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THE UNIVERSITY OF ALBERTA

Amenability and Ideals in the Fourier Algebra of Locally Compact Groups

BY
BRIAN FORREST

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Amenability and Ideals in the Fourier Algebra of Locally Compact Group** submitted by **Brian Forrest** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

Supervisor

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Date: August 28, 1987

To my Mom and Dad, to Jim and to Barb

Abstract

Let G be a locally compact group and let $A(G)$ be the Fourier algebra of G. In this thesis, we shall focus on the relationship between the structure of G and the nature of the closed ideals of $A(G)$.

Necessary and sufficient algebraic and topological conditions are given for a closed ideal of $A(G)$ to possess a bounded approximate identity. It is shown that the existence of such a bounded approximate identity is intimately related to the amenability of G.

For amenable $[SIN]$-groups a complete characterization of the closed ideals in $A(G)$ with a bounded approximate identity in terms of the algebraic structure of their kernels is obtained. This generalizes a result of Liu, van Rooij and Wang.

Amenable locally compact groups are characterized with respect to the structure of their cofinite ideals and to the nature of $VN(G) = A(G)^*$ as a Banach $A(G)$-bimodule. In particular, we prove that G is amenable if and only if every derivation of G into a Banach $A(G)$-bimodule is continuous.

A necessary and sufficient condition for the close ideal $I(H)$ to be complemented in $A(G)$ is given, when H is a closed subgroup of G. It is proved that if the left regular representation of H is completely reducible, then $I(H)$ is complemented in $A(G)$. An example is presented of a group G with a normal abelian closed subgroup H for which $I(H)$ is not complemented in $A(G)$.

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Chapter 1

Introduction

In 1964, Pierre Eymard [19] defined the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$ of an arbitrary locally compact group. $B(G)$ can be identified with the continuous linear functionals on the group C^*-algebra, $C^*(G)$. With the dual norm and pointwise multiplication, $B(G)$ becomes a Banach algebra of continuous functions on G, and $A(G)$ becomes a closed ideal of $B(G)$. When G is abelian, these algebras correspond respectively to the algebras of continuous functions obtained by applying the Fourier transform to $L^1(\widehat{G})$ and the Fourier-Stieltjes transform to $M(\widehat{G})$, where $\widehat{G}$ denotes the dual group of G.

It has been shown by Martin Walter in [64] and [65] that the structure of the group G is completely determined by both $A(G)$ and $B(G)$. In this thesis, we shall be primarily concerned with the relationships between G and $A(G)$. Our principal focus will be the structure of the closed ideals of $A(G)$.

This thesis comprises seven chapters. Chapter 2 contains the definitions of most of the terms used in this introduction and throughout the remainder of the thesis. We also collect much of the notation there for easy reference. Finally, we introduce some important notions from the theory of locally compact groups, from Banach algebra theory and from harmonic analysis.

Chapter 3 concerns itself with the existence of bounded approximate identities in ideals of $A(G)$. Motivated by a theorem of Leptin [39], which states that $A(G)$ has a bounded approximate identity if and only if G is amenable, we establish as the theme of our investigation the deep connection between amenability and the existence of bounded approximate identities. Section 3.1 provides us with the algebraic and topological tools which we will need throughout.

In Section 3.3, we give a complete characterization of the closed ideals in $A(G)$ with bounded approximate identities when G is an amenable $[SIN]$-group. This generalizes [40, Theorem 13].

We close the chapter by determining the structure of closed cofinite ideals of $A(G)$. This will be used in Chapter 6 to characterize amenable groups as the class of locally compact groups for which $A(G)$ has automatically continuous derivations (Theorem 6.1.4).

In Chapter 4, we investigate the properties of $VN(G)$ (the dual of $A(G)$) as a Banach $A(G)$-bimodule. Our main result (Theorem 4.2.2) establishes the connection between the existence of bounded approximate identities in ideals of $A(G)$ and the existence of continuous projections on $VN(G)$ which commute with the module action of $A(G)$ on $VN(G)$. Again amenable groups are characterized. Applications are then given to discrete groups.

Chapter 5 deals with the problem of determining which closed ideals in $A(G)$ are complemented. We will restrict ourselves to ideals of the form $I(H)$, where H is a closed subgroup of G.

In Section 5.1, we show that $I(H)$ is complemented in $A(G)$ if and only if $A(H)$ can be linearly and continuously embedded in $A(G)$. We then show that this can be done for any closed subgroup with a completely reducible left regular representation (for example, all compact subgroups have this property). We also give an example of a closed normal subgroup H of G for which $I(H)$ is not complemented in $A(G)$ [cf. Example 5.1.11].

In Section 5.2, we go further towards the identification of subgroups which determine complemented ideals. We look at the complementation problem for product groups. This leads to applications for certain large classes of groups.

Chapter 6 deals with derivations on $A(G)$. In Section 6.1, we show that G is amenable if and only if every derivation of $A(G)$ into a Banach $A(G)$-bimodule is continuous.

In Section 6.2, we look at commutative $A(G)$-modules. We show that if G is discrete, then $A(G)$ is weakly amenable, a notion introduced and studied recently by Bade, Curtis and Dales [8].

Chapter 7 is a summary of the main results of the thesis. We also mention here some problems for further study.

CHAPTER 2

PRELIMINARIES AND NOTATION

§2.0. Introduction.

This chapter is intended to be a reference for the terms and the notation used throughout the thesis. It is divided into three sections. The first section is an introduction to classes of locally compact groups, while the second section deals with Banach algebras.

Finally, we introduce the principal objects in non-commutative harmonic analysis. We give a very brief summary of [19].

§2.1. Locally Compact Groups.

Let G be a locally compact group. Let $A \subset G$. Then A^-, A^0 and bdy A will denote the closure of A, the interior of A and the boundary of A respectively. Let f be any complex valued function on G. Define

(i) $\operatorname{supp} f = \{x \in G;\ f(x) \neq 0\}^-$,

(ii) $\check{f}(x) = f(x^{-1})$,

(iii) $\tilde{f}(x) = \overline{f(x^{-1})}$,

(iv) $L_x f(y) = f(x^{-1}y)$ for every $y \in G$,

(v) $R_x f(y) = f(yx)$ for every $y \in G$.

Let $CB(G)$ denote the Banach space of bounded complex valued continuous functions on G. Let $C_0(G)$ and $C_{00}(G)$ denote the subspaces of $CB(G)$ consisting of functions which vanish at infinity and which have compact support respectively.

Let $M(G)$ denote the Banach space of bounded regular Borel measures on G with the total variation norm. Then $M(G)$ can be identified with $C_0(G)^*$, the Banach space dual of $C_0(G)$. Let μ_G be a fixed left invariant Haar measure on G. The left invariant Haar integral on $C_{00}(G)$ associated with μ_G will be denoted by

$$\int_G f(x) d\mu_G(x),$$

or if there is no ambiguity, simply by

$$\int_G f(x) dx.$$

If A is a Haar-measurable subset of G, then the Haar measure of A is denoted by $|A|$. If A is any subset of G, then 1_A will denote the characteristic function of A.

For $1 \leq p \leq \infty$, let $(L^p(G), \|\cdot\|_p)$ be the usual Banach spaces associated with G and μ_G. With the inner product

$$\langle f, g \rangle = \int_G f(x)\, \overline{g(x)}\, dx, \qquad f, g \in L^2(G),$$

$L^2(G)$ becomes a Hilbert space.

We will find the following classes of locally compact groups to be of particular interest in subsequent chapters:

[A] *Abelian groups*

[K] *Compact groups*

[V] *Vector groups* = Groups $\mathbb{R}^n$ for $n = 1, 2, \ldots$

[D] *Discrete groups*

[Z] *Central Topological groups* = Groups G such that $G/Z(G)$ is compact, where $Z(G)$ den... ...er of G.

$[FC^-]$ *Topologically Fir... Conjugacy Class groups* = Groups such that the closure of each conjugacy class is compact.

[SIN] *Small Invariant Neighborhood groups* = Groups such that every neighborhood of the identity contains a compact neighborhood which is invariant under all inner automorphisms.

[IN] *Invariant Neighborhood groups* = Groups having a compact neighborhood of the identity which is invariant under all inner automorphisms.

Palmer's survey [48] is an excellent summary of the vast literature on these and other classes of locally compact groups. The diagrams on pages 698 and 699 of [48] will serve as our primary reference for the structure of the elements of each of the above classes and the relationships between these classes. Any such result stated without an explicit reference can be found in [48].

An element $m \in L^\infty(G)^*$ is called a *mean* if

$$1 = \|m\| = m(1_G) \quad \text{and} \quad m(f) \geq 0 \quad \text{whenever} \quad f \geq 0.$$

A mean m is said to be *left invariant* if

$$m(L_x f) = m(f) \quad \text{for every} \quad x \in G, \quad f \in L^\infty(G).$$

Let $LIM(G)$ denote the set of left invariant means on $L^\infty(G)$. Define

[Am] *Amenable groups* = Groups G for which $LIM(G) \neq \emptyset$.

The study of amenable groups goes back to von Neumann [63] who called these groups "meβbar". The term "amenable" is due to Day [15]. Day's work is the starting point for anyone interested in the phenomenon of amenability.

Amenability is a remarkable property. The class of amenable groups can be characterized by many properties which would appear at first glance to be completely independent of the existence of an invariant mean. In addition, various generalized notions of amenability have served useful purposes in other branches of mathematics as well as providing rich theories in and of themselves. At present, the most comprehensive and up-to-date references for amenability are Pier's book [50] and A. Paterson's A.M.S. monograph [49].

$[Am]$ contains all compact groups [cf. 50, p. 112] and all abelian groups [cf. 50, p. 113]. The standard example of a non-amenable group is the free group on two generators [cf. 50, p. 123].

Finally, given any locally compact group G and any subgroup H of G we will let $C_G(H)$ denote the centralizer of H in G.

§2.2. Banach Algebras.

Let $\mathcal{A}$ be a Banach algebra. A net $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ in $\mathcal{A}$ is called a *bounded left [resp. right] approximate identity* if $\lim_\alpha \|u_\alpha u - u\| = 0$ [resp. $\lim_\alpha \|u u_\alpha - u\| = 0$] for every $u \in \mathcal{A}$, and if there exists an M such that $\|u_\alpha\| < M$ for every $\alpha \in \mathfrak{A}$. $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ is a *bounded approximate identity* if it is both a left bounded approximate identity and a right bounded approximate identity.

In many respects, Banach algebras with bounded approximate identities are better behaved than those without. This is illustrated by Cohen's remarkable

Factorization theorem [cf. 29, p. 268].

Let $\mathcal{A}$ be a commutative Banach algebra. Let $\Delta(\mathcal{A})$ denote the maximal ideal space of $\mathcal{A}$. By means of the Gelfand transform, $\mathcal{A}$ can be realized as a subalgebra of $C_0(\Delta(\mathcal{A}))$.

Let I be an ideal in $\mathcal{A}$. Define

$$Z(I) = \{x \in \Delta(\mathcal{A}),\ u(x) = 0 \quad \text{for every } u \in I\}.$$

Then $Z(I)$ is a closed subset of $\Delta(\mathcal{A})$. If E is a closed subset of $\Delta(\mathcal{A})$, define

$$I(E) = \{u \in \mathcal{A},\ u(x) = 0 \quad \text{for every } x \in E\},$$

$$I_0(E) = \{u \in \mathcal{A};\ \text{supp}\ u \in \mathcal{F}(\mathcal{A})\},$$

where $\mathcal{F}(E) = \{K \subset \Delta(\mathcal{A});\ K \text{ is compact and } K \cap E = \emptyset\}$. $I_0(E)$ and $I(E)$ are ideals in $\mathcal{A}$. $I(E)$ is closed. Furthermore, if I is any ideal in $\mathcal{A}$ with $Z(I) = E$, then $I_0(E) \subseteq I \subseteq I(E)$.

A closed subset E of $\Delta(\mathcal{A})$ is said to be a *set of spectral synthesis*, or simply an *S-set*, if $I(E)$ is the only closed ideal I for which $Z(I) = E$. This is equivalent to the density of $I_0(E)$ in $I(E)$ [cf. 29, Theorem 39.18].

$\mathcal{A}$ is said to be *tauberian* if $\emptyset$ is an S-set. $\mathcal{A}$ is said to be *regular* if for every closed set $E \subseteq \Delta(\mathcal{A})$ and each $x \notin E$, there exists a $u \in \mathcal{A}$ such that $u(E) = 0$ and $u(x) = 1$. $\mathcal{A}$ is *semisimple* if whenever $u(x) = v(x)$ for every $x \in \Delta(\mathcal{A})$, then $u = v$.

$\mathcal{A}$ is said to satisfy *Ditkin's Condition* if

(i) for every $u \in A$ and $x \in \Delta(A)$ such that $u(x) = 0$, there exists a sequence $\{v_n\}$ in A such that each v_n vanishes in some neighborhood of x and $\lim_n \|uv_n - u\| = 0$.

(ii) if $\Delta(A)$ is not compact, then, in addition to (i), for every $u \in A$ there exists a sequence $\{v_n\}$ in A such that each v_n has compact support and $\lim_n \|uv_n - u\| = 0$.

§2.3. Harmonic Analysis.

Let $f, g \in L^1(G)$. We define the convolution of f by g as follows:

$$f * g(x) = \int_G f(xy)g(y^{-1})dy.$$

Then $f * g \in L^1(G)$. Furthermore, "$*$" makes $L^1(G)$ into a Banach algebra. We can also define an involution on $L^1(G)$ by

$$f^*(x) = \frac{1}{\Delta(x)} \overline{f(x^{-1})},$$

where $\Delta(x)$ denotes the modular function associated with μ_G. $L^1(G)$ becomes an involutive Banach algebra, called the *Group algebra of G*.

By the Radon-Nikodym theorem, $L^1(G)$ is a closed subspace of $M(G)$. We extend the convolution on $L^1(G)$ to $M(G)$ by

$$\mu * \nu(f) = \int_G \int_G f(xy)d\mu(x)d\nu(y)$$

for each $\mu, \nu \in M(G)$ and $f \in C_0(G)$. We extend the involution to $M(G)$ by

$$d\mu^*(x) = \overline{d\mu(x^{-1})}.$$

Again, $M(G)$ is an involutive Banach algebra containing $L^1(G)$ as a closed two-sided ideal [cf. 17, Chapter 13].

Let Σ_G denote the class of equivalence classes of continuous unitary representations of G. Each $\pi \in \Sigma_G$ induces a continuous non-degenerate $*$-representation of $L^1(G)$ by means of the formula

$$\langle \pi(f)\varsigma, \eta \rangle = \int_G f(x) \langle \pi(x)\varsigma, \eta \rangle \, dx$$

for each $f \in L^1(G)$, $\varsigma, \eta \in \mathcal{H}_\pi$, where $\mathcal{H}_\pi$ is the Hilbert space associated with π. In fact, every non-degenerate $*$-representation arises in this manner. Therefore, we will also denote the class of equivalence classes of non-degenerate $*$-representations on $L^1(G)$ by Σ_G.

If $\pi \in \Sigma_G$, we extend π to a $*$-representation on $M(G)$ by

$$\langle \pi(\mu)\varsigma, \eta \rangle = \int_G \langle \pi(x)\varsigma, \eta \rangle \, d\mu(x) \quad \text{for each } \mu \in M(G), \quad \varsigma, \eta \in \mathcal{H}_\pi.$$

We define a norm $\| \cdot \|_{C^*(G)}$ on $L^1(G)$ as follows:

$$\|f\|_{C^*(G)} = \sup_{\pi \in \Sigma_G} \{ \|\pi(f)\| \},$$

where $\|\cdot\|$ is the usual operator norm in $\mathcal{B}(\mathcal{H}_\pi)$, the Banach space of continuous operators on $\mathcal{H}_\pi$. The completion of $L^1(G)$ with respect to $\| \cdot \|_{C^*(G)}$ is denoted $C^*(G)$. $C^*(G)$ is a C^*-algebra, called the *group C^*-algebra of G.*

The dual of $C^*(G)$ will be denoted by $B(G)$. $B(G)$ is a linear space of continuous functions on G which becomes an algebra with respect to pointwise multiplication. $B(G)$ is called the *Fourier-Stieltjes algebra of G.*

$B(G)$ may be realized either as the space of coefficient functions of Σ_G, that is, functions of the form

$$u(x) = \langle \pi(x)\varsigma, \eta \rangle \qquad \text{for} \quad \pi \in \Sigma_G, \quad \varsigma, \eta \in \mathcal{H}_\pi,$$

or as the span of the continuous positive definite functions on G. A function f on G is *positive definite* if for every $x_1, \ldots, x_n \in G$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$

$$\sum_{i,j} \lambda_i \bar{\lambda}_j f(x_i x_j^{-1}) \geq 0.$$

The continuous positive definite functions on G, denoted by $P(G)$, are the positive linear functionals on $C^*(G)$.

Let $\|\cdot\|_{B(G)}$ denote the norm on $B(G)$ induced by $C^*(G)$ with duality determined by the formula

$$\langle u, f \rangle = \int_G u(x) f(x) dx \qquad \text{for} \quad u \in B(G), \quad f \in L^1(G).$$

If $u \in P(G)$, then $\|u\|_{B(G)} = u(e)$. Since $\|\cdot\|_{C^*(G)} \leq \|\cdot\|_1$, we see that $\|\cdot\|_\alpha$ $\|\cdot\|_{B(G)}$ on $B(G)$. With $\|\cdot\|_{B(G)}$ and pointwise multiplication, $B(G)$ becomes a commutative, regular, semisimple algebra.

Let ρ_G denote the representation of G on $L^2(G)$ defined by

$$\rho_G(x) f = L_x f \qquad \text{for every} \quad x \in G, \quad f \in L^2(G).$$

Then ρ_G is called the *left regular representation* of G. Let $A(G)$ denote the set of coefficient functions of ρ_G. That is

$$A(G) = \left\{ u(x) = \langle \rho_G(x), f, g \rangle = \int_G f(x^{-1}y)\, \overline{g(y)}\, dy; \quad f, g \in L^2(G) \right\}$$
$$= \left\{ (f * \tilde{g})^\vee (x); \quad f, g \in L^2(G) \right\}.$$

Clearly $A(G) \subseteq B(G)$. In fact, $A(G)$ is a $\|\cdot\|_{B(G)}$-closed ideal of $B(G)$. $A(G)$ is called the *Fourier algebra of G.*

$A(G)$ is the norm closure in $B(G)$ of the linear span of $\{(f * \tilde{f})^{\vee}(x);\ f \in C_{00}(G)\}$ and the norm closure in $B(G)$ of $B(G) \cap C_{00}(G)$. We will often write $\|\cdot\|_{A(G)}$ for the restriction of $\|\cdot\|_{B(G)}$ to $A(G)$. With the norm $\|\cdot\|_{A(G)}$, $A(G)$ becomes a commutative, regular, semi-simple, tauberian algebra with $\Delta(A(G))$ homeomorphic to G. Furthermore, $A(G) = B(G)$ if and only if G is compact.

The dual of $A(G)$ will be denoted by $VN(G)$. $VN(G)$ is the von Neumann subalgebra of $\mathcal{B}(L^2(G))$ generated by either $\{L_x;\ x \in G\}$ or $\{L_f;\ f \in L^1(G)\}$; where

$$(L_f)g(x) = f * g(x) \qquad \text{for every} \quad f \in L^1(G), \quad g \in L^2(G).$$

The weak-* topology of $VN(G)$ can be identified with the weak operator topology on $\mathcal{B}(L^2(G))$ restricted to $VN(G)$. The weak operator topology is the topology generated by the family $\{\rho_{f,g};\ f, g \in L^2(G)\}$ of semi-norms on $\mathcal{B}(L^2(G))$, where

$$\rho_{f,g}(T) = \langle Tf, g\rangle \qquad \text{for every} \quad T \in \mathcal{B}(L^2(G)).$$

If $T \in VN(G)$ and $u(x) = (f * \tilde{g})^{\vee}(x) \in A(G)$, then

$$\langle T, u\rangle = \langle Tf, g\rangle_{L^2(G)}.$$

If $T = L_x$ for some $x \in G$, then

$$\langle T, u\rangle = \langle L_x, u\rangle = u(x) \qquad \text{for every} \quad u \in A(G).$$

Let $\widehat{G}$ denote the subset of Σ_G consisting of irreducible representations. If G is abelian, then it is well known [cf. 29, p. 345] that every irreducible representation of G is 1-dimensional. Therefore $\widehat{G}$ consists of the continuous homomorphisms of G into $\boldsymbol{T} = \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$, the circle group. Also, when G is abelian, $\widehat{G}$ with the relative weak-$*$ topology it inherits as a subset of $L^\infty(G)$ and with pointwise multiplication as a binary operation becomes a locally compact abelian group as well. In this case, $\widehat{G}$ is called the *dual group* of G. A famous theorem of Pontryagin [cf. 28, p. 376] asserts that $\hat{\widehat{G}}$ is isomorphic and homeomorphic to G.

If G is abelian and $\mu \in M(G)$, then we define the Fourier-Stieltjes transform $\hat{\mu} \in CB(\widehat{G})$ of μ by

$$\hat{\mu}(\gamma) = \int_G \gamma(x^{-1})\,d\mu(x) \qquad \text{for each} \quad \gamma \in \widehat{G}.$$

One of the fundamental results in abelian harmonic analysis is that the Fourier-Stieltjes transform establishes an isometric isomorphism between $M(G)$ and $B(\widehat{G})$. Moreover, the Fourier transform, the restriction of $\wedge$ to $L^1(G)$, is an isometric isomorphism of $L^1(G)$ onto $A(\widehat{G})$ [cf. 56, p. 32].

The definitions of $A(G)$ and $B(G)$ for non-abelian locally compact groups are due to Eymard in [19]. Eymard's work is the cornerstone of non-abelian Fourier analysis. Any fact about $A(G)$, $B(G)$ or $VN(G)$ stated without specific reference in this or in subsequent chapters can be found in [19].

CHAPTER 3

AMENABILITY AND BOUNDED APPROXIMATE IDENTITIES

§3.0. Introduction.

Liu, van Rooij and Wang proved in [40, p. 479] that if G is a locally compact abelian group and I is a closed ideal in $L^1(G)$, then I has a bounded approximate identity if and only if $I = I(A)$, where A is a closed element of the ring of subsets of $\widehat{G}$ generated by the cosets of subgroups of $\widehat{G}$.

For abelian groups G, this settled once and for all the question of which closed ideals in $L^1(G) \simeq A(\widehat{G})$ have bounded approximate identities. This and similar problems had previously been investigated by, among others, Rosenthal [54], Gilbert [21] and [22], Schreiber [59] and Wik [66].

Leptin in [39] showed that for any locally compact group G, $A(G)$ has a bounded approximate identity if and only if G is amenable. It should be noted that the assumption that the approximate identity be bounded cannot be removed. If G is the free group on two generators, then $A(G)$ has an unbounded approximate identity [cf. 50, p. 156]. By making use of Leptin's result and the fact that $A(G)$ is the predual of a von Neumann algebra, Lau proved in [36] that $I(\{e\})$ has a bounded approximate identity if and only if G is amenable.

In this chapter, we will attempt to determine which closed ideals in $A(G)$ have bounded approximate identities for an arbitrary locally compact group G. Motivated by the results of Leptin and Lau, we show in Section 3.1 that the answer to this question is intimately related to the amenability of G. We also establish topological and algebraic criteria for a closed ideal to be a candidate to possess a bounded approximate identity.

We characterize the weak-$*$ closed ideals having bounded approximate identities in Section 3.2. Theorem 3.3 extends the result of Liu, van Rooij and Wang [40] to amenable $[SIN]$-groups.

Section 3.4 deals with the structure of cofinite ideals in $A(G)$. A number of characterizations of amenable groups are given therein.

---

§3.1. Amenability and Bounded Approximate Identities.

In this section, we establish the connection between amenability and the existence of bounded approximate identities in ideals of $A(G)$. For a closed ideal I in $A(G)$, we will see that the existence of a bounded approximate identity in I is dependent on the algebraic and topological properties of the set $Z(I)$ of common zeros of I.

DEFINITION 3.1.1. Let A, B be closed subsets of G. Let

$$S(A,B) = \{u \in B(G); \quad u(A) \equiv 1, \quad u(B) \equiv 0\},$$

$$s(A,B) = \begin{cases} \inf\{\|u\|_{B(G)}\,; \quad u \in S(A,B)\} & \text{if } S(A,B) \neq \emptyset \\ \infty & \text{if } S(A,B) = \emptyset \end{cases},$$

$$\mathcal{F}(A) = \{K \subset G; \quad K \text{ is compact}, \quad K \cap A = \emptyset\},$$

$$s(A) = \sup\{s(A,K); \quad K \in \mathcal{F}(A)\}.$$

Since $1_G \in B(G)$ and $\|1_G\|_{B(G)} = 1$ it is clear that if $s(A,B) < \infty$, then $s(B,A) < \infty$ and $|s(A,B) - s(B,A)| \leq 1$. If K is compact and A is a closed subset of G

disjoint from K, then $s(A, K) < \infty$ by the regularity of $A(G)$.

Our principal tools in the identification of closed ideals with bounded approximate identities are the next two propositions.

PROPOSITION 3.1.2. *Let G be amenable. Let A be a closed set of spectral synthesis. If $s(A) < \infty$, then $I(A)$ has an approximate identity $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ which satisfies*

(i) $\|u_\alpha\|_{A(G)} \leq 2 + s(A)$ *for every* $\alpha \in \mathfrak{A}$,

(ii) $u_\alpha \in A(G) \cap C_{00}(G)$ *for every* $\alpha \in \mathfrak{A}$,

(iii) if $K \in \mathcal{F}(A)$, *there exists a sequence* $\{u_{K_n}\} \subseteq \{u_\alpha\}_{\alpha\in\mathfrak{A}}$ *such that* $\|vu_{K_n} - v\|_{A(G)} \leq \frac{1}{n}$ *for every* $v \in A(G)$ *with* supp $v \subseteq K$.

Proof. Let $K \in \mathcal{F}(A)$ and $\varepsilon > 0$. Since G is amenable, by a result of H. Leptin [38], there exists a compact set $U = U_{K,\varepsilon} \subseteq G$ such that

$$|U| > 0 \quad \text{and} \quad |KU| \leq (1+\varepsilon)^2 |U|.$$

Define

$$u_{K,\varepsilon}(x) = \frac{1}{(1+\varepsilon)|U|} 1_{KU} * 1_U^{\vee}(x).$$

Then $u_{K,\varepsilon} \in A(G)$ and $\|u_{K,\varepsilon}\|_{A(G)} \leq 1$. Also supp $u_{K,\varepsilon} \subseteq KUU^{-1}$ is compact. If $v \in A(G)$ with supp $v \subseteq K$, then

$$u_{K,\varepsilon} v = \frac{v}{1+\varepsilon}.$$

Suppose that $s(A) < \infty$. Then there exists $w_K \in S(A, K)$ with $\|w_K\|_{B(G)} \leq s(A) + 1$. Define

$$v_{K,\varepsilon} = u_{K,\varepsilon} - u_{K,\varepsilon} w_K.$$

Then $v_{K,\varepsilon} \in I(A)$ and

$$\begin{aligned}\|v_{K,\varepsilon}\|_{B(G)} &\leq \|u_{K,\varepsilon}\|_{A(G)} + \|u_{K,\varepsilon}\|_{A(G)} \|w_K\|_{B(G)} \\ &\leq 2 + s(A).\end{aligned}$$

If $x \in K$, $v_{K,\varepsilon}(x) = u_{K,\varepsilon}(x)$. Therefore, if supp $v \subseteq K$,

$$v_{K,\varepsilon}\, v = \frac{v}{1+\varepsilon}.$$

Define a partial order on $\mathcal{F}(A) \times \mathbb{R}^+$ by $(K,\varepsilon) \geq (K_1,\varepsilon_1)$ if and only if $K \subseteq K_1$ and $\varepsilon \geq \varepsilon_1$.

Let $u \in I(A)$ and $\varepsilon > 0$. Since A is a set of spectral synthesis, $I_0(A)$ is dense in $I(A)$ [29, Theorem 39.18]. Therefore, there exists $v \in A(G)$ with supp $v = K \in \mathcal{F}(A)$, $\|v\|_{A(G)} \leq 2\|u\|_{A(G)}$ and $\|u - v\|_{A(G)} < \varepsilon$. Let $(K_1, \varepsilon_1) \geq (K, \varepsilon)$. Then

$$\begin{aligned}\|u - v_{K_1,\varepsilon_1} u\|_{A(G)} &\leq \|u - v\|_{A(G)} + \|u - v_{K_1,\varepsilon_1} v\|_{A(G)} + \|v_{K_1,\varepsilon_1} v - v_{K_1,\varepsilon_1} u\|_{A(G)} \\ &\leq (s(A) + 2 + 2\|u\|_{A(G)})\varepsilon.\end{aligned}$$

Hence $\{v_{K,\varepsilon}\}_{\mathcal{F}(A)\times\mathbb{R}^+}$ is the desired approximate identity. □

LEMMA 3.1.3. *Let A be a closed subset of G. Suppose that $I(A)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ with $\|u_\alpha\|_{A(G)} \leq M$ for every $\alpha \in \mathfrak{A}$. Then $1_{G\setminus A}$ and 1_A belong to $B(G_d)$, where G_d denotes the algebraic group G with the discrete topology. Furthermore, $\|1_{G\setminus A}\|_{B(G_d)} \leq M$ and $\|1_A\|_{B(G_d)} \leq M + 1$.*

Proof. Let $u \in I(A)$. Since $\lim_\alpha \|u_\alpha u - u\|_{A(G)} = 0$, $\lim_\alpha \|u_\alpha u - u\|_\infty = 0$. Therefore, $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ converges to $1_{G\setminus A}$ in the pointwise topology. For each

$\alpha \in \mathfrak{A}$, $u_\alpha \in B(G_d)$ and $\|u_\alpha\|_{B(G_d)} = \|u_\alpha\|_{A(G)} \leq M$ [19, p. 199]. Since the set $\{v \in B(G_d);\ \|v\|_{B(G_d)} \leq M\}$ is closed in the pointwise topology [19, p. 202], $1_{G\setminus A} \in B(G_d)$ and $\|1_{G\setminus A}\|_{B(G_d)} \leq M$. Consequently, $1_A = 1_G - 1_{G\setminus A} \in B(G_d)$ and $\|1_A\|_{B(G_d)} \leq M + 1$. □

DEFINITION 3.1.4. For any locally compact group G, let $\mathcal{R}(G)$ denote the ring of subsets of G generated by the left cosets of open subgroups of G. $\mathcal{R}(G)$ is called the *coset ring of* G. Define

$$\mathcal{R}_c(G) = \{A \subset G, \quad A \in \mathcal{R}(G_d) \quad \text{and } A \text{ is closed in } G\}.$$

The following result is due to Gilbert [22] and (independently) Schreiber [59] for abelian groups. It is a straight forward albeit somewhat lengthy task to verify that Gilbert's proof is valid for all locally compact groups.

LEMMA 3.1.5. *Let A be a subset of G. Then $A \in \mathcal{R}_c(G)$ if and only if A is of the form*

$$A = \bigcup_{i=1}^{n} x_i(H_i \setminus \Delta_i),$$

where H_i is a closed subgroup of G, $\Delta_i \in \mathcal{R}(H_i)$ and $x_i \in G$.

While Proposition 3.1.2 gives a sufficient topological condition for the existence of a bounded approximate identity in $I(A)$, the next proposition provides us with a necessary algebraic condition.

PROPOSITION 3.1.6. *Let A be a closed subset of G. If $I(A)$ has a bounded approximate identity, then A has the following form:*

$$A = \bigcup_{i=1}^{n} x_i(H_i \setminus \Delta_i),$$

where H_i is a closed subgroup of G, $\Delta_i \in \mathcal{R}(H_i)$ and $x_i \in G$.

Proof. Use Lemma 3.1.3, followed by Host's non-commutative generalization of Cohen's Idempotent theorem [30], and finally Lemma 3.1.5. □

It is an easy observation that $I(A)$ has a bounded approximate identity if and only if $I(xA)$ has a bounded approximate identity. As the sets identified in Proposition 3.1.6 as possible candidates for the zero set of a closed ideal which possesses a bounded approximate identity are built from left cosets, it is important to determine those closed subgroups H for which $I(H)$ has a bounded approximate identity.

LEMMA 3.1.7. *Let G be a locally compact group. Let H be a closed subgroup of G which is either (i) open, (ii) compact or (iii) normal. Then $s(H) = 1$.*

Proof. (i) If H is an open subgroup of G, then $1_H \in B(G)$ and $\|1_H\|_{B(G)} = 1$ [19, p. 205].

(ii) Assume that H is compact. Let $K \in \mathcal{F}(H)$. Since H is a subgroup, we can find an open symmetric neighborhood V of the identity such that V^- is compact and $K^{-1}H \cap HV^2 = \emptyset$. Let

$$u_V(x) = \frac{1_{HV} * 1_{HV}^{\vee}(x)}{|HV|} = \frac{1}{|HV|}\int_G 1_{HV}(xy)1_{HV}(y)dy$$
$$= \frac{|x^{-1}HV \cap HV|}{|HV|}.$$

Since u_V is positive definite [cf. 19, p. 189] and $u_V(e) = 1$, $\|u_V\|_{B(G)} = 1$. If $x \in H$, $u_V(x) = 1$, whereas if $x \in K$, $u_V(x) = 0$. Hence $u_V \in S(H, K)$.

(iii) Assume that H is normal. Let $\pi : G \to G/H$ be the canonical homomorphism. Assume that $K \in \mathcal{F}(H)$. Then $\pi(K) \in \mathcal{F}_{G/H}(\{eH\})$. By (ii) above, there exists $u_0 \in B(G/H)$ with $\|u_0\|_{B(G/H)} = 1$ and $u_0(eH) = 1$, while $u_0(xH) = 0$ for every $x \in K$. Let

$$u(x) = u_0(\pi(x)).$$

Then $u \in B(G)$, $\|u\|_{B(G)} = 1$ [19, p. 199], $u(x) = 1$ if $x \in H$ and $u(x) = 0$ if $x \in K$. □

PROPOSITION 3.1.8. *Let G be an amenable locally compact group. Let H be a closed subgroup of G which is either (i) open, (ii) compact or (iii) normal. Then $I(H)$ has a bounded approximate identity.*

Proof. Closed subgroups are S-sets [61, Theorem 3]. If H satisfies (i), (ii) or (iii), then $s(H) = 1$. By Proposition 3.1.2, $I(H)$ has a bounded approximate identity. □

For non-amenable groups, Proposition 3.1.8 is no longer valid. In fact, Proposition 3.1.8 serves to characterize the groups in [*Am*] (see Theorem 3.1.10).

LEMMA 3.1.9. *Let H be a closed subgroup of G. Then the quotient Banach algebra $A(G)/I(H)$ is isometrically isomorphic to $A(H)$.*

Proof. Let $u \in B(G)$. Let $u_{|H}$ denote the restriction of u to H. Then $u_{|H} \in B(H)$ [19, p. 199]. If $u \in A(G)$, then $u_{|H} \in A(H)$ [19, p. 199]. Furthermore, if $v \in A(H)$, then there exists a $u \in A(G)$ such that $v = u_{|H}$ and $\|v\|_{A(H)} = \inf\{\|u\|_{A(G)} \,;\, u_{|H} = v\}$ with the infimum actually attained [27, Theorem 16].

Define $\psi : A(G)/I(H) \to A(H)$ by

$$\psi(u + I(H)) = u_{|H} \qquad \text{for } u \in A(G).$$

$u + I(H) = v + I(H)$, then $u - v \in I(H)$. Therefore $u_{|H} - v_{|H} = 0$ and ψ is well defined. Clearly, ψ is an algebra homomorphism. If $u_{|H} = 0$, $u \in I(H)$ so ψ is 1–1. Since $A(G)_{|H} = A(H)$, ψ is an algebra isomorphism.

Let $u \in A(G)$. Let $\|\cdot\|_Q$ be the quotient norm on $A(G)/I(H)$. Then

$$\begin{aligned}\|u + I(H)\|_Q &= \inf\{\|v\|_{A(G)}\,; \quad v_{|H} = u_{|H}\}\\ &= \|u_{|H}\|_{A(H)}\,.\end{aligned}$$

□

THEOREM 3.1.10. *Let G be a locally compact group. Then the following are equivalent:*

(i) G is amenable

(ii) $I(H)$ has a bounded approximate identity for some amenable closed subgroup H of G.

Proof. Assume that G is amenable. Let $H = \{e\}$. Then $I(H)$ has a bounded approximate identity by Proposition 3.1.8.

Conversely, assume that H is an amenable closed subgroup of G such that $I(H)$ has a bounded approximate identity. Leptin's theorem [39] implies that $A(H)$ has a bounded approximate identity and hence that $A(G)/I(H)$ has a bounded approximate identity. If $I(H)$ and $A(G)/I(H)$ both have bounded approximate identities, so does $A(G)$ [cf. 18, p. 173]. Therefore G is amenable. □

COROLLARY 3.1.11. *Let G be a locally compact group. Then G is amenable if and only if $I(\{e\})$ has a bounded approximate identity.*

Corollary 3.1.11 is due to A.T. Lau [36, Corollary 4.11]. His techniques are entirely different from ours, in that they rely heavily on the fact that $A(G)$ is a Banach algebra which is also the predual of a von Neumann algebra.

PROPOSITION 3.1.12. *Let G be a locally compact group. Let A be a compact subset of G. If $I_0(A)$ has a bounded approximate identity, then G is amenable.*

Proof. If G is compact, G is amenable. Therefore we may assume that G is non-compact.

Let $K \subset G$ be compact. Let V be a compact neighborhood of e. Let

$$\begin{aligned} v(x) &= \frac{1}{|V|} 1_V * 1_{V^{-1}K}(x) \\ &= \frac{1}{|V|} \int_G 1_V(xy) 1_{V^{-1}K}(y^{-1}) dy. \end{aligned}$$

If $x \in K$, $v(x) = 1$. Also supp v is compact. Let $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ be a bounded approximate identity for $I_0(A)$ with $\|u_\alpha\|_{A(G)} \le M$. By translating A if necessary, we may assume that $A \cap \text{supp } v = \emptyset$. Therefore $v \in I_0(A)$ and $\lim_\alpha \|u_\alpha v - v\|_{A(G)} = 0$. Let $\varepsilon > 0$. There exists $\alpha_0 \in \mathfrak{A}$ such that

$$\inf\{Re\ u_{\alpha_0}(x);\ x \in K\} \ge 1 - \varepsilon.$$

Let $\varphi \in C_{00}(G)$, $\varphi \ge 0$ and supp $\varphi \subseteq K$. Then

$$\begin{aligned} |\langle u_{\alpha_0}, \varphi\rangle| &\le \|L\varphi\|_{cv} \|u_{\alpha_0}\|_{A(G)} \\ &\le M\|L\varphi\|_{cv}, \end{aligned}$$

where $\|L\varphi\|_{cv}$ is the norm of φ as a left convolution operator on $L^2(G)$. However

$$Re\langle u_{\alpha_0},\varphi\rangle = \int_G (Re\ u_{\alpha_0}(x))\varphi(x)dx \geq (1-\varepsilon)\|\varphi\|_1$$

and

$$\|\varphi\|_1 \leq M\|L\varphi\|_{cv}.$$

As K was arbitrary,

$$\|\psi\|_1 \leq M\|L\psi\|_{cv} \quad \text{for every} \quad \psi \in C_{00}(G), \quad \psi \geq 0.$$

Given $\psi \in C_{00}(G)$, $\psi \geq 0$, we have

$$\|\psi\|_1^n = \|\psi^{*n}\|_1 \leq M\|L_{\psi^n}\|_{cv}$$

$$\leq M\|L\psi\|_{cv}{}^n.$$

Therefore,

$$\|\psi\|_1 \leq M^{1/n}\|L\psi\|_{cv} \quad \text{for every } n.$$

It follows that $\|\psi\|_1 = \|L\psi\|_{cv}$ for every $\psi \in C_{00}(G)$, $\psi \geq 0$. This implies that G is amenable [cf. 50, p. 85]. □

It appears that the Fourier algebra of an amenable group is rich in closed ideals with bounded approximate identities while for non-amenable groups many potential candidates are eliminated. Indeed, for non-amenable groups, if $I(A)$ has a bounded approximate identity, then A must be "topologically large."

We cannot remove completely the topological restrictions imposed on A in the statement of Propositon 3.1.12. For example, if G is any discrete group, $1_{\{e\}}$ is an identity for $I(G\backslash\{e\})$.

Leptin's theorem [39] shows that if G is amenable, then $A(G)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ with $\|u_\alpha\|_{A(G)} \leq 1$ for each $\alpha \in \mathfrak{A}$. For ideals in $A(G)$, this is seldom true. In fact, we have:

PROPOSITION 3.1.13. *Let A be a closed subset of G. Then $I(A)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ with $\|u_\alpha\|_{A(G)} \leq 1$ for every $\alpha \in \mathfrak{A}$ if and only if $G\backslash A = xH$ for some $x \in G$ and some open amenable subgroup H of G.*

Proof. Suppose that $G\backslash A = xH$, where H is an open amenable subgroup of G. Since H is an open subgroup, $I(x^{-1}A)$ can be identified with $A(H)$. As H is amenable, $A(H)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ with $\|u_\alpha\|_{A(H)} \leq 1$.

Conversely, assume that $I(A)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ with $\|u_\alpha\|_{A(G)} \leq 1$ for every $\alpha \in \mathfrak{A}$. By Lemma 3.1.3, $1_{G\backslash A} \in B(G_d)$ and $\|1_{G\backslash A}\|_{B(G_d)} = 1$. Let $x \in G\backslash A$. Then $G\backslash A = xH$, where H is a subgroup of G [cf. 23, p. 377]. Since A is closed, H is open. Again, we identify $A(H)$ with $I(x^{-1}A)$. As $I(x^{-1}A)$ has a bounded approximate identity, H must be amenable.

□

In case $A = H$ is a closed subgroup, Proposition 3.1.13 shows that $I(H)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ with $\|u_\alpha\|_{A(G)} \leq 1$ if and only if H is an amenable group and H has index two in G. In particular, $I(\{e\})$ has such an approximate identity if and only if $G = \{e, x\}$.

§3.2. Bounded Approximate Identities in Weak-* Closed Ideals.

Recall that $B(G)$ is the dual of $C^*(G)$. The purpose of this section is to characterize the ideals in $A(G)$ which are closed in the relative weak-$*$ topology that $A(G)$ inherits as a closed subspace of $B(G)$ and at the same time possess bounded approximate identities.

DEFINITION 3.2.1. Let $K \subset G$ be compact. Define

$$A_K(G) = \{u \in A(G),\ \text{supp}\ u \subseteq K\}.$$

It is easy to see that $A_K(G)$ is a closed ideal in $A(G)$.

Assume that $\{u_\alpha\}_{\alpha\in\mathfrak{A}} \subseteq A_K(G)$ and $\{u_\alpha\}$ converges to $u \in B(G)$ in the weak-$*$ topology. Suppose that there exists $x_0 \in G\backslash K$ such that $u(x_0) \neq 0$. Then there exists a compact neighborhood V of x_0 such that $V \cap K = \emptyset$. Let

$$f(x) = 1_V\ \overline{u(x)}.$$

Then $f \in L^1(G)$ and

$$\begin{aligned} 0 < \int_G 1_V\ \overline{u(x)}\ u(x)dx &= \langle f, u\rangle \\ &= \lim_\alpha \langle f, u_\alpha\rangle \\ &= \lim_\alpha \int_G 1_V\ \overline{u(x)}\ u_\alpha(x)dx \\ &= 0 \end{aligned}$$

which is impossible. Therefore supp $u \subseteq K$ and $A_K(G)$ is weak-$*$ closed.

LEMMA 3.2.2. *Let $K \subset G$ be compact. If K is open, then K is the union of finitely many cosets of an open compact subgroup H of G.*

Proof. For every $x \in K$, let V_x be a symmetric neighborhood of e such that

$xV_x^2 \subseteq K$. As K is compact, there exists $\{x_1, \ldots, x_n\}$ such that

$$K = \bigcup_{i=1}^{n} x_i V_{x_i}.$$

Let $W = \bigcap_{i=1}^{n} V_{x_i}$. Let $y \in K$. Then

$$y = x_{i_0} v_{i_0} \quad \text{for some } x_{i_0} \in K, \quad \text{and some } v_{i_0} \in V_{x_{i_0}}.$$

Hence

$$yW \subseteq x_{i_0} V_{x_{i_0}} W \subseteq x_{i_0} V_{x_{i_0}}^2 \subseteq K.$$

It follows that W generates an open compact subgroup H with $KH = K$. As H is open and K is compact, K is the union of finitely many cosets of H. □

We now state the main result of this section.

THEOREM 3.2.3. *Let I be a closed non-zero ideal in $A(G)$ which is weak-∗ closed in $B(G)$. Then the following are equivalent:*

(i) I has an identity,

(ii) I has a bounded approximate identity,

(iii) $I = A_K(G)$ for some compact open subset K of G.

Furthermore, if any of the above holds, then

$$I = \bigoplus_{i=1}^{n} L_{x_i} A(H)$$

for some compact open subgroup H of G and $\{x_1, \ldots, x_n\} \subseteq G$.

<u>Proof</u>. Clearly (i) implies (ii).

Assume that I has a bounded approximate identity $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$. We may assume that $\{u_\alpha\}$ converges in the weak-$*$ topology to some $u \in I$. Let $A = Z(I)$. Since $u \in I$, $u \in I(A)$. As I is non-zero, $G\backslash A$ is non-empty.

If $x \in G\backslash A$, there exists $v \in I$ and an open neighborhood U of x such that $|v(y)| > 0$ for every $y \in U$. Since $\lim_\alpha \|u_\alpha v - v\|_{A(G)} = 0$, $\{u_\alpha\}$ converges uniformly to 1 on U. It follows that $u = 1_{G\backslash A}$. Let $K = G\backslash A$. Then K is open. But $1_K \in A(G) \subseteq C_0(G)$, so K is also compact. It is clear that $I = A_K(G) = 1_K A(G)$. Therefore (ii) $\Rightarrow$ (iii).

Assume that K is compact and open and that $I = A_K(G)$. By Lemma 3.2.2,

$$K = \bigcup_{i=1}^{n} x_i H$$

for some open subgroup H of G. It follows that $K \in \mathcal{R}(G)$ and hence that $1_K \in A(G)$ [30]. Therefore 1_K is an identity for I so (iii) implies (i).

In each case

$$K = \bigcup_{i=1}^{n} x_i H$$

is the disjoint union of finitely many cosets of a compact open subgroup H of G. It is easy to see that

$$A_K(G) = \oplus L_{x_i} A(H).$$

□

If G is compact, then $A(G) = B(G)$ and $A(G)$ is itself a dual Banach space. In [62], K. Taylor showed that if G is a separable group with a completely reducible left regular representation, then $A(G)$ is a dual space. The "$ax+b$" group, which consists of matrices of the form

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}; \quad a \in \mathbb{R}^+, \quad b \in \mathbb{R} \right\},$$

is separable and has a completely reducible left regular representation, but it is not compact [cf. 34]. Therefore, if G is the "$ax+b$" group, then $A(G)$ is a dual space. As G is also amenable, we shall see that this implies the multiplication on $A(G)$ is not weak-$*$ to weak-$*$ separately continuous on bounded spheres.

PROPOSITION 3.2.4. *Let G be a locally compact group for which $A(G)$ is the dual of a Banach space $A_*(G)$. Let A be a closed subset of G for which $I(A)$ has a bounded approximate identity. If for each $u \in A(G)$ the map $v \mapsto uv$ is weak-$*$ to weak-$*$ continuous on bounded spheres, then*

$$G \backslash A^0 = \bigcup_{i=1}^{n} x_i H$$

for some open compact subgroup H of G.

Proof. Let $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ be a bounded approximate identity in $I(A)$. We may assume that $w^* - \lim_\alpha u_\alpha = u$ for some $u \in A(G)$.

Let $T \in A_*(G)$. Let $v \in I(A)$. Then

$$(*) \qquad |\langle uv, T\rangle - \langle v, T\rangle| \le |\langle uv, T\rangle - \langle u_\alpha v, T\rangle| + |\langle u_\alpha v, T\rangle - \langle v, T\rangle|.$$

As $\lim_\alpha \|u_\alpha v - v\| = 0$ and $w^* - \lim_\alpha u_\alpha v = uv$, $(*)$ can be made as small as we like. Hence

$$\langle uv, T\rangle = \langle v, T\rangle \quad \text{for every } T \in A_*(G) \text{ and every } v \in I(A).$$

It follows that $u(G\setminus A) = 1$.

Let $x \in A^0$. There exists $v \in A(G)$ with $\operatorname{supp} v \subseteq A^0$ and $v(x) = 1$. Let $T \in A_*(G)$. Then

$$\langle uv, T\rangle = \lim_{\alpha} \langle u_\alpha v, T\rangle = 0.$$

Therefore, $uv = 0$ and $u =$ [illegible]$_{\setminus A^0}$. As $u \in A(G) \subseteq C_0(G)$, $G\setminus A^0$ is compact and open. Apply Lemma 3.2.2. □

COROLLARY 3.2.5. *Let G be an amenable locally compact group. Assume that $A(G)$ is a dual Banach space. Then the multiplication on $A(G)$ is weak-$*$ to weak-$*$ separately continuous on bounded spheres if and only if G is compact.*

<u>Proof</u>. If G is compact, $A(G) = B(G)$. It is an easy task to verify that multiplication on $B(G)$ is always weak-$*$ to weak-$*$ separately continuous on bounded spheres.

Conversely, since G is amenable, $I(\{e\})$ has a bounded approximate identity (Corollary 3.1.11). If multiplication on $A(G)$ is weak-$*$ to weak-$*$ separately continuous on bounded spheres, then $G\setminus\{e\}^0$ is compact by Proposition 3.2.4. Hence G is compact. □

COROLLARY 3.2.6. *Let G be the "$ax+b$" group. Let $A_*(G)$ be a predual of $A(G)$. Then multiplication on $A(G)$ is not weak-$*$ to weak-$*$ separately continuous on bounded spheres.*

§3.3. Bounded Approximate Identities and $[SIN]$-groups.

For locally compact abelian groups, Liu, van Rooij and Wang [40] proved that a closed ideal I in $A(G)$ has a bounded approximate identity if and only if $I = I(A)$, where $A \in \mathcal{R}_c(G)$. The main purpose of this section is to extend this result to amenable $[SIN]$-groups, a class which contains $[K]$, $[A]$ and $[Am] \cap [D]$.

LEMMA 3.3.1. *Let A, B be closed subsets of G. Suppose that $I(A)$ and $I(B)$ have bounded approximate identities. Then so does $I(A \cup B)$.*

Proof. Let $\{u_i\}_{i\in I}$, $\{v_j\}_{j\in J}$ be approximate identities in $I(A)$ and $I(B)$ with bounds M_1 and M_2 respectively. Let $\{w_1, \ldots, w_n\} \subseteq I(A \cup B)$ and $\varepsilon > 0$. As $I(A \cup B) \subset I(A)$, there exists i_0 such that

$$\|w_k u_{i_0} - w_k\|_{A(G)} < \varepsilon/2 \qquad \text{for} \quad k = 1, 2, \ldots, n.$$

As $\{w_k u_{i_0}\}_{k=1}^n \subset I(B)$, there exists j_0 such that

$$\|w_k u_{i_0} - w_k u_{i_0} v_{j_0}\|_{A(G)} < \varepsilon/2 \qquad \text{for} \quad k = 1, 2, \ldots, n.$$

Then $\|u_{i_0} v_{j_0}\|_{A(G)} \le M_1 M_2$, $u_{i_0}, v_{j_0} \in I(A \cup B)$ and

$$\|w_k - w_k u_{i_0} v_{j_0}\|_{A(G)} \le \varepsilon \qquad \text{for} \quad k = 1, 2, \ldots, n.$$

□

LEMMA 3.3.2. *Let A, B be disjoint closed subsets of G. Assume that there exists $u \in B(G)$ with $u(A) = 1$ and $u(B) = 0$. If G is amenable, then there exists* *a bounded approximate identity in $I(A \cup B)$ if and only if there exists bounded approximate identities in each of $I(A)$ and $I(B)$.*

Proof. Assume that $I(A \cup B)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$. By Leptin's theorem [39], $A(G)$ has a bounded approximate identity $\{v_j\}_{j \in J}$. Then

$$\{u_\alpha u - (1-u)v_j\}$$

is a bounded approximate identity in $I(A)$ and

$$\{(u_\alpha - u_\alpha u) + uv_j\}$$

is a bounded approximate identity in $I(B)$.

The converse is Lemma 3.3.1. □

The next proposition is of independent interest.

PROPOSITION 3.3.3. *Let G be a $[SIN]$-group and H a closed subgroup of G. Then $s(H) = 1$.*

Proof. Let K be a compact subset of G such that $K \cap H = \emptyset$. Let V be a symmetric neighborhood of e such that $\overline{V}$ is compact and

(1) $$V^2 H \cap K^{-1} = \emptyset.$$

As both G and H are unimodular [cf. 48], there exists a G-invariant measure dg on the quotient space G/H [20, p. 267]. We may assume that the Haar measures μ_G and μ_H are chosen such that

$$\int_G f(g)d\mu_G(g) = \int_{G/H} \left[\int_H f(gh)d\mu_H(h)\right] dg \qquad \text{for every } f \in C_{00}(G).$$

By a result of Mosak [46], there exists a continuous non-negative central function v on G such that

$$\operatorname{supp} v \subseteq V$$

and

$$1 = \int_{G/H} \left[\int_H v(gh) d\mu_H(h) \right]^2 dg$$

$$= \int_{G/H} \left[\int_H v(gh') d\mu_H(h') \int_H v(gh) d\mu_H(h) \right] dg$$

$$= \int_{G/H} \left[\int_H v(gh') d\mu_H(h') \int_H v(gh'h) d\mu_H(h) \right] dg$$

(2) $$= \int_G \int_H v(g) v(gh) d\mu_H(h) d\mu_G(g).$$

For every $x \in G$, define

$$u(x) = \int_G \int_H v(g) v(x^{-1} gh) d\mu_H(h) d\mu_G(g).$$

Following an argument of Cowling and Rodway [13, p. 95], we see that u is a coefficient function of the unitary representation obtained by inducing the trivial representation of H to G. Therefore, $u \in B(G)$. Furthermore, $\|u\|_{B(G)} \leq 1$.

If $x \in K$, then $v(g)v(x^{-1} gh) = 0$ for every $g \in G$, $h \in H$ by (1). Therefore, $u(K) = 0$.

Also, if $x \in H$,

$$u(x) = \int_G \int_H v(g) v(x^{-1} gh) d\mu_H(x) d\mu_G(g)$$

$$= \int_G \int_H v(g) v(gh') d\mu_H(h') d\mu_G(g) \qquad \text{(since } v \text{ is central)}$$

$$= 1 \qquad \text{by (2).}$$

Hence $u \in S(H, K)$. Since $K \in \mathcal{F}(H)$ was arbitrary, $s(H) = 1$. □

The proof of Proposition 3.3.3 uses an idea of Reiter [51] which was later modifed by Cowling and Rodway in [13, p. 98] to show that if G is a $[SIN]$-group and H is a closed subgroup of G, then for each $u \in B(H)$ there exists $v \in B(G)$ such that $v_{|H} = u$.

We are now in a position to completely characterize the ideals in the Fourier algebra of an amenable $[SIN]$-group with bounded approximate identities.

THEOREM 3.3.4. *Let G be an amenable $[SIN]$-group. Let I be an ideal in $A(G)$. Then I has a bounded approximate identity if and only if $Z(I) \in \mathcal{R}_c(G)$.*

Moreover, each $A \in \mathcal{R}_c(G)$ is a set of spectral synthesis, so if I is closed, $I = I(Z(I))$. In any case, the bounded approximate identity $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ can be chosen such that

(i) $u_\alpha \in A(G) \cap C_{00}(G)$

(ii) if $K \in \mathcal{F}(A)$ then there exists a sequence $\{u_{K_n}\} \subseteq \{u_\alpha\}_{\alpha \in \mathfrak{A}}$ such that if $v \in I$ and $\operatorname{supp} v \subseteq K$, then

$$\|u_{K_n} v - v\|_{A(G)} \leq \frac{1}{n}.$$

Proof. Let $A \in \mathcal{R}_c(G)$. By Lemma 3.1.5,

$$A = \bigcup_{i=1}^{n} x_i(H_i \backslash \Delta_i),$$

where $x_i \in G$, H_i is a closed subgroup of G and $\Delta_i \in \mathcal{F}(H_i)$.

By Proposition 3.3.3, $s(H_i) = 1$ for each i. Hence $I(H_i)$ has a bounded approximate identity (Proposition 3.1.2).

As $\Delta_i \in \mathcal{R}(H_i)$, $1_{\Delta_i} \in B(H_i)$ by Host's idempotent theorem [30]. By [13, Theorem 2], 1_{Δ_i} extends to a $u \in B(G)$ with $u \in S(\Delta_i,\ H_i\backslash\Delta_i)$. Since G is amenable, $I(H_i\backslash\Delta_i)$ has a bounded approximate identity by Lemma 3.3.2. Hence $I(x_i(H_i\backslash\Delta_i))$ has a bounded approximate identity.

A careful examination of the proofs of each of the results used above shows that the bounded approximate identity in $I(A)$ can be chosen to satisfy both (i) and (ii).

The approximate identity for $I(A)$ lies in $C_{00}(G) \cap A(G)$. Therefore A is an S-set. If I is any ideal with $Z(I) = A$, then $I_0(A) \subseteq I \subseteq I(A)$ [cf. 29, p. 493], so the bounded approximate identity for $I(A)$ is also a bounded approximate identity for I.

Assume that I has a bounded approximate identity. Then $Z(I) \in \mathcal{R}_c(I)$, exactly as in the proof of Lemma 3.1.3. □

COROLLARY 3.3.5. *Let G be either abelian, compact or amenable and discrete. Let $A \in \mathcal{R}_c(G)$. Then $I(A)$ has a bounded approximate identity and A is an S-set.*

□

For an arbitrary locally compact group, it is not true that every $u \in B(H)$ extends to some $v \in B(G)$. Therefore it is not clear that $S(\Delta, H\backslash\Delta) \neq \emptyset$ for every $\Delta \in \mathcal{R}(H)$. It may well be that our technique will not work for groups which are far from $[SIN]$.

§3.4. Cofinite Ideals in $A(G)$.

DEFINITION 3.4.1. Let $\mathcal{A}$ be a Banach algebra. An ideal I in $\mathcal{A}$ is called *cofinite* if the dimension of $\mathcal{A}/I$ is finite. The dimension of $\mathcal{A}/I$ is called the *codimension* of I.

Let $I^2 = \{ \sum_{i=1}^{n} u_i v_i;\ u_i, v_i \in I \}$. Then I^2 is an ideal of $\mathcal{A}$ contained in I. I is said to *factorize weakly* if $I^2 = I$. Such ideals are also called *idempotents*.

Our interest in cofinite ideals in $A(G)$ was motivated by three papers of G. Willis [67], [68] and [69]. Willis succeeded in showing that if G is non-amenable, then no closed cofinite left ideal in $L^1(G)$ has a bounded approximate identity [67]. In contrast to this result, he proved in [68] and [69] that for every locally compact group, every closed codimension one ideal is idempotent and for a large class of groups every codimension two ideal is idempotent. In this section, we will show that while the analogue of Willis' first result holds true for $A(G)$, if G is non-amenable, no closed cofinite ideal in $A(G)$ is idempotent.

We begin with a lemma that may be part of folklore.

LEMMA 3.4.2. *Let G be an amenable locally compact group. Then $A(G)$ satisfies Ditkin's condition.*

Proof. Condition (i) follows immediately from Lemma 3.1.7 and the proof of Proposition 3.1.2. If G is non-compact, then condition (ii) follows from Leptin's theorem [39]. □

PROPOSITION 3.4.3. *Let G be an amenable group. Let A be a closed subset of G. If* bdy (A) *contains no non-empty perfect set, then A is an S-set.*

Proof. The proposition follows immediately from Lemma 3.4.2 and Ditkin's theorem [cf. 29, p. 497]. □

COROLLARY 3.4.4. *Let G be an amenable locally compact group. Let A be a closed discrete subset of G. Then A is an S-set. In particular, every finite subset of G is an S-set and if G is discrete, every subset is an S-set.*

Proof. This is immediate from Proposition 3.4.3. □

COROLLARY 3.4.5. *Let G be an amenable discrete group. Let $A \subset G$. Then $s(A) < \infty$ if and only if $1_A \in B(G)$.*

Proof. If $1_A \in B(G)$, then $s(A) = \|1_A\|_{B(G)}$.

If $s(A) < \infty$, then $I(A)$ has a bounded approximate identity (Proposition 3.1.2) and $1_A \in B(G)$ by Lemma 3.1.3.

COROLLARY 3.4.6. *Let G be an amenable locally compact group. Let I be a closed cofinite ideal in $A(G)$. Then $I = I(A)$ for some finite set $A = \{x_1, \ldots, x_n\}$, where n is the codimension of I. Furthermore $I^2 = I$.*

Proof. Let $A = Z(I)$. Since I is cofinite, A must be finite. Therefore A is an S-set by Corollary 3.4.4 and $I = I(A)$. If $A = \{x_1, \ldots, x_n\}$, let $u_i \in A(G)$ be such that $u_i(x_i) = 1$, $u_i(x_j) = 0$ if $i \neq j$. Then $\{u_i + I(A)\}$ is a basis for $A(G)/I(A)$.

Proposition 3.1.2 implies that $I(A)$ has a bounded approximate identity. By Cohen's factorization theorem [29, p. 268], $I^2(A) = I(A)$. □

LEMMA 3.4.7. *Let G be a non-amenable locally compact group. Let $I = I(\{e\})$. Then I^2 is not closed in $A(G)$.*

Proof. $\{e\}$ is an S-set [19, p. 229] and $Z(I^2) = \{e\}$. Therefore if I^2 is closed, $I^2 = I$. Assume that $I^2 = I$. Let $v \in A(G)$. Let $u \in A(G) \cap C_{00}(G)$ with $u(e) = 1$. Then

$$v = uv + (v - uv)$$

with $v - uv \in I$. Hence

$$v - uv = \sum_{i=1}^{n} w_i v_i \quad \text{for} \quad w_i, v_i \in I.$$

As G is non-compact, there exists $x \in G \backslash \text{supp } u$. Since $uv \in I(\{x\})$ and $I^2(\{x\}) = I(\{x\})$,

$$uv = \sum_{j=1}^{m} t_j m_j \quad \text{for} \quad t_j, m_j \in I(\{x\}).$$

Thus

$$v = \sum_{i=1}^{n} w_i v_i + \sum_{j=1}^{m} t_j m_j \in A^2(G)$$

which is impossible by a result of Losert [42, p. 139]. □

The proof of this lemma can be easily modified to show that if G is a non-amenable locally compact group, no ideal of the form $I(\{x_1, \ldots, x_n\})$ can be idempotent. We contrast this with [68] and [69], where Willis has shown that for any locally compact group G every closed codimension 1 ideal in $L^1(G)$ is idempotent and that there exist non-amenable groups for which every closed codimension 2 ideal of $L^1(G)$ is idempotent.

THEOREM 3.4.8. *Let G be a locally compact group. Then G is amenable if and only if every cofinite ideal is of the form $I(A)$ where A is a finite subset of G.*

Proof. If G is amenable, then every closed cofinite ideal is idempotent by Corollary 3.4.6. By [14, Theorem 2.3], every cofinite ideal is closed and hence is of the form $I(A)$ for some finite subset A of G.

Conversely, if every cofinite ideal is closed, [14, Theorem 2.3] implies that $I^2(\{e\})$ is closed. Therefore G is amenable by Lemma 3.4.7. □

We can now answer the analogue of Willis and Dales' "weak" automatic continuity question [14, p. 397].

THEOREM 3.4.9. *Let G be a locally compact group. Then the following are equivalent:*

(i) G is amenable,

(ii) each homomorphism from $A(G)$ with finite dimensional range is continuous.

Proof. This follows immediately from Theorem 3.4.8 and from [14, Theorem 2.3]. □

CHAPTER 4

INVARIANTLY COMPLEMENTED SUBSPACES AND BOUNDED APPROXIMATE IDENTITIES

In many respects, Banach algebras with bounded approximate identities are better behaved than those without. This fact becomes particularly evident when one looks at their Banach modules.

M. Bekka showed that if G is any locally compact group and I is any closed ideal in $L^1(G)$, then I has a bounded approximate identity if and only if $I^{\perp} = \{g \in L^{\infty}(G); \int_G g(x)f(x)d\mu_G(x) = 0 \text{ for every } f \in I\}$ is the range of a continuous projection on $L^{\infty}(G)$ which commutes with the left module action of $L^1(G)$ on $L^{\infty}(G)$ [10]. We show, in Theorem 4.2.2, that the analog of Bekka's theorem holds for $A(G)$ when G is an amenable group. Moreover, the class of amenable groups can be characterized by the equivalence of these two statements.

We begin the chapter with some facts about general Banach-modules. In Section 4.2, we apply these results to the algebra $A(G)$ and its closed ideals. In particular, we show that G is amenable if and only if the analog of Bekka's result holds.

Finally, we apply the results of this chapter and of the previous chapter to study $A(G)$ for a discrete group G. Here again we are able to characterize amenable discrete groups amongst all discrete groups.

§4.1. Banach Modules.

DEFINITION 4.1.1. Let $\mathcal{A}$ be a Banach algebra. By a *left Banach-$\mathcal{A}$-module* (resp. *right Banach-$\mathcal{A}$-module*) [resp. *Banach-$\mathcal{A}$-bimodule*] we will mean an algebraic left-module (resp. right-module) [resp. bimodule] X which is itself a Banach space and is such that $\|a \cdot x\| \leq \|a\| \, \|x\|$ (resp. $\|x \cdot a\| \leq \|x\| \, \|a\|$) [resp. $\|a \cdot x\| \leq \|a\| \, \|x\|$ and $\|x \cdot a\| \leq \|x\| \, \|a\|$] for every $x \in X, \ a \in \mathcal{A}$.

Let X and Y be left (resp. right) Banach $\mathcal{A}$-modules. A linear map $\Gamma : X \to Y$ is called a *left (resp. right) module homomorphism* if

$$\Gamma(u \cdot x) = u \cdot \Gamma(x) \qquad [\text{resp.} \quad \Gamma(x \cdot u) = \Gamma(x) \cdot u]$$

for every $u \in \mathcal{A}, \ x \in X$.

Let $\mathrm{Hom}_L^{\mathcal{A}}(X,Y)$ [resp. $\mathrm{Hom}_R^{\mathcal{A}}(X,Y)$] denote the continuous left [resp. right] module homomorphisms of X into Y. With respect to the usual operator norm, $\mathrm{Hom}_L^{\mathcal{A}}(X,Y)$ [resp. $\mathrm{Hom}_R^{\mathcal{A}}(X,Y)$] is a Banach space. If X and Y are Banach $\mathcal{A}$-bimodules, then we denote $\mathrm{Hom}_L^{\mathcal{A}}(X,Y) \cap \mathrm{Hom}_R^{\mathcal{A}}(X,Y)$ by $\mathrm{Hom}^{\mathcal{A}}(X,Y)$. In case $X = \mathcal{A}$, $\mathrm{Hom}_L^{\mathcal{A}}(\mathcal{A},Y)$ [resp. $\mathrm{Hom}_R^{\mathcal{A}}(\mathcal{A},Y)$] is the space of *left [resp. right] $(\mathcal{A},Y)$-multipliers.* If Y is a Banach $\mathcal{A}$-bimodule, then $\mathrm{Hom}^{\mathcal{A}}(\mathcal{A},Y)$ is the space of *$(\mathcal{A},Y)$-multipliers.*

Let X be a left [resp. right] Banach $\mathcal{A}$-module. Then X^* becomes a right [resp. left] Banach $\mathcal{A}$-module as follows:

$$\langle T \cdot u, x\rangle = \langle T, u \cdot x\rangle \qquad \text{for every} \quad u \in \mathcal{A}, \quad x \in X, \quad T \in X^*$$

$$[\text{resp.} \quad \langle u \cdot T, x\rangle = \langle T, x \cdot u\rangle \qquad \text{for every} \quad u \in \mathcal{A}, \quad x \in X, \quad T \in X^*].$$

Furthermore, a simple calculation shows that if $\Gamma \in \mathrm{Hom}_L^{\mathcal{A}}(X,Y)$ [resp. $\Gamma \in \mathrm{Hom}_R^{\mathcal{A}}(X,Y)$], then $\Gamma^* \in \mathrm{Hom}_R^{\mathcal{A}}(Y^*,X^*)$ [resp. $\Gamma^* \in \mathrm{Hom}_L^{\mathcal{A}}(X^*,Y^*)$].

PROPOSITION 4.1.2. *Let $\mathcal{A}$ be a Banach algebra with a bounded right [resp. left] approximate identity $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$. Let X be a right [resp. left] Banach $\mathcal{A}$-module. Let $\Gamma \in \mathrm{Hom}_L^{\mathcal{A}}(\mathcal{A},X^*)$ [resp. $\Gamma \in \mathrm{Hom}_R^{\mathcal{A}}(\mathcal{A},X^*)$]. Then there exists $T \in X^*$ such that*

$$\Gamma(u) = u\cdot T \qquad [\text{resp.} \quad \Gamma(u) = T\cdot u]$$

for every $u \in \mathcal{A}$.

<u>Proof</u>. $\Gamma(u) = \lim_\alpha \Gamma(uu_\alpha)$ for every $u \in \mathcal{A}$. As $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ is bounded, we may assume that $\Gamma(u_\alpha)$ converges in the weak-$*$ topology to some $T \in X^*$. Then

$$\begin{aligned}
\langle \Gamma(u), x\rangle &= \langle \lim_\alpha \Gamma(uu_\alpha), x\rangle \\
&= \lim_\alpha \langle \Gamma(uu_\alpha), x\rangle \\
&= \lim_\alpha \langle u\cdot\Gamma(u_\alpha), x\rangle \\
&= \lim_\alpha \langle \Gamma(u_\alpha), x\cdot u\rangle \\
&= \langle T, x\cdot u\rangle \\
&= \langle u\cdot T, x\rangle.
\end{aligned}$$

Therefore, $\Gamma(u) = u\cdot T$. The proof of the second statement is identical. □

DEFINITION 4.1.3. Let X be a left [resp. right] Banach $\mathcal{A}$-module. Let Y be a left [resp. right] Banach $\mathcal{A}$-submodule of X. We say that Y is left [resp.

right] *invariantly complemented* if there exists a projection P from X onto Y such that $P \in \mathrm{Hom}_L^{\mathcal{A}}(X,Y)$ [resp. $P \in \mathrm{Hom}_R^{\mathcal{A}}(X,Y)$]. If X and Y are both Banach $\mathcal{A}$-bimodules, Y is called *invariantly complemented* in X if there exists a projection P from X onto Y with $P \in \mathrm{Hom}^{\mathcal{A}}(X,Y)$.

Let I be a closed subspace of $\mathcal{A}$. Let

$$I^{\perp} = \{\varphi \in \mathcal{A}^*;\ \varphi(u) = 0 \text{ for every } u \in I\}.$$

Let X be a closed subspace of $\mathcal{A}^*$. Let

$$^{\perp}X = \{u \in \mathcal{A};\ \varphi(u) = 0 \text{ for every } \varphi \in X\}.$$

If I is a closed left [resp. right] ideal in $\mathcal{A}$, then $I^{\perp}$ is a weak-$*$ closed right [resp. left] submodule of $\mathcal{A}^*$ and conversely.

PROPOSITION 4.1.4. *Let $\mathcal{A}$ be a Banach algebra with a bounded right [resp. left] approximate identity. Let I be a closed left [resp. right] ideal in $\mathcal{A}$. Then I has a bounded right [resp. left] approximate identity if and only if $I^{\perp}$ is right [resp. left] invariantly complemented.*

<u>Proof</u>. Let $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ be a bounded right approximate identity in I. We may assume that $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ converges in the weak-$*$ topology of $\mathcal{A}^{**}$. Define

$$\langle PT, u\rangle = \langle T, u\rangle - \lim_{\alpha} \langle u_\alpha, T\cdot u\rangle \qquad \text{for} \quad T \in \mathcal{A}^*, \quad u \in A.$$

P is a continuous operator on $\mathcal{A}^*$ with $\|P\| \le 1 + \sup_{\alpha\in\mathfrak{A}}\{\|u_\alpha\|\}$. If $u \in I$, $\lim_{\alpha} \langle u_\alpha, T\cdot u\rangle = \langle T, u\rangle$, so $PT \in I^{\perp}$.

Suppose $T \in I^{\perp}$. Then, if $u \in \mathcal{A}$, $\langle uu_\alpha, T\rangle = 0$ for every $\alpha \in \mathfrak{A}$. Hence

$$\langle PT, u\rangle = \langle T, u\rangle \qquad \text{for every} \quad u \in \mathcal{A}$$

and $PT = T$. Therefore, P is a projection of $\mathcal{A}^*$ onto $I^{\perp}$.

Finally, if $u, v \in \mathcal{A}$ and $T \in \mathcal{A}^*$, then

$$\begin{aligned}\langle (PT)\cdot u, v\rangle &= \langle PT, uv\rangle \\ &= \langle T, uv\rangle - \lim_\alpha \langle u_\alpha, T\cdot uv\rangle \\ &= \langle T\cdot u, v\rangle - \lim_\alpha \langle u_\alpha, (T\cdot u)v\rangle \\ &= \langle P(T\cdot u), v\rangle.\end{aligned}$$

Therefore, $P \in \mathrm{Hom}_R^{\mathcal{A}}(\mathcal{A}^*, I^{\perp})$.

Conversely, assume that $I^{\perp}$ is right invariantly complemented and that $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ is a bounded right approximate identity of $\mathcal{A}$. Let $P \in \mathrm{Hom}_R^{\mathcal{A}}(\mathcal{A}^*, I^{\perp})$ be a projection of $\mathcal{A}^*$ onto $I^{\perp}$. Then $(1-P) \in \mathrm{Hom}_R^{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*)$, where 1 denotes the identity operator on $\mathcal{A}^*$. We have $(1-P)^* \in \mathrm{Hom}_L^{\mathcal{A}}(\mathcal{A}^{**}, \mathcal{A}^{**})$ and $(1-P)^*$ is a projection of $\mathcal{A}^{**}$ onto $(I^{\perp})^{\perp} = I^{-w^*}$, the weak-$*$ closure of I in $\mathcal{A}^{**}$. Since $(1-P)^* \in \mathrm{Hom}_L^{\mathcal{A}}(\mathcal{A}, \mathcal{A}^{**})$, by Proposition 4.1.2, there exists $\Gamma_0 \in \mathcal{A}^{**}$ such that

$$(1-P)^*(u) = u\cdot\Gamma_0 \qquad \text{for every} \quad u \in \mathcal{A}.$$

Furthermore, we may assume that $\Gamma_0 = w^* - \lim_\alpha (1-P)^*(u_\alpha)$ and $\Gamma_0 \in (I^{\perp})^{\perp}$.

Let $u \in I$. If $T \in \mathcal{A}^*$, then

$$\begin{aligned}\langle u \cdot \Gamma_0, T\rangle &= \langle \Gamma_0, T \cdot u\rangle \\ &= \lim_\alpha \langle (1-P)^*(u_\alpha), T \cdot u\rangle \\ &= \lim_\alpha \langle u_\alpha, (1-P)(T \cdot u)\rangle \\ &= \lim_\alpha \langle u u_\alpha, (1-P)T\rangle \\ &= \langle u, (1-P)T\rangle \\ &= \langle u, T\rangle.\end{aligned}$$

Therefore, Γ_0 is a right identity for I^{-w^*}.

There exists a bounded net $\{v_\beta\}_{\beta\in B}$ which converges in the weak-$*$ topology of Γ_0. Therefore $\{v_\beta\}_{\beta\in B}$ is a bounded weak right approximate identity in I. Hence I must also have a bounded approximate identity [cf. 11, p. 58]. □

LEMMA 4.1.5. *Let $\mathcal{A}$ be a Banach algebra with a right [resp. left] bounded approximate identity $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$. Let $\Gamma \in \mathrm{Hom}_L^{\mathcal{A}}(\mathcal{A}, X)$ [resp. $\Gamma \in \mathrm{Hom}_R^{\mathcal{A}}(\mathcal{A}, X)$]. Let $\imath$ be a weak-$*$ limit point of $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ in $\mathcal{A}^{**}$. If $\pi : X \to X^{**}$ is the canonical embedding, then*

$$\pi(\Gamma(u)) = u \cdot \Gamma^{**}(\imath) \qquad [\text{resp.} \quad \pi(\Gamma(u)) = \Gamma^{**}(\imath)\cdot u]$$

for every $u \in \mathcal{A}$.

Proof. Let $\Gamma \in \mathrm{Hom}_L^{\mathcal{A}}(\mathcal{A}, X)$. Then

$$w^* - \lim_\alpha \Gamma^{**}(\pi(u_\alpha)) = \Gamma^{**}(\imath).$$

Hence, for every $u \in A$ and $T \in X^*$

$$\langle u \cdot \Gamma^{**}(i), T \rangle = \lim_{\alpha} \langle u \cdot \Gamma^{**}(\pi(u_\alpha)), T \rangle.$$

Therefore,

$$\pi(\Gamma(u)) = w^* - \lim_{\alpha} \pi(\Gamma(u \cdot u_\alpha)) = w^* - \lim_{\alpha} \pi(u \cdot \Gamma(u_\alpha)) = u \cdot \Gamma^{**}(i).$$

□

The next proposition is due to Gulick, Liu and van Rooij [26, p. 142] for $A = L^1(G)$. It is easy to see that their proof carries over to any Banach algebra A with a bounded approximate identity.

PROPOSITION 4.1.6. *Let A be a Banach algebra with a right [resp. left] approximate identity $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ such that $\|u_\alpha\|_{\alpha\in\mathfrak{A}} \leq C$ for every $\alpha \in \mathfrak{A}$. Then there exists a linear map $\mathcal{M} : \mathrm{Hom}_L^A(A, X) \to (X^* \cdot A)^*$ [resp. $(A \cdot X^*)^*$] such that*

$$\|\Gamma\| \leq \|\mathcal{M}\Gamma\| \leq C\|\Gamma\|$$

for every $\Gamma \in \mathrm{Hom}_L^A(A, X)$. Furthermore, $\mathcal{M}$ is onto if and only if $A \cdot X^{**} \subseteq \pi(X)$ [resp. $X^{**} A \subseteq \pi(X)$].

§4.2. Invariant Projections on $VN(G)$.

We now apply the results of Section 4.1 to the algebra $A(G)$.

PROPOSITION 4.2.1. *Let $A \subset G$ be closed. Suppose that $I(A)$ has a bounded approximate identity. Then there exists a projection P of $VN(G)$ onto $I(A)^{\perp}$ such that $u \cdot P(T) = P(u \cdot T)$ for every $u \in A(G)$, $T \in VN(G)$.*

Proof. This is simply Proposition 4.1.4, if we observe that the existence of a bounded approximate identity for A is not used in the "only if" direction of the proof. □

Our main theorem is:

THEOREM 4.2.2. *Let G be an amenable locally compact group. Let X be a weak-$*$ closed $A(G)$-submodule of $VN(G)$. Then the following are equivalent:*

(i) X is invariantly complemented,

(ii) ${}^{\perp}X$ has a bounded approximate identity.

Furthermore, if G is any locally compact group for which ${}^{\perp}X$ has a bounded approximate identity whenever X is a weak-$*$ closed invariantly complemented submodule of $VN(G)$, then G is amenable.

Proof. The first statement is Proposition 4.1.4. The second statement follows from the observation that $X = \{0\}$ is weak-$*$ closed and invariantly complemented, while $A(G) = {}^{\perp}X$ has a bounded approximate identity if and only if G is amenable. □

DEFINITION 4.2.3. Following Granirer [24], we denote $\langle A(G) \cdot VN(G)\rangle^{-}$ by $UCB(\widehat{G})$. If G is amenable, Cohen's factorization theorem implies that $UCB(\widehat{G}) = A(G) \cdot VN(G)$.

PROPOSITION 4.2.4. *Let G be amenable. Let $\Gamma \in \mathrm{Hom}^{A(G)}(UCB(\widehat{G}), UCB(\widehat{G}))$. Then there exists $\Gamma_0 \in \mathrm{Hom}^{A(G)}(VN(G), VN(G))$ such that*

$$\Gamma_0\big|_{UCB(\widehat{G})} = \Gamma \quad \textit{and} \quad \|\Gamma\| = \|\Gamma_0\|.$$

Proof. Let $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ be a bounded approximate identity for $A(G)$ with $\|u_\alpha\|_{A(G)} \leq 1$ for each α. Given $\alpha \in \mathfrak{A}$ define a bilinear form $\Lambda_\alpha : VN(G) \times A(G) \to \mathbb{C}$ by

$$\Lambda_\alpha(T, u) = \langle \Gamma(u_\alpha \cdot T), u\rangle.$$

Then $\|\Lambda_\alpha\| \leq \|\Gamma\|$ for each $\alpha \in \mathfrak{A}$. It follows from an argument similar to the proof of the Banach-Alaoglu theorem [cf. 57, p. 66] that there exists a subnet $\{\Lambda_{\alpha_k}\}$ of $\{\Lambda_\alpha\}$ and a bilinear form $\Lambda_0 : VN(G) \times A(G) \to \mathbb{C}$ such that $\|\Lambda_0\| \leq \|\Gamma\|$ and Λ_{α_k} converges pointwise to Λ_0.

Define $\Gamma_0 : VN(G) \to VN(G)$ by

$$\langle \Gamma_0(T), u\rangle = \Lambda_0(T, u) \qquad \text{for every} \quad T \in VN(G), \quad u \in A(G).$$

Then $\|\Gamma_0\| \leq \|\Gamma\|$. If $T \in UCB(\widehat{G})$, then $\|u_{\alpha_k} \cdot T - T\|_{VN(G)} \to 0$. Hence,

$$\lim_\alpha \langle \Gamma(u_{\alpha_k} \cdot T), u\rangle = \langle \Gamma(T), u\rangle,$$

so $\Gamma_0\big|_{UCB(\widehat{G})} = \Gamma$. □

PROPOSITION 4.2.5. *Let G be an amenable locally compact group. Let X be a weak-$*$ closed $A(G)$-submodule of $VN(G)$. Then X is invariantly complemented in $VN(G)$ if and only if $X \cap UCB(\widehat{G})$ is invariantly complemented in $UCB(\widehat{G})$.*

Proof. Let P be an invariant projection of $VN(G)$ onto X. Let $T = u \cdot T_1 \in UCB(\widehat{G})$. Then

$$P(T) = P(u \cdot T_1) = u \cdot P(T_1) \in (A(G) \cdot VN(G)) \cap X$$

and hence $P|_{UCB(\widehat{G})}$ is an invariant projection of $UCB(\widehat{G})$ onto $UCB(\widehat{G}) \cap X$.

Conversely, let P be an invariant projection of $UCB(\widehat{G})$ onto $UCB(\widehat{G}) \cap X$. Let P_0 be the extension of P to $VN(G)$ constructed in the proof of Proposition 4.2.4 with respect to the bounded approximate identity $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ of $A(G)$.

Let $u \in {}^{\perp}X$ and $T \in VN(G)$. Then $P(u_\alpha T) \in X$ and

$$\langle P_0(T), u \rangle = \lim_\alpha \langle P(u_\alpha \cdot T), u \rangle = 0.$$

Therefore, $P_0 T \in ({}^{\perp}X)^{\perp} = X$.

If $T \in X$, then $u_\alpha T \in UCB(\widehat{G}) \cap X$. Therefore,

$$\begin{aligned} \langle P_0(T), u \rangle &= \lim_\alpha \langle P(u_\alpha \cdot T), u \rangle \\ &= \lim_\alpha \langle u_\alpha \cdot T, u \rangle \\ &= \langle T, u \rangle \qquad \text{for every} \quad u \in A(G). \end{aligned}$$

Hence P_0 is a projection of $VN(G)$ onto X. □

We do not know whether the assumption that G be amenable is necessary in either Proposition 4.2.4 or 4.2.5.

§4.3. Applications to Discrete Groups.

We close this chapter with some applications to discrete groups. The first result is an analogue of Lau and Losert's [37, Corollary 4].

PROPOSITION 4.3.1. *Let G be a discrete amenable group. Then G has property*

(*) *If X is a weak-$*$ closed invariantly complemented subspace of $VN(G)$, then there exists a weak-$*$ to weak-$*$ continuous projection P from $VN(G)$ onto X such that*

$$P(u \cdot T) = u \cdot P(T) \qquad \text{for every } u \in A(G), \quad T \in VN(G).$$

Conversely, if G is a locally compact group with property (*), then G is discrete.

Proof. Let G be discrete and amenable. Let X be a weak-$*$ closed invariantly complemented subspace of $VN(G)$. By Theorem 4.2.2, ${}^{\perp}X$ has a bounded approximate identity, so ${}^{\perp}X = I(A)$ for some $A \in \mathcal{R}(G)$ and $1_A \in B(G)$ by Theorem 3.3.4 and Lemma 3.1.3. Define $P : VN(G) \to X$ by

$$P(T) = 1_A \cdot T.$$

P is indeed the desired projection.

Suppose that G has property (*). As $\langle L_e \rangle$, the 1-dimensional linear span of $\{L_e\}$, is invariantly complemented [37, Theorem 2], property (*) implies that there exists a weak-$*$ to weak-$*$ continuous invariant projection P_0 of $VN(G)$ onto

$\langle L_e \rangle$. If $u, v \in A(G)$, $P_0^*(u) \in A(G)$ and

$$\begin{aligned}\langle P_0^*(uv), T\rangle &= \langle P_0(T), uv\rangle \\ &= \langle P_0(u \cdot T), v\rangle \\ &= \langle u \cdot P_0^*(v), T\rangle.\end{aligned}$$

Therefore, $P_0^*\big|_{A(G)} \in \mathrm{Hom}^{A(G)}(A(G), A(G))$. There exists a continuous function u_0 on G such that

$$P_0^*(u) = u_0 u \quad \text{for every} \quad u \in A(G).$$

Let $x_0 \in G$. Then

$$\begin{aligned}\langle P_0(L_{x_0}), u\rangle &= \langle P_0^*(u), L_{x_0}\rangle \\ &= \langle uu_0, L_{x_0}\rangle \\ &= u_0(x_0)u(x_0) \\ &= u_0(x_0)\langle L_{x_0}, u\rangle.\end{aligned}$$

As P_0 is a projection onto $\langle L_e \rangle$ and $L_{x_0} \notin \langle L_e \rangle$, we have $u_0(x_0) = 0$. Therefore $u_0 = 1_{\{e\}}$ and G is discrete. □

LEMMA 4.3.2. *Let G be an amenable discrete group and let $\Gamma \in \mathrm{Hom}^{A(G)}(VN(G), VN(G))$. Then Γ is weak-$*$ to weak-$*$ continuous.*

Proof. Let $u \in A(G)$. Since G is amenable, $u = wv$ for some $w, v \in A(G)$.

Let $T \in VN(G)$. Then

$$\begin{aligned} \langle \Gamma^*(u), T\rangle &= \langle \Gamma^*(wv), T\rangle \\ &= \langle w, v\Gamma(T)\rangle \\ &= \langle v \cdot \Gamma^*(w), T\rangle. \end{aligned}$$

By [35, Theorem 3.7], $\Gamma^*(u) = v \cdot \Gamma^*(w) \in A(G)$. □

PROPOSITION 4.3.3. *Let G be an amenable discrete group. Let P be a continuous projection of $VN(G)$ onto a weak-* closed $A(G)$-submodule X of $VN(G)$. If $P \in \mathrm{Hom}^{A(G)}(VN(G), VN(G))$, then $X^{\perp} = I(A)$ for some $A \in R(G)$. Furthermore,*

$$P(T) = 1_A \cdot T \qquad \text{for every} \quad T \in VN(G).$$

Proof. Since G is amenable and discrete, P is weak-* to weak-* continuous by Lemma 4.3.2. Therefore, there exists a function u_0 on G such that

$$P^*(u) = u_0 u \qquad \text{for every} \quad u \in A(G).$$

Let $u \in A(G)$, $T \in VN(G)$. Then

$$\begin{aligned} \langle P(T), u\rangle &= \langle P^*(u), T\rangle \\ &= \langle u_0 u, T\rangle \\ &= \langle u, u_0 \cdot T\rangle, \end{aligned}$$

so $P(T) = u_0 \cdot T$. Since P is a projection, $u_0 = 1_A$ for some A and $A = Z({}^{\perp}X)$. As X is invariantly complemented and G is amenable, Theorem 4.2.2 shows that $I(A)$ has a bounded approximate identity. By Proposition 3.1.6, $A \in \mathcal{R}(G)$. □

PROPOSITION 4.3.4. *Let G be an amenable $[SIN]$-group. Let $A \in \mathcal{R}_c(G)$ and $I = I(A)$. Then there exists a bounded linear map $\mathcal{M}$ from $\mathrm{Hom}^I(I, I)$ into $(I \cdot I^*)^*$. Furthermore, $\mathcal{M}$ is onto if and only if G is discrete.*

Proof. If G is an amenable $[SIN]$-group, then, by Theorem 3.3.5, $I(A)$ has a bounded approximate identity. The existence of $\mathcal{M}$ follows immediately from Proposition 4.1.6.

If G is discrete, $A(G) \cdot A^{**}(G) \subseteq A(G)$ [35, Theorem 3.7]. Cohen's factorization theorem implies that $I^2 = I$. Therefore, $I \cdot I^{**} = I \cdot ((I^{\perp})^{\perp}) \subseteq (I \cdot I) \cdot A(G)^{**} = I \cdot (I \cdot A^{**}(G)) \subseteq I$. By Proposition 4.1.6, $\mathcal{M}$ is onto.

Conversely, assume that $\mathcal{M}$ is onto. By translating A if necessary, we may assume that $e \in G \backslash A$. Let $u_0 \in P(G) \cap I$ with $u(e) = 1$. Following an idea of Lau [35, Theorem 3.7], let

$$K = \{u_0 \cdot \Gamma; \quad \Gamma \in A(G)^{**}, \quad \Gamma \geq 0, \quad \|\Gamma\| = 1\}.$$

Since the map $\Gamma \to u_0 \cdot \Gamma$ is weak-$*$ to weak-$*$ continuous, K is weak-$*$ compact. By Proposition 4.1.6, K is a weakly compact subset of $A(G)$. Let $\Lambda = \{u \in P(G) \cap I;\ u(e) = 1\}$. For each $u \in \Lambda$ define

$$\Lambda_u(v) = uv \qquad \text{for every} \quad v \in K.$$

Then $\{\Lambda_u;\ u \in \Lambda\}$ is a commutative semigroup of continuous maps from $\{K, \text{weak}\}$ into $\{K, \text{weak}\}$. By the Markov-Kakutani fixed-point theorem, there exists some $v_0 \in K$ such that

$$\Lambda_u v_0 = v_0 \qquad \text{for every} \quad u \in \Lambda.$$

If $x \neq e$, there exists $u \in \Lambda$ such that $u(x) \neq 1$. Therefore

$$v_0 = 1_{\{e\}}$$

and G is discrete. □

CHAPTER 5

COMPLEMENTED IDEALS IN $A(G)$

§5.0. Introduction.

The problem of characterizing complemented ideals in the group algebra $L^1(G)$ of a locally compact abelian group has received considerable attention over the last twenty-five years. D.J. Newman showed that if $\boldsymbol{T}$ is the circle group and H^1 is the closed ideal of $L^1(\boldsymbol{T})$ consisting of all those $f \in L^1(\boldsymbol{T})$ such that $\hat{f}(n) = 0$ for every $n < 0$, then H^1 is not complemented in $L^1(\boldsymbol{T})$ [47]. Rudin extended Newman's result by showing that if I is complemented in $L^1(\boldsymbol{T})$, then $I = I(A)$ where $A = \bigcup_{i=1}^{n} a_i Z + b_i$ and conversely [58].

In his memoir [55, p. 20], H. Rosenthal proved that if I is complemented in $L^1(G)$, where G is abelian, then $I = I(A)$ for some $A \in \mathcal{R}_c(\widehat{G})$. Sets belonging to $\mathcal{R}_c(\widehat{G})$ were characterized in [22] and [59].

Alspach and Matheson characterized the closed complemented ideals in $L^1(\mathbb{R})$. They showed that I is complemented if and only if $I = I(A)$ for some

$$A = \bigcup_{i=1}^{n} (\tau_i Z + \beta_i) \backslash F,$$

where the τ_i's are pairwise rationally dependent and F is finite [2].

In [3], Alspach, Matheson and Rosenblatt looked at the complementation problem for arbitrary abelian locally compact groups. They were successful in giving necessary and sufficient conditions for an ideal with a discrete hull to be complemented [3, p. 265]. However, their proof is incorrect, and it was subsequently corrected in [4]. They also gave a complicated inductive procedure for finding complemented ideals. This procedure was exploited by Alspach to identify the complemented ideals in $L^1(\mathbb{R}^2)$ [1]; though it is not likely that it will be of much use for $L^1(\mathbb{R}^n)$ when $n > 2$. Little else is known for the abelian case.

We will attempt to take a small bite from the complementation problem for $A(G)$ of an arbitrary locally compact group. For the most part, we will restrict ourselves to identifying the closed subgroups H of G for which $I(H)$ is complemented. We show that this is equivalent to the existence of a continuous linear map $\Gamma : A(H) \to A(G)$ such that $\Gamma_{u|_H} = u$ for every $u \in A(H)$ (Proposition 5.1.2). We also prove that this criterion is satisfied by a class of subgroups which includes all compact groups. By exploiting a geometric property possessed by $A(G)$ for all groups in the above class, we are able to exhibit a locally compact group G and a closed normal subgroup H such that $I(H)$ is not complemented in $A(G)$.

In Section 5.2, we look at the complementation problem for product groups. By making use of a number of structure theorems, we are able to extend the results of Section 5.1 to larger classes of locally compact groups.

§5.1. The Complemented Ideal Property for Closed Subgroups.

We will show that if H is a closed subgroup of a locally compact group, then $I(H)$ is complemented in $A(G)$ if and only if $A(H)$ can be extended to $A(G)$ in a continuous linear manner. We will also show that for an arbitrary locally compact group every compact subgroup H has this extension property.

DEFINITION 5.1.1. A closed ideal I in $A(G)$ is said to be *complemented* if there exists a continuous projection P of $A(G)$ onto I. Let H be a closed subgroup of G. H is said to have the *complemented ideal property* (*C.I.P.*) in G if $I(H)$ is complemented in $A(G)$.

Let

$$C.I.P.(G) = \{H; \quad H \text{ has } C.I.P. \text{ in } G\}.$$

PROPOSITION 5.1.2. *Let H be an open subgroup of G. Then $H \in C.I.P.(G)$. In particular, if G is a discrete group, then every subgroup of G belongs to $C.I.P.(G)$.*

Proof. $1_{G\setminus H} \in B(G)$. Furthermore $A(G) = 1_H A(G) \oplus 1_{G\setminus H} A(G)$ and $1_{G\setminus H} A(G) = I(H)$. □

The next proposition provides us with our key tool in identifying subgroups in $C.I.P.(G)$.

PROPOSITION 5.1.3. *Let H be a closed subgroup of G. Then $H \in C.I.P.(G)$ if and only if there exists a continuous linear map $\Gamma : A(H) \to A(G)$ such that*

$$\Gamma_{u|_H} = u \qquad \textit{for every} \quad u \in A(H).$$

Proof. Assume $H \in C.I.P.(G)$. Let Q be a continuous projection of $A(G)$ onto $I(H)$. Let $u \in A(H)$. Let $v \in A(G)$ be such that

$$v|_H = u \qquad \text{and} \qquad \|v\|_{A(G)} = \|u\|_{A(H)} \qquad [27, \text{Theorem } 16].$$

Define

$$\Gamma u = v - Qv.$$

Then

$$\|\Gamma u\| \leq \|v\|_{A(G)} + \|Q\| \, \|v\|_{A(G)} = (1 + \|Q\|)\|u\|_{A(H)} \, .$$

Let v_1 be any other extension of u to G. Then

$$(v_1 - Qv_1) - (v - Qv) = (v_1 - v) - Q(v_1 - v) = 0.$$

Therefore, Γ is well defined.

Let $u_1, u_2 \in A(H)$, $\alpha_1, \alpha_2 \in \mathbb{C}$ and v_1, v_2 be extensions of u_1 and u_2 respectively in $A(G)$. As $\alpha_1 v_1 + \alpha_2 v_2$ is an extension of $\alpha_1 u_1 + \alpha_2 u_2$,

$$\begin{aligned}\Gamma(\alpha_1 u_1 + \alpha_2 u_2) &= (\alpha_1 v_1 + \alpha_2 v_2) - Q(\alpha_1 v_1 + \alpha_2 v_2)\\ &= \alpha_1(v_1 - Q(v_1)) + \alpha_2(v_2 - Q(v_2))\\ &= \alpha_1 \Gamma(u_1) + \alpha_2 \Gamma(u_2),\end{aligned}$$

so Γ is linear.

Conversely, assume that $\Gamma : A(H) \to A(G)$ is continuous, linear and $\Gamma u_{|H} = u$ for every $u \in A(H)$. For each $v \in A(G)$ define

$$Q(v) = v - \Gamma(v_{|H}).$$

Q is linear and $\|Q\| \leq 1 + \|\Gamma\|$. Since $\Gamma(v_{|H})_{|H} = v_{|H}$, $Q(v) \in I(H)$. If $v \in I(H)$, $v_{|H} = 0$, so $= v$. □

The next proposition is the starting point of [3]. It is not stated as such, but the essential ideas of the proof are included as a remark [3, p. 257]. We include the details for the sake of mpleteness.

PROPOSITION 5.1.4. *be an abelian locally compact group. Let H be a closed subgroup of G. Then there exists a linear isometry $\Gamma : A(H) \to A(G)$ such that $\Gamma u_{|H} = u$ for every $u \in A(H)$.*

Proof. There exists a closed subgroup $\Lambda_0 \subseteq \widehat{G}$ such that $\widehat{H} = \widehat{G}/\Lambda_0$. Let φ be a locally bounded, locally measurable function on $\widehat{G}$ such that

$$\int_{\Lambda_0} \varphi(xy)d\mu_{\Lambda_0}(y) = 1$$

$\mu_{\widehat{G}/\Lambda_0}$ almost everywhere [cf. 52, Chapter 8].

Let $f \in L^1(\widehat{G}/\Lambda_0)$. Then

$$S_\varphi f(\gamma) = f(\gamma + \Lambda_0)\varphi(\gamma) \qquad \text{for every} \quad \gamma \in \widehat{G}$$

defines a linear isometry of $L^1(\widehat{G}/\Lambda_0)$ into $L^1(\Lambda)$. Let

$$\pi_{\Lambda_0} f(x) = \int_{\Lambda_0} f(xy)d\mu_{\Lambda_0}(y).$$

Then π_{Λ_0} maps $L^1(\widehat{G})$ onto $L^1(\widehat{G}/\Lambda_0)$. Furthermore, $\pi_{\Lambda_0} \circ S_\varphi$ is the identity on $L^1(\widehat{G}/\Lambda_0)$ and if $h \in H$,

$$(S_\varphi f)^\wedge(h) = \hat{f}(h) \qquad \text{for every} \quad f \in L^1(\widehat{G}/\Lambda_0).$$

Let $\Gamma_0 : A(H) \to L^1(\widehat{G}/\Lambda_0)$ be the inverse of the Fourier transform. Define $\Gamma : A(H) \to A(G)$ by

$$\Gamma u = [S_\varphi(\Gamma_0 u)]^\wedge.$$

We have $\|\Gamma u\|_{A(G)} = \|u\|_{A(H)}$ and $\Gamma u_{|H} = u$ for every $u \in A(H)$. □

COROLLARY 5.1.5. *Let G be an abelian locally compact group and let H be a closed subgroup of G. Then $H \in C.I.P.(G)$.*

A representation $\pi \in \Sigma_G$ is said to be *completely reducible* if

$$\pi = \bigoplus_{\alpha \in \mathfrak{A}} \sigma_\alpha,$$

where $\sigma_\alpha \in \widehat{G}$.

It is well known that if G is abelian, then the left regular representation ρ_G is completely reducible if and only if G is compact [cf. 62].

We can now prove the main result of this section.

THEOREM 5.1.6. *Let G be a locally compact group. Let H be a closed subgroup of G such that the left regular representation ρ_H of H is completely reducible. Then there exists a linear isometry $\Gamma : A(H) \to A(G)$ such that $\Gamma u_{|_H} = u$ for every $u \in A(H)$.*

Proof. Let ρ_G denote the left regular representation of G. Since $A(G)_{|_H} = A(H)$ [27, Theorem 16], $\rho_{G_{|_H}}$ is quasi-equivalent to ρ_H [6, p. 27]. Therefore, there exist cardinal numbers n_G and n_H such that $n_G(\rho_{G_{|_H}})$ is unitarily equivalent to $n_H\rho_H$. It follows that there exists a Hilbert space isomorphism $T : n_G\mathcal{H}_{\rho_G} \to n_H\mathcal{H}_{\rho_H}$ such that

$$Tn_G\rho_{G_{|_H}} = n_H\rho_H T \qquad \text{[cf. 17, p. 118]}.$$

As ρ_H is completely reducible, we can write

$$\rho_H = \sum_{\alpha\in\mathfrak{A}} \oplus m_\alpha\sigma_\alpha \qquad \text{and} \qquad \mathcal{H}_{\rho_H} = \sum_{\alpha\in\mathfrak{A}} \oplus m_\alpha\mathcal{H}_{\sigma_\alpha},$$

where $\sigma_\alpha \in \widehat{H}$ for every $\alpha \in \mathfrak{A}$. Each $\mathcal{H}_{\sigma_\alpha}$ is a closed subspace of $\mathcal{H}_{\rho_H}$, so $T^{-1} H_{\sigma_\alpha}$ is a closed subspace of $n_G\mathcal{H}_{\rho_G}$.

Let A_{σ_α} denote the closure in $B(H)$ of the space F_{σ_α} of coefficient functions of σ_α. Since σ_α is irreducible, A_{σ_α} is linearly isometrically isomorphic to $\mathcal{H}_{\sigma_\alpha} \hat{\otimes} \mathcal{H}_{\sigma_\alpha}$

[6, 2-ème partie], where $\hat{\otimes}$ denotes the projective tensor product [cf. 60, p. 188].

For $\varsigma, \eta \in \mathcal{H}_{\sigma_\alpha}$, define

$$\Gamma_{\sigma_\alpha}\langle\sigma_\alpha(\cdot)\varsigma,\eta\rangle = \langle n_G\rho_G(\cdot)T_\varsigma^{-1}, T_\eta^{-1}\rangle.$$

Then Γ_{σ_α} extends to a linear map of $F_{\sigma_\alpha} = \mathcal{H}_{\sigma_\alpha} \otimes \mathcal{H}_{\sigma_\alpha}$ into $An_G\rho_G$ with

$$\begin{aligned}\|\Gamma_{\sigma_\alpha}\sum_{i=1}^{n}\langle\sigma_i(\cdot)\varsigma_i,\eta_i\rangle\|_{B(G)} &\leq \sum_{i=1}^{n}\|T^{-1}\varsigma_i\|_{\mathcal{H}n_G\rho_G}\|T^{-1}\eta_i\|_{\mathcal{H}n_G\rho_G}\\ &= \sum_{i=1}^{n}\|\varsigma_i\|_{\mathcal{H}_{\sigma_\alpha}}\|\eta_i\|_{\mathcal{H}_{\sigma_\alpha}}\\ &= \|\sum_{i=1}^{n}\langle\sigma_\alpha(\cdot)\varsigma_i,\eta_i\rangle\|_{A(H)}.\end{aligned}$$

Therefore, Γ_{σ_α} extends to a norm non-increasing linear map of A_{σ_α} into $An_G\rho_G$.

Since $Tn_G\rho_{G|_H} = n_H\rho_H T$,

$$\begin{aligned}\langle\sigma_\alpha(h)\varsigma,\eta\rangle &= \langle n_G\rho_G(h)T^{-1}\varsigma, T^{-1}\eta\rangle\\ &= \Gamma_{\sigma_\alpha}\langle\sigma_\alpha(\cdot)\varsigma,\eta\rangle(h)\end{aligned}$$

for every $\varsigma, \eta \in \mathcal{H}_{\sigma_\alpha}$. Convergence in $B(H)$ and $B(G)$ implies uniform convergence so

$$\Gamma_{\sigma_\alpha}u_{|_H} = u \qquad \text{for every} \quad u \in A_{\sigma_\alpha}.$$

Furthermore, Γ_{σ_α} is an isometry.

The σ_α's are irreducible and thus pairwise disjoint. Therefore,

$$A(H) = \ell_1\text{-} \oplus A_{\sigma_\alpha} \qquad [6, \text{p. } 39].$$

Define $\Gamma : A(H) \to B(G)$ by

$$\Gamma u = \sum_{\alpha\in\mathfrak{A}}\Gamma_{\sigma_\alpha}u_{\sigma_\alpha},$$

where $u = \sum_{\alpha \in \mathfrak{A}} u_{\sigma_\alpha}$ is the unique direct sum decomposition of u. Then

$$\begin{aligned} \|\Gamma u\|_{B(G)} &= \|\sum_{\alpha\in\mathfrak{A}} \Gamma_{\sigma_\alpha} u_{\sigma_\alpha}\|_{B(G)} \\ &\le \sum_{\alpha\in\mathfrak{A}} \|\Gamma_{\sigma_\alpha} u_{\sigma_\alpha}\|_{B(G)} \\ &= \sum_{\alpha\in\mathfrak{A}} \|u_{\sigma_\alpha}\|_{A(H)} \\ &= \|u\|_{A(H)} . \end{aligned}$$

As $\Gamma_{\sigma_\alpha} u_{\sigma_\alpha}{}_{|H} = u_{\sigma_\alpha}$ for every $\alpha \in \mathfrak{A}$, $\Gamma u_{|H} = u$ for every $u \in A(H)$. Therefore, Γ is a linear isometry of $A(H)$ into $B(G)$.

Finally, $\Gamma_{\sigma_\alpha} u_{\sigma_\alpha} \in A_{n_G \rho_G}$ for each $\alpha \in \mathfrak{A}$, so $\Gamma u \in A_{n_G\rho_G}$. However, $n_G \rho_G$ is quasi-equivalent to ρ_G, so $A_{n_G\rho_G} = A_{\rho_G} = A(G)$ [6, p. 27]. □

COROLLARY 5.1.7. *Let H be a compact subgroup of G. Then there exists a linear isometry $\Gamma : A(H) \to A(G)$ such that $\Gamma u_{|H} = u$ for every $u \in A(H)$. Furthermore, $H \in C.I.P.(G)$.*

Proof. It is well known that if H is compact, ρ_H is completely reducible [cf. 29, p. 29]. □

Let G be a unimodular group. Let

$$\widehat{G}_d = \{\pi \in \widehat{G}; \quad \pi \text{ is square integrable}\}.$$

Let $A_d(G)$ denote the closed subspace of $A(G)$ generated by the extreme points of the unit ball of $A(G)$. Mauceri [45] has shown that

$$A_d(G) = \ell_1 - \bigoplus_{\pi\in\widehat{G}_d} A_\pi$$

and that there exists a closed subspace $A_c(G)$ of $A(G)$ such that

$$A(G) = A_d(G) \oplus A_c(G).$$

Let H be a closed subgroup of G. The proof of Theorem 5.1.7 shows that there exists a linear isometry $\Gamma : A_d(H) \to A(G)$ such that $\Gamma u_{|_H} = u$ for every $u \in A_d(H)$.

Example 5.1.8. (1) Let H be the "$ax+b$" group. We recall that H can be realized as the matrix group

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}; \quad a \in \mathbb{R}^+, \quad b \in \mathbb{R} \right\}.$$

Let $GL(n, \mathbb{R})$ denote the group of invertible $n \times n$ matrices over $\mathbb{R}$ with the topology it inherits as a subspace of $\mathbb{R}^{n^2}$.

For any $n \geq 2$, $G = GL(n, \mathbb{R})$ contains closed subgroups H_α which are isomorphic to H. Let H_1 be such a subgroup. Let $\varphi : H \to H_1$ be an isomorphism. Theorem 3.1.7 shows that there exists a linear isometry $\Gamma : A(H) \to A(G)$ such that

$$\Gamma u(\varphi(h)) = u(h) \qquad \text{for every} \quad u \in A(H), \quad h \in H.$$

(2) Let $H = \mathbb{R}^2 \odot GL(2, \mathbb{R})^+$ be the semidirect product of $\mathbb{R}^2$ with the closed subgroup $GL(2, \mathbb{R})^+$ of $GL(2, \mathbb{R})$ consisting of matrices with positive determinants. Then ρ_H is completely reducible [9].

Let $G = \mathbb{R}^n \odot GL(n, \mathbb{R})$ for $n \geq 2$. Let

$$H_1 = \left\{ \left((x, y, 0, 0, \ldots, 0), \begin{pmatrix} \begin{matrix} A \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & & \\ \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} & & 0 \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix} \right); \quad x, y \in \mathbb{R}, \right.$$

$$\left. A \in GL(2, \mathbb{R})^+ \right\}.$$

Then H_1 is a closed subgroup of G isomorphic to H. It follows that there exists an isomorphism $\varphi : H \to H_1$ and a linear isometry $\Gamma : A(H) \to A(G)$ such that

$$\Gamma u(\varphi(h)) = u(h) \quad \text{for every} \quad u \in A(H), \quad h \in H.$$

Furthermore, $I(H)$ is complemented in $A(G)$.

DEFINITION 5.1.9. A Banach space X is said to possess the Radon-Nikodym Property if for any finite measure space (Ω, Σ, μ) and any μ-continuous vector measure $L : \Sigma \to X$ of bounded total variation, there exists a Bochner integrable function $g : \Omega \to X$ such that

$$L(E) = \int_E g \, d\mu \qquad \text{for every} \quad E \in \Sigma$$

[cf. 16].

In [12, p. 535] Chu showed that the pre-dual of a von-Neumann algebra has the R.N.P. if and only if every closed bounded convex subset is the closed convex hull of its extreme points.

Granirer and Leinert proved that if $K \subset G$ is compact, $A_K(G)$ has the R.N.P. [25, p. 464]. K. Taylor showed that $A(G)$ has the R.N.P. if and only if ρ_G is completely reducible [62]. As it is easy to see that the R.N.P. is preserved by linear isometries, the previous examples show that if G is $GL(n, \mathbb{R})$ or $\mathbb{R}^n \odot GL(n, R)$, then $A(G)$ has closed subspaces with R.N.P. which are not of the form $A_K(G)$, even though $A(G)$ does not have R.N.P. [cf. 62]. However we have:

PROPOSITION 5.1.10. *Let G be a locally compact group. Let H be a closed subgroup of G such that ρ_H is completely reducible. Then $I(H)$ has the Radon-Nikodym Property if and only if $A(G)$ has the Radon-Nikodym Property.*

Proof. If $A(G)$ has the R.N.P., so does every closed subspace.

Conversely, assume that $I(H)$ has the R.N.P. Since ρ_H is completely reducible, $A(H)$ has the R.N.P. It follows from Lemma 3.1.9, that $A(G)/I(H)$ has the R.N.P. and hence, so does $A(G)$ [cf. 16, p. 211]. □

PROPOSITION 5.1.11. *Let G be a locally compact group for which $A(G)$ has the R.N.P. Let H be a closed subgroup of G. If $I(H)$ is complemented in $A(G)$, then $A(H)$ has the R.N.P.*

Proof. Let P be a projection of $A(G)$ onto $I(H)$. Then $(1-P)(A(G))$ has the R.N.P. Define $\Gamma_0 : (1-P)(A(G)) \to A(H)$ by

$$\Gamma_0 u = u_{|H} \quad \text{for every} \quad u \in (1-P)(A(G)).$$

Γ_0 is linear, $\|\Gamma_0\| \leq 1$ and if we let $\Gamma : A(H) \to A(G)$ be the map constructed from P in the proof of Proposition 5.1.3, then $\Gamma = \Gamma_0^{-1}$ and $\|\Gamma\| < \infty$. Therefore, $A(H)$ and $(1-P)(A(G))$ are linearly isomorphic and homeomorphic. It follows that every norm closed convex subset of $A(H)$ is the closed convex hull of its extreme points. Therefore, by Chu's result [12], $A(H)$ has the R.N.P. □

Example 5.1.12. Let $G = \mathbb{R} \odot \mathbb{R}^+$ be the "ax+b" group. Then $A(G)$ has the R.N.P. Also G has a closed normal subgroup H which is isomorphic to $\mathbb{R}$. But $A(\mathbb{R})$ does not have the R.N.P. [cf. 62], Hence Proposition 5.1.11 shows that $I(H)$ is not complemented in $A(G)$.

Example 5.1.12 would seem to imply that the complementation problem is much more complex for arbitrary locally compact groups than it is for abelian locally compact groups.

§5.2. Product Groups.

Let G_1 and G_2 be locally compact groups. Let $u_1 \in A(G_1)$ and $u_2 \in A(G_2)$. We can define an element $u \in A(G_1 \times G_2)$ by

$$u(x,y) = u_1(x)u_2(y).$$

In this manner, we get a linear map $\psi : A(G_1) \hat{\otimes} A(G_2) \to A(G_1 \times G_2)$. Furthermore, $\psi(A(G_1) \hat{\otimes} A(G_2))$ is dense in $A(G_1 \times G_2)$ and $\|\psi\| \leq 1$ [cf. 43].

Losert [43] proved that if the dimension of the irreducible respresentations of G_i are bounded for either G_1 or G_2, then ψ is surjective. Moreover, $\|\psi^{-1}\| = \min(d_1, d_2)$, where $d_i = \sup_{\pi \in \widehat{G}_i} \dim \pi$. Consequently, if G_1 or G_2 is abelian, then ψ is an isometry.

Conversely, if ψ is surjective, then one of the G_i's must have an open abelian subgroup of finite index and ψ is an isometry only if one G_i is abelian.

PROPOSITION 5.2.1. *Let $G = G_1 \times G_2$. Then for each $i = 1, 2$, there exists a linear isometry $\Gamma_i : A(G_i) \to A(G)$ such that $\Gamma_i u_{|G_i} = u$ for every $u \in A(G_i)$. Consequently, $G_i \in C.I.P.(G)$.*

Proof. Let $u_2 \in A(G_2)$ be such that $u_2(e_2) = 1$ and $\|u_2\|_{A(G)} = 1$. Define

$$\Gamma_1 v = \psi(v \otimes u_2) \quad \text{for every} \quad v \in A(G_1).$$

Then $\|\Gamma_1 v\|_{A(G)} \leq \|v\|_{A(G_1)} \|u_2\|_{A(G_2)} \leq \|v\|_{A(G_1)}$. But $\Gamma_1 v_{|G_1} = v$. Hence $\|\Gamma_1 v\|_{A(G)} = \|v\|_{A(G_1)}$. We can define Γ_2 similarly. □

LEMMA 5.2.2. *Let $G = G_1 \times G_2$. Let $A = A_1 \times A_2$, where A_i is closed in G_i. Assume that A is a set of spectral synthesis in G. Then $I_G(A)$ is the closed linear span of*

$$J = \{\psi(I_{G_1}(A_1) \otimes A(G_2)) \cup \psi(A(G_1) \otimes I_{G_2}(A_2))\}.$$

Proof. Let $u_1 \in I_{G_1}(A_1)$ and $u_2 \in A(G_2)$. Then

$$\psi(u_1 \otimes u_2)(x_1, x_2) = 0 \quad \text{if} \quad (x_1, x_2) \in A_1 \times A_2.$$

Similarly, if $u_1 \in A(G_1)$ and $u_2 \in I_{G_2}(A_2)$, then

$$\psi(u_1 \otimes u_2)(x_1, x_2) = 0 \quad \text{if} \quad (x_1, x_2) \in A_1 \times A_2.$$

Let $\langle J \rangle^-$ be the closed linear span of J in $A(G_1 \times G_2)$. Then $\langle J \rangle^- \subseteq I_G(A)$.

Let $v \in A(G)$. Then there exists a net $\{ \sum_{i=1}^{n_\alpha} w_{\alpha_i} \otimes z_{\alpha_i} \}_{\alpha \in \mathfrak{A}}$ in $A(G_1) \otimes A(G_2)$ such that

$$V = \lim_\alpha \psi\Big(\sum_{i=1}^{n_\alpha} w_{\alpha_i} \otimes z_{\alpha_i} \Big).$$

Let $u_1 \in I_{G_1}(A_1)$, $u_2 \in A(G_2)$. Then

$$\psi(u_1 \otimes u_2)v = \lim_{\alpha} \psi\Big((u_1 \otimes u_2)\big(\sum_{i=1}^{n_\alpha} w_{\alpha_i} \otimes z_{\alpha_i}\big)\Big)$$

$$= \lim_{\alpha} \psi\big(\sum_{i=1}^{n_\alpha} (u_1 w_{\alpha_i}) \otimes (u_2 z_{\alpha_i})\big),$$

so $\psi(u_1 \otimes u_2)v \in \langle J \rangle^-$. Similarly, if $u_1 \in A(G_1)$ and $u_2 \in I_{G_2}(A)$ then $\psi(u_1 \otimes u_2)v \in \langle J \rangle^-$. It follows that $\langle J \rangle^-$ is a closed ideal in $A(G)$.

Assume $(x_1, x_2) \notin A_1 \times A_2$. We may assume $x_1 \notin A$. There exists $u_1 \in I_{G_1}(A_1)$ such that $u_1(x_1) = 1$. Let $u_2 \in A(G_2)$ be such that $u_2(x_2) = 1$. Then

$$\psi(u_1 \otimes u_2)(x_1, x_2) = 1.$$

Hence $Z(\langle J \rangle^-) = A_1 \times A_2 = A$. Since A is an S-set, $\langle J \rangle^- = I_G(A)$. □

PROPOSITION 5.2.3. *Let $G = G_1 \times G_2$, where G_1 has an abelian subgroup of finite index. Let $A = A_1 \times A_2$, where each A_i is closed in G_i. Assume that A is a set of spectral synthesis in G. Suppose that $I_{G_i}(A_i)$ is complemented in $A(G_i)$ for $i = 1, 2$. Then $I_G(A)$ is complemented in $A(G)$.*

Proof. Let P_i be a continuous projection of $A(G_i)$ onto $I_{G_i}(A_i)$. Define $P : A(G) \to A(G)$ by

$$Pu = u - \psi((1 - P_1) \otimes (1 - P_2)(\psi^{-1} u)).$$

Since G_1 has an abelian subgroup of finite index, $\psi^{-1} u$ exists and $\|\psi^{-1}\| < \infty$ by [43]. Therefore, P is continuous and linear.

Let $u_1 \in A(G_1)$ and $u_2 \in A(G_2)$. Let $(x_1, x_2) \in A_1 \times A_2$ and $u = \psi(u_1 \otimes u_2)$. Then

$$\begin{aligned} Pu(x_1,x_2) &= u(x_1,x_2) - \psi((1-P_1)\otimes(1-P_2)(\psi^{-1}u))(x_1,x_2) \\ &= u_1(x_1)u_2(x_2) - [((1-P_1)u_1)(x_1)][((1-P_2)u_2)(x_2)] \\ &= u_1(x_1)u_2(x_2) - u_1(x_1)u_2(x_2) \\ &= 0. \end{aligned}$$

Therefore, $Pu \in I_G(A)$ for every $u \in A(G)$.

Let $u_1 \in I_{G_1}(A_1)$ and $u_2 \in A(G_2)$. Let $u = \psi(u_1 \otimes u_2)$. Then

$$\begin{aligned} \psi((1-P_1)\otimes(1-P_2)(\psi^{-1}u)) &= \psi((1-P_1)u_1 \otimes (1-P_2)u_2) \\ &= \psi(0 \otimes (1-P_2)u_2) \\ &= 0. \end{aligned}$$

Therefore $Pu = u$. Similarly, if $u_1 \in A(G_1)$ and $u_2 \in I_{G_2}(A_2)$, then $Pu = u$ for $u = \psi(u_1 \otimes u_2)$. It follows from Lemma 5.2.2 that $Pu = u$ for every $u \in I_G(A)$. □

COROLLARY 5.2.4. *Let $G = G_1 \times G_2$ be a locally compact group. Assume that G_1 has a closed abelian subgroup of finite index. Let H_1 be a closed subgroup of G_1 and let $H_2 \in C.I.P.(G_2)$. Then $H = H_1 \times H_2 \in C.I.P.(G)$.*

Proof. Let J be a closed abelian subgroup of G_1 with finite index. Since J is open, $1_J \in B(G_1)$ and $B(J) \subseteq B(G_1)$. Also $H_1 \cap J$ is of finite index in H_1. Therefore, we can write

$$H_1 = \bigcup_{i=1}^{n} x_j(H_1 \cap J),$$

where the x_i's are distinct coset representatives and $1_{H_1 \cap J} \in B(H_1)$. Let $u \in A(H_1)$. Then

$$u = \sum_{i=1}^{n} L_{x_i} u_i,$$

where $u_i = L_{x_i^{-1}} (1_{x_i(H_1 \cap J)}\ u) \in A(H_1 \cap J)$. Since J is abelian, there exists a linear isometry $\Gamma : A(H_1 \cap J) \to A(J)$ such that $\Gamma v_{|_{H_1 \cap J}} = v$ for every $v \in A(H_1 \cap J)$ (Proposition 5.1.4). Define $\Gamma' : A(H) \to A(G_1)$ by

$$\Gamma' u = \sum_{i=1}^{n} L_{x_i} (\Gamma u_i).$$

Γ' is continuous, linear and $\Gamma' u_{|_{H_1}} = u$ for every $u \in A(H_1)$. Therefore, $H_1 \in C.I.P.(G_1')$. The result now follows immediately from Proposition 5.2.3. □

PROPOSITION 5.2.5. *Let H be a closed subgroup of a separable locally compact group. Let C be a subgroup of $C_G(H)$ and $K = CH$. If K is closed in G, then there exists a linear isometry $\Gamma : A(H) \to A(K)$ such that $\Gamma u_{|_H} = u$ for every $u \in A(H)$. In particular, $H \in C.I.P.(K)$.*

Proof. $K/H = CH/H \cong C/C \cap H$. Let $\varphi : K/H \to C/C \cap H$ be the canonical isomorphism. Let $\psi_1 : C/C \cap H \to C$ be a Borel cross section [cf. 44, p. 102]. Define $\psi : K/H \to K$ by $\psi = \psi_1 \circ \psi$. Then ψ is a Borel cross section and $\psi(\xi) \in C$ for every $\xi \in K/H$.

Let Ω be a compact subset of K/H with $\mu_{K/H}(\Omega) > 0$. Let $x \in K$ and $\xi \in K/H$. Define

$$\xi^x = \pi(x^{-1} y),$$

where $: K \to K/H$ is the canon... homomorphism and $y \in K$ is such that $\pi(y) = \xi$.

Define $\tau : K \times K/H \to H$ by

$$\tau(x, \xi) = \psi(\xi)^{-1} x \psi(\xi^x).$$

For each $u \in A(H) \cap C_{00}(H)$,

$$\Gamma_0 u(x) = \frac{1}{\mu_{K/H}(\Omega)} \int_{K/H} 1_\Omega(\xi^x) 1_\Omega(\xi) u(\tau(x, \xi)) d\mu_{K/H}(\xi).$$

Then supp $\Gamma_0 u$ is compact. Furthermore, $\Gamma_0 u \in A(K)$ with $\|\Gamma_0 u\|_{A(K)} \leq \|u\|_{A(H)}$ [cf. 27, p. 114].

Let $h \in H$. Then

$$\xi^h = \xi,$$

and since $\psi(\xi) \in C$,

$$\tau(h, \xi) = \psi(\xi)^{-1} h \psi(\xi) = h.$$

Therefore

$$\begin{aligned} \Gamma_0 u(h) &= \frac{1}{\mu_{K/H}(\Omega)} \int_{K/H} 1_\Omega(\xi) u(h) d\mu_{K/H}(\xi) \\ &= u(h). \end{aligned}$$

Hence $\Gamma_0 u_{|H} = u$ and $\|\Gamma_0 u\|_{A(K)} = \|u\|_{A(H)}$. As $A(H) \cap C_{00}(H)$ is dense in $A(H)$, Γ_0 can be extended to an isometry Γ of $A(H)$ into $A(K)$ which is such that $\Gamma u_{|H} = u$ for every $u \in A(H)$. □

COROLLARY 5.2.6. *Let G be a separable locally compact group. Then there exists a linear isometry $\Gamma : A(Z(G)) \to A(G)$ such that*

$$\Gamma u_{|Z(G)} = u \qquad \textit{for every} \quad u \in Z(G).$$

In particular, $Z(G) \in C.I.P.(G)$.

COROLLARY 5.2.7. *Let G be a separable locally compact group with an open central subgroup. Let H be a closed subgroup of G. Then there exists a linear isometry $\Gamma : A(H) \to A(G)$ with*

$$\Gamma u|_H = u \qquad \text{for every} \quad u \in A(H).$$

In particular, $H \in C.I.P.(G)$.

Proof. If $Z(G)$ is open, $K = Z(G)H$ is an open subgroup of G, and hence is closed in G. □

We are now in a position to expand our investigation to larger classes of groups.

PROPOSITION 5.2.8. *Let $G \in [Z]$ be almost connected and separable. Let $K = CH$, where $Z(G) \subseteq C \subseteq C_G(H)$ and C is a closed subgroup. If K is closed in G, then $H \in C.I.P.(G)$.*

Proof. Since G is almost connected, there exists a compact subgroup K_1 of G and a vector subgroup V of G such that $G = V \times K$ [48, p. 698].

Assume that K is closed in G. Let $\pi : G = V \times K_1 \to K_1$ be the canonical homomorphism. Since $V \subseteq Z(G) \subseteq K$, $\pi(G)$ is a compact subgroup of K_1. Furthermore, $K = V \times \pi(G)$. By Proposition 5.2.5, $H \in C.I.P.(K)$. However, since $\pi(G) \in C.I.P.(K_1)$ (Corollary 5.1.7), $K \in C.I.P.(G)$ by Proposition 5.2.3.

□

PROPOSITION 5.2.9. *Let $G \in [SIN]$ be separable and almost connected. Let H be a closed subgroup of G. Let $K = CH$, where $Z(G) \subseteq C \subseteq C_G(H)$ and C is a closed subgroup. If K is closed in G, then $H \in C.I.P.(G)$.*

Proof. As G is almost connected, there exists a compact subgroup K_1 in G and a vector subgroup V in G such that $V \times K_1$ is of finite index in G [cf. 48, p. 698]. If K is closed in G, then $K \cap V \times K_1$ is closed in $V \times K$. Furthermore, $K \cap V \times K_1$ is of finite index in K and there exists a compact subgroup K^* of K_1 such that $K \cap V \times K_1 = V \times K^*$. By Proposition 5.2.3, $V \times K^* \in C.I.P.(V \times K_1)$, and hence $V \times K^* \in C.I.P.(G)$.

Let $\Gamma_0 : A(V \times K^*) \to A(G)$ be a linear isometry such that $\Gamma_0 u_{|V \times K^*} = u$ for every $u \in A(V \times K^*)$. Then $K = \bigcup_{i=1}^{n} x_i(V \times K^*)$ and if $u \in A(K)$

$$u = \sum_{i=1}^{n} L_{x_i} u_i,$$

where $u_i = L_{x_i}(1_{x_i(V \times K^*)}\, u)$. Define $\Gamma : A(K) \to A(G)$ by

$$\Gamma u = \sum_{i=1}^{n} L_{x_i} \Gamma(u_\alpha).$$

Then Γ is continuous, linear and $\Gamma u_{|_k} = u$ for every $u \in A(K)$. Therefore, $K \in C.I.P.(G)$. By Proposition 5.2.5, $H \in C.I.P.(K)$, so $H \in C.I.P.(G)$. □

COROLLARY 5.2.10. *Let $G \in [SIN]$ be separable and almost connected. Let H be a closed subgroup of G. If H is abelian, then $H \in C.I.P.(G)$.*

PROPOSITION 5.2.11. *Let $G \in [IN]$ be separable and almost connected. Let V be a closed vector subgroup of G. Then $V \in C.I.P.(G)$.*

Proof. As G is almost connected, G has a compact normal subgroup K such that G/K is an almost-connected $[SIN]$-group [cf. 48, p. 698]. Let $\varphi : G \to G/K$ be the canonical homomorphism. Since $V \cap K = \{e\}$, $\varphi : V \to \varphi(V)$ is a topological isomorphism and $\varphi(V)$ is a vector subgroup of G/K. By Corollary 5.2.10, there exists $\Gamma_0 : A(\varphi(V)) \to A(G/K)$ such that Γ is continuous, linear and

$$\Gamma_0 u_{|_{\varphi(V)}} = u \qquad \text{for every} \quad u \in A(\varphi(V)).$$

For each $u \in A(V)$, we have $u^* \in A(\varphi(v))$ defined by

$$u^*(\varphi(v)) = u(v) \qquad \text{for every} \quad v \in V.$$

Define $\Gamma : A(V) \to A(G)$ by

$$\Gamma u = (\Gamma_0 u^*) \circ \varphi \qquad \text{[cf. 19, p. 21].}$$

Then $\Gamma u_{|_V} = u$ for every $u \in A(V)$. Also Γ is linear and continuous. □

PROPOSITON 5.2.12. *Let G be a separable $[FC]^-$ group. Let H be a closed subgroup of G with no non-trivial compact subgroups. Then $H \in C.I.P.(G)$.*

Proof. Let G be an $[FC]^-$ group. Then G has a compact normal subgroup K such that G/K is the direct product of a vector group and a discrete group [48, p. 698]. It follows that any subgroup which intersects K trivially belongs to $C.I.P.(G)$. □

Up to this point we have concerned ourselves with complementation of ideals of the form $I(H)$, where H is a closed subgroup of G. We close this chapter with a few additional observations on the general complementation problem for $I(A)$, when A is an arbitrary closed subset of G.

PROPOSITION 5.2.13. *Let A, B be closed subsets of G. Suppose that there exists continuous projections P of $A(G)$ onto $I(A)$ and Q of $A(G)$ onto $I(B)$. If $s(A/B, B/A) < \infty$, then there exists a continuous projection Γ of $A(G)$ onto $I(A \cup B)$ with $\|\Gamma\| \leq (s(A\backslash B, B\backslash A) + 2)(\|P\| + \|Q\|)$.*

Proof. Let $u \in S(A\backslash B, B\backslash A)$ with $\|u\|_{B(G)} \leq s(A\backslash B, B\backslash A) + 1$. Let $v \in A(G)$. Define

$$\Gamma v = uPv + (1-u)Qv.$$

It is easy to see that Γ is the desired projection. □

COROLLARY 5.2.14. *Let A be a closed subset of G. Let F be finite. If there exists a continuous projection of $A(G)$ onto $I(A)$, then there exists a continuous projection of $A(G)$ onto $I(A \cup F)$.*

COROLLARY 5.2.15. *Let A, B be disjoint closed subsets of G. Assume that $s(A, B) < \infty$. Then, there exists a continuous projection of $A(G)$ onto $I(A \cup B)$ if and only if there exists continuous projections of $A(G)$ onto $I(A)$ and $I(B)$ respectively.*

Proof. If both $I(A)$ and $I(B)$ are complemented, then by Proposition 5.2.13, $I(A \cup B)$ is complemented.

Conversely, assume that P is a continuous projection of $A(G)$ onto $I(A \cup B)$.

Let $u \in S(A,B)$. Define

$$Qv = P(uv) + v - uv,$$

$$Q_1 v = P((1-u)v) - uv.$$

Then Q is a continuous projection of $A(G)$ onto $I(A)$ and Q_1 is a continuous projection of $A(G)$ onto $I(B)$. $\square$

CHAPTER 6

AMENABILITY AND DERIVATIONS ON $A(G)$

§6.0. Introduction.

In this chapter, we will focus our attention on the nature of the derivations on the Fourier algebra of a locally compact group. We shall concern ourselves with two specific problems.

First, for which locally compact groups are all derivations on $A(G)$ necessarily continuous? We show, in Section 6.1, that every derivation from $A(G)$ to a Banach $A(G)$-bimodule X is continuous if and only if G is amenable.

Recently, Bade, Curtis and Dales [8] introduced the notion of a weakly amenable Banach algebra. In Section 6.2, we examine the derivations of $A(G)$ into a commutative Banach $A(G)$-bimodule. In particular, we show that if G is discrete, then $A(G)$ is weakly amenable (Theorem 6.2.11).

§6.1. Continuity of Derivations on $A(G)$.

DEFINITION 6.1.1. Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. A *derivation* $D : \mathcal{A} \to X$ is a linear map which satisfies

$$D(ab) = aD(b) + D(a)b \qquad \text{for every} \quad a, b \in \mathcal{A}.$$

Much work has been done towards characterizing those groups G for which every derivation D from $L^1(G)$ into an $L^1(G)$-bimodule is continuous [cf. 70]. In this section, we will examine the analogue of this problem for $A(G)$.

LEMMA 6.1.2. *Let G be a non-amenable locally compact group. Then there exists a discontinuous derivation on $A(G)$ with finite-dimensional range.*

Proof. If G is non-amenable, then by Lemma 3.4.6, either (i) $I^2(\{e\})$ is not cofinite, or (ii) $I^2(\{e\})$ is cofinite but not closed. In either case, there exists a non-closed cofinite ideal I of $A(G)$ such that

$$I^2(\{e\}) \subseteq I \subset I(\{e\}).$$

T[illegible] as shown in [14, p. 402], the finite dimensional space $I(\{e\})/I$ can be made into a Banach $A(G)$-bimodule and a discontinuous derivation $D : A(G) \to I(\{e\})/I$ can be constructed. □

LEMMA 6.1.3. *Let G be an amenable locally compact group. Let I be a closed ideal in $A(G)$ with infinite codimension. Then there exists sequences $\{u_n\}$, $\{v_n\}$ in $A(G)$ such that $u_n v_1 \ldots v_{n-1} \notin I$ but $u_n v_1 \ldots v_n \in I$ for all $n \geq 2$.*

Proof. Let $A = Z(I)$. Since G is amenable and I has infinite codimension, A is infinite.

Assume that x is a cluster point of A. Let $x_1 \in A$, $x_1 \neq x$. Let V_1 be a compact neighborhood of x_1 with $x \notin V_1$. Choose $x_2 \in A\backslash V_1$, $x_2 \neq x$. Let V_2 be a compact neighborhood of x_2 such that $x \notin V_2$ and $V_1 \cap V_2 = \emptyset$. Proceeding inductively, we get $x_n \in A\backslash \bigcup_{i=1}^{n-1} V_i$, $x_n \neq x$ and a compact neighborhood V_n of x_n such that $x \notin V_n$ and $V_n \cap \left(\bigcup_{i=1}^{n-1} V_i\right) = \emptyset$.

If A has no cluster points, then there exists a discrete set $\{x_1, x_2, \ldots\} \subseteq A$. Let V_1 be a compact neighborhood of x_1 with $\{x_2, x_3, \ldots\} \cap V_1 = \emptyset$. With V_{n-1}

chosen, let V_n be a compact neighborhood of x_n such that $V_n \cap \{x_{n+1}, x_{n+2}, \dots\} = \emptyset$ and $V_n \cap \left(\bigcup_{i=1}^{n-1} V_i\right) = \emptyset$.

In either case, we get a sequence $\{V_n\}$ of compact neighborhoods of points $\{x_n\}$ in A such that $V_n \cap \left(\bigcup_{i=1}^{n-1} V_i\right) = \emptyset$ for $n \geq 2$.

For each $n = 1, 2, \dots$, let u_n [illegible] function in $A(G)$ with $u_n \geq 0$, $\operatorname{supp} u_n \subseteq V_n$, $u_n(x_n) > 0$ and $\|u_n\|_{A(G)} = 1/2^n$.

For $k = 1, 2, \dots$ define

$$v_k = \sum_{i=k+1}^{\infty} u_i.$$

Since $\sum_{k=1}^{\infty} \|u_k\|_{A(G)} < \infty$, each $v_k \in A(G)$.

If $n \geq 2$, then

$$u_n v_1 \dots v_{n-1}(x_n) > 0.$$

Hence $u_n v_1 \dots v_{n-1} \notin I$. However, $\operatorname{supp} u_n \subseteq V_n$ and $v_n(V_n) = 0$. Therefore,

$$u_n v_1 \dots v_n = 0 \in I.$$

□

THEOREM 6.1.4. *Let G be a locally compact group. Then the following are equivalent:*

(i) G is amenable,

(ii) every derivation of $A(G)$ into a finite dimensional Banach $A(G)$-bimodule is continuous,

(iii) every derivation of $A(G)$ into a Banach $A(G)$-bimodule is continuous.

Proof. That (iii) implies (ii) is trivial. That (ii) implies (i) follows immediately from Lemma 6.1.2.

Assume that G is amenable. Let I be a closed cofinite ideal in $A(G)$. By Corollary 3.4.5, I is idempotent. This, together with Lemma 6.1.3 and [32, Theorem 2], implies that every derivation from $A(G)$ into a Banach $A(G)$-bimodule is continuous. That is, (i) implies (iii). □

§6.2. Weak Amenability of $A(G)$.

DEFINITION 6.2.1. Let φ be a multiplicative linear functional on a Banach algebra $\mathcal{A}$. A point derivation at φ is a linear function $d : \mathcal{A} \to \mathbb{C}$ such that

$$d(uv) = \varphi(u)d(v) + \varphi(v)d(u) \quad \text{for every} \quad u, v \in \mathcal{A}.$$

PROPOSITION 6.2.2. *Let G be a locally compact group. Then $A(G)$ has no continuous non-zero point derivations at any point in the spectrum of $A(G)$.*

Proof. Let $x \in G = \Delta(A(G))$ [cf. 19, p. 229]. Let d be a continuous point derivation at x. Let $v \in A(G)$ be such that $v(x) = 1$ and $\|v\|_{A(G)} = 1$. Then

$$d(v^n) = nd(v) \quad \text{for} \quad n = 1, 2, \dots .$$

Since d is bounded,

$$d(v) = 0.$$

Let $v_1, v_2 \in I(\{x\})$. Then

$$\begin{aligned} d(v_1 v_2) &= v_1(x)d(v_2) + v_2(x)d(v_1) \\ &= 0. \end{aligned}$$

$I^2(\{x\})$ is an ideal in $A(G)$ with $Z(I^2(\{x\})) = \{x\}$. As $\{x\}$ is an S-set, $I^2(\{x\})$ is dense in $I(\{x\})$. Therefore

$$d(u) = 0 \quad \text{for every} \quad u \in I(\{x\}).$$

However, if $u \in A(G)$, then $u = u(x)v + (u - u(x)v)$ and

$$\begin{aligned} d(u) &= u(x)d(v) + d(u - u(x)v) \\ &= 0 \end{aligned}$$

□

LEMMA 6.2.3. *Let I be a closed ideal in $A(G)$ of infinite codimension in $A(G)$. Then $A(G)/I$ is not an integral domain.*

Proof. Since I has infinite codimension, there exist $x, y \in Z(I)$ such that $x \neq y$. Let V_1 and V_2 be compact neighborhoods of x and y respectively with $V_1 \cap V_2 = \emptyset$. Let $u_i \in A(G)$, $i = 1, 2$, $\operatorname{supp} u_i \subseteq V_i$, $u_1(x) \neq 0$ and $u_2(y) \neq 0$. Then $u_i \notin I$, but $0 = u_1 u_2 \in I$.

PROPOSITION 6.2.4. *Let G be a separable compact group. Let A be a commutative Banach algebra with identity and radical X. If $\bigcap_{n=1}^{\infty} X^n = 0$, then every homomorphism from $A(G)$ into A is continuous.*

Proof. If G is separable and compact, $A(G)$ is separable and has an identity. It follows from Theorem 6.1.4 that each point derivation on $A(G)$ is continuous. Lemma 6.2.3 implies that no closed ideal I of infinite codimension in $A(G)$ is such that $A(G)/I$ is an integral domain. As $\bigcap_{n=1}^{n} X^n = 0$, a result of Bade and Curtis

[7, Theorem A] shows that every homomorphism from $A(G)$ into $\mathcal{A}$ is continuous.

□

LEMMA 6.2.5. *Let G be a locally compact group. Let $x \in G$. Then there exists a continuous invariant projection of $VN(G)$ onto $\langle L_x \rangle$.*

Proof. We follow an idea of P. Renaud [53]. Let $\{V_\alpha\}_{\alpha \in \mathfrak{A}}$ be a neighborhood basis at e. Let $u_\alpha \in P(G) \cap A(G)$ be such that $u_\alpha(e) = 1$ and supp $u_\alpha \subseteq V_\alpha$. Let $m \in VN(G)^*$ be a weak-$*$ cluster point of $\{L_x u_\alpha\}$. Then, as in [53, Proposition 3 and Theorem 4],

$$m(u \cdot T) = u(x) m(T) \qquad \text{for every} \quad u \in A(G), \quad T \in VN(G).$$

Furthermore,

$$m(L_x) = 1.$$

Therefore, define

$$P(T) = m(T) L_x.$$

P is an invariant projection of $VN(G)$ onto $\langle L_x \rangle$. □

LEMMA 6.2.6. *Let $D : A(G) \to VN(G)$ be a continuous derivation. Let $x \in G$ and P_x be a continuous invariant projection of $VN(G)$ onto $\langle L_x \rangle$. Define*

$$D_{P_x}(u) = P_x(D(u)) \qquad \textit{for every} \quad u \in A(G).$$

Then D_{P_x} is a continuous point derivation at x. In particular, $D_{P_x} = 0$.

Proof. Let $u, v \in A(G)$. Then

$$D_{P_x}(uv) = P_x(D(uv)) = P_x(u \cdot D(v) + v \cdot D(u))$$

$$= u \cdot P_x(D(v)) + v \cdot P_x(D(u))$$

$$= u(x)D_{P_x}(v) + v(x)D_{P_x}(u).$$

□

DEFINITION 6.2.7. A Banach $\mathcal{A}$-bimodule X is said to be *commutative* if $u \cdot x = x \cdot u$ for every $u \in \mathcal{A}$, $x \in X$.

A Banach algebra $\mathcal{A}$ is called *weakly amenable* [8] if every continuous derivation of $\mathcal{A}$ into a commutative Banach $\mathcal{A}$-bimodule is zero.

The main idea in the next lemma is due to Bade, Curtis and Dales [8].

LEMMA 6.2.8. *Let X be a commutative Banach $A(G)$-bimodule. Let $D : A(G) \to X$ be a continuous derivation. For each $\varphi \in X^*$ and $u \in A(G)$, define $T_{\varphi,u} \in VN(G)$ by*

$$\langle T_{\varphi,u}, v\rangle = \varphi(v \cdot D(u)).$$

Then $\widetilde{D}_\varphi : A(G) \to VN(G)$, defined by

$$\widetilde{D}_\varphi(u) = T_{\varphi,u},$$

is a continuous derivation. Furthermore, if D is non-zero, then for some $\varphi \in X^*$, $\widetilde{D}_\varphi$ is non-zero.

Proof. Let $u, v, w \in A(G)$.

$$
\begin{aligned}
(\widetilde{D}\varphi(uv))(w) = \varphi(w \cdot D(uv)) &= \varphi(w \cdot u \cdot D(v) + w \cdot v \cdot D(u)) \\
&= (\widetilde{D}\varphi(v))(uw) + (\widetilde{D}\varphi(u))(vw) \\
&= [u \cdot \widetilde{D}\varphi(v)](w) + [v \cdot \widetilde{D}\varphi(u)](w).
\end{aligned}
$$

Hence $\widetilde{D}\varphi$ is a derivation. Since $\varphi \in X^*$ and D is continuous, $\widetilde{D}\varphi$ is also continuous.

As $A(G)^2$ is dense in $A(G)$, if D is non-zero, there exist $u, v \in A(G)$ such that $D(uv) \neq 0$. However,

$$
uv = \frac{(u+v)^2 - u^2 - v^2}{2},
$$

so we may assume that $D(u^2) \neq 0$ for some $u \in A(G)$. Let $\varphi \in X^*$ be such that $\varphi(D(u^2)) \neq 0$. Then

$$
[\widetilde{D}\varphi(u)](u) = \varphi(u \cdot D(u)) = \frac{1}{2}\varphi(D(u^2)) \neq 0
$$

$\square$

DEFINITION 6.2.9. Let $T \in VN(G)$. We define the *support of* T by

$$
\text{supp } T = \{x \in G; \quad u(x) = 0 \quad \text{for every} \quad u \in A(G) \quad \text{with} \quad u \cdot T = 0\}
$$

(see [19, p. 224]).

PROPOSITION 6.2.10. *Let X be a commutative Banach $A(G)$-bimodule. Let $\varphi \in X^*$ and $u \in A(G)$. Let $D : A(G) \to X$ be a continuous derivation. If $\widetilde{D}\varphi(u) \neq 0$, then* supp $\widetilde{D}\varphi(u)$ *contains no isolated points.*

Proof. Assume that V is an open neighborhood of $x \in \operatorname{supp} \widetilde{D}\varphi(u)$ with $V \cap \operatorname{supp} \widetilde{D}\varphi(u) = \{x\}$. Let $v \in A(G)$ be such that $v(x) = 1$ and $\operatorname{supp} v \subseteq V$. Define $D_0 : A(G) \to VN(G)$ by

$$D_0(w) = v \cdot \widetilde{D}\varphi(w) \qquad \text{for every} \quad w \in A(G).$$

Then L is a continuous derivation. Since $v(x) = 1$, $x \in \operatorname{supp} D_0(u)$ [19, p. 225]. But $\operatorname{supp} v \cdot \widetilde{D}\varphi(u) \subseteq \operatorname{supp} v \cap \operatorname{supp} \widetilde{D}\varphi(u) = \{x\}$. Hence $\{x\} = \operatorname{supp} D_0(u)$ and $D_0(u) = \lambda L_x$ for some $\lambda \neq 0$ [19, p. 229].

By Lemma 6.2.5, there exists a continuous invariant projection P_x of $VN(G)$ onto $\langle L_x \rangle$. By Lemma 6.2.6, $P_x \circ D_0$ is a non-zero continuous point derivation which contradicts Proposition 6.2.2. Therefore $\operatorname{supp} \widetilde{D}\varphi(u)$ contains no isolated points. □

THEOREM 6.2.11. *Let G be a discrete group. Then $A(G)$ is weakly amenable.*

Proof. This follows immediately from Lemma 6.2.8 and Proposition 6.2.10 if we note that $\operatorname{supp} T = \emptyset$ if and only if $T = 0$ [19, p. 224]. □

THEOREM 6.2.12. *Let G be a discrete group. Then the following are equivalent:*

(i) G is amenable,

(ii) every derivation from $A(G)$ into a commutative Banach $A(G)$-bimodule is zero.

Proof. If G is amenable, then every derivation is continuous (Theorem 6.1.4). Therefore (ii) holds by Theorem 6.2.11. If G is non-amenable, then the proof of

Lemma 6.1.2 gives a non-zero derivation of $A(G)$ into a commutative Banach $A(G)$-bimodule. □

CHAPTER 7

SUMMARY AND OPEN PROBLEMS

In Chapter 3, we exhibited the connection between the amenability of G and the existence of closed ideals with bounded approximate identities. Proposition 3.1.6 states that if $I(A)$ has a bounded approximate identity, then

$$A = \bigcup_{i=1}^{n} x_i(H_i \backslash \Delta_i) \in \mathcal{R}_c(G),$$

where H_i is a closed subgroup of G, $x_i \in G$ and $\Delta_i \in$ coset ring of H_i. In Section 3.3, we showed that if G is an amenable $[SIN]$-group, then the converse also holds.

Problem 1. If G is an amenable locally compact group and if $A \in \mathcal{R}_c(G)$, does $I(A)$ have a bounded approximate identity? If not, then for what classes of locally compact groups must $I(A)$ have a bounded approximate identity if $A \in \mathcal{R}_c(G)$?

Problem 2. For a non-amenable group G, characterize those $A \in \mathcal{R}_c(G)$ for which $I(A)$ has a bounded approximate identity.

We have seen that if G is non-amenable and if $I(A)$ has a bounded approximate identity, then A must be topologically large (see Proposition 3.1.12). We suspect that in this case, if $A = \bigcup_{i=1}^{n} x_i(H_i \backslash \Delta_i)$, then one of the H_i's should have finite index in G and the Δ_i's should be unions of cosets of open amenable subgroups of G.

Also, in Chapter 3, we proved that G is amenable if and only if every homomorphism from $A(G)$ with finite dimensional range is continuous (Theorem 3.4.8).

<u>Problem 3</u>. If G is an amenable locally compact group, is every homomorphism from $A(G)$ into a Banach algebra A continuous?

In [37], Lau and Losert proved that if G is amenable, then a weak-$*$ closed translation invariant subspace X of $L^{\infty}(G)$ is complemented if and only if there exists a continuous projection from $L^{\infty}(G)$ onto X which commutes with left translations. Conversely, they showed that this property characterizes amenable groups. Bekka [10] showed that the same result holds true with left translation replaced by left convolution. Hence, for abelian groups a weak-$*$ closed submodule X of $VN(G)$ is complemented if and only if it is invariantly complemented.

<u>Problem 4</u>. If G is an amenable locally compact group, is every weak-$*$ closed submodule X of $VN(G)$ which is complemented in $VN(G)$ necessarily invariantly complemented?

A positive answer to Problem 4 would prove to be very useful. In particular, if $I(A)$ is complemented in $A(G)$ then $X = I(A)^{\perp}$ is a weak-$*$ closed complemented submodule of $VN(G)$. If X is invariantly complemented, then we can conclude from Theorem 4.2.2 and Proposition 3.1.6 that $I(A)$ has a bounded approximate identity and $A \in \mathcal{R}_c(G)$. This would therefore provide us with an immediate link between chapters 3 and 5.

In Chapter 5, we showed that if H is a closed subgroup of G, then $I(H)$ is complemented in $A(G)$ if and only if $A(H)$ can be linearly and continuously embedded in $A(G)$. Using this fact, we can exhibit a closed normal subgroup H of an amenable group G for which $I(H)$ is not complemented in $A(G)$. This is rather surprising when contrasted with the abelian case (see Proposition 5.1.4) and the compact case (Theorem 5.1.6) where $I(H)$ is complemented for every closed subgroup H of G. In lieu of the remarks following Problem 4 and of the limited success that has been met with in solving the general complementation problem for abelian groups, we feel that any further focus should be on finding some necessary algebraic conditon on A for $I(A)$ to be complemented. In particular:

Problem 5. If $I(A)$ is complemented in $A(G)$, is $A \in \mathcal{R}_c(G)$?

The main result of Chapter 6 is that a locally compact group G is amenable if and only if every derivation of $A(G)$ into a Banach $A(G)$-bimodule is continuous (Theorem 6.1.4).

In Section 6.2, we gave evidence to suggest that if G is any locally compact group and D, any continuous derivation of $A(G)$ into a commutative Banach $A(G)$-bimodule, then $D = 0$. That is, $A(G)$ is weakly amenable [cf. 8].

Problem 6. Find classes of locally compact groups for which $A(G)$ is weakly amenable for every element of these classes. Moreover, is $A(G)$ weakly amenable for every locally compact group G?

We proved (Theorem 6.2.11) that if G is discrete, then $A(G)$ is weakly amenable. It is well known that if G is abelian, then $A(G)$ is weakly amenable [cf. 33, Theorem 2.5].

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