptotically stable to all nonnegative solutions for the equation (3.2.1)-(3.2.2). ☐

3.4 Asymptotic Behavior of (3.2.1)-(3.2.2) when $p > 1$ and $\sigma = \sigma^*$.

Theorem 3.2.1 shows that when $p > 1$, σ^* is a bifurcation point for the equation (3.2.1)-(3.2.2). The equation (3.2.1)-(3.2.2) has zero, one and two positive equilibria when $\sigma < \sigma^*, \sigma = \sigma^*$ and $\sigma > \sigma^*$. Moreover, in the case $\sigma = \sigma^*$, the unique positive equilibrium is degenerate, i.e. the characteristic equation for its linear variational equation has a zero eigenvalue. Hence, the stability of the positive equilibrium becomes very difficult to determine. In this section, we will show that if a nonnegative solution does not approach to the positive equilibrium, it must approach to the disease-free equilibrium. In other words, the feasible region is split into two disjoint regions: the attractive region of the positive equilibrium and the attractive region of the disease-free equilibrium. Furthermore, we give an estimation of the attractive region of the disease-free equilibrium.

When $p > 1$ and $\sigma = \sigma^*$, by Theorem 3.2.2, the equation (3.2.2) has two equilibria: the disease-free equilibrium $y = 0$ and the positive equilibrium $y = y_e = (p-1)/p(1+r)$. In this situation, we have the following

THEOREM 3.4.1. *Let* $y(t)$ *be a nonnegative solution of (3.2.1)-(3.2.2). Then, there are only two possibilities:*

(i) $y(t)$ *is an eventually decreasing function and tends to one of the equilibria,*

(ii) $y(t)$ *is a damped oscillation and* $\lim_{t\to\infty} y(t) = y_e$.

Therefore, there is no periodic solution for the equation (3.2.1)-(3.2.2).

PROOF. If $y(t)$ is an eventually decreasing function, proceeding as in the proof of Theorem 3.3.1, $y(t)$ tends to one of the equilibria. Hence, (i) is true.

If $y(t)$ is not an eventually decreasing function, by Lemma 3.3.3, $\delta_{n+1} > \delta_n$ for $n = 1, 2 \ldots$.

Thus, there is no periodic solution in either case.

We now prove that if $y(t)$ is not eventually decreasing,

$$\lim_{t\to\infty} y(t) = y_e.$$

First,we claim that $\liminf_{t\to\infty} y(t) = y_e$.

Otherwise, since $\{\delta_n\}_{n=1}^{\infty}$ is a strictly increasing sequence, by Lemma 3.3.3, there is a positive μ such that $|\delta_n - I_e| \geq \mu$, $n \geq N$ for some integer N. Noting that the function $y^{p-1} - (1+r)y^p$ takes the unique maximal value $1/\sigma^*$ at $y = y_e$, we have

$$\delta_{n+1}^{p-1} - (1+r)\delta_{n+1}^{p} \leq \frac{1-\varepsilon}{\sigma^*}, \quad n \geq N,$$

for some positive ε.

Thus, from the proof of Lemma 3.3.4,

$$\begin{aligned}\delta_{n+1} - \delta_n &> \frac{1}{\sigma^* r}\left\{1 - \sigma^*\delta_{n+1}^{p-1}[1-(1+r)\delta_{n+1}]\right\}\\ &\geq \frac{\varepsilon}{\sigma^* r} > 0.\end{aligned}$$

Hence, $y(t)$ is unbounded, which is impossible. Therefore, we must have

$$\liminf_{t\to\infty} y(t) = I_e.$$

By defining that

$$\varepsilon_n = \max\{y(t); t \in [t_n - \omega, t_n]\}, \quad n = 1, 2, \ldots,$$

there is $s_n \in [t_n - \omega, t_n]$ such that $y(s_n) = \varepsilon_n$. We now claim $\lim_{n\to\infty} \varepsilon_n = y_e$.

If not, then there is a $\mu > 0$ and a subsequence, assumed to be $\{s_n\}_{n=1}^{\infty}$ itself without loss of generality, such that $y(s_n) > y_e + 3\mu$. Observing from (3.2.2) that $y'(t)$ is bounded, $y(t)$ is a Lipschitz function. Hence, there is a $\alpha > 0$, independent of n, such that $[t_n - \omega, t_n]$ contains a subinterval Δ_n whose length is greater than α and on which $y(t) > y_e + 2\mu$. Since $\lim_{n\to\infty} \delta_n = y_e$, there is an integer N such that $y(t) > \delta_n + \mu$ for $t \in \Delta_{n+1}$, and $n > N$. Thus we have

$$\begin{aligned} y'(t_{n+1}) &= \gamma y(t_{n+1})\{-1 + \sigma^* y^{p-1}(t_{n+1})[1 - y(t_{n+1}) - \gamma \int_{t_{n+1}-\omega}^{t_{n+1}} y(s)ds]\} \\ &< \gamma \delta_{n+1}[-1 + \sigma^* \delta_{n+1}^{p-1}(1 - \delta_{n+1} - r\delta_n - \gamma\alpha\mu)]. \end{aligned}$$

Because

$$\lim_{n\to\infty} [-1 + \sigma^* \delta_{n+1}^{p-1}(1 - \delta_{n+1} - r\delta_n - \gamma\alpha\mu)] = -\sigma^* \gamma\alpha\mu y_e^{p-1} < 0,$$

for sufficiently large n, $y'(t_n) < 0$, which contradicts that $t_{n+1} \neq (n+1)\omega$. Therefore, $\lim_{n\to\infty} \epsilon_n = y_e$. From $t_{n+1} \neq (n+1)\omega$, $n\omega \in [t_{n+1} - \omega, t_{n+1}]$. We in particular have $\lim_{n\to\infty} y(n\omega) = y_e$.

Third, we claim that for any $t_0 \geq 0, \lim_{n\to\infty} y(t_0 + n\omega) = y_e$.

Obviously, the function $\hat{y}(t) = y(t + t_0)$ is a solution of (3.2.2). Hence, the lemmas in the last section and the above statements remain true for $\hat{y}(t)$, especially, $\lim_{n\to\infty} \hat{y}(n\omega) = y_e$. This implies $\lim_{n\to\infty} y(t_0 + n\omega) = y_e$.

Finally, we prove that $\limsup_{t\to\infty} y(t) = y_e$.

If there is $\mu > 0$ such that $\limsup_{t\to\infty} y(t) > y_e + \mu$, then there is a sequence $\{s_n\}_{n=1}^{\infty}$ such that $s_n \to \infty$, as $n \to \infty$, and $y(s_n) > y_e + \mu$. Suppose $s_n = k_n\omega + s'_n$, where $k_n \geq 0$ is an integer and $s'_n \in [0, \omega)$. Then there is $t_0 \in [0, \omega]$ and a subsequence of $\{s'_n\}$, denoted by $\{s'_n\}$ again w.l.o.g., such that $\lim_{n\to\infty} s'_n = t_0$. Hence,

$$\lim_{n\to\infty} |t_0 + k_n\omega - s_n| = 0.$$

Recalling that $y(t)$ is a Lipschitz function,

$$\lim_{n\to\infty} y(t_0 + k_n\omega) = \lim_{n\to\infty} y(s_n) \geq y_e + \mu > y_e.$$

This is impossible because $\lim_{n\to\infty} y(t_0 + n\omega) = y_e$. Therefore, $\limsup_{t\to\infty} y(t) = y_e$.

With $\liminf_{t\to\infty} y(t) = y_e$, we have $\lim_{t\to\infty} y(t) = y_e$. Our proof is completed. □

It is now natural to ask how to locate the attractive regions of both equilibria. In general, it is difficult to do so partially because they depend on $z_0(t)$, the fraction of the initial population which is initially recovered and still stays in the recovered class at time t. But the next theorem give us some information about the attractive region of the disease-free equilibrium.

It is easy to see the equation $y^{p-1} - y^p = \frac{1}{\sigma^*}$ has exact two positive roots. Let y_1 be the smaller root. Then we have

THEOREM 3.4.2. *Let* $y(t)$ *be a nonnegative solution of (3.2.1)-(3.2.2). If there is* t_0 *such that* $y(t_0) \in [0, y_1]$, *then* $y(t) \in [0, y_1]$ *for* $t \geq t_0$ *and* $\lim_{t \to \infty} y(t) = 0$. *In particular, if* $y(0) \in [0, y_1], \lim_{t \to \infty} y(t) = 0$. *Therefore,* $[0, y_1]$ *is a positive invariant, attractive region for the disease-free equilibrium.*

PROOF. Since $y_1 < y_e$, the disease-free equilibrium is the only equilibrium lying in $[0, y_1]$. Hence, it suffices to prove that if $y(t_0) \in [0, y_1]$, then $y(t)$ is a decreasing function for $t \geq t_0$. This is true because for any $y(t) \in [0, y_1]$,

$$\begin{aligned} y'(t) &= -\gamma y(t) + \beta y^p(t)[1 - y(t) - \gamma \int_{t-\omega}^{t} y(s)ds] \\ &\leq \gamma y(t)[-1 + \sigma^* y^{p-1}(t)(1 - y(t))] \leq 0. \end{aligned}$$

The proof is completed. □

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CHAPTER 4

GLOBAL STABILITY OF THE ENDEMIC EQUILIBRIUM IN AN EPIDEMIC MODEL WITH SUBPOPULATIONS

4.1 Introduction.

The following system of $3n$ autonomous ordinary differential equations has been widely used in the study of the spread of infectious diseases (see [5], [13], [10], [6], [12], and [7]). It includes the general SI, SIS, SIR and SIRS models used in mathematical epidemiology and it takes the form

$$\begin{aligned} x_i' &= b_i(N_i - x_i) - x_i \sum_j \lambda_{ij} y_j + \kappa_i z_i \\ y_i' &= -(\gamma_i + b_i) y_i + x_i \sum_j \lambda_{ij} y_j \quad (i = 1, \cdots, n) \\ z_i' &= -(b_i + \kappa_i) z_i + \gamma_i y_i \quad , \end{aligned} \tag{4.1.1}$$

where $x_i(0), y_i(0)$ and $z_i(0) \geq 0$. Here x_i (resp. y_i ; resp. z_i) denotes the number of susceptible (resp. infected; resp. recovered) individuals in the i -th subpopulation. N_i (resp. b_i ; resp, γ_i ; resp. κ_i) is the total size (resp. birth and death rate; resp. recovery rate; resp. rate at which recovered individuals loses immunity) for the i -th subpopulation. λ_{ij} is the effective contact rate between individuals in the i -th subpopulation with individuals in the j -th subpopulation. All the parameters $N_i, b_i, \gamma_i, \kappa_i, \lambda_{ij}$ are assumed to be non-negative.

It was pointed out in [9] that a major unsolved problem in mathematical epidemiology is to determine if the endemic equilibrium of (4.1.1), i.e. an equilibrium of the form (x^*, y^*, z^*) where $x_i^*, y_i^*, z_i^* > 0$, is globally stable. In this paper we will make a contribution to this global stability question by showing if $\frac{\gamma_i}{b_i+\kappa_i} < 1$ for all i , then the endemic equilibrium (if it exists) is globally stable.

The rest of the chapter is organized as follows. In Section 4.2, we set up the necessary notations and state some known results concerning the behaviour of solutions of (4.1.1). We also include new proofs for the existence, uniqueness and local asymptotic stability of the endemic equilibrium. The question of uniform persistence (as introduced in the mathematical population biology literature) will be considered in Section 4.3. It is shown that (4.1.1) is uniformly persistent if and only if the endemic equilibrium exists. In Sections 4.4, the global stability question will be considered. The endemic equilibrium is shown to be globally stable under the assumption of small recovery rates for each subpopulation.

4.2 Some Known Results with New Proofs.

In this section, we will recall some known results concerning (4.1.1) which were proved in [6] and [7]. We will also present new proofs for the existence, uniqueness and local asymptotic stability of the endemic equilibrium.

From now on, we will always make the following assumptions on the parameters:

(H1) $N_i > 0$ for all i ,

(H2) $\gamma_i > 0$ for all i ,

(H3) $b_i + \kappa_i > 0$ for all i , and

(H4) $\lambda_{ij} \geq 0$ for all i,j and $\lambda_{ij} = 0$ if and only if $\lambda_{ji} = 0$.

Also, we will only be interested in (4.1.1) on the positive cone $\mathbb{R}_+^{3n}$. It is easily seen that $\mathbb{R}_+^{3n}$ is positively invariant under (4.1.1).

Let $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, x_n)$, $z = (z_1, \cdots, z_n)$ and $N = (N_1, \cdots, N_n)$. If we set $w = x + y + z$, then by (4.1.1) $w_i' = b_i(N_i - w_i)$ so that $w_i(t) \to N_i$ as $t \to \infty$. Since the set

$$S = \{(x, y, z) \in \mathbb{R}_+^{3n} : x + y + z = N\}$$

is positively invariant under (4.1.1), we can reduce (4.1.1) to a system of $2n$ equations

$$\begin{aligned} y_i' &= -(b_i + \gamma_i)y_i + (N_i - y_i - z_i)\sum_j \lambda_{ij} y_j \\ z_i' &= -(b_i + \kappa_i)z_i + \gamma_i y_i \quad . \end{aligned} \tag{4.2.1}$$

Due to the reduction from (4.1.1) to (4.2.1), we will only be interested in solutions of (4.2.1) lying in

$$B = \{(y, z) \in \mathbb{R}_+^{2n} : y + z \leq N\} \quad .$$

Clearly, the origin $E_0 = (0, 0) \in \mathbb{R}^{2n}$ is an equilibrium for (4.2.1) and it is called the disease-free equilibrium. Let

$$\Lambda = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{pmatrix} \tag{4.2.2}$$

$$\tilde{\Lambda} = \begin{pmatrix} N_1\lambda_{11} & \dots & N_1\lambda_{1n} \\ \vdots & \ddots & \vdots \\ N_n\lambda_{n1} & \dots & N_n\lambda_{nn} \end{pmatrix} \tag{4.2.3}$$

and

$$A = \begin{pmatrix} -(b_1+\gamma_1) & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & -(b_n+\gamma_n) \end{pmatrix} + \tilde{\Lambda} \quad . \tag{4.2.4}$$

It follows from Perron-Frobenius theory that the eigenvalue, $s(A)$, of A with the largest real part is a real number.

THEOREM 4.2.1.[6] *The set* B *is positively invariant under (4.2.1). If* $s(A) \leq 0$ *, the no-disease equilibrium* E_0 *is globally asymptotically stable on* B *. If* $s(A) > 0$ *,* E_0 *is unstable on* B *.*

THEOREM 4.2.2.[7] *Assume* Λ *is irreducible and* $s(A) > 0$ *. Then there exists a unique equilibrium* $E^* = (y^*, z^*)$ *, called the endemic equilibrium, in the interior,* $\dot{B}$ *, of* B *. Furthermore,* E^* *(if it exists) is locally asymptotically stable.*

We will give a new proof for Theorem 4.2.2 by means of the following theorem.

THEOREM 4.2.3. *Given a system of ordinary differential equations*

$$u_i' = F_i(u_1, \cdots, u_n) \quad (i = 1, \cdots, n) \tag{4.2.5}$$

where $F = (F_1, \cdots, F_n)$ *is* C^1 *and* $u = (u_1, \cdots, u_n) \in \dot{\mathbb{R}}^n_+$ *. Assume*

(i) $\frac{\partial F_i}{\partial u_j} \geq 0 \quad (i \neq j)$,

(ii) $u \geq v \geq 0$ *implies* $DF(v) \geq DF(u)$ *, where* $DF(u)$ *denotes the derivative (Jacobian) of* F *at* u *, and*

(iii) given any $\epsilon > 0$ *, there exists a vector* $v \in \mathbf{R}^n$ *such that* $0 < v_i < \epsilon$ *for all* i *and* $F(v) > 0$ *.*

Then (4.2.5) has at most one positive equilibrium. If there is no positive equilibrium, every solution is unbounded. If there is a positive equilibrium, this equilibrium is globally asymptotically stable over $\dot{\mathbf{R}}^n_+$ *.*

REMARK. The statement above is a slight modification of Theorem 2.1 in [14]. It can be proved in a similar way.

PROOF OF THEOREM 4.2.2. To find an endemic equilibrium for (4.2.1) is equivalent to solving

$$-(b_i + \gamma_i)y_i + (N_i - y_i - z_i)\sum_j \lambda_{ij}y_j = 0 \tag{4.2.6}$$

and

$$-(b_i + \kappa_i)z_i + \gamma_i y_i = 0 \tag{4.2.7}$$

where $y_i, z_i > 0$ and $y_i + z_i < N_i$ for all i . From (4.2.7), $z_i = \frac{\gamma_i}{b_i+\kappa_i}y_i$. Substituting this into (4.2.6), we obtain

$$\frac{N_i \sum_k \lambda_{ik}y_k}{(b_i + \gamma_i) + (1 + \frac{\gamma_i}{b_i+\kappa_i})\sum_k \lambda_{ik}y_k} - y_i = 0 \quad . \tag{4.2.8}$$

Denote the left hand side of (4.2.8) by $F_i(y)$ and consider the system

$$y_i' = F_i(y_1, \cdots, y_n) \quad (i = 1, \cdots, n). \tag{4.2.9}$$

We now verify hypotheses (i), (ii) and (iii) in Theorem 4.2.3 for system (4.2.9). First of all,

$$\frac{\partial F_i}{\partial y_j} = \frac{(b_i+\gamma_i)N_i\lambda_{ij}}{[(b_i+\gamma_i)+(1+\frac{\gamma_i}{b_i+\kappa_i})\sum_k \lambda_{ik}y_k]^2} - \delta_{ij} \quad ,$$

where δ_{ij} is the Kronecker delta. Thus (i) is satisfied. Since $\frac{\partial F_i}{\partial y_j}$ is decreasing with respect to each of its variables $y_1, \cdots, y_n$, (ii) is also satisfied. As for (iii), since A is an irreducible matrix with non-negative off-diagonal entries, there exists a positive eigenvector $v = (v_1, \cdots, v_n) > 0$ corresponding to the eigenvalue $s(A)$, i.e.

$$-(b_i+\gamma_i)v_i + N_i\sum_j \lambda_{ij}v_j = s(A)v_i \quad \text{for all} \quad i = 1, \cdots, n.$$

Hence, for any number $p > 0$,

$$F_i(pv) = \frac{pv_i[s(A)N_i - pv_i(1+\frac{\gamma_i}{b_i+\kappa_i})(s(A)+b_i+\gamma_i)]}{(b_i+\gamma_i)N_i + pv_i(1+\frac{\gamma_i}{b_i+\kappa_i})(s(A)+b_i+\gamma_i)} \quad .$$

Now $s(A) > 0$ implies $F_i(pv) > 0$ for p sufficiently small. Thus (iii) is also satisfied. Finally, since $F_i(y_1, \cdots, y_n) \leq N_i - y_i$, (4.2.9) cannot have unbounded solutions. The existence and uniqueness of the endemic equilibrium, E^*, for (4.2.1) now follow immediately from Theorem 3.2.3. Of course, one needs to show $y_i^* + z_i^* < N_i$ for all i but that is clear from (4.2.6).

To demonstrate the local asymptotic stability of the endemic equilibrium, we first note that by (4.2.6)

$$-(b_i+\gamma_i)y_i^* + (N_i - y_i^* - z_i^*)\sum_j \lambda_{ij}y_j^* = 0 \quad (i = 1, \cdots, n). \qquad (4.2.10)$$

Let

$$M = \begin{pmatrix} -(b_1+\gamma_1) & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & -(b_n+\gamma_n) \end{pmatrix} + \begin{pmatrix} N_1 - y_1^* - z_1^* & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & N_1 - y_n^* - z_n^* \end{pmatrix} \Lambda \tag{4.2.11}$$

Then M is irreducible and has non-negative off-diagonal entries. Moreover, by (4.2.10), $My^* = 0$. Thus $s(M) \leq 0$ (c.f. Lemma 4.4.1). Consequently, there exists a diagonal matrix $C = \text{diag}\{c_1, \cdots, c_n\}$ with positive diagonal entries, i.e. $c_i > 0$ for all i , such that $s(CM + M^tC) \leq 0$ (c.f. Lemma 4.4.3). The Jacobian matrix of right hand side of (4.2.1) at E^* is given by

$$Q = \begin{pmatrix} M+J & J \\ \Gamma & K \end{pmatrix} ,$$

where $J =$ diag $\{-\sum_j \lambda_{1j} y_j^*, \cdots, -\sum_j \lambda_{nj} y_j^*\}$, $\Gamma =$ diag $\{\gamma_1, \cdots, \gamma_n\}$ and $K =$ diag $\{-(b_1+\kappa_1), \cdots, -(b_n+\kappa_n)\}$. Define $S = \text{diag}\{C, D\}$, where $D = \text{diag}\{d_1, \cdots, d_n\}$ and $d_{ii} = \frac{c_i}{\gamma_i} \sum_j \lambda_{ij} y_j^* > 0$. Then

$$SQ + Q^tS = \begin{pmatrix} CM + M^tC + 2CJ & 0 \\ 0 & 2DK \end{pmatrix} .$$

Since

$$CJ = \text{diag}\{-c_1 \sum_j \lambda_{1j} y_j^*, \cdots, -c_n \sum_j \lambda_{nj} y_j^*\} ,$$

$$DK = \text{diag}\{-\frac{c_1(b_1+\kappa_1)}{\gamma_1} \sum_j \lambda_{1j} y_j^*, \cdots, -\frac{c_n(b_n+\kappa_n)}{\gamma_n} \sum_j \lambda_{nj} y_j^*\} ,$$

and $s(CM + M^tC) \leq 0$, therefore $SQ + Q^tS$ is a stable matrix. A well known theorem of Liapunov (see [11]) shows that Q is a stable matrix. Hence the endemic equilibrium is locally asymptotically stable. □

Remark. In the case when Λ is reducible, system (4.2.1) decouples into two or more (smaller) irreducible subsystems, by (H4). We can then apply Theorem 4.2.2 to each of these irreducible subsystems. Thus, the endemic equilibrium, if it exists, must be unique and is locally asymptotically stable. Moreover, the endemic equilibrium exists if and only if $s(A_k) > 0$ for all k, where A_k is the A for the k-th irreducible subsystem.

Theorem 4.2.4. *If* Λ *is irreducible and* $s(A) > 0$ *, then there is a unique positive equilibrium* (x^*, y^*, z^*) *of (4.1.1) and it is locally asymptotically stable.*

Proof. The existence and uniqueness of the positive equilibrium is clear. In fact, $x_i^* = N_i - y_i^* - z_i^*$ for all i. To show its asymptotic stability, we use the 'equivalent' system

$$\begin{aligned} w_i' &= b_i(N_i - w_i) \\ y_i' &= -(\gamma_i + b_i)y_i + (w_i - y_i - z_i)\sum_j \lambda_{ij} y_j \quad (i = 1, \cdots, n) \qquad (4.2.12) \\ z_i' &= -(b_i + \kappa_i)z_i + \gamma_i y_i \quad . \end{aligned}$$

Clearly $(x, y, z) = (x^*, y^*, z^*)$ is an asymptotically stable equilibrium for (4.1.1) if and only if $(w, y, z) = (N, y^*, z^*)$ is an asymptotically stable equilibrium for (4.2.12). But the latter follows immediately by linearizing (4.2.12) about (N, y^*, z^*) and making use of Theorem 4.2.2. □

Remark. As was pointed out in the Remark following the proof of Theorem 4.2.2, one can show that the endemic equilibrium for (4.1.1), if it exists, is asymptotically stable, irrespect of whether Λ is irreducible or not.

4.3 Uniform Persistence.

In the last section it was shown that the endemic equilibrium is locally asymptotically stable when it exists. Hereafter, we will study the global asymptotic behavior of solutions of (4.2.1) in the positively invariant set B . In this section, we will show that if the endemic equilibrium exists, then the number of each group (susceptible, infected and removed) in each subpopulation will remain above a certain positive level. In other words, each group in each subpopulation persists. If, in addition, Λ is irreducible and the disease exists in any subpopulation, then it will spread immediately to all subpopulations.

Let

$$B_1 = \{(0,\cdots,0,z_1,\cdots,z_n) : 0 \le z_i \le N_i \quad (i = 1,\cdots,n)\} \quad .$$

Then B_1 is positively invariant under (4.2.1) and is negatively invariant relative to B . B_1 is referred to as the disease-free set.

Our first result says that if the disease exists in any one of the subpopulations, then it will spread immediately to all subpopulations and remains in every subpopulation from then on.

THEOREM 4.3.1. *Assume* Λ *is irreducible. Let* $(y(t), z(t))$ *be a so- of (4.2.1) in* B . *If* $(y(0), z(0)) \in B \setminus B_1$, *then* $(y(t), z(t)) \in \dot{B}$ *for all* $t > 0$, *where* $\dot{B}$ *denotes the interior of* B .

Before we can prove this theorem, we need the following lemmas.

Lemma 4.3.2. *If* $(y(0),z(0)) \in \dot{B}$ *, then* $(y(t),z(t)) \in \dot{B}$ *for all* $t \geq 0$.

Proof. Introduce the sets

$$D_1 = \{(y,z) \in B : 0 \leq z < N, y_i = 0 \quad \text{for some} \quad i\} \quad ,$$

$$D_2 = \{(y,z) \in B : y > 0, z_i = 0 \quad \text{for some} \quad i\} \quad \text{and}$$

$$D_3 = \{(y,z) \in B : y_i + z_i = N_i \quad \text{and} \quad z_i > 0 \quad \text{for some} \quad i\} \quad .$$

Then the boundary, ∂B , of B can be expressed as $\partial B = D_1 \cup D_2 \cup D_3$. We first show that the solution $(y(t),z(t))$ cannot reach the set D_2 at any finite (positive) time. Suppose not, i.e. suppose the solution reaches D_2 at time $t_0 > 0$. Then $y(t_0) > 0$ and $z_i(t_0) = 0$ for some i . Since $z_i(t) \geq 0$ for all $t \geq 0$, $z_i'(t_0) = 0$. On the other hand, $z_i'(t_0) = -(b_i + \kappa_i)z_i(t_0) + \gamma_i y_i(t_0) = \gamma_i y_i(t_0) > 0$, which is a contradiction. Similarly, if the solution meets D_3 at some finite time t_0 , then $y_i(t_0) + z_i(t_0) = N_i$, $z_i(t_0) > 0$ and $y_i'(t_0) + z_i'(t_0) = 0$ for some i . However, $y_i'(t_0) + z_i'(t_0) = -b_i y_i(t_0) - (b_i + \kappa_i)z_i(t_0) = -b_i N_i - \kappa_i z_i(t_0) < 0$, which is also a contradiction. Finally, we show that the solution $(y(t),z(t))$ cannot reach the set D_1 at a finite time. Again suppose not. By permuting the indices $1, \cdots, n$ if necessary, we may assume $y_1(t_0) = \cdots = y_s(t_0) = 0$ and $y_i(t_0) > 0$ for $i = s+1, \cdots, n$, for some $t_0 > 0$, where $1 \leq s \leq n$. If $s = n$, then $y_1(t_0) = \cdots = y_n(t_0) = 0$, i.e. $(y(t_0), z(t_0)) \in B_1$. Since B_1 is negatively invariant relative to B , $(y(0),z(0)) \in B_1$ which is a contradiction. Thus

$s < n$. As before, $y_1'(t_0) = \cdots = y_s'(t_0) = 0$, i.e.

$$-(b_i + \gamma_i)y_i(t_0) + (N_i - y_i(t_0) - z_i(t_0))\sum_j \lambda_{ij}y_j(t_0) = 0 \quad \text{for all} \quad i = 1, \cdots, s.$$

Since for $i = 1, \cdots, s$, $y_i(t_0) = 0$ and $z_i(t_0) < N_i$. Therefore, $\sum_j \lambda_{ij}y_j(t_0) = 0$ for all $i = 1, \cdots, s$. Now $y_j(t_0) > 0$ for all $j = s+1, \cdots, n$, implies that $\lambda_{ij} = 0$ for all $i = 1, \cdots, s$ and $j = s+1, \cdots, n$. This contradicts the assumption Λ is irreducible. Hence the solution $(y(t), z(t))$ cannot reach ∂B at a finite time and thus it must lie in $\dot{B}$. □

LEMMA 4.3.3. *If* $(y(0), z(0)) \in \partial B \setminus B_1$ *, then there exists* $\delta > 0$ *such that*

$$(y(t), z(t)) \in \dot{B} \quad \text{for all} \quad 0 < t < \delta \ . \tag{4.3.1}$$

PROOF. Clearly, $\partial B \setminus B_1 = (D_1 \setminus B_1) \cup D_2 \cup (D_3 \setminus B_1)$. Let

$$D_4 = \{(y, z) \in B : y_i = 0 \quad \text{and} \quad z_i = N_i \quad \text{for some} \quad i\} \ .$$

Since $D_4 \subset D_3$ and $B_1 \cap (D_3 \setminus D_4) = \phi$, we can divide into three cases.

Case 1. $(y(0), z(0)) \in D_2 \cup (D_3 \setminus D_4)$. (4.3.1) follows easily because $y(0) > 0$, therefore $z_i'(0) > 0$ whenever $z_i(0) = 0$ and $y_i'(0) + z_i'(0) < 0$ whenever $y_i(0) + z_i(0) = N_i$ and $z_i(0) > 0$.

Case 2. $(y(0), z(0)) \in D_1 \setminus B_1$. By reordering the indices $1, \cdots, n$ if necessary, we can assume $y_1(0) = \cdots = y_s(0) = 0$ for some $1 \leq s < n$ and $y_i(0) > 0$ for all $i = s+1, \cdots, n$. Then there exists a $\delta > 0$ such that $y_i(t) > 0$, for all $0 < t < \delta$ and $i = s+1, \cdots, n$. Claim: $y_i(t) > 0$ for all $0 < t < \delta$ and $i = 1, \cdots, s$. Suppose not, then there is $0 < t' < \delta$ such that $y_i(t') = 0$ for some $1 \leq i \leq s$. By reordering the indices $1, \cdots, s$ if necessary, we can assume $y_1(t') = \cdots = y_v(t') = 0$ for some $1 \leq v \leq s$ and $y_i(t') > 0$ for all $i = v+1, \cdots, s$. Using the same argument as in Lemma 4.3.2, we obtain $y_1'(t') = \cdots = y_v'(t') = 0$. This implies $\lambda_{ij} = 0$ for all $i = 1, \cdots, v$ and $j = v+1, \cdots, n$, which contradicts the irreducibility of Λ .

Case 3. $(y(0), z(0)) \in D_4 \setminus B_1$. By reordering the indices $1, \cdots, n$ if necessary, we can assume $y_1(0) = \cdots = y_s(0) = 0$, $z_1(0) = N_i, \cdots z_s(0) = N_s$, and $z_i(0) < N_i$ for all $i = s+1, \cdots, n$. Since $z_i'(0) = -(b_i + \kappa_i)N_i < 0$ for all $i = 1, \cdots, s$, there is a $\delta > 0$ such that $z_i(t) < N_i$ for all $0 < t < \delta$ and for all $i = 1, \cdots, n$. If (3.3.1) does not hold, there exists $0 < t' < \delta$ such that $y_i(t') = 0$ for some $1 \leq i \leq s$. By reordering the indices $1, \cdots, s$ if necessary, we can assume $y_1(t') = \cdots = y_v(t') = 0$ for some $1 \leq v \leq s$. Repeating the argument as in case 2, we obtain a contradiction. $\square$

Proof of Theorem 4.3.1. Let $(y(0), z(0)) \in B \setminus B_1$. If $(y(0), z(0)) \in \dot{B}$, then $(y(t), z(t)) \in \dot{B}$ for $t \geq 0$ by Lemma 4.3.2. If $(y(0), z(0)) \in$

$\partial B \setminus B_1$, by Lemma 4.3.3 there exists $\delta > 0$ such that $(y(t), z(t)) \in \dot{B}$ for all $0 < t < \delta$. Hence $(y(t), z(t)) \in \dot{B}$ for all $t > 0$. □

It is known that if $s(A) \leq 0$ then the no-disease equilibrium E_0 is the unique equilibrium and it is globally asymptotically stable on B . When $s(A) > 0$, we have the following result.

THEOREM 4.3.4. *If* Λ *is irreducible and* $s(A) > 0$ *, then (4.2.1) is uniformly persistent in* B *with respect to* ∂B *. That is, there is a* $\eta > 0$ *such that* $\liminf_{t\to\infty} y_i(t) \geq \eta$, $\liminf_{t\to\infty} z_i(t) \geq \eta$ *, and* $\limsup_{t\to\infty} y_i(t) + z_i(t) \leq N_i - \eta$ *, for all solution* $(y(t), z(t))$ *with initial condition in* $B \setminus B_1$.

The biological interpretation of Theorem 4.3.4 is that if the threshold, $s(A)$, exceeds zero, the disease will not only exist in every subpopulation but in fact the number of individuals in each group (susceptible, infectious and removed) will always remain beyond a certain positive level η . The proof of Theorem 4.3.4 depends on a theorem in [8] which we will state below for the sake of easy reference. (See also [1] and [4]).

Let $\mathcal{X}$ be a metric space with metric d , $f : \mathcal{X} \to \mathcal{X}$ be continuous and $\mathcal{Y} \subset \mathcal{X}$ be closed with $f(\mathcal{X} \setminus \mathcal{Y}) \subset \mathcal{X} \setminus \mathcal{Y}$. Suppose $\mathcal{X}$ has a compact global attractor X and let M be the maximal compact invariant set in $\mathcal{Y}$. Then we have

Theorem 4.3.5.[8] *f is uniformly persistent with respect to $\mathcal{Y}$ if and only if*

1. *M is isolated in X, and*
2. *$W^s(M) \subset \mathcal{Y}$, where $W^s(M)$ denotes the stable set of M.*

Proof of Theorem 4.3.4. Let $\mathcal{X} = B$, $\mathcal{Y} = \partial B$ and f be the time one map of the flow defined by (4.2.1). It follows from Theorem 4.3.1 that $f(\mathcal{X} \setminus \mathcal{Y}) \subset \mathcal{X} \setminus \mathcal{Y}$. Clearly $X = \omega(B)$, where $\omega(B)$ is the ω-limit set of B is a global attractor of $\mathcal{X}$. Let M be the maximal compact invariant set in $\mathcal{Y}$.

Claim: $M = \{E_0\}$.

Suppose not, then there exists $(y^0, z^0) \in M$ and either (i) $(y^0, z^0) \in B \setminus B_1$ or (ii) $(y^0, z^0) \in B_1 \setminus \{E_0\}$. Let $(y(t), z(t))$ be the solution with initial condition (y^0, z^0). If (i) holds then $(y(t), z(t) \in \dot{B}$, by Theorem 4.3.1, contradicting $M \subset \partial B$. On the other hand, if (ii) holds then the solution must take the form

$$(y(t), z(t)) = (0, \cdots, 0, c_1 e^{-(b_1+\kappa_1)t}, \cdots, c_1 e^{-(b_n+\kappa_n)t})$$

where $c_1, \cdots, c_n$ are not all zero. Clearly for t sufficiently negative, $(y(t), z(t)) \notin B$, which contradicts the invariance of M.

In order to show uniform persistence, it suffices to verify conditions 1 and 2 in Theorem 4.3.5. We will do this by constructing a suitable Lyapunov function. Let $V(y) = v^t y$ where $v = (v_1, \cdots, v_n)$ is a positive eigenvector of A^t corresponding to the eigenvalue $s(A)$. Then there exists $a > 0$ such that $V(y) \geq a\|y\|$ for all $y \geq 0$, where $\|y\| = \max_i\{|y_i|\}$. Since $s(A) > 0$ and the derivative of V along solutions is

$$V' = s(A)v^t y - \sum_i v_i(y_i + z_i)\sum_j \lambda_{ij} y_j \quad ,$$

$V' > 0$ in a neighbourhood N of E_0 relative to $B \setminus B_1$. It follows that any solution in N must leave N at a finite time. Consequently, M is isolated and the stable set of M , $W^s(M)$, is equal to B_1 . □

Remark. In the case when Λ is reducible, by using Theorem 4.3.4 and follow the same line of reasoning as in the Remark following Theorem 4.2.2, one can easily show that if the endemic equilibrium exists, then (4.2.1) is uniformly persistent with respect to ∂B .

Theorem 4.3.6. If the endemic equilibrium (x^*, y^*, z^*) exists, then (4.1.1) is uniformly persistent, i.e. there exists $\eta > 0$ such that for all i we have, $\liminf_{t\to\infty} x_i(t) \geq \eta$, $\liminf_{t\to\infty} y_i(t) \geq \eta$, and $\liminf_{t\to\infty} z_i(t) \geq \eta$, for all solution $(x(t), y(t), z(t))$ of (1.1) with $(x(0), y(0), z(0)) \in R^{3n}_+ \setminus S_0$, where

$$S_0 = \{(x, y, z) \in R^{3n}_+ : y = 0\} \quad .$$

4.4 Global Stability of the Endemic Equilibrium.

For the sake of simplicity of notation, we will make the following assumption for this section.

Standing Assumption. $N_i = 1$ for all $i = 1, \cdots, n$.

This assumption can always be achieved by a suitable rescaling of the variables x_i, y_i, z_i $(i = 1, \cdots, n)$, namely, $x_i \mapsto \frac{x_i}{N_i}$, $y_i \mapsto \frac{y_i}{N_i}$ and $z_i \mapsto \frac{z_i}{N_i}$. Note that b_i, γ_i and κ_i are not changed by this rescaling.

We will show that the endemic equilibrium is globally stable under the assumption

$$\frac{\gamma_i}{b_i + \kappa_i} < 1 \quad (i = 1, \cdots, n) \quad . \tag{4.4.1}$$

Before we can prove that, we need the following lemmas.

Let $M = (m_{ij})$ be a n by n (real) matrix with non-negative off-diagonal entries.

LEMMA 4.4.1. (c.f. [2]) *Assume* M *is irreducible. Then* $s(M) \leq 0$ *if and only if there exists a vector* $v \in \mathbb{R}^n$ *with* $v > 0$ *such that* $Mv \leq 0$.

LEMMA 4.4.2. *Assume* M *is irreducible and* $s(M) \leq 0$. *Then there exists a diagonal matrix* C *with positive diagonal entries such that* $s(CM + M^t C) \leq 0$.

Proof. Since M has non-negative off-diagonal entries and is irreducible, so is M^t. By Lemma 1, there exist two vectors $v = (v_1, \cdots, v_n)^t > 0$ and $u = (u_1, \cdots, u_n)^t > 0$ such that $Mv \le 0$ and $M^t u \le 0$. Let $C = \mathrm{diag}\{\frac{u_1}{v_1}, \cdots, \frac{u_n}{v_n}\}$. Then $(CM + M^t C)v = CMv + M^t Cv = CMv + M^t u \le 0$. By Lemma 1 again, $s(CM + M^t C) \le 0$. □

Lemma 4.4.3. (c.f. [2]) *Assume* $s(M) < 0$. *Then there exists a diagonal matrix* $D = \mathrm{diag}\{d_1, \cdots, d_n\}$ *with* $d_i > 0$ *for all* $i = 1, \cdots, n$ *such that* MD *is diagonal dominant, that is,* $\sum_j m_{ij} d_j < 0$.

Lemma 4.4.4. *Assume* $s(M) \le 0$. *Then, for all* $\epsilon > 0$, *there is a diagonal matrix* $D = \mathrm{diag}\{d_1, \cdots, d_n\}$ *with* $d_i > 0$ *for all* $i = 1, \cdots, n$ *such that* $\frac{1}{d_i} \sum_j m_{ij} d_j < \epsilon$.

Proof. Consider the matrix $\tilde{M} = M - \epsilon I$. Since $\tilde{M}$ has non-negative off-diagonal entries and $s(\tilde{M}) \le \epsilon < 0$, we can apply Lemma 5.3 to $\tilde{M}$ and obtain a diagonal matrix $D = \mathrm{diag}\{d_1, \cdots, d_n\}$ with positive diagonal entries such that $\sum_j m_{ij} d_j - \epsilon d_i < 0$. Thus, $\frac{1}{d_i} \sum_j m_{ij} d_j - \epsilon < 0$, or $\frac{1}{d_i} \sum_j m_{ij} d_j < \epsilon$. □

Lemma 4.4.5. *If* $\Lambda = (\lambda_{ij})$ *is irreducible and* $s(A) > 0$, *then* $\liminf_{t \to \infty} \sum_j \lambda_{ij} y_j(t) > 0$ *for all* $i = 1, \cdots, n$, *where* $(y(t), z(t))$ *is a solution of (4.2.1) with* $(y(0), z(0)) \in B \setminus B_1$.

Proof. By Theorems 4.3.1 and 4.3.4, $\liminf_{t\to\infty} y_j(t) > 0$ for all $j = 1, \cdots, n$. Since Λ is irreducible, for each i there is a j_i such that $\lambda_{ij_i} > 0$. Hence, $\liminf_{t\to\infty} \sum_j \lambda_{ij} y_j(t) \geq \liminf_{t\to\infty} \lambda_{ij_i} y_{j_i}(t) > 0$. □

Lemma 4.4.6. *Let* $y(t)$ *be a real-valued continuous function defined on* $[0, \infty)$. *Define* $z(t) = ae^{-\theta t} + b\int_0^t e^{-\theta(t-s)} y(s)ds$, *where* $\theta > 0$, a *and* b *are constants. If* $\lim_{t\to\infty} y(t) = 0$, *then* $\lim_{t\to\infty} z(t) = 0$.

Proof. It suffices to show $\lim_{t\to\infty} \int_0^t e^{-\theta(t-s)} y(s)ds = 0$. Given any $\epsilon > 0$, there exists $T_1 > 0$ such that $|y(s)| < \frac{\theta\epsilon}{2}$, whenever $s \geq T_1$. Since $y(t)$ is bounded on $\mathbf{R}_+$, $M = \sup_{t\in\mathbf{R}_+} |y(t)| < \infty$. Choose any $T_2 > \frac{2}{\theta} \ln \frac{2M}{\epsilon\theta}$. For $t \geq T_2$, we have

$$|\int_0^{\frac{t}{2}} e^{-\theta(t-s)} y(s)ds| \leq \frac{M}{\theta} e^{-\frac{\theta}{2}t} < \frac{\epsilon}{2} \quad .$$

Thus, for $t > 2\max(T_1, T_2)$

$$|\int_0^t e^{\theta(t-s)} y(s)ds| \leq |\int_0^{\frac{t}{2}} e^{-\theta(t-s)} y(s)ds| + |\int_{\frac{t}{2}}^t e^{-\theta(t-s)} y(s)ds| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad .$$

□

We are now ready to state and prove the main result of this section.

Theorem 4.4.7. *Assume* Λ *is irreducible and* $s(A) > 0$. *If (4.4.1) holds, then for any solution* $(y(t), z(t))$ *with* $(y(0), z(0)) \in B \setminus B_1$, *we have,* $\lim_{t\to\infty}(y(t), z(t)) = E^*$. *Hence, the endemic equilibrium* E^* *is globally stable over* $B \setminus B_1$.

PROOF. Integrating $z_i' = -(b_i + \kappa_i)z_i + \gamma_i y_i$, we have

$$z_i(t) = z_i(0)e^{-(b_i+\kappa_i)t} + \gamma_i \int_0^t e^{-(b_i+\kappa_i)(t-s)} y_i(s)ds \tag{4.4.2}$$

and hence by (4.2.1)

$$\begin{aligned} y_i' &= -(b_i + \gamma_i)y_i \\ &\quad + (1 - y_i - \gamma_i \int_0^t e^{-(b_i+\kappa_i)(t-s)} y_i(s)ds) \sum_j \lambda_{ij} y_j - z_i(0)e^{-(b_i+\kappa_i)t} \sum_j \lambda_{ij} y_j \quad . \end{aligned}$$

Let $\hat{y}_i = y_i - y_i^*$. Then

$$\begin{aligned} \hat{y}_i' &= -(b_i + \gamma_i)\hat{y}_i + (1 - y_i^* - z_i^*) \sum_j \lambda_{ij} \hat{y}_j \\ &\quad - (\hat{y}_i + \gamma_i \int_0^t e^{-(b_i+\kappa_i)(t-s)} \hat{y}_i(s)ds) \sum_j \lambda_{ij} y_j \\ &\quad - (-\frac{\gamma_i}{b_i + k_i} y_i^* + z_i(0))e^{-(b_i+k_i)t} \sum_j \lambda_{ij} y_j \quad . \end{aligned} \tag{4.4.3}$$

By (4.4.1) and Lemma 4.4.5, there is $d > 0$ and $T_1 > 0$ such that

$$(1 - \frac{\gamma_i}{b_i + \kappa_i}) \sum_j \lambda_{ij} y_j(t) \geq d \tag{4.4.4}$$

for all $t \geq T_1$ and for all i . Since $My^* = 0$ and $y^* > 0$, where M was defined in (4.2.11), therefore $s(M) = 0$, by Lemma 4.4.1. Choosing $\epsilon = \frac{d}{5}$ in Lemma 3.4.4, we obtain a diagonal matrix $D = \operatorname{diag}\{d_1, \cdots, d_n\}$ with $d_i > 0$ for all i such that

$$-(b_i + \gamma_i) + \frac{1 - y_i^* - z_i^*}{d_i} \sum_j \lambda_{ij} d_j < \frac{d}{5} \quad \text{for all} \quad i = 1, \cdots, n. \tag{4.4.5}$$

Set $y_i = d_i u_i$, $y_i^* = d_i u_i^*$ and $\hat{u}_i = u_i - u_i^*$ for all i . Then (4.4.3) becomes

$$\hat{u}_i' = -(b_i + \gamma_i)\hat{u}_i + \frac{1 - y_i^* - z_i^*}{d_i} \sum_j \lambda_{ij} d_j \hat{u}_j$$
$$- (\hat{u}_i + \gamma_i \int_0^t e^{-(b_i+\kappa_i)(t-s)} \hat{u}_i(s) ds) \sum_j \lambda_{ij} y_j \tag{4.4.6}$$
$$- \frac{1}{d_i}(-\frac{\gamma_i}{b_i + \kappa_i} y_i^* + z_i(0)) e^{-(b_i+\kappa_i)t} \sum_j \lambda_{ij} y_j \quad .$$

In order to show global stability of E^* , we need to show $\limsup_{t\to\infty} |\hat{u}_i(t)| = 0$ for all i . Suppose not, i.e. suppose $\max_i\{\limsup_{t\to\infty} |\hat{u}_i(t)|\} = p > 0$. Then $\limsup_{t\to\infty} |\hat{u}_{i_0}(t)| = p$ for some i_0 . Choose any $0 < \epsilon' < 1$ such that

$$\frac{2\epsilon'}{d_{i_0}} \sum_j \lambda_{i_0 j} d_j \le \frac{1}{5} d(1 - \epsilon') \quad . \tag{4.4.7}$$

Let $T_2 > 0$ be sufficiently large such that whenever $t \ge T_2$, we have

$$|\hat{u}_i(t)| \le p(1 + \epsilon') \quad \text{for all } i, \tag{4.4.8}$$

$$\frac{\gamma_{i_0}}{d_{i_0}(b_{i_0} + \kappa_{i_0})} e^{-\frac{b_{i_0}+\kappa_{i_0}}{2}t} \le \frac{1}{5}(1 - \frac{\gamma_{i_0}}{b_{i_0} + \kappa_{i_0}}) p(1 - \epsilon') \tag{4.4.9}$$

and

$$\frac{1}{d_{i_0}}(\frac{\gamma_{i_0}}{b_{i_0} + \kappa_{i_0}} y_{i_0}^* + z_{i_0}(0)) e^{-(b_{i_0}+\kappa_{i_0})t} \sum_j \lambda_{i_0 j} \le \frac{1}{5} dp(1 - \epsilon') \quad . \tag{4.4.10}$$

Let $T = 2\max(T_1, T_2)$. Given any $t \ge T$, we can choose t' sufficiently large so that $\max_{s\in[\frac{t}{2},t']} |\hat{u}_{i_0}(s)| \ge p(1 - \epsilon')$. Let $t_0 \in [\frac{t}{2}, t']$ be a maximal point of $|\hat{u}_{i_0}(s)|$, i.e. $|\hat{u}_{i_0}(t_0)| = \max_{s\in[\frac{t}{2},t']} |\hat{u}_{i_0}(s)|$. Then

$$|\hat{u}_{i_0}(t_0)| \ge p(1 - \epsilon') \tag{4.4.11}$$

and

$$
\begin{aligned}
&|\hat{u}_{i_0}(t_0)| - |\gamma_{i_0} \int_0^{t_0} e^{-(b_{i_0}+\kappa_{i_0})(t_0-s)} \hat{u}_{i_0}(s) ds| \\
&\geq |\hat{u}_{i_0}(t_0)| - \gamma_{i_0} \int_{\frac{t}{2}}^{t_0} e^{-(b_{i_0}+\kappa_{i_0})(t_0-s)} |\hat{u}_{i_0}(s)| ds - \frac{\gamma_{i_0}}{d_{i_0}} \int_0^{\frac{t}{2}} e^{-(b_{i_0}+\kappa_{i_0})(t_0-s)} ds \\
&\geq |\hat{u}_{i_0}(t_0)| - \frac{\gamma_{i_0}}{b_{i_0}+\kappa_{i_0}} |\hat{u}_{i_0}(t_0)| - \frac{\gamma_{i_0}}{d_{i_0}(b_{i_0}+\kappa_{i_0})} e^{-\frac{b_{i_0}+\kappa_{i_0}}{2} t} \\
&\geq (1 - \frac{\gamma_{i_0}}{b_{i_0}+\kappa_{i_0}}) |\hat{u}_{i_0}(t_0)| - \frac{1}{5}(1 - \frac{\gamma_{i_0}}{b_{i_0}+\kappa_{i_0}}) p(1-\epsilon')
\end{aligned}
$$

by (4.4.11). Thus,

$$
\begin{aligned}
&|\hat{u}_{i_0}(t_0)| - |\gamma_{i_0} \int_0^{t_0} e^{-(b_{i_0}+\kappa_{i_0})(t_0-s)} \hat{u}_{i_0}(s) ds| \\
&\geq \frac{4}{5}(1 - \frac{\gamma_{i_0}}{b_{i_0}+\kappa_{i_0}}) |\hat{u}_{i_0}(t_0)| \quad .
\end{aligned}
\tag{4.4.12}
$$

Therefore, $(\hat{u}_{i_0}(t_0) \quad +\gamma_{i_0} \int_0^{t_0} e^{-(b_{i_0}+\kappa_{i_0})(t_0-s)} \hat{u}_{i_0}(s))ds) \quad \sum_j \lambda_{i_0 j} y_j(t_0)$ has the same sign as $\hat{u}_{i_0}(t_0)$ and its absolute value is, by (4.4.4), greater than or equal to $\frac{4}{5} d |\hat{u}_{i_0}(t_0)|$.

Consider the i_0 -th equation of (4.4.6):

$$
\begin{aligned}
\hat{u}'_i(t_0) = &-(b_i+\gamma_i)\hat{u}_i(t_0) + \frac{1-y_i^*-z_i^*}{d_i} \sum_j \lambda_{ij} d_j \hat{u}_j(t_0) \\
&- (\hat{u}_i(t_0) + \gamma_i \int_0^t e^{-(b_i+\kappa_i)(t-s)} \hat{u}_i(s) ds) \sum_j \lambda_{ij} y_j(t_0) \\
&- \frac{1}{d_i}(-\frac{\gamma_i}{b_i+\kappa_i} y_i^* + z_i(0)) e^{-(b_i+\kappa_i)t} \sum_j \lambda_{ij} y_j(t_0) \quad .
\end{aligned}
\tag{4.4.13}
$$

If $\hat{u}_{i_0}(t_0) > 0$, then

$$\begin{aligned}
\hat{u}'_{i_0}(t_0) &\le [-(b_{i_0}+\gamma_{i_0}) + (1-y^*_{i_0}-z^*_{i_0})\lambda_{i_0i_0}]\hat{u}_{i_0}(t_0) \\
&+ \frac{1-y^*_{i_0}-z^*_{i_0}}{d_{i_0}} \sum_{j\ne i_0} \lambda_{i_0j}d_j|\hat{u}_j(t_0)| - \frac{4}{5}d\hat{u}_{i_0}(t_0) \\
&- \frac{1}{d_{i_0}}(-\frac{\gamma_{i_0}}{b_{i_0}+\kappa_{i_0}}y^*_{i_0} + z_{i_0}(0))e^{-(b_{i_0}+\kappa_{i_0})t_0} \sum_j \lambda_{i_0j} \\
&\le [-(b_{i_0}+\gamma_{i_0}) + \frac{1-y^*_{i_0}-z^*_{i_0}}{d_{i_0}} \sum_j \lambda_{i_0j}d_j]\hat{u}_{i_0}(t_0) \\
&+ \frac{2p(1-y^*_{i_0}-z^*_{i_0})}{d_{i_0}} \sum_{j\ne i_0} \lambda_{i_0j}d_j\epsilon' \\
&- \frac{4}{5}d\hat{u}_{i_0}(t_0) + \frac{1}{5}dp(1-\epsilon') \quad .
\end{aligned}$$

By (4.4.7),

$$\begin{aligned}
\hat{u}'_{i_0}(t_0) &\le \frac{d}{5}\hat{u}_{i_0}(t_0) + \frac{2}{5}dp(1-\epsilon') - \frac{4}{5}d\hat{u}_{i_0}(t_0) \\
&\le -\frac{1}{5}\hat{u}_{i_0}(t_0) \le \frac{1}{5}p(1-\epsilon') < 0 \quad .
\end{aligned} \tag{4.4.14}$$

Since t_0 is a maximal point of $|\hat{u}_{i_0}(s)|$ on $[\frac{t}{2}, t']$, it follows that t_0 must be the left hand endpoint of the interval, i.e. $t_0 = \frac{t}{2}$. On the other hand, if $\hat{u}_{i_0}(t_0) < 0$, from (4.4.13) we have

$$\begin{aligned}
\hat{u}'_{i_0}(t_0) &= [-(b_{i_0}+\gamma_{i_0}) + (1-y^*_{i_0}-z^*_{i_0})\lambda_{i_0i_0}]\hat{u}_{i_0}(t_0) \\
&+ \frac{1-y^*_{i_0}-z^*_{i_0}}{d_{i_0}} \sum_{j\ne i_0} \lambda_{i_0j}d_j\hat{u}_j(t_0) \\
&- (\hat{u}_{i_0}(t_0) + \gamma_{i_0}\int_0^{t_0} e^{-(b_{i_0}+\kappa_{i_0})(t_{i_0}-s)}\hat{u}_{i_0}(s)ds) \sum_j \lambda_{i_0j}y_j(t_0) \\
&- \frac{1}{d_{i_0}}(-\frac{\gamma_{i_0}}{b_{i_0}+\kappa_{i_0}}y^*_{i_0} + z_{i_0}(0))e^{-(b_{i_0}+\kappa_{i_0})t_0} \sum_j \lambda_{i_0j}y_j(t_0) \\
&\ge -[-(b_{i_0}+\gamma_{i_0}) + (1-y^*_{i_0}-z^*_{i_0})\lambda_{i_0i_0}]|\hat{u}_{i_0}(t_0)| \\
&- \frac{1-y^*_{i_0}-z^*_{i_0}}{d_{i_0}} \sum_{j\ne i_0} \lambda_{i_0j}d_j|\hat{u}_j(t_0)| \\
&+ \frac{4}{5}d|\hat{u}_{i_0}(t_0)| - \frac{1}{5}dp(1-\epsilon')
\end{aligned}$$

where the estimate for the third term, the one with the integral, follows from (4.4.12) and (4.4.4) and the estimate for the fourth term comes from (4.4.10). Since $|\hat{u}_j(t_0)| \le |\hat{u}_{i_0}(t_0)| + 2p\epsilon'$, the sum of the first and second terms is

$$\ge -[-(b_{i_0} + \kappa_{i_0}) + \frac{1 - y^*_{i_0} - z^*_{i_0}}{d_{i_0}} \sum_j \lambda_{i_0 j} d_j] |\hat{u}_{i_0}(t_0)| - \frac{2\epsilon' p}{d_{i_0}} \sum_j \lambda_{i_0 j} d_j \quad .$$

Therefore

$$\begin{aligned} \hat{u}'_{i_0}(t_0) &\ge -\frac{d}{5}|\hat{u}_{i_0}(t_0)| - \frac{2\epsilon' p}{d_{i_0}} \sum_j \lambda_{i_0 j} d_j + \frac{4}{5} d |\hat{u}_{i_0}(t_0)| - \frac{1}{5} d |\hat{u}_{i_0}(t_0)| \\ &\ge \frac{2}{5} d |\hat{u}_{i_0}(t_0)| - \frac{1}{5} dp(1 - \epsilon') \qquad (4.4.15) \\ &\ge \frac{1}{5} dp(1 - \epsilon') > 0 \quad . \end{aligned}$$

This again implies $t_0 = \frac{t}{2}$. Therefore, for any $t \ge T$, we can choose t' so large that the maximal point of $|\hat{u}_{i_0}(s)|$, $s \in [\frac{t}{2}, t']$ is the left hand endpoint $\frac{t}{2}$. From this, one deduces that $|\hat{u}_{i_0}(t)|$ is a decreasing function on $[\frac{T}{2}, \infty)$.

Using the fact $|\hat{u}_{i_0}(t)|$ is decreasing, we get

$$\lim_{t \to \infty} |\hat{u}_{i_0}(t)| = \limsup_{t \to \infty} |\hat{u}_{i_0}(t)| = p > 0.$$

Consequently, $\hat{u}_{i_0}(t)$ is of one sign for t large.

Case 1. $\hat{u}_{i_0}(t)$ is positive for t large. By (4.4.14), $\lim_{t \to \infty} \hat{u}_{i_0}(t) = -\infty$, which is impossible.

Case 2. $\hat{u}_{i_0}(t)$ is negative for t large. By (4.4.15), $\lim_{t \to \infty} \hat{u}_{i_0}(t) = +\infty$, which is also impossible.

Therefore, $\lim_{t\to\infty} |\hat{u}_{i_0}(t)| = 0$ and hence $\lim_{t\to\infty} \hat{u}_i(t) = 0$ for all i. This implies $\lim_{t\to\infty} \hat{y}_i(t) = 0$ for all i and hence $\lim_{t\to\infty} y_i(t) = y_i^*$ for all i.

Lastly, since $z_i^* = \frac{\gamma_i}{b_i+\kappa_i} y_i^*$, by (4.4.2) we have,

$$z_i(t) - z_i^* = (z_i(0) - \frac{\gamma_i y_i^*}{b_i + \kappa_i})e^{-(b_i+\kappa_i)t} + \gamma_i \int_0^t e^{-(b_i+\kappa_i)(t-s)}(y_i(s) - y_i^*)ds$$

Using Lemma 4.4.6, we conclude that $\lim_{t\to\infty} z_i(t) = z_i^*$ and the proof is complete. □

THEOREM 4.4.8. *If the endemic equilibrium* E^* *for (4.2.1) is globally stable over* $\dot{B}$, *then the endemic equilibrium* (x^*, y^*, z^*) *for (4.1.1) is globally stable over* $\dot{\mathbb{R}}_+^{3n}$.

PROOF. Let us first assume Λ is irreducible. Given any solution $(x(t), y(t), z(t))$ of (4.1.1) with $(x(0), y(0), z(0)) \in \dot{\mathbb{R}}_+^{3n}$, its ω-limit set Ω must be contained in S, since $x(t) + y(t) + z(t) \to N$ as $t \to \infty$. We will show that $\Omega = \{(x^*, y^*, z^*)\}$.

Claim. $\Omega \neq \{(N, 0, 0)\}$. Suppose not, then $x(t) \to N$ and $y(t) \to 0$ as $t \to \infty$. Let $V(y) = v^t y$ be the Lyapunov function used in the proof of Theorem 4.2.4. Then

$$V' = s(A)v^t y - \sum_i v_i(N_i - x_i) \sum_j \lambda_{ij} y_j \quad .$$

Since $y(t) \geq 0$ and $y(t) \neq 0$ for all $t > 0$, therefore $V' > 0$ for sufficiently large t. This contradicts $y(t) \to 0$ as $t \to \infty$.

Hence, Ω must contain a point $(x^0, y^0, z^0) \in S$ with either (i) $y^0 = 0$ or (ii) $y^0 > 0$. If (i) holds, since Ω is invariant, it contains the negative orbit through (x^0, y^0, z^0). However, this orbit is unbounded and this contradicts the compactness of Ω. Thus (i) is impossible. On the other hand, if (ii) holds, since $(N, 0, 0)$ is globally stable over $S \setminus S_1$, where

$$S_1 = \{(x, y, z) \in \mathrm{R}_+^{3n} : y = 0\} \quad ,$$

therefore $(x^*, y^*, z^*) \in \Omega$. However, since Ω is chain transitive and (x^*, y^*, z^*) is asymptotically stable, we have $\Omega = \{(x^*, y^*, z^*)\}$, as desired.

In the case when Λ is reducible, all we need is to apply the above argument to each of the irreducible subsystems as was discussed in the remark following the proof of Theorem 4.2.2. □

Using Theorems 4.4.7 and 4.4.8, we can conclude

THEOREM 4.4.9. If the endemic equilibrium (x^*, y^*, z^*) for (4.1.1) exists and $\frac{\gamma_i}{b_i + \kappa_i} < 1$ for all i, then it is globally stable over $\mathrm{R}_+^{3n} \setminus S_0$.

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CHAPTER 5

QUALITATIVE ANALYSIS OF AN HIV/AIDS TRAMSMISSION MODEL WITH SUBPOPULATIONS

5.1 Introduction.

In recent years, there is considerable interest in the study of AIDS epidemic in mathematical epidemiology. In fact, the research on modelling the transmission dynamics of HIV/AIDS epidemic has become a major research topic in mathematical epidemiology. A number of mathematical models have been developed to study the transmission dynamics of the human immunodeficiency virus (HIV) and AIDS epidemic. Anderson et al. [1] and May and Anderson [15] considered the AIDS spread in homosexual males which were divided into groups by different contact rates with new partners. Proportionate mixing was chosen in their model. Hyman and Stanley [10] used a similar model as that of Anderson and May to study the spread of HIV in a sexual active and drug using community. Hethcote [7] formulated a model to simulate the spread of the HIV in metropolitan areas. Castillo-Chavez et al. [2],[3] used models to identify the role played by the long period of incubation for homogeneous and heterogeneous populations. A similar work has been done by Dietz [4] and Dietz and Hadeler [5]. Hsieh [9] studied an AIDS model with random screening to describe the spread of HIV in a male homosexual population.

The HIV transmission model discussed here has been proposed by Jacquez, Simon, Koopman, Sattenspiel and Perry [11]. In this model, they divided a homosexual population into subpopulations based on sexual activity, the number of contacts per unit time. It is assumed that there is no migration between subpopulations but they interact by way of sexual contacts. For each subpopulation, HIV infection take finite number of stages and finally leads to AIDS. Preferred mixing, which includes restricted mixing and proportional mixing as special cases, was chosen as the pattern of contacts between subpopulations and the transmission probability from susceptibles to infecteds was assumed to depend on the stage of infection. In their paper [11], Jacquez et al. gave a threshold condition so that below the threshold the disease-free equilibrium is the only equilibrium and above the threshold a unique endemic equilibrium occurs. However, no connections have been shown between the threshold condition and the stability of the no-disease equilibrium, which is common for most epidemic models.

The purpose of this chapter is to extend the work of Jacquez et al. on the qualitative analysis of the HIV/AIDS transmission model. In the next section we will briefly describe the model and the results obtained in [11]. A new threshold condition will be presented in Section 5.3. We will show that like most other epidemic models the threshold can be chosen to be the stability modulus of the Jacobian matrix at the disease-free equilibrium. Hence, the instability of the disease-free equilibrium is a key factor to trigger the epidemic of the disease.

Since we employ a new method to prove the above results, more information about the stability of the endemic equilibrium is obtained, which could lead to a proof of the local asymptotic stability of the endemic equilibrium. At the end of Section 5.3, we show the equivalence of our threshold condition and that obtained by Jacquez et al. in [11]. Because the model we are studying here is very complicated and subpopulations are incorporated, it becomes very difficult or may be impossible to show the (local or global) stability of the endemic equilibrium above the threshold. Nevertheless, we can still expect the model to have some stability-like property such as persistence. We will show in Section 5.4 that above the threshold the dynamics derived from the model persists uniformly. In other words, above the threshold if there is some initial infection, the number of susceptibles and infecteds not only in every subpopulation but also in each stage of infection will eventually remain above a certain positive level, which does not depend on the number of susceptibles who are initially infected. In Section 5.5, we will consider the case of restricted mixing. We show that above the threshold the endemic equilibrium is locally asymptotically stable provided that the number of stages is less than three or the transfer rate from one stage to the next stage is relatively small. In [11], Jacquez et al. also gave a proof on the local asymptotic stability of the endemic equilibrium. Unfortunately, the proof is incomplete because the Descarte's rule of signs only deals with the number of real roots, not real part of complex roots. Moreover, the global asymptotic stability of the endemic equilibrium will be considered.

We conclude that if the transfer rate from one stage to the next does not exceed the mortality rate from all other sources the endemic equilibrium is globally asymptotically stable. In Section 5.6, we drop the restriction on the transmission fractions and a sufficient condition for the existence of the endemic equilibrium is given.

5.2 The HIV/AIDS Transmission Model.

In this section, we state the HIV/AIDS transmission model proposed by Jacquez et al. in [11]. Suppose that a population is divided into n subpopulations by sexual activity. Sexual contacts between subpopulations are assumed but there is no emigation between them. For each subpopulation, say, the i-th subpopulation, it is again divided into three groups: susceptibles, infecteds and persons with AIDS. Let us denote by x_i the number of susceptibles, y_i the number of infecteds, and z_i the number of persons with AIDS respectively in the i-th subpopulation. Furthermore, each infected is assumed to pass through m stages from initial infection to having AIDS so that he or she no longer takes part in transmission. Hence, we can partition the infecteds in the i-th subpopulation into m subgroups. We have $y_i = \sum_{r=1}^{m} y_{ir}$, where y_{ir} is the number of infected in the r-th stage. Under the above assumptions, the transitions among these subpopulations are described by the following system of ordinary differential equations:

$$x_i' = \mu(U_i - x_i) - c_i x_i \sum_{j=1}^{n} \rho_{ij} \sum_{r=1}^{m} \frac{\beta_{ijr} y_{jr}}{x_j + y_j}, \tag{5.2.1}$$

$$y_{i1}' = -(k+\mu) y_{i1} + c_i x_i \sum_{j=1}^{n} \rho_{ij} \sum_{r=1}^{m} \frac{\beta_{ijr} y_{jr}}{x_j + y_j}, \tag{5.2.2}$$

$$y_{ir}' = k y_{i,r-1} - (k+\mu) y_{ir}, \quad r = 2, \ldots, m, \tag{5.2.3}$$

$$z_i' = k y_{im} - \delta z_i, \tag{5.2.4}$$

for $i = 1, \ldots, n$.

Here, all parameters but ρ_{ij} are constants and the parameters have the following definitions:

$U_i =$ the expected subpopulation size without HIV infection;

$c_i =$ the number of persons contacted per person in the i-th subpopulation per unit time, all c_i are nonzero;

$\rho_{ij} =$ the proportion of contacts of a person in the i-th subpopulation with persons in j-th subpopulation. For preferred mixing it takes the form

$$\rho_{ii} = \rho_i + (1 - \rho_i) \frac{c_i(1-\rho_i)(x_i + y_i)}{\sum_{k=1}^{n} c_k(1-\rho_k)(x_k + y_k)} \tag{5.2.5}$$

and

$$\rho_{ij} = (1 - \rho_i) \frac{c_j(1-\rho_j)(x_j + y_j)}{\sum_{k=1}^{n} c_k(1-\rho_k)(x_k + y_k)}, \quad j \neq i \tag{5.2.6}$$

where $0 \le \rho_i \le 1$, $i = 1, \ldots, n$, are constants. If $\rho_i = 0$ for all i, we have proportionate mixing. If $\rho_i = 1$ for all i, we have restricted mixing;

$\beta_{ijr} =$ the transmission fraction between a susceptible in the i-th subpopulation and an infected in the j-th subpopulation at the r-th stage. From this section to Section 5.5, we always assumed β_{ijr} only depends on stage r, not on i and j, that is, $\beta_{ijr} = \beta_r$ and $\beta_r > 0$ for all r ;

$k =$ the fractional transfer rate from $y_{i,r-1}$ to y_{ir};

$\mu =$ the mortality rate from all other sources. In this paper it is assumed to be positive;

$\delta =$ the mortality rate for persons with AIDS.

REMARK. The model stated here has been modified slightly by replacing U_i with μU_i in equation (5.2.1) for simplicity. When μ is positive, the dynamics for both models are exactly the same. We will not discuss the case $\mu = 0$ in this paper.

Since z_i is not involved in the first three equations (5.2.1)-(5.2.3), it suffices to study the model without the equation (5.2.4).

Some properties have been obtained by Jacquez et al. in [11].

(i) The compact convex set

$$C = \{(x,y);\ x_i + y_i \leq U_i,\ x_i \geq 0,\ y_{ir} \geq 0,\ r = 1,\ldots,m,\ i = 1,\ldots,n\},$$

where $x = (x_1,\ldots,x_n)$, $y = (y_{11},\ldots,y_{nm})$, is a positively invariant set. Every solution of the system (5.2.1)-(5.2.3) will reach C after a certain time and remain in C.

(ii) The point $E_0 : x_i = U_i,\ y_{ir} = 0,\ r = 1,\ldots,m;\ i = 1,\ldots,n,$ is always an equilibrium of the system (5.2.1)-(5.2.3) and the set

$$C_0 = \{(x,y);\ y_{ir} = 0 \quad \text{for all} \quad r \quad \text{and} \quad i\},$$

is an invariant set of the system (5.2.1)-(5.2.3). We call the equilibrium E_0 the disease-free equilibrium and C_0 the disease-free set. It is shown that the disease-free equilibrium is globally asymptotically stable with respect to C_0. Therefore, without initial infection of HIV there is always no infection and population size will approach the expected population size.

(iii) For all cases of preferred mixing (except restricted mixing) there exists a threshold

$$H = \frac{\sum\limits_{i=1}^{n} c_i(1-\rho_i)U_i}{\theta_\mu \beta_\mu \sum\limits_{i=1}^{n} c_i^2(1-\rho_i)^2 U_i \, \frac{1}{\max\{0, 1-\theta_\mu\beta_\mu c_i\rho_i, i=1,\ldots,n,\}}}, \tag{5.2.7}$$

where

$$\theta_\mu = \frac{1-(\frac{k}{k+\mu})^m}{\mu} \quad \text{and} \quad \beta_\mu = \frac{\sum\limits_{r=1}^{m} \frac{k^{r-1}}{(k+\mu)^r}\beta}{\frac{1}{\mu}[1-(\frac{k}{k+\mu})^m]}\,. \tag{5.2.8}$$

When $H \geq 1$ the disease-free equilibrium is the only equilibrium and when $H < 1$ there is a unique interior endemic equilibrium.

(iv) For the case of restricted mixing, the threshold is

$$H = \min\{\frac{1}{c_i\theta_\mu\beta_\mu}, \quad i = 1, \ldots, n\}. \tag{5.2.9}$$

When $H \geq 1$ the disease-free equilibrium is globally asymptotically stable.

5.3 Threshold Condition and Stability of Equilibria.

We now establish a connection of the stability of the disease-free equilibrium with a threshold condition. First we simplify the system (5.2.1)-(5.2.3) by defining $w_i = U_i - x_i - y_{i1}, \quad i = 1, \ldots, n$. We obtain an equivalent system

$$w_i' = -\mu w_i + k y_{i1}, \tag{5.3.1}$$

$$\begin{aligned} y_{i1}' = c_i(U_i - w_i - y_{i1})[&\frac{\rho_i \sum_{r=1}^{m} \beta_r y_{ir}}{U_i - w_i + y_{i2} + \cdots + y_{im}} \\ &+ \frac{(1-\rho_i)\sum_{j=1}^{n} c_j(1-\rho_j)\sum_{r=1}^{m}\beta_r y_{jr}}{\sum_{k=1}^{n} c_k(1-\rho_k)(U_k - w_k + y_{k2} + \cdots + y_{km})}] - (k+\mu)y_{i1}, \end{aligned} \tag{5.3.2}$$

$$y_{ir}' = k y_{i,r-1} - (k+\mu)y_{ir}, \quad r = 2, \ldots, m. \tag{5.3.3}$$

for $i = 1, \ldots, n$.

REMARK: Since there are now only n nonlinear equations, namely (5.3.2), the calculation of the Jacobian matrix for this system becomes easier. Also, the disease-free equilibrium has been brought to the origin $w_i = 0, y_{ir} = 0, r =$

$1, \ldots, m; i = 1, \ldots, n$. Some other advantages to use this system can be seen in the proof of persistence (Section 5.4).

If we suppose that $(w_1^s, \ldots, w_n^s, y_{11}^s, \ldots, y_{nm}^s)$ is an equilibrium of the system (5.3.1)-(5.3.3), then they satisfy the following identities

$$w_i^s = \frac{k(1-\alpha)}{\mu(1-\alpha^m)}\, y_i^s, \qquad y_{ir}^s = \frac{\alpha^{r-1}(1-\alpha)}{1-\alpha^m}\, y_i^s, \tag{5.3.4}$$

and

$$U_i - w_i^s + y_{i2}^s + \cdots + y_{im}^s = U_i - \frac{\alpha^m}{1-\alpha^m}\, y_i^s, \tag{5.3.5}$$

where $y_i^s = y_{i1}^s + \cdots + y_{im}^s$, $\alpha = \frac{k}{k+\mu}$.

Substituting the expressions (5.3.4)-(5.3.5) to the right hand side of (5.3.2), we have, for $i = 1, \ldots, n$,

$$\begin{aligned} f_i(y_1^s, \ldots, y_n^s) = {} & -\theta_\mu^{-1} y_i^s + \beta_\mu c_i\Big(U_i - \frac{1}{1-\alpha^m}\, y_i^s\Big) \\ & \times \Big[\frac{\rho_i y_i^s}{U_i - \frac{\alpha^m}{1-\alpha^m}\, y_i^s} + \frac{(1-\rho_i)\sum_{k=1}^{n} c_k(1-\rho_k) y_k^s}{\sum_{k=1}^{n} c_k(1-\rho_k)(U_k - \frac{\alpha^m}{1-\alpha^m}\, y_k^s)}\Big], \end{aligned} \tag{5.3.6}$$

where θ_μ, β_μ are defined by (5.2.8).

Remark: Most of the notations used in our paper follow those in [11].

We now introduce a new system of ordinary differential equations

$$y_i' = f_i(y_1, \ldots, y_n), \quad i = 1, \ldots, n, \tag{5.3.7}$$

where f_i is defined by (5.3.6).

It is easy to see that the equilibria of the systems (5.3.1)-(5.3.3) and (5.3.7) can be obtained from one another by the identities (5.3.4)-(5.3.5). Moreover, if we denote the right hand side of (5.3.2) by F_i, then

$$\frac{\partial f_i}{\partial y_j} = \frac{1-\alpha}{1-\alpha^m}\left(\frac{k}{\mu}\frac{\partial F_i}{\partial w_j} + \sum_{r=1}^{m} \alpha^{r-1} \frac{\partial F_i}{\partial y_{jr}}\right) \tag{5.3.8}$$

provided that w_i, y_i and y_{ir}, $r = 1, \ldots, m$, satisfy (5.3.4)-(5.3.5).

Therefore, we can study the equilibrium distribution of the system (5.3.1)-(5.3.3) and related properties(existence, uniqueness, stability, etc.), by means of the system (5.3.7).

For the system (5.3.7), it is easy to see that

$$S = \{(y_1, \ldots, y_n); \quad 0 \le y_i \le (1-\alpha^m)U_i, \quad i = 1, \ldots, n\}$$

is a positive invariant set.

Let A_0 be the Jacobian matrix of the system (5.3.7) at the origin and $s(A_0)$ be the stability modulus of A_0, which is the maximum of the real parts of the eigenvalues of A_0. Then we have

THEOREM 5.3.1. *If* $s(A_0) \le 0$, *the origin is the only nonnegative equilibrium for (5.3.7) and it is globally asymptotically stable in* S . *If* $s(A_0) > 0$, *there is a unique, positive, globally asymptotically stable equilibrium in* $S \setminus$

$\{0\}$. *Moreover, the stability modulus of the Jacobian matrix at the positive equilibrium is negative, i.e. the Jacobian matrix is stable.*

PROOF. We split the proof into six lemmas.

LEMMA 5.3.2. *The system (5.3.7) has at most one positive equilibrium.*

PROOF. Following the idea of Jacquez et al. in [11], we introduce a nonnegative variable K and consider the following $n+1$ equations

$$-y_i + \frac{\theta_\mu\beta_\mu c_i\rho_i(U_i - \frac{1}{1-\alpha^m}\, y_i)y_i}{U_i - \frac{\alpha^m}{1-\alpha^m}\, y_i} + \theta_\mu\beta_\mu c_i(1-\rho_i)(U_i - \frac{1}{1-\alpha^m}\, y_i)K = 0$$

$$i = 1, \ldots, n, \tag{5.3.9}$$

and

$$\frac{\sum\limits_{k=1}^{n} c_k(1-\rho_k)y_k}{\sum\limits_{k=1}^{n} c_k(1-\rho_k)(U_k - \frac{\alpha^m}{1-\alpha^m}\, y_k)} = K. \tag{5.3.10}$$

Obviously, a positive equilibrium exists if and only if there is a positive K such that the n equations (5.3.9) and (5.3.10) hold.

Let $K = K_i(y_i)$ be the function satisfying (5.3.9). Then differentiating the both sides of (5.3.9) with respect to y_i, we have

$$\begin{aligned}-1 &+ \frac{\theta_\mu\beta_\mu c_i\rho_i(U_i - \frac{1}{1-\alpha^m}\, y_i)}{U_i - \frac{\alpha^m}{1-\alpha^m}\, y_i} - \frac{\theta_\mu\beta_\mu c_i\rho_i U_i y_i}{(U_i - \frac{\alpha^m}{1-\alpha^m}\, y_i)^2} \\ &- \frac{\theta_\mu\beta_\mu c_i(1-\rho_i)K_i(y_i)}{1-\alpha^m} + \theta_\mu\beta_\mu c_i(1-\rho_i)(U_i - \frac{1}{1-\alpha^m}\, y_i)K_i'(y_i) = 0.\end{aligned} \tag{5.3.11}$$

Thus for $K > 0$, by (5.3.9)

$$-1 + \frac{\theta_\mu \beta_\mu c_i \rho_i (U_i - \frac{1}{1-\alpha^m}\, y_i)}{U_i - \frac{\alpha^m}{1-\alpha^m}\, y_i} < 0,$$

then, $K_i'(y_i) > 0.$

Hence there exists the inverse function of $K = K_i(y_i)$, denoted by $y_i = y_i(K)$, and $y_i'(K) = \frac{1}{K_i'(y_i)} > 0.$

We now prove

$$G(K) = \frac{\sum_{k=1}^{n} c_k(1-\rho_k) y_k(K)}{\sum_{k=1}^{n} c_k(1-\rho_k)(U_k - \frac{\alpha^m}{1-\alpha^m}\, y_k(K))}$$

is concave down, i.e. $G''(K) < 0$ for $K > 0$.

By differentiation,

$$G' = \frac{[\sum_{k=1}^{n} c_k(1-\rho_k)U_k)][\sum_{k} c_k(1-\rho_k) y_k'(K)]}{[\sum_{k=1}^{n} c_k(1-\rho_k)(U_k - \frac{\alpha^m}{1-\alpha^m}\, y_k(K))]^2} \tag{5.3.12}$$

and

$$G'' = [\sum_{k=1}^{n} c_k(1-\rho_k)U_k][\sum_{k=1}^{n} c_k(1-\rho_k)(U_k - \frac{\alpha^m}{1-\alpha^m} y_k)]$$
$$\times \frac{\sum_{k=1}^{n} c_k(1-\rho_k)[y_k''(K) + \frac{2\alpha^m y_k'^2(K)}{(1-\alpha^m)(U_k - \frac{\alpha^m}{1-\alpha^m} y_k)}]}{[\sum_{k=1}^{n} c_k(1-\rho_k)(U_k - \frac{\alpha^m}{1-\alpha^m}\, y_k)]^3}.$$

So, if

$$y_k''(K) + \frac{2\alpha^m y_k'^2(K)}{(1-\alpha^m)(U_k - \frac{\alpha^m}{1-\alpha^m}\, y_k)} < 0 \quad \text{for} \quad k = 1, \ldots, n.$$

then $G'' < 0$.

Define

$$\phi_i(y_i) = \frac{U_i - \frac{1}{1-\alpha^m}\, y_i}{U_i - \frac{\alpha^m}{1-\alpha^m}\, y_i}.$$

From (5.3.9) again, we have

$$\theta_\mu \beta_\mu c_i (1-\rho_i) K_i(y_i) = \frac{y_i}{U_i - \frac{1}{1-\alpha^m}\, y_i} - \frac{\theta_\mu \beta_\mu c_i \rho_i y_i}{U_i - \frac{\alpha^m}{1-\alpha^m}\, y_i}.$$

Therefore,

$$\begin{aligned} y_i'(K) &= \frac{1}{K_i'(y_i)} \\ &= \frac{\theta_\mu \beta_\mu c_i (1-\rho_i)(U_i - \frac{1}{1-\alpha^m}\, y_i)^2}{U_i(1-\theta_\mu \beta_\mu c_i \rho_i \phi_i^2)}, \\ y_i''(K) &= -2 y_i'(K) \\ &\quad \times \frac{\theta_\mu \beta_\mu c_i (1-\rho_i)(U_i - \frac{1}{1-\alpha^m}\, y_i)(1-\alpha^m \theta_\mu \beta_\mu c_i \rho_i \phi_i^3)}{(1-\alpha^m) U_i (1-\theta_\mu \beta_\mu c_i \rho_i \phi_i^2)^2}. \end{aligned} \tag{5.3.13}$$

Thus

$$\begin{aligned} y_i''(K) + \frac{2\alpha^m y_i'^2(K)}{(1-\alpha^m)(U_i - \frac{\alpha^m}{1-\alpha^m}\, y_i)} &= \frac{2 y_i'(K) \theta_\mu \beta_\mu c_i (1-\rho_i)(U_i - \frac{1}{1-\alpha^m}\, y_i)}{(1-\alpha^m) U_i (1-\theta_\mu \beta_\mu c_i \rho_i \phi_i^2)} \\ &\quad \times \left(-\frac{1-\alpha^m \theta_\mu \beta_\mu c_i \rho_i \phi_i^3}{1-\theta_\mu \beta_\mu c_i \rho_i \phi_i^2} + \alpha^m \phi_i\right) < 0 \end{aligned}$$

since

$$-\frac{1-\alpha^m \theta_\mu \beta_\mu c_i \rho_i \phi_i^3}{1-\theta_\mu \beta_\mu c_i \rho_i \phi_i^2} + \alpha^m \phi_i = \frac{-1+\alpha^m \phi_i}{1-\theta_\mu \beta_\mu c_i \rho_i \phi_i^2} < 0.$$

Therefore $G''(K) < 0$ for $K > 0$. It follows from $G(K) \geq 0$ for $K > 0$ that $G(K) = K$ has at most one positive solution. □

REMARK. $G(K)$ can take two values at $K = 0$. We denote $G(0)$ to be the value satisfying $G(0) = \lim_{K \to 0+} G(K)$. It is easy to verify that $G(0) = 0$ if and only if $1 - \theta_\mu \beta_\mu c_i \rho_i > 0$ for all i. In this case, $G(K)$, $K \geq 0$, is well-defined and $G'(0)$ exists.

LEMMA 5.3.3. *$G(K) = K$ has no positive solutions if and only if $G(0) = 0$ and $G'(0) \leq 1$.*

PROOF. It is trivial. □

LEMMA 5.3.4. *the system (5.3.8) is cooperative in S, e.g. $\frac{\partial f_i}{\partial y_j} \geq 0, i \neq j$ at any point in S.*

PROOF. Since for $j \neq i$

$$\frac{\partial f_i}{\partial y_j} = \frac{\beta_\mu c_i(1-\rho_i)(U_i - \frac{1}{1-\alpha^m}\, y_i)c_j(1-\rho_j)\sum_{k=1}^{n} c_k(1-\rho_k)U_k}{[\sum_{k=1}^{n} c_k(1-\rho_k)(U_k - \frac{\alpha^m}{1-\alpha^m}\, y_k)]^2} \geq 0 \quad (5.3.14)$$

we complete the proof. □

LEMMA 5.3.5. *Let E_s be a positive equilibrium and A_s be the Jacobian matrix at E_s. Then $s(A_s) < 0$. Therefore, any positive equilibrium is locally asymptotically stable.*

PROOF. We have already known that $E_s = (y_1^s, \ldots, y_n^s)$ exists if and only if there exists $K_s > 0$ such that they satisfy (5.3.9) and (5.3.10). Thus $G(K_s) = K_s$. From the concavity of $G(K)$, $G'(K_s) < 1$.

Since $y_1^s, \dots, y_n^s, K_s$ satisfy (5.3.10) and (5.3.11),

$$\frac{\partial f_i}{\partial y_i} = -\beta_\mu c_i(1-\rho_i)(U_i - \frac{1}{1-\alpha^m}\, y_i^s)K_i'(y_i^s) + \frac{\beta_\mu c_i^2(1-\rho_i)^2(U_i - \frac{1}{1-\alpha^m}\, y_i^s)\sum_{k=1}^{n} c_k(1-\rho_k)U_k}{[\sum_{k=1}^{n} c_k(1-\rho_k)(U_k - \frac{\alpha^m}{1-\alpha^m}\, y_k^s)]^2}\,. \tag{5.3.15}$$

Let

$$B_s = \operatorname{diag}\{\frac{1}{\beta_\mu c_i(1-\rho_i)(U_i - \frac{1}{1-\alpha^m}\, y_i^s)K_i'(y_i^s)}\,,\ i = 1, \dots, n\} \times A_s.$$

It follows from (5.3.14) and (5.3.15) that

$$B_s = -I + \frac{\sum_{k=1}^{n} c_k(1-\rho_k)U_k}{[\sum_{k=1}^{n} c_k(1-\rho_k)(U_k - \frac{\alpha^m}{1-\alpha^m}\, y_k)]^2} \begin{pmatrix} y_1'(K_s) \\ \vdots \\ y_n'(K_s) \end{pmatrix} (c_1(1-\rho_1), \dots, c_n(1-\rho_n)).$$

Hence

$$s(B_s) = -1 + \frac{[\sum_{k=1}^{n} c_k(1-\rho_k)U_k]\sum_{k=1}^{n} c_k(1-\rho_k)y_k'(K_s)}{[\sum_{k=1}^{n} c_k(1-\rho_k)(U_k - \frac{\alpha^m}{1-\alpha^m}\, y_k^s)]^2} = -1 + G'(K_s) < 0.$$

Finally, since the stability modulus of A_s and B_s have the same sign, we obtain $s(A_s) < 0$. □

LEMMA 5.3.6. *Let* A_0 *be the Jacobian matrix at the origin. Then* $s(A_0) \leq 0$ *if and only if* $G(0) = 0$ *and* $G'(0) \leq 1$.

PROOF. If $G(0) > 0$, by the remark after Lemma 5.3.2, $1 - \theta_\mu\beta_\mu c_i\rho_i \leq 0$ for some i. Thus, at least one of the diagonal elements of A_0 is positive. This implies $s(A_0) > 0$. Hence, we assume $G(0) = 0$. Under this assumption $G'(K), K \geq 0$ is well-defined and $G'(0)$ exists. An argument similar to the proof of Lemma 5.3.5 can show that the sign of $s(A_0)$ is the same as that of $-1+G'(0)$. Therefore $s(A_0) \leq 0$ if and only if $G'(0) \leq 1$. □

LEMMA 5.3.7. *If* $s(A_0) \leq 0$, *the origin is the only equilibrium, which is globally asymptotically stable in* S. *If* $s(A_0) > 0$, *there is a unique positive equilibrium, which is globally asymptotically stable in* $S \setminus \{0\}$.

PROOF. The existence and uniqueness are an immediate corollary of Lemma 5.3.3 and Lemma 5.3.6.

Let $\phi(t)$ be the solution of the system (5.3.7) with

$$\phi(0) = ((1-\alpha^m)U_1, \ldots, (1-\alpha^m)U_n).$$

Then $f_i(\phi(0)) < 0$ for all i. By Lemma 5.3.4 and a theorem of Selgrade [16], $\phi(t)$ tends to an equilibrium in S as $t \to \infty$.

If $s(A_0) \leq 0$, the origin is the only equilibrium. Thus $\lim_{t\to\infty} \phi(t) = 0$.

Let $\gamma(t)$ be any solution with $\gamma(0) \in S, \gamma(0) \neq 0$. Then, $\gamma(0) \leq \phi(0)$. Thus $0 \leq \gamma(t) \leq \phi(t)$ for $t \geq 0$ by Lemma 5.3.4. Letting $t \to \infty$, we have $\lim_{t\to\infty} \gamma(t) = 0$. It follows that the origin is globally asymptotically stable.

Not we suppose $s(A_0) > 0$ and E_s is the positive equilibrium. Then, $\lim_{t\to\infty} \phi(t) = E_s$. From $0 \le \gamma(t) \le \phi(t)$, the ω-limit set of $\gamma(t)$

$$\omega(\gamma) \subset [0, E_s] = \{y \in R^n;\ 0 \le y \le E_s\}.$$

On the other hand, since $s(A_0) > 0$ there is a positive eigenvector u of A_0 corresponding to $s(A_0)$; here u is chosen so small that $u \le \gamma(0)$ (without the loss of generality, we assume $\gamma(0) > 0$ because $y_i = 0$ implies $f_i > 0$). Let $\psi(t)$ be the solution with $\psi(0) = u$, then $f_i(\psi(0)) > 0$ for all i. Thus $\lim_{t\to\infty} \psi(t) = E_s$.

By the same argument as before, $\psi(t) \le \gamma(t)$ for all $t \ge 0$. Hence, $\omega(\gamma) \subset [E_s, \phi(0)]$. Thus $\omega(\gamma) \subset [0, E_s] \cap [E_s, \phi(0)] = \{E_s\}$. That is, $\lim_{t\to\infty} \gamma(t) = E_s$. Therefore E_s is globally asymptotically stable in $S \backslash \{0\}$.
□

Theorem 5.3.1 shows that the system (5.3.7) has the same properties as the Lajmanovich-Yorke's model.

We now turn to the system (5.3.1)-(5.3.3). Let A be the Jacobian matrix of the system (5.3.1)-(5.3.3) at the disease-free equilibrium and $s(A)$ be the stability modulus of A. With the help of Theorem 5.3.1, we have the following results for system (5.3.1)-(5.3.3).

THEOREM 5.3.8. *If* $s(A) \le 0$, *the disease-free equilibrium is a unique equilibrium for system (5.3.1)-(5.3.3). If* $s(A) > 0$, *the disease-free equilibrium is* *unstable. In this case there are only two equilibria: the disease-free equilibrium and an interior, endemic equilibrium. Thus the stability modulus of the Jacobian matrix at the disease-free equilibrium is a threshold. Furthermore, there exists a nonnegative matrix* D *with* $\det D = 1$ *such that the product of the Jacobian matrix at the endemic equilibrium and* D *is a stable matrix.*

PROOF. By Theorem 5.3.1, to prove all but the last statement it suffices to prove that $s(A) \le 0$ if and only if $s(A_0) \le 0$. Rearranging the equations in the system (5.3.1)-(5.3.3) in order $w_1, \ldots, w_n,\ y_{11}, \ldots, y_{n1}, \ldots, y_{1m}, \ldots, y_{nm}$, we can write the matrix A as

$$A = \begin{pmatrix} -\mu I & kI & 0 & \cdots & 0 \\ 0 & M_1 & M_2 & \cdots & M_m \\ 0 & kI & -(k+\mu)I & \cdots & 0 \\ & & & \ddots & \\ & & \ddots & & \\ 0 & & & kI & -(k+\mu)I \end{pmatrix}$$

where I is the $n \times n$ identity matrix, $M_r = (\frac{\partial F_i}{\partial y_{jr}})|_{w_i=0, y_{jr}=0}$. Hence, A is off-diagonal nonnegative and from (5.3.8), $A_0 = \frac{1-\alpha}{1-\alpha^m} \sum_{r=1}^{m} \alpha^{r-1} M_r$.

Now, let

$$D = \begin{pmatrix} I & \frac{k}{\mu} I & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & \alpha I & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \alpha^{m-1} I & \cdots & \alpha I & I \end{pmatrix}$$

Then $D \ge 0$ and $\det D = 1$. Moreover,

$$AD = \begin{pmatrix} -\mu I & 0 & 0 & \cdots & 0 \\ 0 & \frac{1-\alpha^m}{1-\alpha} A_0 & A_2 & \cdots & A_m \\ 0 & 0 & -(k+\mu)I & \cdots & 0 \\ & & \ddots & \ddots & \\ 0 & & & 0 & -(k+\mu)I \end{pmatrix}$$

where $A_r = \sum_{s=r}^{m} \alpha^{s-r} M_s \geq 0,\ r = 2, \ldots, m.$

Thus, if $s(A_0) \leq 0$, then $s(AD) \leq 0$. Since AD is off-diagonal nonnegative, there is a vector $u > 0$ such that $ADu \leq 0$. Let $v = Du$, then $v > 0$ and $Av \leq 0$, so $s(A) \leq 0$ (Theorem 5.4, [6]).

Inversely, assume $s(A) \leq 0$. Then we have $v > 0$ such that $Av \leq 0$. Define $u = D^{-1}v$ and rewrite u and v as $u = (u^0, u^1, \ldots, u^m)$, $v = (v^0, v^1, \ldots, v^m)$, where $u^r, v^r, r = 0, \ldots, m$ are n-vectors. Since

$$D^{-1} = \begin{pmatrix} I & -\frac{k}{\mu} & 0 & \ldots & 0 \\ 0 & I & 0 & \ldots & 0 \\ 0 & -\alpha I & \ddots & & \\ & \ddots & \ddots & \ddots & \\ 0 & & 0 & -\alpha I & I \end{pmatrix}$$

and $Av \leq 0$, we have $u^1 = v^1 > 0, u^r = -\alpha v^{r-1} + v^r \geq 0$. Therefore

$$\frac{1-\alpha^m}{1-\alpha} A_0 u^1 \leq \frac{1-\alpha^m}{1-\alpha} A_0 u^1 + \sum_{r=2}^{m} A_r u^r = \sum_{r=1}^{m} M_r v^r \leq 0.$$

This implies $A_0 u^1 \leq 0$. It follows that $s(A_0) \leq 0$.

Finally, let B be the Jacobian matrix at the endemic equilibrium. The same argument can show

$$BD = \begin{pmatrix} -\mu I & 0 & 0 & \ldots & 0 \\ * & \frac{1-\alpha^m}{1-\alpha} A_1 & * & \ldots & * \\ 0 & 0 & -(k+\mu)I & \ldots & 0 \\ & & \ddots & \ddots & \\ 0 & & & 0 & -(k+\mu)I \end{pmatrix}$$

where A_1 is the Jacobian matrix of the system (5.3.7) at the corresponding interior equilibrium, which is stable by Theorem 5.3.1. Therefore, BD is a stable matrix. We complete the proof. □

In the next theorem, we will show that the threshold condition in Theorem 5.3.8 and the one obtained by Jacquez et al. in [11] are equivalent.

THEOREM 5.3.9. *Let* H *be defined by (5.2.7) or (5.2.9) and* $s(A)$ *be defined in Theorem 5.3.8. Then,* $H \geq 1$ *if and only if* $s(A) \leq 0$.

PROOF. From the proof of Theorem 5.3.8, we have shown that $s(A) \leq 0$ if and only if $s(A_0) \leq 0$. In Lemma 5.3.6 we showed $s(A_0) \leq 0$ if and only if $G(0) = 0$ and $G'(0) \leq 1$. Hence it suffices to prove $G(0) = 0$ and $G'(0) \leq 1$ if and only if $H \geq 1$.

If $G(0) = 0$, $G'(0) \leq 1$, then $1 - \theta_\mu \beta_\mu c_i \rho_i > 0$ for all i. Hence

$$H = \frac{\sum_{k=1}^{n} c_k(1-\rho_k)U_k}{\theta_\mu \beta_\mu \sum_{k=1}^{n} \frac{c_k^2(1-\rho_k)^2 U_k}{1-\theta_\mu \beta_\mu c_k \rho_k}}.$$

On the other side, $y_i(0) = 0$ and from (5.3.13), $y_i'(0) = \frac{\theta_\mu \beta_\mu c_i(1-\rho_i)U_i}{1-\theta_\mu \beta_\mu c_i \rho_i}$.

Thus,

$$\begin{aligned} 1 \geq G'(0) &= \frac{[\sum_{k=1}^{n} c_k(1-\rho_k)U_k][\sum_{k=1}^{n} c_k(1-\rho_k)y_k'(0)]}{[\sum_{k=1}^{n} c_k(1-\rho_k)(U_k - \frac{\alpha^m}{1-\alpha^m}\, y_k(0))]^2} \\ &= \frac{\theta_\mu \beta_\mu \sum_{k=1}^{n} \frac{c_k^2(1-\rho_k)^2 U_k}{1-\theta_\mu \beta_\mu c_k \rho_k}}{\sum_{k=1}^{n} c_k(1-\rho_k)U_k} = \frac{1}{H}. \end{aligned}$$

Hence $H \geq 1$.

Conversely, suppose $H \geq 1$. Then $1 - \theta_\mu \beta_\mu c_i \rho_i > 0$ for all i. Thus $G(0) = 0$. and $G'(0)$ exists. Repeating the above procedure, we still have $G'(0) = \frac{1}{H}$. So $G'(0) \leq 1$. □

5.4 HIV Epidemic.

The mathematical notion of persistence has been utilized in the literature of mathematical biology. Under the definition of persistence, if the dynamic system modelling an ecosystem persists uniformly all the species in the ecosystem will survive. In the biological point of view, it is of importance to study persistence-extinction phenomena. The same idea can also be applied to mathematical epidemiology. Persistence of an epidemiological model here means that the disease is endemic and the number of persons in each group: susceptible, infected or removed (recovered) remains eventually above a positive lower bound. On the other hand, when we model the dynamics of epidemic for heterogeneous populations the model we obtain becomes so complicated that it is almost impossible to study the global, even local, stability of the endemic equilibrium if it exists. Hence, persistence can be regarded as an appropriated global property to describe the epidemic of the disease.

In this section, we will study the persistence of system (5.2.1)-(5.2.3). We will show that above the threshold system (5.2.1)-(5.2.3) is uniformly persistent with respect to C. More precisely, we have

THEOREM 5.4.1. *Let* A *be the Jacobian matrix at the disease-free equilibrium and* $s(A)$ *be the stability modulus of* A. *If* $s(A) > 0$, *there exists a* $\eta > 0$ *such that for any solution* $(x_1(t), \ldots, x_n(t), y_{11}(t), \ldots, y_{nm}(t))$ *with* $(x_1(0), \ldots, x_n(0), y_{11}(0), \ldots, y_{nm}(0)) \in C \setminus C_0$, *we have*

$$\liminf_{t \to \infty} \; x_i(t) \geq \eta$$

$$\liminf_{t \to \infty} \; y_{ir}(t) \geq \eta, \quad r = 1, \ldots, m$$

$$\limsup_{t \to \infty} \; (x_i(t) + y_i(t)) \leq U_i - \eta,$$

for all i, *where* $y_i(t) = y_{i1}(t) + \cdots + y_{im}(t)$.

In Section 5.3, after defining $\omega_i = U_i - x_i - y_{i1}$, we came up with the equivalent system (5.3.1)-(5.3.3) to system (5.2.1)-(5.2.3). For system (5.3.1)-(5.3.3), the compact convex set C can be rewritten as

$$C = \{(w, y);\; y_{i2} + \cdots + y_{im} \leq w_i,\; w_i + y_{i1} \leq U_i,$$
$$y_{ir} \geq 0,\; r = 1, \ldots, m;\; i = 1, \ldots, n\}$$

where $w = (w_1, \ldots, w_n)$, $y = (y_{11}, \ldots, y_{nm})$. Hence, Theorem 5.4.1 can be rewritten as the following theorem.

THEOREM EQ. *Let* A *be the Jacobian matrix of the system (5.3.1)-(5.3.3) at the disease-free equilibrium and* $s(A)$ *be the stability modulus of* A.

If $s(A) > 0$, *there exists a number* $\eta > 0$ *such that for any solution* $(w(t), y(t))$ *with* $(w(0), y(0)) \in C \setminus C_0$,

$$\liminf_{t\to\infty} \quad y_{ir}(t) \geq \eta, \quad r = 1, \ldots, m, \tag{5.4.1}$$

$$\liminf_{t\to\infty} \quad [w_i(t) - y_{i1}(t) - \cdots - y_{im}(t)] \geq \eta, \tag{5.4.2}$$

and

$$\liminf_{t\to\infty} \quad [U_i - w_i(t) - y_{i1}(t)] \geq \eta, \tag{5.4.3}$$

for $i = 1, \ldots, n$.

Before proving Theorem EQ, we introduce some notations and establish some preliminary results. Hereafter, we always denote the interior of C by *int* C and the boundary of C by ∂C.

LEMMA 5.4.2. *Let* $(w(t), y(t))$ *be a solution in* C. *If* $y(0) \neq 0$, *then* $(w(t), y(t)) \in$ *int* C *for all* $t > 0$.

PROOF. Suppose not. Then there is a number $t_0 > 0$ such that $(w(t_0), y(t_0)) \in \partial C$. Let

$$D_1 = \{(w, y);\ w_i + y_{i1} = U_i \quad \text{for some} \quad i\}$$

$$D_2 = \{(w, y);\ y \neq 0 \quad \text{but} \quad y_{ir} = 0 \quad \text{for some } i \text{ and } r\}$$

and

$$D_3 = \{(w, y);\ w_i = y_{i2} + \cdots + y_{im} \quad \text{for some} \quad i\}.$$

Thus $\partial C \subset C_0 \cup D_1 \cup D_2 \cup D_3$.

If $(w(t_0), y(t_0)) \in C_0$, it follows from the invariance of C_0 that $(w(0), y(0)) \in C_0$, i.e. $y(0) = 0$. This is a contradiction. Hence, $(w(t_0), y(t_0)) \notin C_0$.

If $(w(t_0), y(t_0)) \in D_1$, then $w_i(t_0) + y_{i1}(t_0) = U_i$ for some i. Since C is positive invariant, $w_i'(t_0) + y_{i1}'(t_0) = 0$. But from the equation (5.3.1) and (5.3.2),

$$w_i'(t_0) + y_{i1}'(t_0) = -\mu(w_i(t_0) + y_{i1}(t_0)) = -\mu U_i < 0,$$

an contradiction. So $(w(t_0), y(t_0)) \notin D_1$.

If $(w(t_0), y(t_0)) \in D_2 \setminus D_1$, then $y_{ir}(t_0) = 0$ for some i and r. Thus, $y_{ir}'(t_0) = 0$. From the equation (5.3.3), $y_{i,r-1}(t_0) = 0$, which implies $y_{i,r-1}'(t_0) = 0$. After $r-1$ steps, we have $y_{i1}(t_0) = 0$ and $y_{i1}'(t_0) = 0$. Since $w_i(t_0) + y_{i1}(t_0) \neq U_i$,

$$\sum_{r=1}^{m} \beta_r y_{jr}(t_0) = 0 \quad \text{for} \quad j = 1, \ldots, n.$$

Hence, $y_{jr}(t_0) = 0$ for $r = 1, \ldots, m;\ j = 1, \ldots, n$. Noting that C_0 is invariant, we have $y(0) = 0$, which is a contradiction too.

Finally, if $(w(t_0), y(t_0)) \in D_3 \setminus D_2$, then, for some i,

$$w_i(t_0) = y_{i2}(t_0) + \cdots + y_{im}(t_0) \quad \text{and} \quad w_i'(t_0) = y_{i2}'(t_0) + \cdots + y_{im}'(t_0).$$

On the other hand, from the equation (5.3.1) and (5.3.3),

$$-\mu w_i(t_0) + k y_{i1}(t_0) = k y_{i1}(t) - (k+\mu) y_{i2}(t_0) + \cdots + k y_{i,m-1}(t_0) - (k+\mu) y_{im}(t).$$

Hence $y_{im}(t_0) = 0$.

Thus, $(w(t_0), y(t_0)) \in D_2$. This is also impossible. Therefore $(w(t_0), y(t_0)) \notin \partial C$, which contradicts our assumption. Hence we conclude that

$(y(t), w(t)) \in int\, C$ for all $t > 0$. □

LEMMA 5.4.3. *Let* $(w(t), y(t))$ *be a solution which intersects with* $\partial C \setminus \{0\}$. *Then there exists a number* t_0 *such that* $(w(t_0), y(t_0)) \notin C$.

PROOF. We divide the proof into two parts.

(i) $(w(t), y(t))$ intersects with $C_0 \setminus \{0\}$. In this case, the solution is written as $w(t) = w(0)e^{-\mu t}$ and $y(t) = 0$, where $w(0) \neq 0$. Thus, $w(t)$ is unbounded on $(-\infty, 0]$, so is $(w(t), y(t))$. From the boundness of C, we can choose $t_0 \in (-\infty, 0]$ such that $(w(t_0), y(t_0)) \notin C$.

(ii) $(w(t), y(t))$ intersects with $\partial C \setminus C_0$. Suppose $(w(t'), y(t')) \in \partial C \setminus C_0$. Pick a number $t_0 < t'$. If $(w(t_0), y(t_0)) \notin C$, we complete the proof. If $(w(t_0), y(t_0)) \in C$, we have $y(t_0) \neq 0$ because $y(t') \neq 0$. Considering the solution $(\hat{w}(t), \hat{y}(t)) = (w(t+t_0), y(t+t_0))$, we have $\hat{y}(0) = y(t_0) \neq 0$, $(\hat{w}(0), \hat{y}(0)) = (w(t_0), y(t_0)) \in C$ but $(\hat{w}(t'-t_0), \hat{y}(t'-t_0)) = (w(t'), y(t')) \in \partial C$. That contradicts to Lemma 5.4.2. So we prove Lemma 5.4.3. □

Lemma 5.4.4. *The origin is the only compact invariant set in the boundary of C. If $s(A) > 0$, the origin is repellent with respect to int C. Hence, the origin is an isolated invariant set in C and its stable manifold lies in ∂C.*

Proof. Let M be the maximal invariant set in ∂C. If $M \neq \{0\}$, there is a point $(w_0, y_0) \in M, (w_0, y_0) \neq 0$. By Lemma 5.4.3, we have a solution $(w(t), y(t))$ with $w(0) = w_0, y(0) = y_0$ such that there exists a number t_0 such that $(w(t_0), y(t_0)) \notin C$. That is contradictory to the invariance of M. Hence $M = \{0\}$.

We now assume $s(A) > 0$. Since A is off-diagonal nonnegative, there is a positive eigenvector v of A^t to $s(A)$. Define $V(w, y) = v^t \binom{w}{y}$. Then it is not difficult to show that along solutions in a sufficient small neighbourhood of the origin in *int* C, $V'(w(t), y(t)) > 0$. Therefore the origin is repellent with respect to *int* C. □

Lemma 5.4.5 [8]. *Let $\phi(t, x)$ be a continuous dynamic system in a metric space, X be a compact positive invariant set and $Y \subset X$ be closed with $\phi(t, X \backslash Y) \subset X \backslash Y$ for all $t \geq 0$. Let M be the maximal compact invariant set in Y. Then $\phi(t, x)$ is uniformly persistent if and only if*

(i) M is isolated in X, and

(ii) The stable manifold of M lies in Y.

REMARK. Lemma 5.4.5 is a version of Theorem 4.1 in [8] for continuous dynamic system. It can be proved easily by using Theorem 4.1 in [8].

PROOF OF THEOREM EQ. Let $\phi(t,x)$ be the dynamic system generated by the system (5.3.1)-(5.3.3), $X = C$, $Y = \partial C$ and $M = \{0\}$. By Lemma 5.4.4, the conditions (i) and (ii) in Lemma 5.4.5 are satisfied. Hence $\phi(t,x)$ is uniformly persistent with respect to ∂C. By the definition of uniform persistence, for any $(w(t), y(t))$ with $(w(0), y(0)) \in int\ C$, (5.4.1), (5.4.2) and (5.4.3) hold. For any solution $(w(t), y(t))$ with $(w(0), y(0)) \in C \setminus C_0$, $(w(t), y(t)) \in int\ C$ for $t > 0$ by Lemma 5.4.2. Thus (5.4.1), (5.4.2) and (5.4.3) still hold. Therefore we finish our proof. ☐ We now restate the above results in terms of biological interest.

BIOTHEOREM. *Above the threshold* $s(A)$, *the disease will remain epidemic for all further time if there is initial infection in at least one subpopulation. Moreover, the number of susceptibles and infecteds of each subpopulation at each stage will be greater than a positive constant level, which is independent of the initial levels.*

5.5 Restricted Mixing.

We now consider the stability of the system (5.2.1)-(5.2.3) in the case of restricted mixing. Jacquez et al. have shown in [11] that below the threshold the disease-free equilibrium is globally asymptotically stable. In the previous sections we showed that above the threshold the system (5.2.1)-(5.2.3) has a unique endemic equilibrium and it is uniformly persistent. However, the stability of the endemic equilibrium still remains unknown. In this section, we will present sufficient conditions for the local stability and the global stability of the endemic equilibrium in the case of restricted mixing.

For restricted mixing, the pattern of contacts takes the form

$$\rho_{ii} = 1 \quad \text{and} \qquad \rho_{ij} = 0, \quad j \neq i.$$

Hence, there are no contacts between subpopulations. Without the loss of generality, we can consider the system (5.2.1)-(5.2.3) with one population. In this case, the system (5.2.1)-(5.2.3) reduces to the following system

$$\begin{aligned} x' &= \mu(U - x) - \frac{cx}{x+y} \sum_{r=1}^{m} \beta_r y_r, \\ y_1' &= -(k+\mu) y_1 + \frac{cx}{x+y} \sum_{r=1}^{m} \beta_r y_r, \\ y_r' &= k y_{r-1} - (k+\mu) y_r, \quad r = 2, \dots, m \end{aligned} \tag{5.5.1}$$

where $y = y_1 + \cdots + y_m$.

For the system (5.5.1), combining the results in Jacquez et al. [11] and the previous sections we obtain

THEOREM 5.5.1. *If* $c\theta_\mu \beta_\mu \leq 1$, *the disease-free equilibrium is globally asymptotically stable in* C. *If* $c\theta_\mu \beta_\mu > 1$, *the disease-free equilibrium is unstable* *and there is a unique endemic equilibrium. Moreover, the system (5.5.1) is uniformly persistent with respect to* C.

We now give a sufficient condition for the local stability of the endemic equilibrium.

THEOREM 5.5.2. *Assume that* $c\theta_\mu\beta_\mu > 1$. *If* $\sum_{r=1}^{m-1}\alpha^r < c\theta_\mu\beta_\mu$, *then the endemic equilibrium is locally asymptotically stable.*

To prove Theorem 5.5.2, we begin with the following lemma.

LEMMA 5.5.3. *Given a* $n \times n$ *matrix* $M = (m_{ij})$. *Let*

$$\hat{M} = \begin{pmatrix} m_{11} & |m_{12}| & \dots & |m_{1n}| \\ |m_{21}| & m_{22} & \dots & |m_{2n}| \\ \vdots & & \ddots & \vdots \\ |m_{n1}| & \dots & \dots & m_{nn} \end{pmatrix}.$$

If $s(\hat{M}) < 0$, *then* $s(M) < 0$. *Moreover, there is a positive diagonal matrix* $D =$ *diag*$\{d_1, \dots, d_n\}$ *such that* $DM + M^tD$ *is negative definite.*

PROOF. Since $\hat{M}$ is off-diagonal nonnegative and $s(\hat{M}) < 0$, there is a positive diagonal matrix $D =$ diag$\{d_1, \dots, d_n\}$ such that $D\hat{M} + \hat{M}^tD$ is negative definite. We now show that $DM + M^tD$ also is negative definite.

For any n-vector $u = (u_1, \dots, u_n)^t \neq 0$, let $\hat{u} = (|u_1|, \dots, |u_n|)$. Then

$$u^t(DM + M^tD)u = 2\sum_{i=1}^{n} d_i m_{ii} u_i^2 + \sum_{j\neq i} d_i m_{ij} u_i u_j$$

$$\leq 2\sum_{i=1}^{n} d_i m_{ii} |u_i|^2 + \sum_{j\neq i} d_i |m_{ij}|\, |u_i|\, |u_j| = \hat{u}^t(D\hat{M} + \hat{M}^tD)\hat{u} < 0.$$

Hence $DM+M^tD$ is negative definite. By the Lyapunov theorem, $s(M)<0$ (cf. [13]). □

Proof of Theorem 5.5.2. Introducing a new variable $w=U-x-y_1$, we rewrite the system (5.5.1) by

$$\begin{aligned} w' &= -\mu w + k y_1, \\ y_1' &= -(k+\mu)y_1 + \frac{c(U-w-y_1)}{U-w+y_2+\cdots+y_m}\sum_{r=1}^{m}\beta_r y_r \\ y_r' &= k y_{r-1} - (k+\mu)y_r, \quad r=2,\ldots,m, \end{aligned} \tag{5.5.2}$$

where $y=y_1+\cdots+y_m$.

Suppose $(w,y_1,\ldots,y_m)$ is the endemic equilibrium. Then,

$$-(k+\mu)y_1 + \frac{c(U-w-y_1)}{U-w+y_2+\cdots+y_m}\sum_{r=1}^{m}\beta_r y_r = 0$$

and $w,y_1,\ldots,y_m$ satisfy the identities (5.3.4)-(5.3.5). Hence, by calculation,

$$\begin{aligned} y_1 &= \frac{U(1-\alpha)(c\theta_\mu\beta_\mu-1)}{c\theta_\mu\beta_\mu-\alpha^m}, \\ w &= \frac{k}{\mu}\,y_1 \quad \text{and} \quad y_r=\alpha^{r-1}y_1, \quad r=2,\ldots,m. \end{aligned}$$

Let B be the Jacobian matrix at the endemic equilibrium. Then

$$B=\left(\begin{array}{c:cccc} -\mu & k & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\ell & & & & \\ 0 & & M & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right)$$

where $\ell = \frac{(c\theta_\mu\beta_\mu - 1)^2}{c\theta_\pi^2\beta_\mu^2}$ and

$$M = \begin{pmatrix} -(k+\mu) - \frac{c\theta_\mu\beta_\mu - 1}{\theta_\mu} + \frac{\beta_1}{\theta_\mu\beta_\mu}, & -\frac{c\theta_\mu\beta_\mu - 1}{c\theta_\mu^2\beta_\mu} + \frac{\beta_2}{\theta_\mu\beta_\mu}, & \dots, & -\frac{c\theta_\mu\beta_\mu - 1}{c\theta_\mu^2\beta_\mu} + \frac{\beta_m}{\theta_\mu\beta_\mu} \\ k & -(k+\mu) & & 0 \\ & \ddots & \ddots & \\ 0 & & k & -(k+\mu) \end{pmatrix}.$$

Since

$$\hat{M} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^{m-1} \end{pmatrix} = \frac{c\theta_\mu\beta_\mu - 1}{c\theta_\mu^2\beta_\mu} \begin{pmatrix} \sum_{r=1}^{m-1} \alpha^r - c\theta_\mu\beta_u \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leq 0,$$

$$s(\hat{M}) \leq 0.$$

Moreover,

$$\begin{aligned} \det \hat{M} &= \det[\hat{M} \times \begin{pmatrix} 1 & & 0 \\ \alpha & \ddots & \\ \vdots & \ddots & \\ \alpha^{m-1} & \dots \ \alpha & 1 \end{pmatrix}] \\ &= -\frac{(k+\mu)^{m-1}(c\theta_\mu\beta_\mu - 1)(\sum_{r=1}^{m-1} \alpha^r - c\theta_\mu\beta_\mu)}{c\theta_\mu^2\beta_\mu} \neq 0. \end{aligned}$$

Thus $s(\hat{M}) < 0$.

By Lemma 5.5.3, there is a positive diagonal matrix $D_1 = \operatorname{diag}\{d_1, \dots, d_m\}$ such that $D_1 M + M^t D_1$ is negative definite. Define $D = \operatorname{diag}\{d_0, d_1, \dots, d_m\}$, where $d_0 = \frac{d_1 \ell}{k} > 0$. Then,

$$DB + B^t D = \begin{pmatrix} -2d_0\mu & 0 \\ 0 & D_1 M + M^t D_1 \end{pmatrix}$$

is negative definite. Therefore $s(B) < 0$, which implies the endemic equilibrium is locally asymptotically stable. □

Corollary 5.5.4. *If* $m < 3$, *that is, the number of the infection stages is less than three, the endemic equilibrium is locally asymptotically stable.*

Proof. It is trivial for $m = 1$.

When $m = 2$, then

$$\sum_{r=1}^{m-1} \alpha^r = \alpha = \frac{k}{k+\mu} \leq 1 < c\theta_\mu \beta_\mu.$$

By Theorem 5.5.2, we complete the proof. □

Corollary 5.5.5. *If* $k \leq \min\{c\beta_1, \ldots, c\beta_{m-1}\}$, *then the endemic equilibrium is locally asymptotically stable.*

Proof. From $k \leq \min\{c\beta_1, \ldots, c\beta_{m-1}\}$,

$$k \sum_{r=1}^{m-1} \alpha^{r-1} \leq c \sum_{r=1}^{m-1} \alpha^{r-1} \beta_r < c \sum_{r=1}^{m} \alpha^{r-1} \beta_r.$$

The last inequality implies $\sum_{r=1}^{m-1} \alpha^r < c\theta_\mu \beta_\mu$. □

Corollary 5.5.6. *If* $k \leq \mu$, *then the endemic equilibrium is locally asymptotically stable.*

Proof. Since $k \leq \mu$, $\alpha = \frac{k}{k+\mu} \leq \frac{1}{2}$. Thus, $\sum_{r=1}^{m-1} \alpha^r \leq 1 < c\theta_\mu \beta_\mu$.

□

Corollary 5.5.6 showed that if the transfer rate is not greater than the mortality rate from all other sources, then the endemic equilibrium is locally asymptotically stable. In the next theorem we will see that in this particular case the endemic equilibrium is not only locally but globally asymptotically stable.

THEOREM 5.5.7. *If* $k \le \mu$, *then the endemic equilibrium is globally asymptotically stable in* $C \setminus C_0$.

In the proof of Theorem 5.5.7 we need the following lemma.

LEMMA 5.5.8. *Let* $\phi(t)$ *be a bounded continuous function on* $[0, \infty)$ *and* $\psi(t)$ *be a solution of the differential equation* $\psi' = a\phi(t) - b\psi$, *where* $a \ge 0, b > 0$. *Then,*

$$\frac{a}{b} \liminf_{t\to\infty} \phi(t) \le \liminf_{t\to\infty} \psi(t) \le \limsup_{t\to\infty} \psi(t) \le \frac{a}{b} \limsup_{t\to\infty} \phi(t). \tag{5.5.3}$$

PROOF. Evidently,

$$\psi(t) = \psi(0)e^{-bt} + a\int_0^t e^{-b(t-s)}\phi(s)ds.$$

For any $1 > \varepsilon > 0$, there is a $T > 0$ such that when $t \ge T$,

$$|\psi(0)|e^{-bt} \le \frac{\varepsilon}{2},$$
$$\frac{2aM}{b} e^{-bt} \le \frac{\varepsilon}{2} \quad \text{and} \quad \phi(t) \le \limsup_{t\to\infty} \phi(t) + \varepsilon,$$

where $1 + \frac{a}{b} + |\phi(t)| \le M$.

Hence for $t \geq 2T$,

$$\psi(t) \leq \frac{\varepsilon}{2} + aM \int_0^T e^{-b(t-s)} ds + a \int_T^t e^{-b(t-s)} (\limsup_{t \to \infty} \phi(t) + \varepsilon) ds$$
$$\leq \varepsilon + \frac{a}{b} \limsup_{t \to \infty} \phi(t).$$

Therefore, $\limsup_{t \to \infty} \psi(t) \leq \frac{a}{b} \limsup_{t \to \infty} \phi(t)$.

A similar argument can prove the first half of (5.5.2). $\square$

PROOF OF THEOREM 5.5.7. Let $(w^s, y_1^s, \ldots, y_m^s)$ be the endemic equilibrium and $(w(t), y(t))$ be a solution of (5.5.2) with $(w(0), y(0)) \in C \setminus C_0$. By Theorem EQ in Section 5.4, we have

$$\liminf_{t \to \infty} y_r(t) > \eta > 0, \quad r = 1, 2, \ldots, m.$$

We now define $\hat{w}(t) = w(t) - w^s, \hat{y}_r(t) = y_r(t) - y_r^s$, $r = 1, \ldots, m$. Then it follows from Lemma 5.5.8 that

$$\frac{k}{\mu} \liminf_{t \to \infty} \hat{y}_1(t) \leq \liminf_{t \to \infty} \hat{w}(t) \leq \limsup_{t \to \infty} \hat{w}(t) \leq \frac{k}{\mu} \limsup_{t \to \infty} \hat{y}_1(t)$$

$$\text{and} \quad \alpha^{r-1} \liminf_{t \to \infty} \hat{y}_1(t) \leq \liminf_{t \to \infty} \hat{y}_r(t) \leq \limsup_{t \to \infty} \hat{y}_r(t) \leq \alpha^{r-1} \limsup_{t \to \infty} \hat{y}_1(t)$$

$$r = 1, \ldots, m. \tag{5.5.4}$$

Hence, for a given $\varepsilon > 0$, whose value will be chosen later, there is a $T > 0$ such that for $t \geq T$, $i = 1, \ldots, m$,

$$\frac{k}{\mu} \liminf_{t \to \infty} \hat{y}_1(t) - \frac{\varepsilon}{2} < \hat{w}(t) < \frac{k}{\mu} \limsup_{t \to \infty} \hat{y}_1(t) + \frac{\varepsilon}{2}$$

$$\alpha^{r-1} \liminf_{t \to \infty} \hat{y}_1(t) - \frac{\varepsilon}{2} < \hat{y}_r(t) < \alpha^{r-1} \limsup_{t \to \infty} \hat{y}_1(t) + \frac{\varepsilon}{2} \tag{5.5.5}$$

$$y_r(t) \geq \eta,$$

and

$$\begin{aligned} &\sup_{s\geq t}\hat{y}_1(s) > \limsup_{t\to\infty}\hat{y}_1(t) - \frac{\varepsilon}{2} \\ &\inf_{s\geq t}\hat{y}_1(s) < \liminf_{t\to\infty}\hat{y}_1(t) + \frac{\varepsilon}{2}, \end{aligned} \tag{5.5.6}$$

Suppose $\limsup_{t\to\infty}\hat{y}_1(t) > \delta > 0$. Now for a fixed $t \geq T$, we can find an arbitrary large $t' > t$ such that $\max_{t'\geq s\geq t}\hat{y}_1(s) \geq \limsup_{t\to\infty}\hat{y}_1(t) - \frac{\varepsilon}{2}$.

Let t_0 be the maximal point. Then at $t = t_0$,

$$\begin{aligned} \hat{y}_1'(t_0) = {} & -(k+\mu)\hat{y}_1(t_0) + \frac{c(U - w^s - y_1^s)}{U - w^s + y_2^s + \cdots + y_m^s}\sum_{r=1}^{m}\beta_r\hat{y}_r(t_0) \\ & - c\,\frac{y^s(\hat{w}(t_0) + \hat{y}_1(t_0)) + (U - w^s - y_1^s)(\hat{y}_1(t_0) + \cdots + \hat{y}_m(t_0))}{(U - w^s + y_2^s + \cdots + y_m^s)(U - w(t_0) + y_2(t_0) + \cdots + y_m(t_0))}\sum_{r=1}^{m}\beta_r y_r(t_0). \end{aligned}$$

By (5.5.5) and (5.5.6),

$$\begin{aligned} \hat{y}_1'(t_0) \leq {} & \frac{c(U - w^s - y_1^s)}{U - w^s + y_2^s + \cdots + y_m^s}\sum_{r=1}^{m}\beta_r\varepsilon - \frac{c}{U^2}\sum_{r=1}^{m}\beta_r\eta \times [y^s(1 - \frac{k}{\mu})\hat{y}_1(t_0) - \\ & - y^s\varepsilon + (U - w^s - y_1^s)(1 - \alpha - \cdots - \alpha^{m-1})\hat{y}_1(t_0) - (m-1)(U - w^s - y_1^s)\varepsilon] \\ \leq {} & - \frac{c\delta[y^s(1 - \frac{k}{\mu}) + (U - w^s - y_1^s)(1 - \alpha - \cdots - \alpha^{m-a})]\sum_{r=1}^{m}\beta_r\eta}{U^2} \\ & + c\varepsilon[\frac{(U - w^s - y_1^s)}{U - w^s + y_2^s + \cdots + y_m^s}\sum_{r=1}^{m}\beta_r + \frac{y^s + (m-1)(U - w^s - y_1^s)}{U^2}]. \end{aligned}$$

Thus we can choose ε so small that $\hat{y}_1'(t_0) < 0$. Therefore, we have shown that there exists a number $T > 0$ such that for any $t \geq T$, there is a $t' > t$ such that the maximal point of $\hat{y}_1(s)$ on $[t, t']$ is t. That implies that $\hat{y}_1(t)$ is monotonically decreasing on $[T, \infty)$ and its derivative is less than a negative value. But that is impossible because of the boundness of $\hat{y}_1(t)$. Hence, $\limsup_{t\to\infty}\hat{y}_1(t) \leq 0$.

Similarly, we can prove $\liminf_{t\to\infty} \hat{y}_1(t) \geq 0$.

So $\lim_{t\to\infty} \hat{y}_1(t) = 0$. Finally, using the inequalities in (5.5.4), we obtain

$$\lim_{t\to\infty} \hat{w}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \hat{y}_r(t) = 0, \quad r = 1, \ldots, m.$$

That implies the system (5.5.2) or (5.5.1) is globally asymptotically stable. □

5.6 Results for General Models.

In the previous sections we analyzed the system (5.2.1)-(5.2.3) under the assumption that β_{ijr} only depends on r. However, the transitions between susceptibles and infected may vary from subpopulations to subpopulations, based on many factors such as age, race, geographic separation, etc. Hence, in some circumstances it is more realistic to remove the restriction from the transmission fractions. But this situation makes the model tremendously difficult to deal with. The methods taken in [11] can no longer be used. Nevertheless, some mathematical properties of the system (5.2.1)-(5.2.3) can be obtained by using the techniques in Section 5.3 and 5.4 in this paper and the theory of monotonic flows.

First, we notice that the system

$$\begin{aligned} y_i' = {} & -\theta_\mu^{-1} y_i + c_i\left(U_i - \frac{1}{1-\alpha^m}\, y_i\right) \\ & \times \left[\frac{\rho_i \beta_{ii} y_i}{U_i - \frac{\alpha^m}{1-\alpha^m}\, y_i} + \frac{(1-\rho_i)\sum_{k=1}^{n} c_k(1-\rho_k)\beta_{ik} y_k}{\sum_{k=1}^{n} c_k(1-\rho_k)\left(U_k - \frac{\alpha^m}{1-\alpha^m}\, y_k\right)}\right], \end{aligned} \tag{5.6.1}$$

$i = 1, \ldots, n$

where $\beta_{ij} = \frac{1-\alpha}{1-\alpha^m} \sum_{r=1}^{m} \alpha^{r-1} \beta_{ijr}$, has the same equilibrium distribution as the system (5.2.1)-(5.2.3) provided that they satisfy the identities (5.3.4) and (5.3.5).

It follows from Lemma 5.3.4 that the system (5.6.1) is cooperative and from the proof of Theorem 5.3.1 that the disease-free equilibrium is unstable if and only if the origin of (5.6.1) is unstable. Hence if we pick up a solution of (5.6.1), whose initial value is close to zero and is on the eigenvector of the Jacobian matrix corresponding to the stability modulus at the origin, the solution tends to a positive equilibrium. This equilibrium is the minimal equilibrium. On the other hand, if we pick up the solution with the initial value $(1-\alpha^m)(U_1, \dots, U_m)$, the solution tends to an equilibrium, which is maximal. In this case, the set of all positive equilibria is a partial order set with a minimal point and a maximal point.

Next, we turn back to the system (5.2.1)-(5.2.3). Looking at the proof of Theorem EQ in Section 5.4, we see that the proof does not use any properties of β_{ijr}. In other words, Theorem EQ is true no matter whether β_{ijr} depends on i and j or not. Thus if the disease-free equilibrium is unstable the system (5.2.1)-(5.2.3) persists uniformly.

Summarizing the above argument, we can conclude as follows for this general model.

THEOREM 5.6.1. *If the disease-free equilibrium is unstable then there exists at least one endemic equilibrium and this general system is uniformly persistent. Furthermore, when there are more than one endemic equilibria we have two endemic equilibria* (x^1, y^1) *and* (x^2, y^2) *such that for any endemic equilibrium* (x^s, y^s), $x^2 \leq x^s \leq x^1$ *and* $y^1 \leq y^s \leq y^2$.

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CHAPTER 6

ON THE UNIQUENESS AND NONUNIQUENESS OF ENDEMIC EQUILIBRIA OF AN HIV/AIDS TRANSMISSION MODEL WITH SUBPOPULATIONS

6.1 Introduction.

In this chapter, we study the following HIV/AIDS transmission model,

$$\begin{aligned} x_i' &= U_i - \mu x_i - c_i x_i \sum_{j=1}^{n} \rho_{ij}\beta_{ij} \frac{y_j}{x_j + y_j}, \\ y_i' &= -(k_i + \mu) y_i + c_i x_i \sum_{j=1}^{n} \rho_{ij}\beta_{ij} \frac{y_j}{x_j + y_j}, \qquad (6.1.1) \\ z_i' &= k_i y_i - \delta_i z_i, \end{aligned}$$

for $i = 1, \ldots, n$.

In this model, a population is divided into n subpopulations by sexual activity. Then each subpopulation is redivided into three epidemiological classes: susceptibles, HIV infecteds and AIDS infecteds. Moreover, we denote the number of persons in each class by x_i, y_i and z_i respectively. It is assumed that individuals having AIDS no longer take part in transmission. The parameters in the system (6.1.1) are defined as follows:

$U_i =$ the constant recruitment rate of susceptibles into the i-th subpopulation;

$c_i =$ the number of persons contacted per person in the i-th subpopulation per unit time;

$\rho_{ij} =$ the proportion of the contacts of a person in the i-th subpopulation with persons in j-th subpopulation. In this chapter, we will only consider proportionate mixing. So ρ_{ij} takes form

$$\rho_{ij} = \frac{c_j(x_j + y_j)}{\sum_{k=1}^{n} c_k(x_k + y_k)};$$

$\beta_{ij} =$ the transmission rate between a susceptible in the i-th subpopulation and an infected in the j-th subpopulation;

$k_i =$ the transfer rate from y_i to z_i;

$\mu =$ the mortality rate from all other sources;

$\delta_i =$ the mortality rate for persons with AIDS in the i-th subpopulation.

The n-group model (6.1.1) and its generalizations have been widely used for studying the spread of HIV/AIDS by many authors. In a homogeneous population $(n = 1)$, Anderson, Medley, May and Johnson [1] considered a model similar to (6.1.1) except that they assume HIV infection does not always lead to AIDS. Hence, a class of non-infectious seropositives was introduced. The same model was studied by Castillo-Chavez, Cooke, Huang and Levin [2]. They showed that the stability modulus of the Jacobian at the no-disease equilibrium is a threshold. Below the threshold the no-disease equilibrium is unique and is globally asymptotically stable, and above it there is a unique positive endemic equilibrium, which is locally asymptotically stable and globally asymptotically stable under certain conditions. Many results have been obtained for a heterogeneous population $(n > 1)$. Anderson et al. [1] studied the model (6.1.1) with no recruitment $(U_i = 0)$, zero competing mortality rate $(\mu = 0)$ and group-independent transmission rate $(\beta_{ij} = \beta)$. Jacquez, Simon, Koopman, Sattenspiel and Perry [8] also investigated the model (6.1.1) but assumed that the HIV infecteds pass through m infective stages from initial infection to having AIDS, and the transmission rates only vary with the stage of infection. A threshold was found such that below it the no-disease equilibrium is the only equilibrium and above it a (unique) positive endemic equilibrium occurs. We extended the work of Jacquez et al. in the last chapter by showing that the stability modulus of the Jacobian at the disease-free equilibrium can be chosen as a threshold. Hence, the existence of the positive endemic equilibrium completely depends on the stability of the no-disease equilibrium. Dropping the restriction on the transmission rates, we also prove that there always exists a positive endemic equilibrium provided the disease-free equilibrium is unstable. Recently, some impressive results have been obtained for the model (6.1.1) by Castillo-Chavez et al. [2,3]. For $\mu = \mu_0$, at which the stability modulus of the Jacobian at the disease-free equilibrium is zero, they introduced an expression $h(\mu_0)$. If $h(\mu_0) \neq 0$, μ_0 is a bifurcation point. If $h(\mu_0) > 0$ $(h(\mu_0) < 0)$, there is a unique positive endemic equilibrium continuously depending on μ

when μ is slightly less (greater) than μ_0. In the case when μ is slightly greater than μ_0 and $h(\mu_0) < 0$, they showed that the model (6.1.1) has at least two positive endemic equilibria. They also showed that for $\mu \in (0, \mu_0)$, the model (6.1.1) has a positive endemic equilibrium (this result has been extended by us to more general models. See Section 5.5 of Chapter 5). However, there are still some questions remaining unanswered. For instance, is the positive endemic equilibrium unique when $\mu \in (0, \mu_0)$? If the answer is no, what kind of conditions can guarantee the uniqueness of the positive endemic equilibrium?

The purpose of the present chapter is to answer these questions.

The remainder of the chapter is organized as follows. In the next section, some basic properties of the theory of monotone flows are presented. In Section 6.3, we give several conditions under which the positive endemic equilibrium exists and is unique. These conditions can be verified easily as long as the parameters in (6.1.1) are known. In Section 6.4, we present a two-group model. By properly choosing the parameters, we show that this model has at least three positive endemic equilibria.

6.2 Preliminaries.

In this section, we will recall some known results on the theory of monotone flows which will be essential to our work.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. For any two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we write

$$x < y \quad \text{if} \quad x_i < y_i \quad \text{for} \quad i = 1, \ldots, n,$$

$$x \leq y \quad \text{if} \quad x_i \leq y_i \quad \text{for} \quad i = 1, \ldots, n.$$

Consider a system of ordinary differential equations in $\mathbb{R}^n$,

$$x' = f(x) \tag{6.2.1}$$

where $x \in \mathbb{R}^n$, $f = (f_1, \cdots, f_n)$ is continuously differentiable.

We say the system (6.2.1) is cooperative if at any point,

$$\frac{\partial f_i}{\partial x_j} \geq 0 \quad \text{for} \quad i \neq j.$$

It is well-known that the flow generated by a cooperative system is a monotone flow: for any solutions $x(t)$ and $y(t)$ of (6.2.1), if $x(0) \leq y(0)$, then $x(t) \leq y(t)$ for all $t \geq 0$.

Let x^* be an equilibrium of (6.2.1) and A^* be the Jacobian of (6.2.1) at x^*. We denote $s(A^*)$ to be the stability modulus of A^*, which is the maximum of the real part of the eigenvalues of A^*.

An equilibrium x^* is called a sink if $s(A^*) < 0$, and a trap if there is a (non-empty) open set N in $\mathbb{R}^n$ such that all solutions with their initial values in N converge to x^* as $t \to \infty$. It is easy to see if x^* is a trap,then $s(A^*) \leq 0$. If $s(A^*) > 0$, we say x^* is strongly unstable.

The following theorems are due to Hirsch and Selgrade. We omit their proofs and refer the interested readers to [4], [5] and [12].

THEOREM 6.2.1. *Suppose system (6.2.1) is cooperative and let* $\phi(t)$ *be a bounded solution of (6.2.1) with* $\phi(0) = x$. *If one of the following conditions holds:*

(i) $f(x) \geq 0 \quad \big(f(x) \leq 0\big)$;

(ii) $\phi(T) > x \quad \big(\phi(T) < x\big)$ *for some* $T > 0$.

Then the solution $\phi(t)$ *tends to an equilibrium, i.e.*

$$\lim_{t\to\infty} \phi(t) = x^* \quad \text{for some equilibrium} \quad x^*.$$

THEOREM 6.2.2. *Let* $\phi(t)$ *be defined in Theorem 6.2.1. If* $\lim_{t\to\infty} \phi(t) = x^*$ *and* $x^* < x$ *or* $x^* > x$, *then* x^* *is a trap.*

THEOREM 6.2.3. *Let* x^* *and* x^{**} *be two equilibria of (6.2.1) and* $x^* < x^{**}$. *If* x^* *and* x^{**} *both are sinks, there is an equilibrium* $\hat{x}$ *such that* $x^* < \hat{x} < x^{**}$ *and* $\hat{x}$ *is unstable. If* x^* *and* x^{**} *are strongly unstable, there is an equilibrium* $\bar{x}$ *such that* $x^* < \bar{x} < x^{**}$ *and* $\bar{x}$ *is a trap.*

6.3 Sufficient Conditions for the Uniqueness of Endemic Equilibrium.

Since the first two equations in (6.1.1) are independent of z_i and the third equation in (6.1.1) is a linear equation in z_i (hence easily solved once $y_i(t)$ is known), it suffices to consider the first two equations of (6.1.1) for the moment. We begin this section with introducing an equivalent system of (6.1.1)

$$v_i' = -\mu v_i + k_i y_i, \tag{6.3.1a}$$

$$y_i' = -(k_i+\mu)y_i + c_i(U_i - \mu v_i - \mu y_i) \times \frac{\sum\limits_{j=1}^{n} c_j \beta_{ij} y_j}{\sum\limits_{k=1}^{n} c_k(U_k - \mu v_k)}, \tag{6.3.1b}$$

for $i = 1, \ldots, n$.

Obviously, the system (6.1.1) and (6.3.1) can be derived from one another by the relations

$$v_i = \frac{U_i}{\mu} - x_i - y_i, \qquad i = 1, \ldots, n.$$

Hence, for the rest of this section, we will study (6.3.1) instead of (6.1.1).

Let

$$K = \operatorname{diag}\{k_1, \ldots, k_n\},$$

$$L = \frac{1}{\sum\limits_{k=1}^{n} c_k U_k} \begin{pmatrix} c_1^2 U_1 \beta_{11} & \ldots & c_1 U_1 c_n \beta_{1n} \\ \vdots & & \vdots \\ c_n U_n c_1 \beta_{n1} & \ldots & c_n^2 U_n \beta_{nn} \end{pmatrix},$$

and

$$A(\mu) = -K - \mu E + L,$$

where E is the $n \times n$ identity matrix. Thus, the Jacobian of (6.3.1) at the disease-free equilibrium: $(v, y) = (0, 0)$ is

$$\begin{pmatrix} -\mu E & K \\ 0 & A(\mu) \end{pmatrix},$$

and the stability of the disease-free equilibrium is decided by the stability of $A(\mu)$.

Assume that at $\mu = \mu_0$, $s(A(\mu_0)) = 0$. Then

$$s(A(\mu)) = s(A(\mu_0)) - (\mu - \mu_0) > 0,$$

for $0 < \mu < \mu_0$ and $s(A(\mu)) < 0$ for $\mu > \mu_0$.

The model (6.1.1) or (6.3.1) has been examined by many authors. We summarize some of their results here for reference.

We introduce the expression

$$h(\mu_0) = \sum_{i=1}^{n} \bar{I}_i I_i \sum_{j=1}^{n} (c_i \beta_{ij} - k_j) c_j I_j,$$

where $I = (I_1, \ldots, I_n)$ (resp. $\bar{I} = (\bar{I}_1, \ldots, \bar{I}_n)$) is a positive eigenvector of $(\mu_0 E + K)^{-1} A(\mu_0)$ (resp. $A^T(\mu_0)(\mu_0 E + K)^{-1}$) corresponding to the eigenvalue one. When L is irreducible, Castillo-Chavez et al. proved

THEORÈM 6.3.1. *([6]) If* $h(\mu_0) \neq 0$, μ_0 *is a bifurcation point. Moreover, if* $h(\mu_0) > 0$ (resp. $h(\mu_0) < 0$), *there is an* $\varepsilon > 0$ *and unique differentiable functions* x *and* y *mapping* $(\mu_0 - \varepsilon, \mu_0] \to \mathbb{R}^n_+$ (resp. $[\mu_0, \mu_0 + \varepsilon) \to \mathbb{R}^n_+$)

such that $(x(\mu_0), y(\mu_0)) = (\frac{U_1}{\mu}, \ldots, \frac{U_n}{\mu}, 0, \ldots, 0)$ *and* $(x(\mu), y(\mu))$ *is a positive endemic equilibrium of (6.1.1). Furthermore, the endemic equilibrium is locally asymptotically stable for* $\mu \in (\mu_0 - \varepsilon, \mu_0)$ *(resp. unstable for* $\mu \in (\mu_0, \mu_0 + \varepsilon)$*).*

THEOREM 6.3.2. *([6] & Chapter 5 in this thesis) For each* $\mu \in (0, \mu_0)$, *(6.1.1) has a positive equilibrium.*

THEOREM 6.3.3. *([6]) If* $h(\mu_0) < 0$, *then there is an* $\varepsilon > 0$ *such that (6.1.1) has at least two positive equilibria for each* $\mu \in (\mu_0, \mu_0 + \varepsilon)$.

If we suppose $\beta_{ij} = \beta$, i.e. β_{ij} is independent of i and j in (6.1.1), a threshold condition is given.

THEOREM 6.3.4. *([8] & Chapter 5 in this thesis) If* $s(A(\mu)) < 0$, *then the disease-free equilibrium is locally asymptotically stable and there are no positive endemic equilibria. If* $s(A(\mu)) > 0$ *and* L *is irreducible, the disease-free equilibrium is unstable and (6.1.1) has a unique positive endemic equilibrium.*

We now turn to our main results in this section. The standing assumptions hereafter are that: the transmission rates β_{ij} are arbitrary nonnegative constants and L is irreducible.

THEOREM 6.3.5. *If* $A(\mu)$ *is a nonnegative matrix, then the system (6.3.1) has a unique positive equilibrium.*

REMARK. The nonnegativity of $A(\mu)$ is equivalent to

$$\mu + k_i \leq \frac{c_i^2 U_i \beta_{ii}}{\sum_{k=1}^{n} c_k U_k}, \quad i = 1, \dots, n. \tag{6.3.2}$$

Our method in the proof of Theorem 6.3.5 also can be also used for preferred mixing (cf. [8] for the definition). The condition turns out to be

$$\mu + k_i \leq \frac{c_i^2 (1-\rho_i)^2 U_i \beta_{ii}}{\sum_{k=1}^{n} c_k (1-\rho_k) U_k}, \qquad i = 1, \dots, n. \tag{6.3.3}$$

THEOREM 6.3.6. *If*

$$c_i \beta_{ij} - k_j > 0 \quad \text{for all} \quad i \quad \text{and} \quad j, \tag{6.3.4}$$

then the system (6.3.1) exists a unique positive endemic equilibrium.

REMARK. If $c_i \beta_{ij} - k_j > 0$ for all i and j, we automatically have $h(\mu_0) > 0$.

We now begin to prove our results. First, we need the following lemmas.

LEMMA 6.3.7. *Introduce a system of* n *ordinary differential equations,*

$$y_i' = -(k_i + \mu) y_i + c_i \big(U_i - (k_i + \mu) y_i\big) \frac{\sum_{j=1}^{n} c_j \beta_{ij} y_j}{\sum_{\ell=1}^{n} c_\ell (U_\ell - k_\ell y_\ell)} \tag{6.3.5}$$

$$i = 1, \dots, n.$$

Then, the number of positive endemic equilibria of (6.3.1) is equal to the number of equilibria of (6.3.5) in the region

$$G = \{0 < y_i < \frac{U_i}{k_i + \mu}\,;\; i = 1, \ldots, n\}.$$

PROOF. If $\binom{v^*}{y^*}$ is a positive equilibrium of (6.3.1), where $v^* = (v_1^*, \ldots, v_n^*)^t$, $y^* = (y_1^*, \ldots, y_n^*)^t$, then, for $i = 1, \ldots, n$

$$v_i^* = \frac{k_i}{\mu}\, y_i^*, \quad v_i^* \le \frac{U_i}{\mu}\,, \tag{6.3.6}$$

$$-(k_i + \mu) y_i^* + c_i(U_i - \mu v_i^* - \mu y_i^*)\, \frac{\sum_{j=1}^{n} c_j \beta_{ij} y_j^*}{\sum_{\ell=1}^{n} c_\ell(U_\ell - \mu v_\ell^*)} = 0. \tag{6.3.7}$$

Replacing v_i^* of (6.3.7) by (6.3.6), we have

$$-(k_i + \mu) y_i^* + c_i\big(U_i - (k_i + \mu) y_i^*\big)\, \frac{\sum_{j=1}^{n} c_j \beta_{ij} y_j^*}{\sum_{\ell=1}^{n} c_\ell(U_\ell - k_\ell y_\ell^*)} = 0.$$

Hence y^* is an equilibrium of (6.3.5). It is easy to see $y^* \in G$. Conversely, if $y^* \in G$ is an equilibrium of (6.3.5), let

$$v^* = \frac{1}{\mu}\, K y^*.$$

Then $\binom{v^*}{y^*}$ is a positive endemic equilibrium of (6.3.1). □

Lemma 6.3.8. *The system (6.3.5) is cooperative on* G.

Proof. Denote the right-hand side of (6.3.5) by f_i. Then for $i \neq j$,

$$\frac{\partial f_i}{\partial y_j} = c_i\big(U_i - (k_i + \mu)y_i\big)\, \frac{[\sum_{\ell=1}^{n} c_\ell(U_\ell - k_\ell y_\ell)]c_j\beta_{ij} + [\sum_{\ell=1}^{n} c_\ell\beta_{i\ell}y_\ell]c_jk_j}{[\sum_{\ell=1}^{n} c_\ell(U_\ell - k_\ell y_\ell)]^2}$$

$$\geq 0.$$

□

Lemma 6.3.9. *If the matrix* $\begin{pmatrix} \mu E & K \\ B & A \end{pmatrix}$ *is the Jacobian of (6.3.1) at an equilibrum* $\binom{v^*}{y^*}$, *then* $\mu^{-1}BK + A$ *is the Jacobian of (6.3.5) at the corresponding equilibrium* y^*. *Consequently* $A(\mu)$ *is the Jacobian of (6.3.5) at* $y^* = 0$.

Proof. Let us denote the right-hand side of (6.3.1b) and (6.3.5) by $F(v, y)$ and $f(y)$, respectively. Then we have the identity

$$F(\mu^{-1}Ky, y) \equiv f(y)$$

$$Df(y) = \mu^{-1}D_1F(\mu^{-1}Ky, y)K + D_2F(\mu^{-1}Ky, y).$$

Thus, at $\binom{v^*}{y^*}$,

$$Df(y^*) = \mu^{-1}D_1F(\mu^{-1}Ky^*, y^*)K + D_2F(\mu^{-1}Ky^*, y^*)$$

$$= \mu^{-1}D_1F(v^*, y^*)K + D_2F(v^*, y^*) = \mu^{-1}BK + A.$$

□

Lemma 6.3.10. *Let* y^* *be a positive equilibrium of (6.3.5) in* G. *If* $A(\mu)y^* > 0$, *then* y^* *is a sink.*

Proof. By calculations, the Jacobian of (6.3.5) at y^* is

$$Df(y^*) = \frac{1}{\sum\limits_{\ell=1}^{n} c_\ell(U_\ell - k_\ell y_\ell^*)} \begin{pmatrix} -c_1(k_1+\mu)\sum\limits_{k=1}^{n} c_k\beta_{1k}y_k^* & & 0 \\ & \ddots & \\ 0 & & -c_n(k_n+\mu)\sum\limits_{k=1}^{n} c_k\beta_{nk}y_k^* \end{pmatrix},$$

$$+\frac{1}{\left[\sum\limits_{\ell=1}^{n} c_\ell(U_\ell - k_\ell y_\ell^*)\right]^2}$$

$$\times \begin{pmatrix} c_1^2(U_1-(k_1+\mu)y_1^*)k_1\sum\limits_{k=1}^{n} c_k\beta_{1k}y_k^* & \cdots & c_1(U_1-(k_1+\mu)y_1^*)c_nk_n\sum\limits_{k=1}^{n} c_k\beta_{1k}y_k^* \\ \vdots & & \vdots \\ c_n(U_n-(k_n+\mu)y_n^*)c_1k_1\sum\limits_{k=1}^{n} c_k\beta_{nk}y_k^* & \cdots & c_n^2(U_n-(k_n+\mu)y_n^*)k_n\sum\limits_{k=1}^{n} c_k\beta_{nk}y_k^* \end{pmatrix}$$

$$-K-\mu E+\frac{1}{\sum\limits_{k=1}^{n} c_\ell(U_\ell - k_\ell y_\ell^*)}$$

$$\times \begin{pmatrix} c_1^2(U_1-(k_1+\mu)y_1^*)\beta_{11} & \cdots & c_1(U_1-(k_1+\mu)y_1^*)c_n\beta_{1n} \\ \vdots & & \vdots \\ c_n(U_n-(k_n+\mu)y_n^*)c_1\beta_{n1} & \cdots & c_n^2(U_n-(k_n+\mu)y_n^*)\beta_{nn} \end{pmatrix}.$$

Thus, the i-th component of $Df(y^*)y^*$ is

$$\frac{\sum_{k=1}^{n} c_k\beta_{ik}y_k^*}{\sum_{\ell=1}^{n} c_\ell(U_\ell - k_\ell y_\ell^*)}\Big[-c_i(k_i+\mu)y_i^* + c_i\big(U_i - (k_i+\mu)y_i^*\big)\frac{\sum_{\ell=1}^{n} c_\ell k_\ell y_\ell^*}{\sum_{\ell=1}^{n} c_\ell(U_\ell - k_\ell y_\ell^*)}\Big]$$

$$= \frac{\sum_{k=1}^{n} c_k\beta_{ik}y_k^*}{\sum_{\ell=1}^{n} c_\ell(U_\ell - k_\ell y_\ell^*)}$$

$$\times\Big[-c_i(k_i+\mu)y_i^* + c_i\big(U_i - (k_i+\mu)y_i^*\big)\frac{\sum_{\ell=1}^{n} c_\ell U_\ell - \sum_{\ell=1}^{n} c_\ell(U_\ell - k_\ell y_\ell^*)}{\sum_{\ell=1}^{n} c_\ell(U_\ell - k_\ell y_\ell^*)}\Big]$$

$$= -\frac{\sum_{\ell=1}^{n} c_\ell U_\ell}{\sum_{\ell=1}^{n} c_\ell(U_\ell - k_\ell y_\ell^*)}\Big[-(k_i+\mu)y_i^* + c_iU_i\frac{\sum_{k=1}^{n} c_k\beta_{ik}y_k^*}{\sum_{\ell=1}^{n} c_\ell U_\ell}\Big].$$

Hence

$$Df(y^*)y^* = -\frac{\sum_{\ell=1}^{n} c_\ell U_\ell}{\sum_{\ell=1}^{n} c_\ell(U_\ell - k_\ell y_\ell^*)}A(\mu)y^* < 0.$$

It follows from the M-matrix theory that $s\big(Df(y^*)\big) < 0$. Consequently, y^* is a sink. □

PROOF OF THEOREM 6.3.5. Existence of an equilibrium is obvious for $s\big(A(\mu)\big) > 0$. Now, since $A(\mu)$ is nonnegative and irreducible, for any positive equilibrium y^* in G, $A(\mu)y^* > 0$. Hence every equilibrium in G is a sink. If there are two equilibria in G, it is easy to show that there are two equilibria y^* and y^{**} such that $y^* < y^{**}$, because G is positive invariant and (6.3.5) is cooperative on G. By Theorem 6.2.3, there exists an unstable equilibrium in G, which is a contradiction. We complete the proof.

□

PROOF OF THEOREM 6.3.6. From Lemma 6.3.10 and the proof of Theorem 6.3.5, it suffices to show that for any equilibrium in G, we have $A(\mu)y^* > 0$.

Since y^* is an equilibrium,

$$-(k_i+\mu)y_i^* + c_i(U_i-(k_i+\mu)y_i^*)\frac{\sum_{j=1}^{n} c_j\beta_{ij}y_j^*}{\sum_{j=1}^{n} c_j(U_j-k_jy_j^*)}=0,$$

$i=1,\ldots,n$, where $y^*=(y_1^*,\ldots,y_n^*)$. Thus,

$$-(k_i+\mu)y_i^*\sum_{j=1}^{n} c_jU_j + c_iU_i\sum_{j=1}^{n} c_j\beta_{ij}y_j^*$$

$$= c_i(k_i+\mu)y_i^*\sum_{j=1}^{n} c_j\beta_{ij}y_j^* - (k_i+\mu)y_i^*\sum_{j=1}^{n} c_jK_jy_j^*$$

$$= (k_i+\mu)y_i^*\sum_{k=1}^{n}(c_i\beta_{ij}-k_j)c_jy_j^* > 0.$$

This implies $A(\mu)y^* > 0$. □

6.4 A Two-group Model with at least Three Positive Endemic Equilibria.

In the last section, we presented several sufficient conditions under which the positive endemic equilibrium of the model (6.1.1) is unique if it exists. In this section, we will see if these conditions are violated, the multiple group model (6.1.1) may have multiple positive endemic equilibria, even in the case that the disease-free equilibrium is unstable $(\mu < \mu_0)$. This disprove a conjecture by Huang [7].

We now consider a two-group model $(n = 2)$. Let $U_i = k_i = c_i = 1$, $i = 1, 2$, and $\mu = \frac{1}{2}$. For given $0 < \varepsilon < \frac{1}{3}$, whose value will be chosen later, define

$$\begin{aligned}
\beta_{12} &= \frac{3}{\varepsilon},\\
\beta_{11} &= 3(3-\varepsilon) - \varepsilon\beta_{12},\\
\beta_{22} &= \frac{9}{4},\\
\beta_{21} &= \frac{3\varepsilon(3-\varepsilon)}{4-3\varepsilon} - \varepsilon\beta_{22}.
\end{aligned}$$

We obtain a two-group model:

$$\begin{aligned}
x_i' &= 1 - \frac{1}{2}\,x_i - x_i\,\frac{\beta_{i1}y_1 + \beta_{i2}y_2}{(x_1+y_1)+(x_2+y_2)},\\
y_i' &= -\frac{3}{2}\,y_i + x_i\,\frac{\beta_{i1}y_1 + \beta_{i2}y_2}{(x_1+y_1)+(x_2+y_2)},
\end{aligned} \tag{6.4.1}$$

for $i = 1, 2$.

By Lemma 6.3.7, the system (6.4.1) has the same number of positive equilibria as the following system

$$
\begin{aligned}
y_1' &= -\frac{3}{2}\,y_1 + \left(1 - \frac{3}{2}\,y_1\right) \times \frac{\beta_{11}y_1 + \beta_{12}y_2}{(1-y_1)+(1-y_2)}\,, \\
& \hspace{20em} (6.4.2) \\
y_2' &= -\frac{3}{2}\,y_2 + \left(1 - \frac{3}{2}\,y_2\right) \times \frac{\beta_{21}y_1 + \beta_{22}y_2}{(1-y_1)+(1-y_2)}\,.
\end{aligned}
$$

in the region $G : 0 < y_1 < \frac{2}{3}\,,\ 0 < y_2 < \frac{2}{3}\,.$

Hence, it suffices to show the system (6.4.2) has at least three positive equilibria in G.

CLAIM 1: the origin 0 and the point $y^* = (\frac{1}{2}\,, \frac{\varepsilon}{2}\,)$ are equilibria of the system (6.4.2).

It is obvious for the origin.

Let $y_1^* = \frac{1}{2}\,, y_2^* = \frac{\varepsilon}{2}\,.$ Then

$$
\begin{aligned}
-\frac{3}{2}\,y_1^* + \left(1 - \frac{3}{2}\,y_1^*\right) \times \frac{\beta_{11}y_1^* + \beta_{12}y_2^*}{(1-y_1^*)+(1-y_2^*)} &= -\frac{3}{4} + \frac{1}{4} \times \frac{\frac{1}{2}(\beta_{11} + \varepsilon\beta_{12})}{\frac{(3-\varepsilon)}{2}} \\
&= -\frac{3}{4} + \frac{1}{4} \times \frac{3(3-\varepsilon)}{3-\varepsilon} = 0
\end{aligned}
$$

$$
\begin{aligned}
-\frac{3}{2}\,y_2^* + \left(1 - \frac{3}{2}\,y_2^*\right) \times \frac{\beta_{21}y_1^* + \beta_{22}y_2^*}{(1-y_1^*)+(1-y_2^*)} &= -\frac{3\varepsilon}{4} + \frac{4-3\varepsilon}{4} \times \frac{\frac{1}{2}(\beta_{21} + \varepsilon\beta_{22})}{\frac{(3-\varepsilon)}{2}} \\
&= -\frac{3\varepsilon}{4} + \frac{4-3\varepsilon}{4} \times \frac{\frac{3\varepsilon(3-\varepsilon)}{4-3\varepsilon}}{3-\varepsilon} = 0.
\end{aligned}
$$

Thus $(\frac{1}{2}\,, \frac{\varepsilon}{2}\,)$ is an equilibrium.

CLAIM 2: Both equilibria 0 and y^* are unstable for sufficiently small $\varepsilon > 0$.

Let $A_0(\varepsilon)$ and $A^*(\varepsilon)$ be the Jacobians of the system (6.4.2) at the origin and the point $(\frac{1}{2}, \frac{\varepsilon}{2})$, respectively.

Then

$$A_0(\varepsilon) = \begin{pmatrix} -\frac{3}{2} + \frac{\beta_{11}}{2} & \frac{\beta_{12}}{2} \\ \frac{\beta_{21}}{2} & -\frac{3}{2} + \frac{\beta_{22}}{2} \end{pmatrix}$$

and

$$A^*(\varepsilon) = \begin{pmatrix} -6 + \frac{3+\beta_{11}}{2(3-\varepsilon)} & \frac{3+\beta_{12}}{2(3-\varepsilon)} \\ \frac{3\varepsilon+(4-3\varepsilon)\beta_{21}}{2(3-\varepsilon)} & -\frac{3}{2} - \frac{9\varepsilon}{2(4-3\varepsilon)} + \frac{3\varepsilon+(4-3\varepsilon)\beta_{22}}{2(3-\varepsilon)} \end{pmatrix}.$$

Since

$$-\frac{3}{2} + \frac{\beta_{11}}{2} = \frac{1}{2}(3-3\varepsilon) > 0,$$

$s(A_0) > 0$, i.e. the origin is unstable.

Noticing

$$\beta_{11} = 3(2-\varepsilon), \qquad \beta_{12} = \frac{3}{\varepsilon},$$
$$\beta_{21} = \frac{15\varepsilon^2}{4(4-3\varepsilon)}, \qquad \beta_{22} = \frac{9}{4},$$

$$A^*(\varepsilon) = \begin{pmatrix} -\frac{9}{2} & \frac{3(1+\frac{1}{\varepsilon})}{2(3-\varepsilon)} \\ \frac{3\varepsilon(4+5\varepsilon)}{8(3-\varepsilon)} & -\frac{3}{2} - \frac{9\varepsilon}{2(4-3\varepsilon)} + \frac{3(12-5\varepsilon)}{8(3-\varepsilon)} \end{pmatrix}.$$

Thus, the determinant of $A^*(\varepsilon)$

$$\det A^*(\varepsilon) = -\frac{9}{2}\left[-\frac{3}{2} - \frac{9\varepsilon}{2(4-3\varepsilon)} + \frac{3(12-5\varepsilon)}{8(3-\varepsilon)}\right]$$

$$-\frac{9\varepsilon(4+5\varepsilon)}{16(3-\varepsilon)^2} - \frac{9(4+5\varepsilon)}{16(3-\varepsilon)^2}.$$

Let $\varepsilon \to 0$. We have

$$\lim_{\varepsilon\to 0} \det A^*(\varepsilon) = -\frac{1}{4} < 0.$$

Hence, we can choose sufficient small $\varepsilon > 0$ such that

$$\det A^*(\varepsilon) < 0.$$

By the M-matrix theory, both eigenvalues of $A^*(\varepsilon)$ must be real. With

$\det A^*(\varepsilon) < 0$, we conclude that one of the eigenvalues of $A^*(\varepsilon)$ is positive and another is negative. Therefore, $A^*(\varepsilon)$ is an unstable matrix, which implies the equilibrium y^* is unstable.

CLAIM 3: There are two positive equilibria, denoted by $\hat{y}$ and $\bar{y}$, such that

$$0 < \hat{y} < y^*, \qquad y^* < \bar{y} < (\frac{2}{3}, \frac{2}{3}).$$

Therefore, the system (6.4.3) has at least three positive equilibria $\hat{y}$, y^*, and $\bar{y}$.

Since 0 and y^* are both strongly unstable and the system (6.4.2) is cooperative, by Theorem 6.2.3, there exists a sink $\hat{y}$ such that $0 < \hat{y} < y^*$.

To show there is an equilibrium between y^* and the point $(\frac{2}{3}, \frac{2}{3})$, we pick the solution with its initial value being $(\frac{2}{3}, \frac{2}{3})$. Then the ω-limit set of this solution is an equilibrium $\bar{y}$ because $y_i' < 0, i = 1, 2,$ at $(\frac{2}{3}, \frac{2}{3})$. But this equilibrium cannot coincide with y^*. Otherwise y^* is a trap which contradicts to that y^* is strongly unstable. Thus $\bar{y} > y^*$.

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