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On the Hamiltonian Structure, Stability Characteristics and Finite Amplitude Evolution of Geostrophic Fronts

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **On the Hamiltonian Structure, Stability Theory and Finite Amplitude Evolution of Geostrophic Fronts** submitted by **Carol G. Slomp** in partial fulfillment of the requirements for the degree of **Master of Science** in **Applied Mathematics**.

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Date: August 31, 1995

Abstract

A reduced gravity model with an outcropping interface developed by Cushman-Roisin is presented to describe the evolution of geostrophic fronts. This model describes a two layer system where only the uppermost layer is active and each outcrop line describes a front.

This model is found to possess a time invariant Hamiltonian. This Hamiltonian is used to show that all fronts described by this model are linearly stable. Nonlinear stability cannot be unconditionally established. However, a specific geostrophic wedge–like front is found to satisfy these conditions and thus found to be nonlinearly stable.

A weakly nonlinear wave packet theory is developed to describe the evolution of this wedge–like front using neutrally stable modes found by Cushman–Roisin. It is shown that the evolution of the wave packets is governed by the Nonlinear Schrödinger equation.

Solutions for the Nonlinear Schrödinger equation are found. The Stokes wave solution is also found to be modulationally stable. Plane wave solutions as well as a snoidal wave solution and a solitary wave solution are found. Further research areas are discussed.

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Chapter 1

Introduction

An oceanographic front is the boundary between two water masses with differing temperature and/or density (Bowman and Esaias, 1978). They are dynamically active zones where rapid spatial changes in water properties are observed. The characterizing feature of a front is its large horizontal gradient of certain oceanographic properties, e.g., the density (this large horizontal gradient being measured in comparison with the relatively uniform distribution of the rest of the ocean). Upper ocean fronts are also characterized by a strong convergence or divergence of surface currents.

Fronts occur on a wide range of scales varying from the smaller scale fronts found in estuaries caused by outflowing fresh water into sea water, to the planetary scale fronts associated with warm salty water from lower latitudes entering the colder waters of the higher latitudes (Bowman and Esaias, 1978). A smaller scale front, for example, would be that created by the fresh water of the Mississippi River forming a plume over the delta. An example of a planetary scale front would be that caused by the Gulf Stream or Kuroshio (Federov, 1983).

Fronts play an important role in the dynamics of the ocean. They affect the distribution of temperature, salinity and chemical constituents that in turn affect the biological productivity of that area (Bowman and Esaias, 1978). They can increase the biological productivity by extending the depth of light penetration thereby increasing the concentration of phytoplankton, the base of the oceanic food chain. In addition, the frontal water may have a lower salinity which also leads to increased biological productivity in these waters.

Because of the convergence of the surface currents associated with fronts, they hinder the ability of coastal waters to horizontally transfer such things as heat and momentum (Bowman and Esaias, 1978). This convergence also effects the transfer of pollutants and chemicals. Oil slicks have commonly been observed to align themselves with the surface convergence of a front. Fronts may also hinder the disposal of radioactive wastes to the outer ocean.

However, mixing across fronts does lead to the exchange of coastal and offshore waters (Federov, 1983) and thus to the exchange of any energy, pollutants or chemicals. Hence the dynamics leading to this mixing are important to identify so that, for example, the disposal of wastes is only done in situations that will lead to their eventual dispersion to the outer ocean. Some of the dynamics that may be involved in this mixing include the mean flow associated with the front, shear effects of the front and the baroclinic instability of the front. The role of this latter property has led to the study of the causes of the instability of fronts.

The instability of oceanic fronts is important to determine as it governs the release of energy in the form of eddies and rings and the possible mixing of, for example, pollutants and chemicals with adjoining water masses.

Several attempts have been made to model the dynamics of ocean fronts. Most earlier theoretical work was based on quasi–geostrophic theory. Quasigeostrophic theory examines situations in which departures from geostrophy, where geostrophy is defined as the balance between the Coriolis force and the buoyancy force, are small (Cushman–Roisin *et al.*, 1992). This 'smallness' is measured by the Rossby number which is the nondimensional parameter characterizing the ratio between the natural time scale associated with the rotation of the earth and the dynamical time scale of the motion. The small Rossby number assumption models situations in which the dynamical time scale is much larger than the natural time scale of the earth's rotation. This characterizes much of the dynamics in the ocean. Quasigeostrophic theory also makes the assumption that the thickness variations of the layer of fluid are small in comparison to the mean thickness.

Because of this assumption, that the relative thickness variations of the layer of fluid must be small, quasigeostrophic theory does not apply to the evolution of fronts. For fronts, the thickness variation is on the order of the mean depth of the front itself. In fact, for fronts in the coastal regions of the oceans, the isopycnals usually intersect either the surface of the ocean or the ocean bottom.

Despite this shortcoming, quasi-geostrophic theory was still initially used to examine frontal dynamics. Orlanski (1968) used a quasi-geostrophic approximation in describing two active density layers separated by a steep sloping interface. Orlanski found that this model predicted baroclinic instabilities similar to those instabilities found in other quasi-geostrophic models. Smith (1976) used a quasi-geostrophic model to describe a two layer system where only the lower layer was considered active imitating the overflow of Norwegian sea water through the Denmark strait. His model was able to explain the energetics of this situation qualitatively but not quantitatively. He reasoned that this might be due to nonlinear effects.

The limitations of quasi-geostrophic theory though became apparent when Griffiths and Linden (1981) conducted experiments where dyed water was released into a salt water solution. The instabilities they found could not be explained using quasi-geostrophic models. This led Griffiths, Killworth and Stern (1982) to develop a model that was not quasi-geostrophic to describe a reduced-gravity system where the lower active layer was parabolic in shape (a coupled front). Their model showed that all linearized perturbations were unstable even when there was no extremum in the potential vorticity as is required by quasi-geostrophic theory (LeBlond and Mysak, 1978). Their model, however, still was deficient in explaining some of the instabilities that they had observed in their experiments. Killworth and Stern (1982) used a modified model in which they positioned a rigid boundary on one side of a coastal current with the front on the other side of the current. They found that this front was unstable to small amplitude perturbations when the potential vorticity increased towards the coast. Their model showed better agreement with experiments. Paldor (1983) showed that a single layer with uniform potential vorticity was always stable.

Although these single layer models did explain some of the results from experiments they still did not explain all instabilities that were observed. Killworth and Stern (1982) hoped they could explain these instabilities by examining a two layer system where the lower layer was considered motionless but finitely deep. This two layer system showed that the second layer had significant effects on the stability of the model and therefore could not be neglected.

Despite this finding, Cushman-Roisin (1986) considered the much simpler reduced gravity system where only the uppermost layer was active. He reasoned that the earlier studies proved that a wealth of knowledge about fronts could still be obtained from a single active layer system. He represented his surface front by an outcropping interface or, in other words, an interface that intersected the surface of the ocean. Cushman–Roisin assumed that the Rossby number was small, not by demanding that deflections in the upper layer were small, but by demanding that the length scale of the motion was much larger than the deformation radius of the earth which is defined as the distance in which fluid can flow before being affected by the Coriolis force (LeBlond and Mysak, 1978). His model was applicable to three different situations under certain limits. For the limiting case where the beta effect was negligible, Cushman–Roisin found a small amplitude solution for the stability problem where the wedge front, $h_0(y) = \alpha y$, was considered the basic state. He suggested that this solution could serve as a basis for examining the finite amplitude evolution of frontal waves. Using this model Cushman-Roisin also found the wedge–front to be linearly stable.

Swaters (1993) modified this model by changing the length scale to a more intermediate scale and also by including baroclinic processes. His system involved two active layers over a linearly sloping bottom. He chose a length scale that was much less than the internal deformation radius associated with the bottom layer but much greater than the internal deformation radius associated with the top layer. He also considered the linear stability problem for the barotropic limit of his model and found all fronts to be linearly stable.

Most studies on frontal dynamics up to this point were limited to linear stability analysis. Experiments however have showed that finite amplitude effects are important considerations on the stability of flows (Griffiths and Linden, 1981). Swaters (1993) considered the nonlinear effects of his model by developing and exploiting a Hamiltonian formulation of his model. He was able to show nonlinear stability under certain conditions. This nonlinear stability analysis however was somewhat limited as it required the use of a certain Poincaré inequality.

Karsten and Swaters (1995) further examined the Hamiltonian formulation of this model in an attempt to eliminate the restriction of the Poincaré inequality. They were able to show nonlinear stability under certain conditions without the assumption of the Poincaré inequality by considering an alternate invariant of the Hamiltonian system. The implications of this analysis for the barotropic limit of their model will be used in this thesis.

The principle purpose of this thesis is to develop a finite amplitude instability theory for the wedge–front profile examined by Cushman–Roisin (1986). We will derive and examine the Cushman–Roisin (1986) model in the limiting case where the beta effect is negligible using the asymptotic expansion ideas of Swaters (1993). Furthermore, we will review the Hamiltonian formulation of this model as outlined by Swaters (1993) and the linear stability analysis of this model using the Hamiltonian formulation. As already discussed, all fronts are shown to be linearly stable. We will then examine the nonlinear stability problem by using this Hamiltonian formulation. Using both the Hamiltonian invariant used by Swaters (1993) and the invariant used by Karsten and Swaters (1995), nonlinear stability will be proven but only under certain conditions. Specifically we will examine the wedge–front introduced by Cushman–Roisin (1986) and show that it satisfies these conditions and thus is nonlinearly stable.

Despite being a crude representation of a front, the wedge–shaped front still captures the essential dynamics of the physics of this model. In addition, the wedge–front profile is simplified enough so that analytical solutions for the linear stability problem can be found (Cushman–Roisin, 1986). However an analytical solution for the full nonlinear stability problem cannot be found. We will examine, however, the nonlinear evolution of the flow by using a multiple scales analysis. As suggested by Cushman–Roisin (1986), his linear stability solution will serve as the basis for the analysis. This multiple scales analysis will involve introducing slow space and time variables into the nonlinear stability problem. The frontal height will then be asymptotically expanded about a small parameter, ϵ, which characterizes the nondimensional amplitude of the frontal perturbations. We will consider the problems up to the $O(\epsilon^2)$. Solvability conditions for the $O(\epsilon^2)$ problem will show that the perturbation amplitude of the flow introduced in the $O(1)$ problem must satisfy the Nonlinear Schrödinger equation.

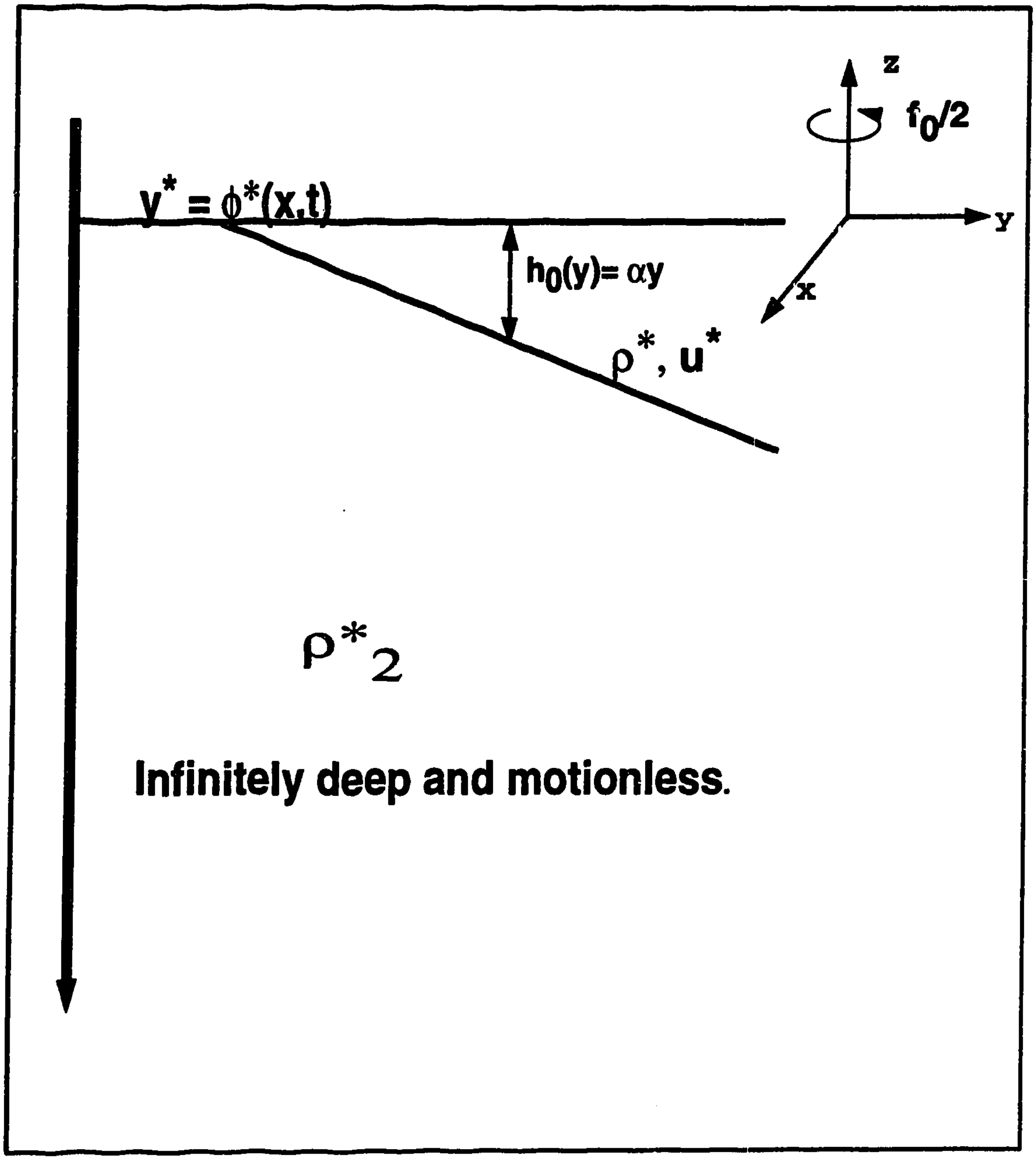
We will examine solutions to this Nonlinear Schrödinger equation. The Stokes wave solution in particular will be examined and shown to be modulationally stable reinforcing the conclusion made using the Hamiltonian formulation of the model that this model is nonlinearly stable. Other solutions to the Nonlinear Schrödinger equation that will be found are a snoidal wave solution and a solitary wave solution.

The outline of this thesis is as follows. In chapter 2, the derivation of the governing equations for the barotropic model is given using the reasoning given by Swaters (1993). The linear solution found by Cushman-Roisin (1986) is reviewed. The Hamiltonian structure of the model is discussed and its linear and nonlinear stability is also studied following the theorems given by Swaters (1993) and Swaters and Karsten (1994). Specifically it is shown that all fronts are linearly stable and that the wedge–front, in particular, is nonlinearly stable.

In chapter 3, the equations for the weakly nonlinear evolution of the wedge front modes are derived using scaling arguments and slow space and time variables. An asymptotic expansion is done and the various problems up to $O(\epsilon^2)$ are discussed. The perturbation amplitude is shown to satisfy the Nonlinear Schrödinger equation.

In chapter 4, several solutions to the Nonlinear Schrödinger equation are derived. Among these solutions is a snoidal wave solution as well as a solitary wave solution. In chapter 5, the thesis is summarized and future research topics are discussed.

Figure 1–1: The Physical Model



Chapter 2

Governing Equations

2.1 Derivation of the frontal model

The derivation of the governing equations for this model begins with the shallow water equations (Pedlosky, 1979) and follows closely the reasoning presented by Swaters (1993). The basic model is a reduced gravity shallow water system assumed to be hydrostatic, homogeneous, non–viscous and incompressible. This model can be derived from a two layer system in which the lower layer is assumed to be infinitely deep (see Fig. 1–1), motionless and in hydrostatic balance.

A review of the derivation of the reduced gravity shallow water equations, following Pedlosky, (1979) is first given beginning with the dimensional, inviscid, incompressible equations for a homogeneous rotating fluid:

$$\nabla^{*}\cdot\mathbf{u}^{*} = 0, \tag{2.1}$$

$$(\partial_{t^*} + \mathbf{u}^{*}\cdot\nabla^{*})\, u^{*} - f_0 v^{*} = -\widetilde{p}^{*}_{x^*}/\rho^{*}, \tag{2.2}$$

$$(\partial_{t^*} + \mathbf{u}^{*}\cdot\nabla^{*})\, v^{*} + f_0 u^{*} = -\widetilde{p}^{*}_{y^*}/\rho^{*}, \tag{2.3}$$

$$(\partial_{t^*} + \mathbf{u}^{*}\cdot\nabla^{*})\, w^{*} = -\widetilde{p}^{*}_{z^*}/\rho^{*}, \tag{2.4}$$

where t^* is time, x^* the along–front coordinate, y^* the cross–front coordinate and z^* the vertical coordinate; (u^*, v^*, w^*) are the velocity components in the (x^*, y^*, z^*) directions respectively; $\partial_{t^*} + \mathbf{u}^* \cdot \nabla^*$ is the material derivative describing the time evolution following the flow and will be denoted by $\frac{D}{Dt^*}$ (Kundu, 1990); f_0 is the constant Coriolis parameter, ρ^* is the density, and $\tilde{p}^*$ is the dynamic pressure, that is, the total pressure, denoted p^*, minus the hydrostatic component, that is,

$$p^*(x^*, y^*, z^*, t^*) = -\rho^* g z^* + \tilde{p}^*(x^*, y^*, z^*, t^*). \tag{2.5}$$

In order to derive the reduced gravity equations the parameter, $\delta = \frac{D}{L}$, where D characterizes the vertical lengthscale and L characterizes the horizontal lengthscale is introduced and assumed to satisfy the inequality, $\delta \ll 1$. This reflects that in the ocean mesoscale disturbances have much longer horizontal lengthscales than vertical lengthscales. Therefore, if W and U characterize the vertical and horizontal velocity scale, respectively, from the continuity equation (2.1) the following scaling is derived,

$$w^*_{z^*} \simeq O\left(\frac{W}{D}\right) \simeq O\left(\frac{U}{L}\right), \tag{2.6}$$

which suggests

$$W \simeq O\left(\frac{DU}{L}\right) = O(\delta U). \tag{2.7}$$

It follows that

$$w^* u^*_{z^*} \simeq \frac{UW}{D} = O\left(\frac{U}{D} \cdot \frac{DU}{L}\right) = O\left(\frac{U^2}{L}\right). \tag{2.8}$$

Assuming that the pressure is dynamically relevant at leading order in the horizontal momentum equations, it follows from the horizontal momentum equations that

$$\begin{aligned} P &= \rho^* L \cdot O\left(\frac{U}{T}, \frac{U^2}{L}, f_0 U\right)_{\max} \\ &= \rho^* U \cdot O\left(\frac{L}{T}, U, f_0 L\right)_{\max}, \end{aligned} \tag{2.9}$$

where P is the scale for the dynamic pressure, $\tilde{p}$.

Turning to the vertical momentum equation, the following is derived:

$$
\begin{aligned}
O\left(\frac{\rho^* \frac{Dw^*}{Dt^*}}{\frac{\partial \tilde{p}^*}{\partial z^*}}\right) &= \frac{\rho^* \left[\frac{W}{T}, \frac{UW}{L}\right]_{\max}}{\frac{P}{D}} \\
&= \frac{D\left[\frac{W}{T}, \frac{UW}{L}\right]_{\max}}{U\left[\frac{L}{T}, U, f_0 L\right]_{\max}} \\
&= \frac{DW}{LU} \frac{\left[\frac{1}{T}, \frac{U}{L}\right]_{\max}}{\left[\frac{1}{TL}, \frac{U}{L^2}, \frac{f_0}{L}\right]_{\max}} \\
&= \delta^2 \frac{\left[\frac{1}{T}, \frac{U}{L}\right]_{\max}}{\left[\frac{1}{T}, \frac{U}{L}, f_0\right]_{\max}}.
\end{aligned} \tag{2.10}
$$

Examining equation (2.10), it is evident that if $\frac{U}{f_0 L}$ or $\frac{1}{Tf_0}$ is $O(1)$ or greater than this ratio is of $O(\delta^2)$. If these terms are smaller than $O(1)$ then this ratio is smaller than $O(\delta^2)$. Therefore, the term, $\frac{Dw^*}{Dt^*}$, in equation (2.4) may be neglected in comparison to the pressure gradient and the vertical momentum equation (2.1) reduces, to leading order, to the hydrostatic balance,

$$\frac{\partial \tilde{p}^*}{\partial z^*} = 0. \tag{2.11}$$

It follows from (2.11) that

$$\tilde{p}^* \equiv A(x^*, y^*, t^*), \tag{2.12}$$

where A is an arbitrary function of its arguments. It therefore follows that the total pressure, given by equation (2.5) is

$$p^* = -\rho^* g z^* + A(x^*, y^*, t^*). \tag{2.13}$$

The dynamic pressure, $A(x^*, y^*, t^*)$, must be related to the thickness of the surface layer, denoted by $h^*(x^*, y^*, t^*)$. The dynamic boundary condition on the interface between the upper and lower layers is that the total pressure must be continuous across the interface. Since the lower layer is assumed to be motionless and in hydrostatic balance the total pressure in the lower layer is given simply by $-\rho_2 g z^*$, where ρ_2 is the density of the lower layer. There is no explicit (x^*, y^*, t^*) dependence in the lower layer. If there was, then it would follow that there would be horizontal motion in the lower layer.

Consequently, continuity of the total pressure across the interface $z^* = -h^*$ (see Fig. 1–1) implies

$$-\rho_2 g h^*(x^*, y^*, t^*) = \rho^* g h^*(x^*, y^*, t^*) + A^*(x^*, y^*, t^*), \tag{2.14}$$

which can be rearranged into the form

$$A^*(x^*, y^*, t^*) = -g' \rho^* h^*(x^*, y^*, t^*), \tag{2.15}$$

where

$$g' = g\frac{\rho_2 - \rho^*}{\rho^*} > 0, \tag{2.16}$$

is the reduced gravity which vanishes in the lower layer. The total pressure in the upper layer is therefore given by

$$p^* = -\rho^* g z^* - g' \rho^* h^*(x^*, y^*, t^*). \tag{2.17}$$

Differentiating equation (2.17) with respect to x^* and y^* implies, respectively,

$$\frac{\partial \tilde{p}^*}{\partial x^*} = \rho^* g' \frac{\partial h^*}{\partial x^*}, \tag{2.18}$$

$$\frac{\partial \tilde{p}^*}{\partial y^*} = \rho^* g' \frac{\partial h^*}{\partial y^*}. \tag{2.19}$$

Since $\tilde{p}^*$ is independent of z^*, horizontal accelerations will be independent of z^* and the horizontal velocities are assumed to be independent of z^*, in other words,

$$\frac{\partial u^*}{\partial z^*} = \frac{\partial v^*}{\partial z^*} = 0. \tag{2.20}$$

Substituting equations (2.18) and (2.19) into the horizontal momentum equations (2.2) and (2.3) and using equation (2.20) leads to

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} - f_0 v^* = -g' \frac{\partial h^*}{\partial x^*}, \tag{2.21}$$

$$\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} + f_0 u^* = -g' \frac{\partial h^*}{\partial y^*}, \tag{2.22}$$

or in vector form

$$\left[\partial_{t^*} + \mathbf{u}^* \cdot \nabla^* + f_0 \hat{\mathbf{e}}_3 \times\right] \mathbf{u}^* = -g' \nabla^* h^*, \tag{2.23}$$

where $\mathbf{u}^*$ is the horizontal velocity vector in the upper layer. These equations are valid as long as the wavelengths being studied are much longer than the depth of the fluid so that vertical velocities are much smaller than horizontal velocities.

Following Pedlosky (1979), since u^* and v^* are independent of z^*, the continuity equation (2.1) can be integrated from $z^* = -h^*$ to $z^* = 0$ to give

$$w^*(x^*, y^*, 0, t^*) - w(x^*, y^*, -h^*, t^*) + h^* \left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} \right) = 0. \tag{2.24}$$

This is simplified by using the kinematic boundary condition on the interface given by the equation

$$w(x^*, y^*, -h^*, t^*) = -\frac{Dh^*}{Dt^*}. \tag{2.25}$$

In addition, on the surface of the frontal layer, the rigid lid approximation is used so that

$$w^*(x^*, y^*, 0, t^*) = 0. \tag{2.26}$$

Substituting these equations (2.25) and (2.26) into the continuity equation (2.24) leads to

$$\frac{\partial h^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* h^* + h^* \nabla^* \cdot \mathbf{u}^* = 0, \tag{2.27}$$

or

$$\frac{\partial h^*}{\partial t^*} + \nabla^* \cdot (\mathbf{u}^* h^*) = 0. \tag{2.28}$$

In conclusion, the following reduced gravity shallow water equations hold for the upper active frontal layer:

$$[\partial_{t^*} + \mathbf{u}^* \cdot \nabla^* + f_0 \hat{\mathbf{e}}_3 \times]\, \mathbf{u}^* = -g' \nabla^* h^*, \tag{2.29}$$

$$\partial_{t^*} h^* + \nabla^* \cdot [\mathbf{u}^* h^*] = 0, \tag{2.30}$$

where $\mathbf{u}^*$ is the horizontal velocity vector for the upper layer, and h^* is the thickness of the upper layer or the depth at which the front lies (Pedlosky, 1979).

It is now beneficial to nondimensionalize the model equations (2.29) and (2.30). Kundu (1990) explains that by nondimensionalizing these equations, they can be used to describe many dynamically different situations. In other words, different situations may have different velocities and length scales than those initially examined but if certain relationships still hold between these quantities, expressed as nondimensional parameters, the same nondimensional equations can be used to describe both situations. Pedlosky (1979) also explains that the size of these specific nondimensional parameters determined by the physical situation being described, determine the relative sizes of each of the terms in the equations and are helpful in determining which terms can be neglected so that the equations may be examined analytically.

Following methods used in Pedlosky (1979) the nondimensionalization is carried out by first introducing the following variables:

$$(x^*, y^*) = L\,(x, y)\,, \tag{2.31}$$

$$t^* = Tt, \tag{2.32}$$

$$(u^*, v^*) = U\,(u, v)\,, \tag{2.33}$$

$$h^* = h_0 h, \tag{2.34}$$

where L, T, and U characterize the magnitude of the length, time and velocity scales of the motion, respectively, and where h_0 is a characteristic thickness of the front.

These variables are introduced first into the x–direction momentum equation (2.21),

$$\frac{U}{T}\frac{\partial u}{\partial t} + \frac{U^2}{L}u\frac{\partial u}{\partial x} + \frac{U^2}{L}v\frac{\partial u}{\partial y} - f_0 U v = -\frac{g' h_0}{L}\frac{\partial h}{\partial x}, \tag{2.35}$$

which when divided through by U and f_0 gives

$$\frac{1}{f_0 T}\frac{\partial u}{\partial t} + \frac{U}{f_0 L}u\frac{\partial u}{\partial x} + \frac{U}{f_0 L}v\frac{\partial u}{\partial y} - v = -\frac{g' h_0}{f_0 U L}\frac{\partial h}{\partial x}. \tag{2.36}$$

The nondimensionalization is achieved by examining balances found in physical situations. The first scaling is a direct result of the fact that the formation of the front results primarily from a balance between the Coriolis force and the buoyancy forces known as the geostrophic balance. This geostrophic balance is modelled by making the last term on the left of equation (2.36) of the same order of magnitude as the term on the right. This gives the relation

$$h_0 = \frac{f_0 L U}{g'}. \tag{2.37}$$

Secondly, fronts in physical situations show the approximate length and velocity scales given by $L \simeq 100\ km$ and $U \simeq 10\ cm/s$ (Cushman-Roisin, 1994). In addition, the size of the Coriolis parameter, f_0, is approximately given as

$f_0 \simeq 10^{-4}$. This suggests the introduction of the Rossby number, R_0, defined as

$$R_0 = \frac{U}{f_0 L} \ll 1. \tag{2.38}$$

Rearranging this relation gives the following scaling for the magnitude of the velocity,

$$U = R_0 f_0 L. \tag{2.39}$$

This scaling makes the nonlinear momentum terms small compared to the Coriolis acceleration terms because of the small Rossby number. Substituting the scaling for U into the geostrophic balance, equation (2.37), gives the scaling for the horizontal lengthscale,

$$L^2 = \frac{g' h_0 L}{f_0 U} = \frac{g' h_0}{R_0 f_0^2} = R_0^{-1} L_R^2, \tag{2.40}$$

where L_R^2 is the internal Rossby deformation radius defined as

$$L_R^2 = \frac{g' h_0}{f_0^2}, \tag{2.41}$$

which describes the distance over which a geophysical fluid disturbance can travel before being significantly affected by the Coriolis force (LeBlond and Mysak, 1978).

In order to examine only planetary or subinertial motions and filter out inertial oscillations and Poincaré waves the time scale is assumed to satisfy the inequality

$$T \gg \frac{1}{f_0}, \tag{2.42}$$

so that only slowly evolving motions are considered. Since fronts evolve on a time scale of days to weeks (Cushman–Roisin, 1994) the following scaling is suggested:

$$T = \frac{1}{R_0^2 f_0}, \tag{2.43}$$

This completes the nondimensionalization with the new nondimensional (unasterisked) variables introduced as follows:

$$(x^*, y^*) = L\,(x, y) \text{ where } L = \frac{(g' h_0/R_0)^{1/2}}{f_0}, \tag{2.44}$$

$$t^* = \left(\frac{1}{f_0 R_0^2}\right) t, \tag{2.45}$$

$$(u^*, v^*) = (f_0 L R_0)\ \ (u, v)\,, \tag{2.46}$$

$$h^* = h_0 h, \tag{2.47}$$

$$R_0 = \frac{L_R^2}{L^2}. \tag{2.48}$$

The new nondimensional equations can now be derived. Substituting equations (2.44)–(2.48) into equation (2.36) gives the x–direction momentum equation,

$$R_0^2\,\frac{\partial u}{\partial t} + R_0 u\frac{\partial u}{\partial x} + R_0 v\frac{\partial u}{\partial y} - v = -\frac{\partial h}{\partial x}. \tag{2.49}$$

The derivation for the y–direction momentum equation follows similarly. The nondimensional vector form of the momentum equations can be written in vector form as

$$R_0^2\frac{\partial \mathbf{u}}{\partial t} + R_0 \mathbf{u}\cdot\nabla\mathbf{u} + \hat{\mathbf{e}}_3\times\mathbf{u} + \nabla h = \mathbf{0}. \tag{2.50}$$

Putting the nondimensional variables in the continuity equation (2.30) for the upper layer gives the nondimensional equation,

$$f_0 R_0^2\frac{\partial h}{\partial t} + \frac{f_0 L R_0}{L}\nabla\cdot(\mathbf{u}h) = 0. \tag{2.51}$$

Dividing through by $f_0 R_0$ gives the simplified equation

$$R_0 \frac{\partial h}{\partial t} + \nabla \cdot (\mathbf{u}h) = 0. \tag{2.52}$$

In this equation the time derivative is an order of magnitude larger than in the momentum equations. This corresponds to purely frontal dynamics since fronts evolve much slower than quasi–geostrophic dynamics (Cushman–Roisin, 1986).

In summary, the nondimensional equations for the top layer are as follows:

$$R_0^2 \frac{\partial \mathbf{u}}{\partial t} + R_0 \mathbf{u} \cdot \nabla \mathbf{u} + \hat{\mathbf{e}}_3 \times \mathbf{u} + \nabla h = \mathbf{0}, \tag{2.53}$$

$$R_0 \frac{\partial h}{\partial t} + \nabla \cdot (\mathbf{u}h) = 0. \tag{2.54}$$

This model can be compared to some other models of fronts. Flierl (1984) in describing the dynamics of a strongly nonlinear warm eddy considered situations where the Rossby number was small and thus the horizontal velocity was small but not negligibly small, that is, he considered the Rossby number to be of the order $R_0 < 1$, but not $R_0 \ll 1$. This assumption resulted in a model in which the nonlinear momentum terms were of the same order as the Coriolis acceleration terms. Flierl also considered situations where the relative acceleration terms were almost negligible – similar to the assumption made here.

Swaters and Flierl (1991) and Swaters (1991) considered a two layer system where both layers were active over a sloping bottom to describe the motion of cold–core mesoscale eddies. The nonlinear momentum terms in these studies were much smaller than the leading order geostrophic motion in the horizontal momentum equation and thus their model had somewhat weaker nonlinearities than the model given here. In their continuity equation also the changes in the eddy thickness were as important as the horizontal divergence whereas in this model the changes in the eddy thickness are of the order, $O(R_0)$, of the horizontal divergence terms.

The smallness of the Rossby number suggests that a solution to the model equations (2.53) and (2.54) can be constructed in the form of an asymptotic expansion given by

$$(h, \mathbf{u}) \sim \left(h^{(0)}, \mathbf{u}^{(0)}\right) + R_0 \left(h^{(1)}, \mathbf{u}^{(1)}\right) + O\left(R_0^2\right), \tag{2.55}$$

(Bender and Orszag, 1978). Substituting the expansions (2.55) into the nondimensional equations (2.53) and (2.54) gives the $O(1)$ problem,

$$\hat{\mathbf{e}}_3 \times \mathbf{u}^{(0)} + \nabla h^{(0)} = \mathbf{0}, \tag{2.56}$$

$$\nabla \cdot \left(\mathbf{u}^{(0)} h^{(0)}\right) = 0, \tag{2.57}$$

In component form equation (2.56) can be written as

$$v^{(0)} = \frac{\partial h^{(0)}}{\partial x}, \tag{2.58}$$

$$u^{(0)} = -\frac{\partial h^{(0)}}{\partial y}, \tag{2.59}$$

where $u^{(0)}$ and $v^{(0)}$ are the velocities in the x and y directions, respectively. This shows that the leading order velocity is geostrophically determined, that is,

$$\mathbf{u}^{(0)} = \hat{\mathbf{e}}_3 \times \nabla h^{(0)}. \tag{2.60}$$

If equation (2.60) is substituted into equation (2.57) it follows that

$$\begin{aligned} \nabla \cdot \left(\mathbf{u}^{(0)} h^{(0)}\right) &= \left(u^{(0)} h^{(0)}\right)_x + \left(v^{(0)} h^{(0)}\right)_y \\ &= -\left(h_y^{(0)} h^{(0)}\right)_x + \left(h_x^{(0)} h^{(0)}\right)_y \\ &= -\frac{1}{2}\left(h^{(0)}\right)_{yx} + \frac{1}{2}\left(h^{(0)}\right)_{yx} = 0. \end{aligned} \tag{2.61}$$

Equation (2.57) is therefore identically satisfied and the $O(1)$ equations are said to be geostrophically degenerate.

Therefore, in order to obtain a complete system, the $O(R_0)$ equations must be considered. These equations are given as

$$\mathbf{u}^{(0)} \cdot \nabla \mathbf{u}^{(0)} + \hat{\mathbf{e}}_3 \times \mathbf{u}^{(1)} + \nabla h^{(1)} = \mathbf{0}, \tag{2.62}$$

$$\frac{\partial h^{(0)}}{\partial t} + \nabla \cdot \left(\mathbf{u}^{(0)} h^{(1)} + \mathbf{u}^{(1)} h^{(0)} \right) = 0. \tag{2.63}$$

Substituting equation (2.60) into equation (2.62) gives

$$\hat{\mathbf{e}}_3 \times \mathbf{u}^{(1)} + \nabla h^{(1)} = -\left(\hat{\mathbf{e}}_3 \times \nabla h^{(0)} \right) \cdot \nabla \left(\hat{\mathbf{e}}_3 \times \nabla h^{(0)} \right). \tag{2.64}$$

The x–direction component of equation (2.64) is given by

$$v^{(1)} = \frac{\partial h^{(1)}}{\partial x} + \frac{\partial h^{(0)}}{\partial y} \frac{\partial^2 h^{(0)}}{\partial y \partial x} - \frac{\partial h^{(0)}}{\partial x} \frac{\partial^2 h^{(0)}}{\partial y^2} = \frac{\partial h^{(1)}}{\partial x} + J\left(\frac{\partial h^{(0)}}{\partial y}, h^{(0)} \right), \tag{2.65}$$

where $J(A, B) = A_x B_y - A_y B_x$ with subscripts denoting partial differentiation. Following similarly for the y–direction component gives

$$u^{(1)} = -\frac{\partial h^{(1)}}{\partial y} + J\left(\frac{\partial h^{(0)}}{\partial x}, h^{(0)} \right). \tag{2.66}$$

Equation (2.65) and (2.66) may be written in the vector form

$$\mathbf{u}^{(1)} = \hat{\mathbf{e}}_3 \times \nabla h^{(1)} + J\left(\nabla h^{(0)}, h^{(0)} \right). \tag{2.67}$$

In this equation, the $\hat{\mathbf{e}}_3 \times \nabla h^{(1)}$ term is the perturbation geostrophic flow and the $J\left(\nabla h^{(0)}, h^{(0)} \right)$ term is the contribution to $\mathbf{u}^{(1)}$ from the nonlinear momentum terms.

Substituting these equations (2.60) and (2.67) into equation (2.63) implies

$$\frac{\partial h^{(0)}}{\partial t}+\nabla\cdot\left(\left(\hat{\mathbf{e}}_3\times\nabla h^{(0)}\right)h^{(1)}+\left(\hat{\mathbf{e}}_3\times\nabla h^{(1)}+J\left(\nabla h^{(0)},h^{(0)}\right)\right)h^{(0)}\right)=0. \quad (2.68)$$

Expanding this equation (2.68) and rearranging gives

$$\frac{\partial h^{(0)}}{\partial t}+J\left(h^{(0)}\Delta h^{(0)}+\frac{1}{2}\nabla h^{(0)}\cdot\nabla h^{(0)},h^{(0)}\right)=0, \quad (2.69)$$

which describes the leading order evolution of the thickness of the front. This equation is the same equation derived by Cushman-Roisin (1986) for a model where the beta effect was negligible and the length scale was much greater than the deformation radius.

The dimensional position of the outcropping, that is, the position where the front intersects the surface, $z^* = 0$, is given by $y^* = \phi^*(x^*, t^*)$. Since this variable represents a horizontal quantity it can be nondimensionalized using L such that

$$y=\phi(x,t)=\frac{\phi^*(x^*,t^*)}{L} \text{ on } z=0. \quad (2.70)$$

The dimensional kinematic boundary condition is that all particles on the boundary will remain on that boundary, i.e.,

$$\frac{D}{Dt}(y^*-\phi^*)=0, \quad (2.71)$$

on $y^* = \phi^*$. Simplified, equation (2.71) becomes

$$\frac{Dy^*}{Dt^*}=v^*=\frac{D\phi^*}{Dt}=\frac{\partial\phi^*}{\partial t}+u^*\frac{\partial\phi^*}{\partial x^*}, \quad (2.72)$$

on $y^* = \phi^*$. Using the nondimensional variables defined by equations (2.44)–(2.48) gives the nondimensional equation

$$R_0 f_0 L v = L f_0 R_0^2 \frac{\partial \phi}{\partial t} + \frac{R_0 f_0 L^2}{L} u \frac{\partial \phi}{\partial x}, \tag{2.73}$$

which when simplified produces

$$v = R_0 \frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} \text{ on } y = \phi(x,t). \tag{2.74}$$

The dynamic boundary condition is that the thickness of the front must be zero on the outcropping, that is,

$$h(x,y,t) = 0 \text{ on } y = \phi(x,t). \tag{2.75}$$

Just as the other variables were expanded asymptotically with respect to R_0, ϕ is also expanded asymptotically to give

$$\phi(x,t) \sim \phi^{(0)}(x,t) + R_0 \phi^{(1)}(x,t) + O\left(R_0^2\right). \tag{2.76}$$

Substituting this equation (2.76) into the kinematic condition, equation (2.72), along with the asymptotic expansion for $\mathbf{u}$ (2.55) produces the following equation:

$$R_0 \phi_t^{(0)} + u^{(0)} \phi_x^{(0)} + R_0 u^{(0)} \phi_x^{(1)} + R_0 u^{(1)} \phi_x^{(0)} = v^{(0)} + R_0 v^{(1)} + O\left(R_0^2\right), \tag{2.77}$$

on $y = \phi(x,t) = \phi^{(0)}(x,t) + R_0 \phi^{(1)}(x,t) + O(R_0^2)$. Before writing the $O(1)$ boundary conditions the terms in (2.77) are Taylor expanded about $y = \phi^{(0)}(x,t)$ so that

$$\left(u^{(n)}, v^{(n)}\right)(x,\phi,t) \simeq \left(u^{(n)}, v^{(n)}\right)\left(x,\phi^{(0)},t\right)$$
$$+R_0 \left(u_y^{(n)}, v_y^{(n)}\right)\left(x,\phi^{(0)},t\right)\phi^{(1)} + O\left(R_0^2\right). \tag{2.78}$$

Using this Taylor expansion (2.78) in equation (2.77) gives

$$R_0\left[\phi_t^{(0)}+v^{(0)}\phi_x^{(1)}+u^{(1)}\phi_x^{(0)}+u_y^{(0)}\phi^{(1)}\phi_x^{(1)}-v_y^{(0)}\phi^{(1)}-v^{(1)}\right]$$

$$+v^{(0)}\phi_x^{(0)}-v^{(0)}=0+O\left(R_0^2\right), \tag{2.79}$$

on $y=\phi^{(0)}(x,t)$.

Equating all terms proportional to $O(1)$ gives the leading order boundary condition,

$$v^{(0)}=u^{(0)}\frac{\partial\phi^{(0)}}{\partial x} \text{ on } y=\phi^{(0)}(x,t). \tag{2.80}$$

The leading order momentum equations (2.58) and (2.59), however, are given as

$$u^{(0)}=-\frac{\partial h^{(0)}}{\partial y} \text{ and } v^{(0)}=\frac{\partial h^{(0)}}{\partial x}, \tag{2.81}$$

so that equation (2.80) may be written in the form

$$\frac{\partial h^{(0)}}{\partial x}+\frac{\partial h^{(0)}}{\partial y}\frac{\partial\phi^{(0)}}{\partial x}=0 \text{ on } y=\phi^{(0)}(x,t). \tag{2.82}$$

Substituting the asymptotic expansions (2.55) and (2.76) into the dynamic boundary condition (2.75) and Taylor expanding about $y=\phi^{(0)}(x,t)$ leads to

$$h^{(0)}\left(x,\phi^{(0)},t\right)+R_0\left[h^{(1)}\left(x,\phi^{(0)},t\right)+h_y^{(0)}\left(x,\phi^{(0)},t\right)\phi^{(1)}\right]=0+O\left(R_0^2\right), \tag{2.83}$$

and the $O(1)$ dynamic boundary condition is given by

$$h^{(0)}\left(x,\phi^{(0)}(x,t),t\right)=0\,. \tag{2.84}$$

However, and this is an important point, the boundary conditions (2.82) and (2.84) are *not* independent and in fact (2.82) follows from (2.84). Note differentiating (2.84) with respect to x gives

$$0 = \frac{d}{dx} h^{(0)} \left(x, \phi^{(0)}(x,t), t\right)$$

$$= h_x^{(0)} \left(x, \phi^{(0)}(x,t), t\right) + h_y^{(0)} \left(x, \phi^{(0)}(x,t), t\right) \phi_x^{(0)}, \tag{2.85}$$

which is just equation (2.82).

For the most part, the working domain for this thesis will be unbounded in the positive y–direction. It will be required then that the velocity field be bounded, that is, as $y \to \infty$, $\mathbf{u}^{(0)} < \infty$. By using the leading order velocity equations, (2.58) and (2.59), this means

$$\left|\nabla h^{(0)}\right| < \infty \text{ as } y \to \infty. \tag{2.86}$$

In summary, the reduced gravity model is given by

$$\frac{\partial h^{(0)}}{\partial t} + J\left(h^{(0)} \Delta h^{(0)} + \frac{1}{2} \nabla h^{(0)} \cdot \nabla h^{(0)}, h^{(0)}\right) = 0, \tag{2.87}$$

with the leading order boundary conditions,

$$h^{(0)}\left(x, \phi^{(0)}, t\right) = 0, \tag{2.88}$$

$$\left|\nabla h^{(0)}\right| < \infty, \quad \text{as} \quad y \to \infty. \tag{2.89}$$

An exact nonlinear solution to these equations is given as

$$h^{(0)} = h_0(y), \quad \phi^{(0)} = a, \tag{2.90}$$

where a is a constant, $h_0(a) = 0$ and $|h_{0y}| < \infty$ in order to meet the boundary conditions. This is easily shown as $h_{0t} = h_{0x} = 0$ so that when equation (2.90) is substituted into equation (2.87) it becomes

$$\frac{\partial h^{(0)}}{\partial t} + J\left(h^{(0)}\Delta h^{(0)} + \frac{1}{2}\nabla h^{(0)} \cdot \nabla h^{(0)}, h^{(0)}\right) =$$

$$0 + \left(h^{(0)}\Delta h^{(0)} + \frac{1}{2}\nabla h^{(0)} \cdot \nabla h^{(0)}\right)_x h_y^{(0)} - \left(h^{(0)}\Delta h^{(0)} + \frac{1}{2}\nabla h^{(0)} \cdot \nabla h^{(0)}\right)_y h_x^{(0)}$$

$$= \left(h^{(0)}h_{yy}^{(0)} + \frac{1}{2}\left(h_y^{(0)}\right)^2\right)_x h_y^{(0)} - \left(h^{(0)}h_{yy}^{(0)} + \frac{1}{2}\left(h_y^{(0)}\right)^2\right)_y \cdot 0$$

$$= 0 \cdot h_y^{(0)} = 0. \tag{2.91}$$

2.2 Potential Vorticity Dynamics

In this Section, it is shown that the governing equation (2.87) is the leading order potential vorticity equation for the flow. For the reduced gravity shallow water equations, the conservation of potential vorticity is derived using the methods demonstrated by Pedlosky (1979) which involves first examining the dimensional horizontal momentum equation (2.29), in component form,

$$u_{t^*}^* + u^* u_{x^*}^* + v^* u_{y^*}^* - f_0 v^* = -g' h_{x^*}^*, \tag{2.92}$$

$$v_{t^*}^* + u^* v_{x^*}^* + v^* v_{y^*}^* + f_0 u^* = -g' h_{y^*}^*. \tag{2.93}$$

Taking the derivative with respect to x of the y–momentum equation (2.93) and subtracting the derivative with respect to y of the x–momentum equation (2.92) gives the equation

$$\frac{D}{Dt}\left(f_0 + v_{x^*}^* - u_{y^*}^*\right) + \left(f_0 + v_{x^*}^* - u_{y^*}^*\right)\left(u_{x^*}^* + v_{y^*}^*\right) = 0. \tag{2.94}$$

This is simplified by using the continuity equation (2.30), written in the form,

$$\frac{Dh^*}{Dt^*} + h^*\left(u_{x^*}^* + v_{y^*}^*\right) = 0, \tag{2.95}$$

to give

$$\frac{D}{Dt^*}\left(f_0 + v^*_{x^*} - u^*_{y^*}\right) - \frac{\left(f_0 + v^*_{x^*} - u^*_{y^*}\right)}{h^*}\frac{Dh^*}{Dt^*} = 0. \tag{2.96}$$

Multiplying equation (2.96) through by $1/h^*$ leads to

$$\frac{D}{Dt^*}\left(\frac{f_0 + v^*_{x^*} - u^*_{y^*}}{h^*}\right) = 0. \tag{2.97}$$

Therefore for the shallow water equations, the potential vorticity given by

$$\mathrm{PV} = \frac{f_0 + v^*_{x^*} - u^*_{y^*}}{h^*}, \tag{2.98}$$

is conserved following the motion.

In order to consider the potential vorticity for the frontal equations, equation (2.97) is nondimensionalized using relations (2.44)–(2.48) to give

$$R_0\frac{\partial}{\partial t}\left(\frac{1 + R_0\left(v_x - u_y\right)}{h}\right) + \mathbf{u}\cdot\nabla\left(\frac{1 + R_0\left(v_x - u_y\right)}{h}\right) = 0. \tag{2.99}$$

In examining the leading order potential vorticity, the equation is first rewritten as

$$h\left(R_0\frac{\partial}{\partial t}\left(1 + R_0\left(v_x - u_y\right)\right) + \mathbf{u}\cdot\nabla\left(1 + R_0\left(v_x - u_y\right)\right)\right)$$
$$+ \left(1 + R_0\left(v_x - u_y\right)\right)\left(R_0\frac{\partial h}{\partial t} + \mathbf{u}\cdot\nabla h\right) = 0. \tag{2.100}$$

The frontal thickness and horizontal velocities are now expanded in the Rossby number, R_0, following that done in equation (2.55) and substituted into equation (2.100) giving the leading order problem,

$$-u^{(0)}h^{(0)}_x - v^{(0)}h^{(0)}_y = 0. \tag{2.101}$$

However if the leading order momentum equations (2.58) and (2.59) are used to simplify this equation (2.101), it follows that

$$-u^{(0)}h_x^{(0)} - v^{(0)}h_y^{(0)} = h_y^{(0)}h_x^{(0)} - h_x^{(0)}h_y^{(0)} = 0, \tag{2.102}$$

and the leading order potential vorticity is identically zero. Therefore, the first nontrivial terms in the potential vorticity equation (2.99) are the $O(R_0)$ terms, given by

$$h_t^{(0)} - h^{(0)}\mathbf{u}^{(0)} \cdot \nabla \left(v_x^{(0)} - u_y^{(0)}\right) + \mathbf{u}^{(1)} \cdot \nabla h^{(0)} + \mathbf{u}^{(0)} \cdot \nabla h^{(1)} = 0, \tag{2.103}$$

where equation (2.101) has been used. Equations (2.58) and (2.59) can again be used to simplify this equation along with equation (2.67) to give

$$h_t^{(0)} - h^{(0)}\left(\hat{\mathbf{e}}_3 \times \nabla h^{(0)}\right) \cdot \nabla\left(\Delta h^{(0)}\right) + \left(\hat{e}_3 \times \nabla h^{(1)}\right) \cdot \nabla h^{(0)}$$

$$+J\left(\nabla h^{(0)}, h^{(0)}\right) \cdot \nabla h^{(0)} + \left(\hat{e}_3 \times \nabla h^{(0)}\right) \cdot \nabla h^{(1)} = 0. \tag{2.104}$$

This can be simplified further by considering the quantity $(\hat{\mathbf{e}}_3 \times \nabla A) \cdot \nabla B$ which can be written as follows,

$$(\hat{\mathbf{e}}_3 \times \nabla A) \cdot \nabla B = (-A_y, A_x) \cdot (B_x, B_y) = A_x B_y - A_y B_x = J(A, B). \tag{2.105}$$

Using this identity in equation (2.104) gives

$$h_t^{(0)} - h^{(0)} J\left(h^{(0)}, \Delta h^{(0)}\right) + J\left(h^{(1)}, h^{(0)}\right)$$

$$+J\left(\nabla h^{(0)}, h^{(0)}\right) \cdot \nabla h^{(0)} + J\left(h^{(0)}, h^{(1)}\right) = 0. \tag{2.106}$$

Two additional identities that can be used to simplify this equation are given as

$$J(A, B) = A_x B_y - A_y B_x = -\left(A_y B_x - A_x B_y\right) = -J(A, B), \tag{2.107}$$

$$AJ(A,B) = A(A_xB_y - A_yB_x)$$

$$= \frac{1}{2}\left(A^2\right)_x B_y - \frac{1}{2}\left(A^2\right)_y B_x = J\left(\frac{1}{2}A^2, B\right). \tag{2.108}$$

These identities simplify equation (2.106) further to

$$0 = h_t^{(0)} - h^{(0)}J\left(h^{(0)}, \Delta h^{(0)}\right) + J\left(\frac{1}{2}\nabla h^{(0)} \cdot \nabla h^{(0)}, h^{(0)}\right)$$

$$-J\left(h^{(0)}, h^{(1)}\right) + J\left(h^{(0)}, h^{(1)}\right)$$

$$= h_t^{(0)} + h^{(0)}J\left(\Delta h^{(0)}, h^{(0)}\right) + J\left(\frac{1}{2}\nabla h^{(0)} \cdot \nabla h^{(0)}, h^{(0)}\right). \tag{2.109}$$

The final identity that is required to simplify this expression is given by

$$AJ(B,A) = A(B_xA_y - B_yA_x)$$

$$= AB_xA_y + A_xBA_y - A_xBA_y - AB_yA_x$$

$$= (AB)_x A_y - (AB)_y A_x = J(AB, A). \tag{2.110}$$

When this identity is used in equation (2.109) the resulting equation is given as

$$h_t^{(0)} + J\left(h^{(0)}\Delta h^{(0)} + \frac{1}{2}\nabla h^{(0)} \cdot \nabla h^{(0)}, h^{(0)}\right) = 0, \tag{2.111}$$

which is just equation (2.87), the leading order governing equation of motion for the frontal model being considered. Thus the conditions for the conservation of potential vorticity leads to the identical dynamic relationship for the frontal thickness as the derivations using the shallow water equations. In what follows the (0) superscript in the model equation (2.111) will be deleted.

2.3 Hamiltonian Formulation

Over the last several years successful attempts have been made to formulate the equations of motion for fluid dynamics in Hamiltonian form. Salmon (1988) explains that development of the Hamiltonian equations for fluid mechanics initially did not seem beneficial since the Eulerian equations for fluids are generally much simpler than the Hamiltonian equations. However, Salmon (1988) points out that there have proven to be some benefits to using Hamiltonian systems. First of all, Salmon points out that the Hamiltonian formulation is a succinct way of describing the dynamics of the motion of fluids. Secondly, there is a connection between the symmetry properties of the Hamiltonian and the conservation laws of the dynamics and thus approximate conservation laws of the system can often be deduced from these symmetries. Finally, Salmon (1988) points out that the Hamiltonian mechanics are not tied to a particular choice of coordinates.

Olver (1982) demonstrated these benefits when he put the Euler equations for inviscid–incompressible fluid flow into Hamiltonian form. He demonstrated some symmetries of the system as well as some of the conservation laws using this Hamiltonian formulation. Holm *et al.* (1985) used Hamiltonian systems to examine the stability of two dimensional flow systems, three dimensional flow systems as well as plasma systems. The stability algorithm they develop is similar to the one that will be used here.

Continuous systems, such as fluid flows, are infinite dimensional dynamical systems. This is because each variable in the system can be Fourier expanded and thus describing the system involves solving for the infinite number of Fourier coefficients of the Fourier expansion (Courant and Hilbert, 1953). Infinite dimensional systems are described by systems of partial differential equations where the dependent variables are functions of space and time defined over some domain.

A system of partial differential equations is Hamiltonian if it can be written in the form (Olver, 1982; Morrison, 1982),

$$\mathbf{q}_t = \mathcal{D}\frac{\delta H(\mathbf{q})}{\delta \mathbf{q}}. \tag{2.112}$$

where the $\mathbf{q}$ variable is a column vector of n dependent variables; the variable $H(\mathbf{q})$ is a functional (that is, a function of a function, as the dependent variable $\mathbf{q}$ for a continuous system is itself a function) known as the Hamiltonian functional and which must be a conserved quantity; and $\mathcal{D}$ is a matrix of differential operators.

In addition to being able to write the dynamical system in the form given in equation (2.112), it can also be described using a Poisson bracket for a Hamiltonian system defined by (Morrison, 1982)

$$[F,G] \equiv \left\langle \frac{\delta F}{\delta \mathbf{q}}, \mathcal{D}\frac{\delta G}{\delta \mathbf{q}} \right\rangle, \tag{2.113}$$

where F and G are arbitrary functionals depending on $\mathbf{q}$ and where an appropriate inner product is chosen for the phase space being described. This bracket must display the following algebraic properties (Arnold, 1978; Morrison, 1982):

$$[F,F] = 0, \tag{2.114}$$

$$[F,G] = -[G,F], \tag{2.115}$$

$$[\alpha F + \beta G, Q] = \alpha [F,Q] + \beta [G,Q], \tag{2.116}$$

$$[FG,Q] = F[G,Q] + [F,Q]G, \tag{2.117}$$

$$[F,[G,Q]] + [G,[Q,F]] + [Q,[F,G]] = 0. \tag{2.118}$$

This last property is called the Jacobi identity and is usually the most difficult property to prove because it involves taking variations of inner products and then using these variations in another inner product. These algebraic properties put constraints on the matrix $\mathcal{D}$.

The model just derived in Section 2.2 can be rewritten using this Hamiltonian formulation. The derivation of this Hamiltonian formulation makes use of the Jacobian formula, $J(A,B) = A_x B_y - A_y B_x$, and several of its properties. These properties are demonstrated in the following lemma:

Lemma 1 *The Jacobian* $J(A,B) = A_x B_y - A_y B_x$ *exhibits the following properties:*

1.

$$J(A,B) = -J(B,A), \tag{2.119}$$

2.

$$AJ(A,B) = \frac{1}{2} J\left(A^2, B\right), \tag{2.120}$$

3.

$$J(AB,C) = BJ(A,C) + AJ(B,C), \tag{2.121}$$

4.

$$J(A,B) = \nabla \cdot \left[\left(\hat{\mathbf{e}}_3 \times \nabla A\right) B\right], \tag{2.122}$$

5.

$$J(A,B) = \hat{\mathbf{e}}_3 \cdot \left(\nabla A \times \nabla B\right), \tag{2.123}$$

6.

$$J(A,B)\, J(C,D) + J(D,A)\, J(C,B) + J(B,D)\, J(C,A) = 0. \tag{2.124}$$

Proof.

1.

$$J(A,B) = A_x B_y - A_y B_x = -(B_x A_y - B_y A_x) = -J(B,A). \qquad (2.125)$$

2.

$$AJ(A,B) = A(A_x B_y - A_y B_x) = \frac{1}{2}\left(\left[A^2\right]_x B_y - \left[A^2\right]_y B_x\right)$$

$$= \frac{1}{2} J\left(A^2, B\right). \qquad (2.126)$$

3.

$$J(AB,C) = (AB)_x C_y - (AB)_y C_x$$

$$= (A_x B C_y - A_y B C_x) + (A B_x C_y - A B_y C_x)$$

$$= B(A_x C_y - A_y C_x) + A(B_x C_y - B_y C_x)$$

$$= BJ(A,C) + AJ(B,C). \qquad (2.127)$$

4.

$$J(A,B) = A_x B_y - A_y B_x$$

$$= A_x B_y + A_{xy} B - A_{xy} B - A_y B_x$$

$$= (-A_y B)_x + (A_x B)_y$$

$$= \nabla \cdot B(-A_y, A_x)$$

$$= \nabla \cdot (B\hat{e}_3 \times \nabla A). \qquad (2.128)$$

5.

$$J(A,B) = A_x B_y - A_y B_x$$

$$= \hat{\mathbf{e}}_3 \cdot ([A_x B_y - A_y B_x]\, \hat{\mathbf{e}}_3)$$

$$= \hat{\mathbf{e}}_3 \cdot [\nabla A \times \nabla B]. \qquad (2.129)$$

6.

$$J(A,B)\,J(C,D) + J(D,A)\,J(C,B) + J(B,D)\,J(C,A)$$

$$= (A_x B_y - A_y B_x)(C_x D_y - C_y D_x) + (D_x A_y - D_y A_x)(C_x B_y - C_y B_x)$$

$$+ (B_x D_y - B_y D_x)(C_x A_y - C_y A_x)$$

$$= A_x B_y C_x D_y - A_x B_y C_y D_x - A_y B_x C_x D_y + A_y B_x C_y D_x$$

$$+ A_y B_y C_x D_x - A_y B_x C_y D_x - A_x B_y C_x D_y + A_x B_x C_y D_y$$

$$+ A_y B_x C_x D_y - A_x B_x C_y D_y - A_y B_y C_x D_x + A_x B_y C_y D_x = 0. \qquad (2.130)$$

■

Before a description of the Hamiltonian formulation of this model (equation (2.87)) can be given, the spatial domain also must be specified. This spatial domain is denoted as Ω and it is the periodic channel given by

$$\Omega = \{\,(x,y)|-x_0 < x < x_0, \phi(x,t) \leq y < L \leq \infty\}. \qquad (2.131)$$

It is assumed that $h(x,y,t)$ and $\phi(x,t)$ are smoothly periodic and that, of course, $h(x,\phi,t) = 0$. On the other wall located at $y = L$ it is assumed that

$$h(x,L,t) = h_L \equiv \text{ constant}, \qquad (2.132)$$

$$|\nabla h|(x,L,t) < \infty. \qquad (2.133)$$

The boundary of this domain is denoted by $\partial\Omega$.

Theorem 1 *The frontal model,*

$$h_t + J\left(h\Delta h + \frac{1}{2}\nabla h \cdot \nabla h, h\right) = 0, \tag{2.134}$$

with boundary conditions,

$$h\left(x, \phi, t\right) = 0, \tag{2.135}$$

$$h\left(x, L, t\right) = h_L \equiv \text{ constant,}$$

$$\left|\nabla h\right|\left(x, L, t\right) < \infty, \tag{2.136}$$

can be written as a scalar infinite dimensional Hamiltonian system in the form (2.112) with

$$q = h, \tag{2.137}$$

$$H\left(q\right) = -\frac{1}{2}\iint_{\Omega} h\nabla h \cdot \nabla h \; dxdy, \tag{2.138}$$

$$\mathcal{D}* \;\; = J\left(q, *\right), \tag{2.139}$$

where J is the Jacobian identity given by $J\left(A, B\right) = A_x B_y - A_y B_x$ and where the associated Poisson bracket is

$$\left[F, G\right] = \iint_{\Omega} \frac{\delta F}{\delta h} J\left(h, \frac{\delta G}{\delta h}\right) dxdy, \tag{2.140}$$

with Ω given by equation (2.131).

Proof. In order to prove this theorem, it is first shown that the Hamiltonian function is invariant. Then it is shown that this system can be written in the form given in equation (2.112). This involves calculating the first variational derivative of the prescribed Hamiltonian function. Finally it is shown that the corresponding Poisson bracket of this system satisfies the five properties required of a Poisson bracket given in equations (2.114)–(2.118).

First, it is demonstrated that H is conserved by showing that $\partial H/\partial t = 0$:

$$\frac{\partial H}{\partial t} = -\frac{1}{2}\iint\limits_{\Omega} h_t \nabla h \cdot \nabla h \; dxdy - \iint\limits_{\Omega} h \nabla h \cdot \nabla h_t \; dxdy + \frac{1}{2}\left\{ \int_{-x_0}^{x_0} \left[h \nabla h \cdot \nabla h \right]_{y=\phi} \phi_t dx \right\}$$

$$= -\frac{1}{2}\iint\limits_{\Omega} h_t \nabla h \cdot \nabla h \; dxdy - \iint\limits_{\Omega} h \nabla h \cdot \nabla h_t \; dxdy, \qquad (2.141)$$

since $h = 0$ on $y = \phi$. Continuing with the calculation gives

$$\frac{\partial H}{\partial t} = -\frac{1}{2}\iint\limits_{\Omega} h_t \nabla h \cdot \nabla h \; dxdy - \frac{1}{2}\iint\limits_{\Omega} \nabla h^2 \cdot \nabla h_t dxdy$$

$$= -\frac{1}{2}\iint\limits_{\Omega} h_t \nabla h \cdot \nabla h \; dxdy - \int_{\partial\Omega} h_t h \left(\mathbf{n} \cdot \nabla h \right) ds + \frac{1}{2}\iint\limits_{\Omega} h_t \Delta h^2 dxdy, \qquad (2.142)$$

where Green's identity has been used and $\mathbf{n}$ is the outward normal to the boundary, $\partial\Omega$. Working only with the boundary integral in (2.142) yields,

$$\int_{\partial\Omega} h_t h \left(\mathbf{n} \cdot \nabla h \right) ds = \int_{\phi(-x_0,t)}^{\infty} \left[h_t h \mathbf{n} \cdot \nabla h \right]_{x=-x_0} ds + \int_{-x_0}^{x_0} \left[h_t h \mathbf{n} \cdot \nabla h \right]_{y=\phi(x,t)} ds$$

$$+ \int_{\phi(x_0,t)}^{\infty} \left[h_t h \mathbf{n} \cdot \nabla h \right]_{x=x_0} ds + \int_{-x_0}^{x_0} \left[h_t h \left(\mathbf{n} \cdot \nabla h \right) \right]_{y=L} ds$$

$$= \int_{\phi(-x_0,t)}^{\infty} \left[h_t h \left(-h_x \right) \right]_{x=-x_0} dy + \int_{-x_0}^{x_0} \left[h_t h \mathbf{n} \cdot \nabla h \right]_{y=\phi(x,t)} ds$$

$$+ \int_{\phi(x_0,t)}^{\infty} \left[h_t h h_x \right]_{x=x_0} dy + \int_{-x_0}^{x_0} \left[h_t h \left(\mathbf{n} \cdot \nabla h \right) \right]_{y=L} ds. \qquad (2.143)$$

The second and the fourth integral given here vanishes, respectively, since $h = 0$ on $y = \phi(x,t)$ and $\frac{\partial h_L}{\partial t} \equiv 0$. The remaining two integrals exactly cancel one another since h and ϕ are smoothly periodic on the boundaries, $x = \pm x_0$. Therefore the boundary integral in equation (2.142) vanishes resulting in the integral,

$$\frac{\partial H}{\partial t} = -\frac{1}{2}\iint\limits_{\Omega} h_t \nabla h \cdot \nabla h \; dxdy + \frac{1}{2}\iint\limits_{\Omega} h_t \Delta h^2 dxdy$$

$$= -\frac{1}{2} \iint\limits_{\Omega} h_t \nabla h \cdot \nabla h \, dxdy + \frac{1}{2} \iint\limits_{\Omega} h_t \left(\nabla h \cdot \nabla h + h \Delta h \right) dxdy$$

$$= \iint\limits_{\Omega} h_t \left(\frac{1}{2} \nabla h \cdot \nabla h + h \Delta h \right) dxdy. \qquad (2.144)$$

This integral can be simplified yet further by using equation (2.134) and substituting in for h_t to give

$$\frac{\partial H}{\partial t} = - \iint\limits_{\Omega} J \left(h\Delta h + \frac{1}{2} \nabla h \cdot \nabla h, h \right) \left(\frac{1}{2} \nabla h \cdot \nabla h + h \Delta h \right) dxdy$$

$$= - \iint\limits_{\Omega} \frac{1}{2} J \left(\left[h\Delta h + \frac{1}{2} \nabla h \cdot \nabla h \right]^2, h \right) dxdy, \qquad (2.145)$$

on using the identity (2.120). This can be simplified further by using the identities (2.119) and (2.122) to give

$$\frac{\partial H}{\partial t} = \iint\limits_{\Omega} \nabla \cdot \left[(\hat{e}_3 \times \nabla h) \left(h\Delta h + \frac{1}{2} \nabla h \cdot \nabla h \right)^2 \right] dxdy$$

$$= \int_{\partial\Omega} \left(h\Delta h + \frac{1}{2} \nabla h \cdot \nabla h \right)^2 (\hat{e}_3 \times \nabla h) \cdot \mathbf{n} ds$$

$$= \int_{\partial\Omega} \left(h\Delta h + \frac{1}{2} \nabla h \cdot \nabla h \right)^2 \nabla h \cdot \mathbf{T} ds, \qquad (2.146)$$

where the divergence theorem has been used and the identity $(\hat{e}_3 \times \nabla h) \cdot \mathbf{n} = \nabla h \cdot \mathbf{T}$ where $\mathbf{T}$ is the tangent vector to the boundary, $\partial\Omega$. This last boundary integral can be simplified in much the same manner as the boundary integral in equation (2.142) giving

$$\frac{\partial H}{\partial t} = \int_{\phi(-x_0,t)}^{\infty} \left[\left(h\Delta h + \frac{1}{2} \nabla h \cdot \nabla h \right)^2 (-h_y) \right]_{x=-x_0} dy$$

$$+ \int_{-x_0}^{x_0} \left[\left(h\Delta h + \frac{1}{2} \nabla h \cdot \nabla h \right)^2 \nabla h \cdot \mathbf{T} \right]_{y=\phi(x,t)} ds$$

$$+\int_{\phi(x_0,t)}^{\infty}\left[\left(h\Delta h+\frac{1}{2}\nabla h\cdot\nabla h\right)^2 h_y\right]_{x=x_0} dy$$

$$+\int_{-x_0}^{x_0}\left[\left(h\Delta h+\frac{1}{2}\nabla h\cdot\nabla h\right)^2 h_x\right]_{y=L} dx. \qquad (2.147)$$

Using the smooth periodicity of h and ϕ on the boundaries $x = \pm x_0$, the first and third integrals in this equation (2.147) exactly cancel one another. The last integral also vanishes since $\frac{\partial h_L}{\partial x} = 0$. The remaining second integral with $\mathbf{T} = (1, \phi_x)$ becomes

$$\frac{\partial H}{\partial t} = \int_{-x_0}^{x_0}\left[\left(h\Delta h+\frac{1}{2}\nabla h\cdot\nabla h\right)^2 (h_x + h_y\phi_x)\right]_{y=\phi(x,t)} dx = 0, \qquad (2.148)$$

by using the boundary condition, $h_x + h_y\phi_x = 0$ on $y = \phi(x,t)$. Therefore since $\partial H/\partial t = 0$, H is conserved.

Secondly, it is shown that the model (2.134) can be written in the form (2.112). This involves first calculating the variational derivative of H. In order to do this the first variation of the Hamiltonian is calculated to give

$$\delta H = -\frac{1}{2}\iint_\Omega \delta h\nabla h\cdot\nabla h\; dxdy - \iint_\Omega h\nabla h\cdot\nabla\delta h\; dxdy$$

$$+\frac{1}{2}\int_{-x_0}^{x_0}(h\nabla h\cdot\nabla h)_{y=\phi(x,t)}\,\delta\phi dx. \qquad (2.149)$$

In this equation (2.149), the last integral vanishes since $h = 0$ on $y = \phi(x,t)$ and Green's theorem can be used on the second integral to give

$$\delta H = -\frac{1}{2}\iint_\Omega \delta h\nabla h\cdot\nabla h\; dxdy - \frac{1}{2}\iint_\Omega \nabla h^2\cdot\nabla\delta h dxdy$$

$$= -\frac{1}{2}\iint_\Omega \delta h\nabla h\cdot\nabla h\; dxdy - \int_{\partial\Omega} h\delta h\,(\mathbf{n}\cdot\nabla h)\,ds + \frac{1}{2}\iint_\Omega \delta h\Delta h^2 dxdy. \qquad (2.150)$$

Before proceeding further boundary conditions on δh must be specified. It is assumed that δh is smoothly periodic at $x = \pm x_0$ and that $\delta h = 0$ on $y = L$. The boundary integral

$$\int_{\partial\Omega} h\delta h\,(\mathbf{n}\cdot\nabla h)\,ds = 0, \tag{2.151}$$

because $h = 0$ on $y = \phi(x,t)$, $\delta h = 0$ on $y = L$ and because of the periodicity at $x = \pm x_0$. The first variation of H then simplifies to

$$\begin{aligned}\delta H &= -\frac{1}{2}\iint\limits_{\Omega} \delta h \nabla h\cdot\nabla h\,dxdy + \frac{1}{2}\iint\limits_{\Omega} \delta h \Delta h^2 dxdy \\ &= -\frac{1}{2}\iint\limits_{\Omega} \delta h\nabla h\cdot\nabla h\,dxdy + \iint\limits_{\Omega}\delta h\,(\nabla h\cdot\nabla h + h\Delta h)\,dxdy \\ &= \iint\limits_{\Omega}\delta h\left(\frac{1}{2}\nabla h\cdot\nabla h + h\Delta h\right)dxdy.\end{aligned} \tag{2.152}$$

The first variational derivative of H is therefore given by

$$\frac{\delta H}{\delta h} = \frac{1}{2}\nabla h\cdot\nabla h + h\Delta h. \tag{2.153}$$

It follows that

$$\begin{aligned}\mathcal{D}\frac{\delta H(\mathbf{q})}{\delta\mathbf{q}} &= J\left(h, \frac{\delta H}{\delta h}\right) \\ &= J\left(h, \frac{1}{2}\nabla h\cdot\nabla h + h\Delta h\right) \equiv h_t.\end{aligned} \tag{2.154}$$

Finally, it is shown that the Poisson bracket satisfies the five properties necessary for a Hamiltonian formulation. It is assumed that all functionals must meet the condition,

$$\left.\frac{\delta F}{\delta h}\right|_{\partial\Omega} = f(h), \tag{2.155}$$

when evaluated on $y = \phi$ or $y = L$, where f is a smooth function of its arguments. In other words this property states that the first variation of these functionals must be an explicit function of h on the boundaries. It is also assumed that the variational derivatives are smooth functions of their arguments.

The first property of the Poisson bracket, i.e. (2.114), is easily proven:

$$[F,F] \equiv \left\langle \frac{\delta F}{\delta h}, \mathcal{D}\frac{\delta F}{\delta h} \right\rangle = \iint_\Omega \frac{\delta F}{\delta h} J\left(h, \frac{\delta F}{\delta h}\right) dxdy$$

$$= \frac{1}{2}\iint_\Omega J\left(h, \left[\frac{\delta F}{\delta h}\right]^2\right) dxdy$$

$$= \iint_\Omega \nabla \cdot \left[(\hat{e}_3 \times \nabla h)\left(\frac{\delta F}{\delta h}\right)^2\right] dxdy, \tag{2.156}$$

where the Jacobian identities (2.120) and (2.122) have been used. The divergence theorem can now be used to give

$$[F,F] = \frac{1}{2}\int_{\partial\Omega} [(\hat{e}_3 \times \nabla h)\cdot \mathbf{n}] \left(\frac{\delta F}{\delta h}\right)^2 ds$$

$$= \frac{1}{2}\int_{\partial\Omega} \mathbf{T}\cdot\nabla h \left(\frac{\delta F}{\delta h}\right)^2 ds$$

$$= \frac{1}{2}\int_{\phi(-x_0,t)}^{\infty} \left[\left(\frac{\delta F}{\delta h}\right)^2 (-h_y)\right]_{x=-x_0} dy$$

$$+ \frac{1}{2}\int_{-x_0}^{x_0} \left[\left(\frac{\delta F}{\delta h}\right)^2 (h_x + h_y\phi_x)\right]_{y=\phi(x,y)} dx$$

$$+ \frac{1}{2}\int_{\phi(x_0,t)}^{\infty} \left[\left(\frac{\delta F}{\delta h}\right)^2 h_y\right]_{x=x_0} dy + \frac{1}{2}\int_{-x_0}^{x_0} \left[\left(\frac{\delta F}{\delta h}\right)^2 h_x\right]_{y=L} dx = 0, \tag{2.157}$$

since the first and third integrals cancel due to the periodicity of h on the boundaries $x = \pm x_0$, since $\frac{\partial h_L}{\partial x} = 0$ and since $h(x,\phi,t) = 0$ implies $h_x + h_y\phi_x = 0$ on $y = \phi$.

The second property, i.e. (2.115), is the skew symmetry property:

$$[F,G] \equiv \left\langle \frac{\delta F}{\delta h}, \mathcal{D}\frac{\delta G}{\delta h} \right\rangle = \iint_\Omega \frac{\delta F}{\delta h} J\left(h, \frac{\delta G}{\delta h}\right) dxdy$$

$$= -\iint_\Omega \frac{\delta F}{\delta h} J\left(\frac{\delta G}{\delta h}, h\right) dxdy$$

$$= -\iint_\Omega J\left(\frac{\delta F}{\delta h}\frac{\delta G}{\delta h}, h\right) dxdy + \iint_\Omega \frac{\delta G}{\delta h} J\left(\frac{\delta F}{\delta h}, h\right) dxdy, \tag{2.158}$$

where the Jacobian identities given in equations (2.119) and (2.121) have been used. Using equation (2.119) once more gives

$$[F, G] = -\iint_\Omega J\left(\frac{\delta F}{\delta h}\frac{\delta G}{\delta h}, h\right) dxdy - \iint_\Omega \frac{\delta G}{\delta h} J\left(h, \frac{\delta F}{\delta h}\right) dxdy$$

$$= \iint_\Omega \nabla \cdot \left[(\hat{e}_3 \times \nabla h)\frac{\delta F}{\delta h}\frac{\delta G}{\delta h}\right] - \iint_\Omega \frac{\delta G}{\delta h} J\left(h, \frac{\delta F}{\delta h}\right) dxdy$$

$$= \int_{\partial\Omega} \frac{\delta F}{\delta h}\frac{\delta G}{\delta h}(\hat{e}_3 \times \nabla h) \cdot \mathbf{n} ds - \iint_\Omega \frac{\delta G}{\delta h} J\left(h, \frac{\delta F}{\delta h}\right) dxdy, \tag{2.159}$$

where the identity (2.122) and the divergence theorem have been used. The boundary integral again disappears using the same reasoning as used for equation (2.157) leaving the identity

$$[F, G] = -\iint_\Omega \frac{\delta G}{\delta h} J\left(h, \frac{\delta F}{\delta h}\right) dxdy = -[G, F]. \tag{2.160}$$

The third property, i.e. (2.116), is the distributive property:

$$[\alpha F + \beta G, Q] \equiv \iint_\Omega \frac{\delta}{\delta h}(\alpha F + \beta G) J\left(h, \frac{\delta Q}{\delta h}\right) dxdy$$

$$= \iint_\Omega \alpha \frac{\delta F}{\delta h} J\left(h, \frac{\delta Q}{\delta h}\right) dxdy + \iint_\Omega \beta \frac{\delta G}{\delta h} J\left(h, \frac{\delta Q}{\delta h}\right) dxdy$$

$$\equiv \alpha [F, Q] + \beta [G, Q]. \tag{2.161}$$

The proof of the associative property, i.e. (2.117), is as follows:

$$[FG,Q] \equiv \iint\limits_{\Omega} \frac{\delta}{\delta h}(FG)\, J\left(h, \frac{\delta Q}{\delta h}\right) dxdy$$

$$= \iint\limits_{\Omega} \left(\frac{\delta F}{\delta h}G + F\frac{\delta G}{\delta h}\right) J\left(h, \frac{\delta Q}{\delta h}\right) dxdy,$$

$$= G\iint\limits_{\Omega} \frac{\delta F}{\delta h} J\left(h, \frac{\delta Q}{\delta h}\right) dxdy + F\iint\limits_{\Omega} \frac{\delta G}{\delta h} J\left(h, \frac{\delta Q}{\delta h}\right) dxdy,$$

$$\equiv G\,[F,Q] + F\,[G,Q]. \tag{2.162}$$

Finally, the Jacobi identity must be proven, that is,

$$[F,[G,Q]] + [G,[Q,F]] + [Q,[F,G]] = 0. \tag{2.163}$$

To begin this proof, it is noted that

$$[F,[G,Q]] = -\left[[G,Q],F\right] = -\left\langle \frac{\delta}{\delta h}\left\langle \frac{\delta G}{\delta h}, J\left(h, \frac{\delta Q}{\delta h}\right)\right\rangle, J\left(h, \frac{\delta F}{\delta h}\right)\right\rangle. \tag{2.164}$$

In order to simplify the first term of this inner product, the first variation of the inside inner product is found:

$$\delta\left\langle \frac{\delta G}{\delta h}, J\left(h, \frac{\delta Q}{\delta h}\right)\right\rangle = \delta \iint\limits_{\Omega} \frac{\delta G}{\delta h} J\left(h, \frac{\delta Q}{\delta h}\right) dxdy$$

$$= \iint\limits_{\Omega} \left[\frac{\delta^2 G}{\delta h^2}\delta h J\left(h, \frac{\delta Q}{\delta h}\right) + \frac{\delta G}{\delta h} J\left(\delta h, \frac{\delta Q}{\delta h}\right) + \frac{\delta G}{\delta h} J\left(h, \frac{\delta^2 Q}{\delta h^2}\delta h\right)\right] dxdy$$

$$- \int_{-x_0}^{x_0} \left[\frac{\delta G}{\delta h} J\left(h, \frac{\delta Q}{\delta h}\right)\right]_{y=\phi(x,t)} \delta\phi dx. \tag{2.165}$$

This last boundary integral can be evaluated using the fact that the first variational derivatives of the functionals must be explicit functions of h on $y = \phi$. This means that on the $y = \phi(x,t)$ where $h = 0$, the variational derivatives must be constant, that is,

$$\left.\frac{\delta F}{\delta h}\right|_{y=\phi(x,t)} = f\,(t) = f\,(0) = \text{constant}, \tag{2.166}$$

where F is an arbitrary functional. However, away from the boundary, the first variation of these functionals can be thought of as functions of x, y, and t, that is,

$$\frac{\delta F}{\delta h} = f\,(x, y, t)\,, \tag{2.167}$$

where f is some smooth function of its argument. Assuming that $\frac{\delta F}{\delta h}$ approaches its boundary value smoothly, it follows from (2.166) and (2.167) that

$$\begin{aligned} 0 &= \frac{d}{dx}\left(\left[\frac{\delta F}{\delta h}\right]_{y=\phi}\right) \\ &= \left[\left(\frac{\delta F}{\delta h}\right)_x + \left(\frac{\delta F}{\delta h}\right)_y \phi_x\right]_{y=\phi}. \end{aligned} \tag{2.168}$$

This boundary condition (2.168) can be used in the boundary integral in (2.165) to give

$$\begin{aligned} &\int_{-x_0}^{x_0} \left[\frac{\delta G}{\delta h} J\left(h, \frac{\delta Q}{\delta h}\right)\right]_{y=\phi(x,t)} \delta\phi dx dy \\ &= \int_{-x_0}^{x_0} \left[\frac{\delta G}{\delta h}\left(h_x\left(\frac{\delta Q}{\delta h}\right)_y - h_y\left(\frac{\delta Q}{\delta h}\right)_x\right)\right]_{y=\phi(x,t)} \delta\phi dx dy \\ &= \int_{-x_0}^{x_0} \left[\frac{\delta G}{\delta h}\left(h_x\left(\frac{\delta Q}{\delta h}\right)_y + h_y\left(\frac{\delta Q}{\delta h}\right)_y \phi_x\right)\right]_{y=\phi(x,t)} \delta\phi dx dy \\ &\int_{-x_0}^{x_0} \left[\frac{\delta G}{\delta h}\left(\frac{\delta Q}{\delta h}\right)_y (h_x + h_y\phi_x)\right]_{y=\phi(x,t)} \delta\phi dx dy = 0, \end{aligned} \tag{2.169}$$

since $h_x\,(x, \phi, t) + h_y\,(x, \phi, t)\,\phi_x = 0$. Therefore the boundary integral vanishes and equation (2.165) simplifies to

$$\delta\left\langle \frac{\delta G}{\delta h}, J\left(h, \frac{\delta Q}{\delta h}\right)\right\rangle$$

$$= \iint\limits_{\Omega} \frac{\delta^2 G}{\delta h^2}\delta h J\left(h, \frac{\delta Q}{\delta h}\right) + \frac{\delta G}{\delta h} J\left(\delta h, \frac{\delta Q}{\delta h}\right) + \frac{\delta G}{\delta h} J\left(h, \frac{\delta^2 Q}{\delta h^2}\delta h\right) dxdy$$

$$= \iint\limits_{\Omega} \frac{\delta^2 G}{\delta h^2}\delta h J\left(h, \frac{\delta Q}{\delta h}\right) + J\left(\frac{\delta G}{\delta h}\delta h, \frac{\delta Q}{\delta h}\right) - \delta h J\left(\frac{\delta G}{\delta h}, \frac{\delta Q}{\delta h}\right) dxdy$$

$$- \iint\limits_{\Omega} J\left(\frac{\delta G}{\delta h}\frac{\delta^2 Q}{\delta h^2}\delta h, h\right) - \frac{\delta^2 Q}{\delta h^2}\delta h J\left(\frac{\delta G}{\delta h}, h\right) dxdy, \qquad (2.170)$$

where the Jacobian identities (2.119) and (2.121), have been used.

Simplifying further using identity (2.122) gives

$$\delta\left\langle \frac{\delta G}{\delta h}, J\left(h, \frac{\delta Q}{\delta H}\right)\right\rangle = \iint\limits_{\Omega} \frac{\delta^2 G}{\delta h^2}\delta h J\left(h, \frac{\delta Q}{\delta h}\right) - \delta h J\left(\frac{\delta G}{\delta h}, \frac{\delta Q}{\delta h}\right) dxdy$$

$$+ \iint\limits_{\Omega} \frac{\delta^2 Q}{\delta h^2}\delta h J\left(\frac{\delta G}{\delta h}, h\right) dxdy + \iint\limits_{\Omega} \nabla\cdot\left[\frac{\delta Q}{\delta h}\hat{\mathbf{e}}_3 \times \nabla\left(\frac{\delta G}{\delta h}\delta h\right)\right] dxdy$$

$$- \iint\limits_{\Omega} \nabla\cdot\left[h\hat{\mathbf{e}}_3 \times \nabla\left(\frac{\delta G}{\delta h}\frac{\delta^2 Q}{\delta h^2}\delta h\right)\right] dxdy$$

$$= \iint\limits_{\Omega} \left[\frac{\delta^2 G}{\delta h^2} J\left(h, \frac{\delta Q}{\delta h}\right) - J\left(\frac{\delta G}{\delta h}, \frac{\delta Q}{\delta h}\right) + \frac{\delta^2 Q}{\delta h^2} J\left(\frac{\delta G}{\delta h}, h\right)\right] \delta h dxdy$$

$$+ \int_{\partial\Omega} \left[\frac{\delta Q}{\delta h}\hat{\mathbf{e}}_3 \times \nabla\left(\frac{\delta G}{\delta h}\delta h\right) - h\hat{\mathbf{e}}_3 \times \nabla\left(\frac{\delta G}{\delta h}\frac{\delta^2 Q}{\delta h^2}\delta h\right)\right]\cdot \mathbf{n} ds, \qquad (2.171)$$

where the divergence theorem has been used on the last two integrals. The boundary integral again simplifies to

$$\int_{\partial\Omega} \left[\frac{\delta Q}{\delta h}\hat{\mathbf{e}}_3 \times \nabla\left(\frac{\delta G}{\delta h}\delta h\right) - h\hat{\mathbf{e}}_3 \times \nabla\left(\frac{\delta G}{\delta h}\frac{\delta^2 Q}{\delta h^2}\delta h\right)\right]\cdot \mathbf{n} ds$$

$$= \int_{\partial\Omega} \frac{\delta Q}{\delta h} \left(- \left[\frac{\delta G}{\delta h} \delta h \right]_y , \left[\frac{\delta G}{\delta h} \delta h \right]_x \right) \cdot \mathbf{n} ds$$

$$- \int_{\partial\Omega} h \left(- \left[\frac{\delta G}{\delta h} \frac{\delta^2 Q}{\delta h^2} \delta h \right]_y , \left[\frac{\delta G}{\delta h} \frac{\delta^2 Q}{\delta h^2} \delta h \right]_x \right) \cdot \mathbf{n} ds$$

$$= \int_{\phi(-x_0,t)}^{\infty} \frac{\delta Q}{\delta h} \left[\frac{\delta G}{\delta h} \delta h \right]_y \Bigg|_{x=-x_0} dy - \int_{\phi(x_0,t)}^{\infty} \frac{\delta Q}{\delta h} \left[\frac{\delta G}{\delta h} \delta h \right]_y \Bigg|_{x=x_0} dy$$

$$+ \int_{-x_0}^{x_0} \frac{\delta Q}{\delta h} \left[\frac{\delta G}{\delta h} \delta h \right]_x \Bigg|_{y=L} dx$$

$$- \int_{-x_0}^{x_0} \left[\frac{\delta Q}{\delta h} \left(\left(\frac{\delta G}{\delta h} \delta h \right)_y \phi_x + \left(\frac{\delta G}{\delta h} \delta h \right)_x \right) \right]_{y=\phi(x,t)} dx$$

$$- \int_{\phi(-x_0,t)}^{\infty} h \left(\frac{\delta G}{\delta h} \frac{\delta^2 Q}{\delta h^2} \delta h \right)_y \Bigg|_{x=-x_0} dy + \int_{\phi(x_0,t)}^{\infty} h \left(\frac{\delta G}{\delta h} \frac{\delta^2 Q}{\delta h^2} \delta h \right)_y \Bigg|_{x=x_0} dy$$

$$+ \int_{-x_0}^{x_0} h \left[\frac{\delta G}{\delta h} \frac{\delta^2 Q}{\delta h^2} \delta h \right]_x \Bigg|_{y=L} dx$$

$$+ \int_{-x_0}^{x_0} h \left[\left(\frac{\delta G}{\delta h} \frac{\delta^2 Q}{\delta h^2} \delta h \right)_y \phi_x + \left(\frac{\delta G}{\delta h} \frac{\delta^2 Q}{\delta h^2} \delta h \right)_x \right] \Bigg|_{y=\phi(x,t)} dx$$

$$= \int_{-x_0}^{x_0} h \left[\frac{\delta G}{\delta h} \frac{\delta^2 Q}{\delta h^2} \delta h \right]_x + \frac{\delta Q}{\delta h} \left[\frac{\delta G}{\delta h} \delta h \right]_x \Bigg|_{y=L} dx$$

$$- \int_{-x_0}^{x_0} \left[\frac{\delta Q}{\delta h} \left(\left(\frac{\delta G}{\delta h} \right)_y \phi_x + \left(\frac{\delta G}{\delta h} \right)_x \right) \delta h \right]_{y=\phi(x,t)} dx$$

$$- \int_{-x_0}^{x_0} \left[\frac{\delta Q}{\delta h} \frac{\delta G}{\delta h} \left(\delta h_y \phi_x + \delta h_x \right) \right]_{y=\phi(x,t)} dx, \tag{2.172}$$

where the smooth periodicity at $x = \pm x_0$ has been used and the fact that $h = 0$ on $y = \phi(x,t)$. The first remaining integral is also zero since $\frac{\delta Q}{\delta h}$, $\frac{\delta G}{\delta h}$, δh and h are constant on $y = L$. Using equation (2.168) shows that the first term in this second integral vanishes. The second term in the second integral can be rewritten as

$$-\int_{-x_0}^{x_0}\left[\frac{\delta Q}{\delta h}\frac{\delta G}{\delta h}\left(\delta h_y\phi_x+\delta h_x\right)\right]_{y=\phi(x,t)}dx$$

$$=-\left[\frac{\delta Q}{\delta h}\frac{\delta G}{\delta h}\right]_{y=\phi(x,t)}\int_{-x_0}^{x_0}\frac{d}{dx}\left(\delta h|_{y=\phi(x,t)}\right)dx=0, \tag{2.173}$$

using the periodicity of δh.

Therefore the boundary integral vanishes so that the first variation (2.171) is given by

$$\delta\left\langle\frac{\delta G}{\delta h},J\left(h,\frac{\delta Q}{\delta H}\right)\right\rangle$$

$$=\iint_{\Omega}\left[\frac{\delta^2 G}{\delta h^2}J\left(h,\frac{\delta Q}{\delta h}\right)-J\left(\frac{\delta G}{\delta h},\frac{\delta Q}{\delta h}\right)+\frac{\delta^2 Q}{\delta h^2}J\left(\frac{\delta G}{\delta h},h\right)\right]\delta h dx dy.$$

Therefore the first variational derivative of this inner product is given as

$$\frac{\delta}{\delta h}\left\langle\frac{\delta G}{\delta h},J\left(h,\frac{\delta Q}{\delta H}\right)\right\rangle$$

$$=\frac{\delta^2 G}{\delta h^2}J\left(h,\frac{\delta Q}{\delta h}\right)+J\left(\frac{\delta Q}{\delta h},\frac{\delta G}{\delta h}\right)+\frac{\delta^2 Q}{\delta h^2}J\left(\frac{\delta G}{\delta h},h\right). \tag{2.174}$$

Similarly, the other required variational derivatives associated with the inner products in the Jacobi identity (2.163) are given by

$$\frac{\delta}{\delta h}\left\langle\frac{\delta Q}{\delta h},J\left(h,\frac{\delta F}{\delta h}\right)\right\rangle$$

$$=\frac{\delta^2 Q}{\delta h^2}J\left(h,\frac{\delta F}{\delta h}\right)+J\left(\frac{\delta F}{\delta h},\frac{\delta Q}{\delta h}\right)+\frac{\delta^2 F}{\delta h^2}J\left(\frac{\delta Q}{\delta h},h\right), \tag{2.175}$$

$$\frac{\delta}{\delta h}\left\langle\frac{\delta F}{\delta h},J\left(h,\frac{\delta G}{\delta h}\right)\right\rangle$$

$$=\frac{\delta^2 F}{\delta h^2}J\left(h,\frac{\delta G}{\delta h}\right)+J\left(\frac{\delta G}{\delta h},\frac{\delta F}{\delta h}\right)+\frac{\delta^2 G}{\delta h^2}J\left(\frac{\delta F}{\delta h},h\right). \tag{2.176}$$

It therefore follows that

$$[F,[G,Q]]+[G,[Q,F]]+[Q,[F,G]]$$

$$=-[[G,Q],F]-[[Q,F],G]-[[F,G],Q],$$

$$=-\left\langle \frac{\delta}{\delta h}\left\langle \frac{\delta G}{\delta h}, J\left(h, \frac{\delta Q}{\delta h}\right)\right\rangle, J\left(h, \frac{\delta F}{\delta h}\right)\right\rangle - \left\langle \frac{\delta}{\delta h}\left\langle \frac{\delta Q}{\delta h}, J\left(h, \frac{\delta F}{\delta h}\right)\right\rangle, J\left(h, \frac{\delta G}{\delta h}\right)\right\rangle$$

$$-\left\langle \frac{\delta}{\delta h}\left\langle \frac{\delta F}{\delta h}, J\left(h, \frac{\delta G}{\delta h}\right)\right\rangle, J\left(h, \frac{\delta Q}{\delta h}\right)\right\rangle$$

$$=\left\langle \frac{\delta^2 G}{\delta h^2} J\left(h, \frac{\delta Q}{\delta h}\right), J\left(\frac{\delta F}{\delta h}, h\right)\right\rangle - \left\langle J\left(\frac{\delta Q}{\delta h}, \frac{\delta G}{\delta h}\right), J\left(h, \frac{\delta F}{\delta h}\right)\right\rangle$$

$$-\left\langle \frac{\delta^2 Q}{\delta h^2} J\left(\frac{\delta G}{\delta h}, h\right), J\left(h, \frac{\delta F}{\delta h}\right)\right\rangle + \left\langle \frac{\delta^2 Q}{\delta h^2} J\left(h, \frac{\delta F}{\delta h}\right), J\left(\frac{\delta G}{\delta h}, h\right)\right\rangle$$

$$-\left\langle J\left(\frac{\delta F}{\delta h}, \frac{\delta Q}{\delta h}\right), J\left(h, \frac{\delta G}{\delta h}\right)\right\rangle - \left\langle \frac{\delta^2 F}{\delta h^2} J\left(\frac{\delta Q}{\delta h}, h\right), J\left(h, \frac{\delta G}{\delta h}\right)\right\rangle$$

$$+\left\langle \frac{\delta^2 F}{\delta h^2} J\left(h, \frac{\delta G}{\delta h}\right), J\left(\frac{\delta Q}{\delta h}, h\right)\right\rangle - \left\langle J\left(\frac{\delta G}{\delta h}, \frac{\delta F}{\delta h}\right), J\left(h, \frac{\delta Q}{\delta h}\right)\right\rangle$$

$$-\left\langle \frac{\delta^2 G}{\delta h^2} J\left(\frac{\delta F}{\delta h}, h\right), J\left(h, \frac{\delta Q}{\delta h}\right)\right\rangle. \tag{2.177}$$

However since

$$\left\langle \frac{\delta^2 A}{\delta h^2} J\left(h, \frac{\delta B}{\delta h}\right), J\left(\frac{\delta C}{\delta h}, h\right)\right\rangle = \iint_{\Omega} \frac{\delta^2 A}{\delta h^2} J\left(h, \frac{\delta B}{\delta h}\right), J\left(\frac{\delta C}{\delta h}, h\right) dxdy$$

$$= \left\langle \frac{\delta^2 A}{\delta h^2}, J\left(h, \frac{\delta B}{\delta h}\right) J\left(\frac{\delta C}{\delta h}, h\right)\right\rangle, \tag{2.178}$$

for any functionals A, B, C, the Jacobi identity can be rewritten as

$$[F,[G,Q]]+[G,[Q,F]]+[Q,[F,G]]$$

$$=\left\langle \frac{\delta^2 G}{\delta h^2}, J\left(h, \frac{\delta Q}{\delta h}\right) J\left(\frac{\delta F}{\delta h}, h\right)\right\rangle - \left\langle J\left(\frac{\delta Q}{\delta h}, \frac{\delta G}{\delta h}\right), J\left(h, \frac{\delta F}{\delta h}\right)\right\rangle$$

$$-\left\langle \frac{\delta^2 Q}{\delta h^2}, J\left(\frac{\delta G}{\delta h}, h\right) J\left(h, \frac{\delta F}{\delta h}\right)\right\rangle + \left\langle \frac{\delta^2 Q}{\delta h^2}, J\left(h, \frac{\delta F}{\delta h}\right) J\left(\frac{\delta G}{\delta h}, h\right)\right\rangle$$

$$-\left\langle J\left(\frac{\delta F}{\delta h}, \frac{\delta Q}{\delta h}\right), J\left(h, \frac{\delta G}{\delta h}\right)\right\rangle - \left\langle \frac{\delta^2 F}{\delta h^2}, J\left(\frac{\delta Q}{\delta h}, h\right) J\left(h, \frac{\delta G}{\delta h}\right)\right\rangle$$

$$+\left\langle \frac{\delta^2 F}{\delta h^2}, J\left(h, \frac{\delta G}{\delta h}\right) J\left(\frac{\delta Q}{\delta h}, h\right)\right\rangle - \left\langle J\left(\frac{\delta G}{\delta h}, \frac{\delta F}{\delta h}\right), J\left(h, \frac{\delta Q}{\delta h}\right)\right\rangle$$

$$-\left\langle \frac{\delta^2 G}{\delta h^2}, J\left(\frac{\delta F}{\delta h}, h\right) J\left(h, \frac{\delta Q}{\delta h}\right)\right\rangle. \tag{2.179}$$

Simplifying then gives

$$[F,[G,Q]] + [G,[Q,F]] + [Q,[F,G]]$$

$$= -\left\langle J\left(\frac{\delta Q}{\delta h}, \frac{\delta G}{\delta h}\right), J\left(h, \frac{\delta F}{\delta h}\right)\right\rangle - \left\langle J\left(\frac{\delta F}{\delta h}, \frac{\delta Q}{\delta h}\right), J\left(h, \frac{\delta G}{\delta h}\right)\right\rangle$$

$$-\left\langle J\left(\frac{\delta G}{\delta h}, \frac{\delta F}{\delta h}\right), J\left(h, \frac{\delta Q}{\delta h}\right)\right\rangle$$

$$= -\iint_\Omega \left[J\left(\frac{\delta Q}{\delta h}, \frac{\delta G}{\delta h}\right) J\left(h, \frac{\delta F}{\delta h}\right) + J\left(\frac{\delta F}{\delta h}, \frac{\delta Q}{\delta h}\right) J\left(h, \frac{\delta G}{\delta h}\right)\right] dxdy$$

$$-\iint_\Omega \left[J\left(\frac{\delta G}{\delta h}, \frac{\delta F}{\delta h}\right) J\left(h, \frac{\delta Q}{\delta h}\right)\right] dxdy = 0, \tag{2.180}$$

where the last Jacobian identity (2.124), has been used. Therefore, the Jacobi identity holds for this system and the bracket satisfies the five properties necessary for a Hamiltonian formulation. This completes the proof of the theorem. ■

The time evolution of any functional F, can be described by the following property (Olver, 1982):

$$\frac{dF}{dt} = [F, H]. \tag{2.181}$$

The proof is simple:

$$\frac{dF}{dt} = \frac{\delta F}{\delta q_i} \cdot \frac{dq_i}{dt} = \frac{\delta F}{\delta q_i} \mathcal{D}_{ij} \frac{\delta H}{\delta q_j} = \left\langle \frac{\delta F}{\delta q}, \mathcal{D} \frac{\delta H}{\delta q}\right\rangle = [F, H], \tag{2.182}$$

where the summation convention is applied. Therefore the evolution of $h(x,y,t)$ can be written using the Poisson bracket notation as

$$h_t = [h, H], \tag{2.183}$$

provided h is interpreted as the functional,

$$h(x,y,t) = \iint_{\Omega} \delta(x-x')\,\delta(y-y')\,h(x',y',t)\,dx'dy'. \tag{2.184}$$

This is easily shown as follows. The Poisson bracket is given as

$$[h, H] = \iint_{\Omega'} \frac{\delta h}{\delta h} J'\left(h, \frac{\delta H}{\delta h}\right) dx'dy', \tag{2.185}$$

where Ω' and J' are written with respect to the variables (x', y'). The first variational derivative of h calculated from equation (2.184) is given as

$$\frac{\delta h}{\delta h} = \delta(x-x')\,\delta(y-y'). \tag{2.186}$$

Using this equation (2.186) and the variational derivative of H (2.153) in the Poisson bracket (2.185) gives

$$\begin{aligned}
[h, H] &= \iint_{\Omega'} \delta(x-x')\,\delta(y-y')\,J'\left(h, \frac{1}{2}\nabla h\cdot\nabla h + h\Delta h\right) dx'dy' \\
&= J\left(h, \frac{1}{2}\nabla h\cdot\nabla h + h\Delta h\right) \\
&= h_t.
\end{aligned} \tag{2.187}$$

Every Hamiltonian system possibly possesses two types of invariants. The first type is related to underlying transformational symmetries of the system. The relationship between the conserved quantity and the invariants of the Hamiltonian system is exposed through a special case of Noether's Theorem (Courant and Hilbert, 1953):

Theorem 2 *If the Hamiltonian formulation is invariant under translations in the variable s and if the functional M satisfies*

$$J\frac{\delta M}{\delta \mathbf{q}} = -\mathbf{q}_s, \tag{2.188}$$

then M is invariant in time (Shepherd, 1990).

Since the Hamiltonian formulation given for this system has no explicit dependence on the variable x then it must be invariant under translations in x (modulo a factor of $2x_0$ to account for the periodicity in x). Therefore, if a functional M exists such that

$$J\left(h, \frac{\delta M}{\delta h}\right) = -h_x, \tag{2.189}$$

then M will be invariant in time. The functional, M, is given by

$$M = -\iint_{\Omega} yh dx dy, \tag{2.190}$$

since

$$\delta M = -\iint_{\Omega} y \delta h \, dx dy, \tag{2.191}$$

so that

$$\frac{\delta M}{\delta h} = -y, \tag{2.192}$$

and

$$J\left(h, \frac{\delta M}{\delta h}\right) = J\left(h, -y\right) = -h_x. \tag{2.193}$$

The second type of invariants are known as Casimirs and are due to the non-invertibility of the $\mathcal{D}$ matrix operator (Shepherd, 1990). Casimirs are defined as functionals that satisfy the relation

$$0 = [F, C] \equiv \left\langle \frac{\delta F}{\delta \mathbf{q}}, \mathcal{D}\frac{\delta C}{\delta \mathbf{q}} \right\rangle, \tag{2.194}$$

where F can be any functional of $\mathbf{q}$. Thus, since $\frac{\delta F}{\delta \mathbf{q}}$ is arbitrary, it follows that

$$\mathcal{D}\frac{\delta C}{\delta \mathbf{q}} = 0. \tag{2.195}$$

If the inverse of the operator $\mathcal{D}$ exists then it follows

$$\frac{\delta C}{\delta \mathbf{q}} = \mathbf{0},$$

and C would be the trivial solution. However, for this Hamiltonian formulation (Theorem 1), $\mathcal{D}^{-1}$ does not exist and thus there are nontrivial Casimirs. From the definition of a Casimir (2.195) with the matrix $\mathcal{D}$ given by equation (2.139) , the following equation results

$$J\left(h, \frac{\delta C}{\delta h}\right) = 0. \tag{2.196}$$

In order to solve for the first variation of the Casimir, this equation is expanded showing that

$$\begin{aligned} 0 = J\left(h, \frac{\delta C}{\delta h}\right) &= h_x \left(\frac{\delta C}{\delta h}\right)_y - h_y \left(\frac{\delta C}{\delta h}\right)_x \\ &= \nabla h \times \nabla \frac{\delta C}{\delta h}. \end{aligned} \tag{2.197}$$

This demonstrates that the normal surfaces to h and $\frac{\delta C}{\delta h}$ are parallel meaning that whenever h is constant, $\frac{\delta C}{\delta h}$ is constant. Therefore the first variation of the Casimir must be a function of h, or in other words,

$$\frac{\delta C}{\delta h} = f(h), \tag{2.198}$$

where $f(*)$ at this point is any function of h. Written in integral form this means that

$$\delta C = \iint_{\Omega} f(h)\, \delta h dx dy, \tag{2.199}$$

so that the Casimir can be written as

$$C(h) = \iint_{\Omega} [\Phi(h) - \Phi(0)]\, dx dy, \tag{2.200}$$

where $\Phi'(h) = f(h)$.

It is also important to note that C is an invariant of the motion since

$$\frac{dC}{dt} = [H, C] = 0, \tag{2.201}$$

on using the definition of a Casimir. Since these Casimirs are nontrivial this system is said to be noncanonical (Holm *et al.*, 1985).

2.4 Linear Stability Theorems

2.4.1 The normal mode problem

In order to examine the linear stability of the steady solution to the model (see equation (2.87)), a normal mode analysis following the derivations given by Swaters (1993) is initially used. In equations (2.90)–(2.91) it was shown that a steady parallel shear flow solution of the governing equation was given by

$$h = h_0(y), \tag{2.202}$$

with boundary condition,

$$h_0(0) = 0 \quad \text{on} \quad \phi = 0, \tag{2.203}$$

where it is assumed $a \equiv 0$. It is noted that in this section the stability analysis is restricted to parallel shear flows but that in Section 2.4.2, a linear stability analysis for a more general class of steady solutions is presented.

In order to study the linear stability of this solution, this steady state is perturbed such that the total frontal thickness and boundary are given by

$$h(x,y,t) = h_0(y) + h'(x,y,t), \tag{2.204}$$

$$\phi = \phi'(x,t), \tag{2.205}$$

where h' and ϕ' are considered small quantities. Roughly speaking, if $h'(x,y,t)$ and $\phi'(x,t)$ do not grow with time then this steady state is stable. Consequently the time evolution of $h'(x,y,t)$ and $\phi'(x,t)$ must be determined.

Substituting equation (2.204) into the governing equation (2.87) gives the following equation:

$$h_t + (h_0 + h)\, h_{xxx}(h_{0y} + h_y) + (h_0 + h)\, h_{yyx}(h_{0y} + h_y) + h_x h_{xx}(h_{0y} + h_y)$$

$$+ (h_{0y} + h_y)\, h_{yx}(h_{0y} + h_y) - (h_0 + h)\, h_{xxy} h_x - (h_{0y} + h_y)(h_{0yy} + h_{yy})\, h_x$$

$$- (h_0 + h)(h_{0yyy} + h_{yyy})\, h_x - h_x h_{xy} h_x = 0, \tag{2.206}$$

where the prime has been neglected. Since the perturbation, h, is assumed to be small, quadratic and higher order terms of h can be neglected to form the linear stability problem,

$$h_t + h_0 h_{0y} h_{xxx} + h_0 h_{0y} h_{yyx} + (h_{0y})^2 h_{yx} - h_{0y} h_{0yy} h_x - h_0 h_{0yyy} h_x = 0. \tag{2.207}$$

Equations (2.204) and (2.205) are also substituted into the boundary condition (2.88) to give

$$h_0 + h = 0 \text{ on } y = \phi(x,t), \tag{2.208}$$

where primes have again been neglected. The steady state frontal thickness and the perturbed frontal thickness can be Taylor expanded about $y = 0$ to give

$$[h_0(y), h(x,y,t)]$$

$$= [h_0(0), h(x,0,t)] + [h_{0y}(0), h_y(x,0,t)]\,\phi(x,t) + O\left(\phi^2\right), \tag{2.209}$$

so that when substituted into equation (2.208) the following relation is given:

$$h_0 + h_{0y}\phi + h + h_y\phi + O\left(\phi^2\right) = 0 \text{ on } y = 0. \tag{2.210}$$

Assuming that ϕ and h are both small, the quadratic terms of ϕ and h can be neglected to form the linearized boundary condition

$$h + h_{0y}\phi = 0 \text{ on } y = 0, \tag{2.211}$$

where $h_0(0) = 0$ has been used. Equation (2.89) gives the final boundary condition

$$|\nabla h| < \infty \text{ as } y \to \infty, \tag{2.212}$$

in the case where $L = \infty$.

Since the coefficients of the linear stability problem (2.207) and the boundary conditions (2.211) are independent of x and t, Drazin and Reid (1981) explain that this problem can be solved using a Laplace transform in t and a Fourier transform in x. The perturbed frontal thickness can then be expressed in the form

$$h(x,y,t) = \int_{-\infty}^{\infty} \int_{Br} \tilde{h}(y;k,s) \exp(st + ikx)\, ds dk, \tag{2.213}$$

where k is the real–valued along front wavenumber, s is the Laplace transform variable and where Br is the Bromwich contour.

If this equation (2.213) is substituted into the linear stability equations, a forced ordinary differential equation for $\tilde{h}(y;k,s)$ in which the forcing is proportional to the initial condition, $h(x,y,0)$, would be obtained. The coefficients of this ordinary differential equation would depend on the parameters k and s. Once the ordinary differential equation is solved for $\tilde{h}(y;k,s)$ the frontal thickness, $h(x,y,t)$, could be computed from equation (2.213). In general, of course, there would be poles associated with the singularities in $\tilde{h}(y;k,s)$. The residues associated with these poles are, in general, proportional to $\exp(s(k)t + ikx)$ where the Laplace transform variable is a function of the wavenumber, k. These relations, that is, $s = s(k)$, are in fact the eigenrelations associated with the homogeneous solutions to the ordinary differential equation for $\tilde{h}(y;k,s)$ and the accompanying homogeneous boundary conditions. This relationship is crucial in determining, as will be shown momentarily, the stability of $h_0(y)$.

The eigenrelations can in fact be determined by making the normal mode substitution,

$$h(x,y,t) = \tilde{h}(y)\exp[ik(x-ct)] + c.c., \tag{2.214}$$

where k is the real valued x-direction wavenumber, c is the complex-valued phase speed, and *c.c.* denotes complex conjugate, into the linear stability problem (2.207). Note that in the normal mode substitution the coefficient of the time term is written as $-ikc$. This is done to emphasize the wave–like interpretation of the perturbations. The same justification is used for assuming a normal mode solution for the perturbed boundary, $\phi(x,t)$, so that the substitution,

$$\phi(x,t) = \tilde{\phi}\exp(ik(x-ct)) + c.c., \tag{2.215}$$

in addition to the normal mode expansion for h, equation (2.214), is made into the linearized boundary conditions (2.211).

By examining the form of the normal mode solution (2.214), it can be seen that as long as c is strictly real then this solution will merely oscillate in time. However, if c has a positive imaginary component, then $h\,(x, y, t)$ will grow exponentially in time and thus the flow will be unstable. The stability analysis will involve, then, examining whether the eigenvalue, c, is strictly real or has a positive imaginary component for any values of the wavenumber, k and any other parameters in the problem.

This stability analysis will only determine if modes are growing exponentially with time. Drazin and Reid (1981) explain that there could also be modes growing algebraically, that is, proportional to t^{β} where $\beta > 0$. However, exponentially growing terms will ultimately dominate any algebraically growing terms and thus the behavior of the exponential terms is more important. If exponentially growing modes are not found, however, normal mode analysis will not rule out instabilities due to algebraically growing terms. These must be dealt with using alternate stability arguments (see Section 2.3.2).

The normal mode substitutions (2.214) and (2.215) then are made into the linear stability problem given by equation (2.207) and into the boundary conditions given by equations (2.211) and (2.212) resulting in the equation

$$h_0 h_{0y}\left(\tilde{h}'' - k^2\tilde{h}\right) + \left(h_{0y}\right)^2 \tilde{h}' - \left(c + h_{0y}h_{0yy} + h_0 h_{0yyy}\right)\tilde{h} = 0, \qquad (2.216)$$

with boundary conditions

$$\begin{aligned} &\tilde{h} + h_{0y}\tilde{\phi} = 0 \text{ on } y = 0, \\ &|h|\,, |h_y| < \infty \text{ as } y \to \infty. \end{aligned} \qquad (2.217)$$

In order to obtain general stability conditions from the normal mode equations it is advantageous, following Swaters (1993), to introduce the function $\tilde{F}\,(y)$ defined by

$$\tilde{h} = h_{0y}\tilde{F}(y). \qquad (2.218)$$

with its derivatives,

$$\tilde{h}' = h_{0yy}\tilde{F} + h_{0y}\tilde{F}', \tag{2.219}$$

$$\tilde{h}'' = h_{0yyy}\tilde{F} + 2h_{0yy}\tilde{F}' + h_{0y}\tilde{F}'', \tag{2.220}$$

which when substituted into equation (2.216) yields, after a little algebra

$$\left[h_0 \left(h_{0y}\right)^2 \tilde{F}'\right]' - \left(k^2 h_0 h_{0y} + c\right) h_{0y}\tilde{F} = 0. \tag{2.221}$$

The substitution (equation (2.218)) is also made into the boundary conditions (2.217) to give

$$\tilde{F} + \tilde{\phi} = 0 \text{ on } y = 0, \tag{2.222}$$

$$\left|\tilde{F}\right|, \left|h_{0yy}\tilde{F}\right|, \left|\tilde{F}'\right| < \infty \text{ as } y \to \infty. \tag{2.223}$$

For the steady solutions that will be examined in this thesis, (i.e. the wedge–like front given by $h_0(y) = \alpha y$), it follows from the differential equation (2.221) that $\tilde{F} \propto \exp(-ky)$ for $y \to \infty$. Therefore it is assumed in further derivations that $\left|\tilde{F}\right| \to 0$ as $y \to \infty$.

Further analysis is facilitated by multiplying equation (2.221) by the complex conjugate, F^*, and then integrating over the domain to give the balance

$$\int_0^\infty \left(F^* \left[h_0 \left(h_{0y}\right)^2 \tilde{F}'\right]' - \left(k^2 h_0 h_{0y} + c\right) h_{0y}\left|\tilde{F}\right|^2\right) dy = 0. \tag{2.224}$$

Integrating once by parts and using the boundary conditions, $h_0(0) = 0$ and $\tilde{F}(\infty) = 0$ produces

$$\int_0^\infty \left(h_0 \left(h_{0y}\right)^2 \left|\tilde{F}'\right|^2 + \left(k^2 h_0 h_{0y} + c\right) h_{0y}\left|\tilde{F}\right|^2\right) dy = 0. \tag{2.225}$$

Since c is complex it is of the form $c = c_R + ic_I$. As previously discussed any normal mode of the frontal thickness, $h(x, y, t)$, will be unstable if $c_I > 0$. If

$c = c_R + ic_I$ is substituted into the integral (2.225) the imaginary part is given by

$$c_I \int_0^\infty h_{0y} \left|\tilde{F}\right|^2 dy = 0. \tag{2.226}$$

Assuming instability, i.e. $c_I > 0$, this equality can only hold if

$$\int_0^\infty h_{0y} \left|\tilde{F}\right|^2 dy = 0. \tag{2.227}$$

This can be used in the real part of equation (2.225) given by

$$\int_0^\infty h_0 \left(h_{0y}\right)^2 \left|\tilde{F}'\right|^2 dy + \int_0^\infty k^2 h_0 \left(h_{0y}\right)^2 |F|^2 dy + c_R \int_0^\infty h_{0y} |F|^2 dy = 0. \tag{2.228}$$

to give the simplification

$$\int_0^\infty h_0 \left(h_{0y}\right)^2 \left(|F'|^2 + k^2 |F|^2\right) dy = 0. \tag{2.229}$$

It follows from (2.229) that $\tilde{F} \equiv 0$ (unless h_0 or h_{0y} is zero which is physically uninteresting). It therefore follows that there are only trivial unstable solutions to the linear stability equations. The above argument also applies to assuming $c_I < 0$ (i.e. asymptotically stable). Hence all steady parallel shear flows of the form $h_0(y)$ are neutrally stable (i.e. $c_I \equiv 0$).

This analysis can also give some information about the character of the normal modes. Since $c_I = 0$ it follows from (2.228) that

$$c_R \equiv \frac{-\int_0^\infty h_0 \left(h_{0y}\right)^2 \left(|F'|^2 + k^2 |F|^2\right) dy}{\int_0^\infty h_{0y} |F|^2 dy}. \tag{2.230}$$

The numerator is positive definite and so the sign of the phase speed will depend on the sign of the denominator. Therefore the phase speed will depend on the slope of the front or h_{0y}. If the slope is everywhere positive, i.e. $h_{0y} > 0$, c_R will be negative and the waves will be travelling with the frontal outcropping to their left. If the slope is everywhere negative, i.e. $h_{0y} < 0$, c_R will be positive and the waves will be travelling with the frontal outcropping to their right.

2.4.2 Formal Stability

The Hamiltonian formulation of a model can also be used to study the stability of fluid flow. Arnold (1965, 1969) made important developments in this area by using definitions of stability proposed by Liapunov. Holm *et al.* (1985) expanded these ideas into a "stability algorithm" which gave a systematic way in which the stability or instability of a system could be determined using the Hamiltonian formulation.

In the first part of this stability algorithm, Holm *et al.* (1985) outline methods in which formal stability is established. Holm *et al.* define that an equilibrium point, u_e, of a dynamical system is *formally stable* if a conserved quantity of the system is found whose first variation equals zero at this equilibrium point and whose second variation is either positive or negative definite at the equilibrium point. If such an equilibrium point is found then these derivations can be used to prove linear stability in the sense of Liapunov.

Definition 1 *The steady solution, $\varphi_s(x,y)$, is said to be* ***linearly stable in the sense of Liapunov*** *with respect to the norm $\|*\|$, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|\delta\Psi_0\| < \delta$ then $\|\delta\Psi\| < \epsilon$ for all $t \geq 0$ where $\delta\Psi_0(x,y) \equiv \delta\Psi(x,y,0)$ and where $\delta\Psi(x,y,t)$ solves the linear stability problem.*

Formal stability is more general than normal mode stability. If formal stability can be established then algebraic and normal mode instabilities are excluded. Also the normal mode stability analysis presented last section was restricted to parallel shear flows of the form, $h_0(y)$ and $\phi_0 = a$. The analysis presented here will apply to a more general class of steady solutions given by $h_0(x,y)$ and $\phi_0(x)$.

In this subsection, following the stability algorithm presented by Holm *et al.*, (1985) linear stability in the sense of Liapunov is established. Establishing linear stability involves first establishing the linear stability problem involving perturbations around the basic steady state or equilibrium point, $h_0(x,y)$. Secondly, it involves determining a conserved quantity whose first variation vanishes at this steady state, h_0. In this case this conserved quantity is composed of a linear combination of the Hamiltonian defined in equation (2.138) and a Casimir. A specific Casimir is determined so that the first variation of this conserved quantity does indeed vanish at the steady state, h_0. Finally the second variation of this conserved quantity is shown to be negative definite. Using the conserved norm established by the second variation, linear stability in the sense of Liapunov is proven.

First of all, the linear stability equation is formed by assuming that the steady state is perturbed slightly such that the total flow takes the form,

$$h(x,y,t) = h_0(x,y) + \delta h(x,y,t), \tag{2.231}$$

where $h_0(x,y)$ is a steady state solution satisfying

$$J\left(\frac{1}{2}\nabla h_0 \cdot \nabla h_0 + h_0 \Delta h_0, h_0\right) = 0, \tag{2.232}$$

and where $\delta h(x,y,t)$ is initially a small perturbation. Previously in the normal mode analysis this perturbation was denoted as $h'(x,y,t)$ (see equation (2.204)). However, this notation is used in this analysis to symbolize the variation of the frontal thickness. The outcropping takes the form

$$\phi(x,t) = \phi_0(x) + \delta\phi(x,t), \tag{2.233}$$

where $\delta\phi$ is a small perturbation. The boundary condition at $y = L$ is given by

$$h_0(x, L) = h_L, \tag{2.234}$$

so that

$$\delta h(x, L, t) = 0. \tag{2.235}$$

Also all fields are assumed smoothly periodic at $x = \pm x_0$.

The steady state equation (2.232) can be integrated using the same geometrical argument used to establish equation (2.198). Equation (2.230) states that lines of constant h_0 and lines of constant, $\frac{1}{2}\nabla h_0 \cdot \nabla h_0 + h_0 \Delta h_0$, are parallel so it follows that

$$\frac{1}{2}\nabla h_0 \cdot \nabla h_0 + h_0 \Delta h_0 = F(h_0), \tag{2.236}$$

where $F(h_0)$ is some function. Once the total flow (2.231) has been substituted into the governing equation (2.87) quadratic and higher order terms in the perturbation, δh, are dropped. This is valid since in small time intervals δh is assumed to remain small and thus quadratic and higher order terms are considered negligibly small. Therefore the general linear stability problem is given as

$$\delta h_t + J\left(h_0 \Delta \delta h + \delta h \Delta h_0 + \nabla h_0 \cdot \nabla \delta h - F'(h_0)\,\delta h, h_0\right) = 0, \tag{2.237}$$

on using equations (2.232) and (2.236) and where $F'(h_0)$ is the derivative of $F(h_0)$ with respect to its argument h_0. This equation is a generalization of the linear stability problem used in the normal mode problem (2.207) where equation (2.236) has been used.

Secondly, a conserved quantity whose first variation vanishes at the steady state must be determined. Swaters (1993) found that such a conserved quantity can be formed by constructing a constrained Hamiltonian defined as

$$\mathcal{H}(h) = H(h) + C(h), \tag{2.238}$$

where $H(h)$ is the Hamiltonian given by equation (2.138) and $C(h)$ is the Casimir defined in equation (2.200). This constrained Hamiltonian is invariant in time since both $H(h)$ and $C(h)$ are invariant in time, i.e.,

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{H}(h) &= \frac{\partial}{\partial t}(H(h) + C(h)) \\ &= \frac{\partial H}{\partial t} + \frac{\partial C}{\partial t} = 0 + 0 = 0,\end{aligned} \tag{2.239}$$

by equations (2.148) and (2.201). Explicitly, the constrained Hamiltonian is given by

$$\mathcal{H}(h) = \iint\limits_{\Omega} \left[-\frac{1}{2}h\nabla h \cdot \nabla h + \Phi(h) - \Phi(0)\right] dxdy, \tag{2.240}$$

where equations (2.138) and (2.200) have been used.

A specific Casimir must now be determined so that the first variation of the constrained Hamiltonian vanishes at the steady state solution $h_0(x, y)$. First, the first variation is calculated as

$$\begin{aligned}\delta\mathcal{H}(h) = &\iint\limits_{\Omega} \left[-\frac{1}{2}\delta h\nabla h \cdot \nabla h - h\nabla h \cdot \nabla\delta h + \Phi'(h)\,\delta h\right] dxdy \\ &- \int_{x_0}^{x_0} \left[-\frac{1}{2}h\nabla h \cdot \nabla h + \Phi(h) - \Phi(0)\right]_{y=\phi(x,t)} \delta\phi dx \\ = &\iint\limits_{\Omega} \delta h\left(-\frac{1}{2}\nabla h \cdot \nabla h + \Phi'(h)\right) dxdy - \frac{1}{2}\iint\limits_{\Omega} \nabla h^2 \cdot \nabla\delta h dxdy,\end{aligned} \tag{2.241}$$

where the boundary integral is identically zero since $h = 0$ on $y = \phi(x, t)$. It is restated that δh and $\delta\phi$ are assumed smoothly periodic at $x = \pm x_0$ and that

$\delta h = 0$ on $y = L \leq \infty$. Using Green's theorem to simplify the last integral gives

$$\delta \mathcal{H}(h) = \iint_{\Omega} \delta h \left(-\frac{1}{2}\nabla h \cdot \nabla h + \Phi'(h)\right) dxdy - \int_{\partial\Omega} h\delta h (\mathbf{n} \cdot \nabla h)\, ds$$

$$+\frac{1}{2}\iint_{\Omega} \delta h \Delta h^2 dxdy$$

$$= \iint_{\Omega} \left(-\frac{1}{2}\nabla h \cdot \nabla h + \Phi'(h) + \nabla h \cdot \nabla h + h\Delta h\right) \delta h dxdy$$

$$= \iint_{\Omega} \left(\frac{1}{2}\nabla h \cdot \nabla h + h\Delta h + \Phi'(h)\right) \delta h dxdy, \tag{2.242}$$

where the smooth periodicity of h and δh at $x = \pm x_0$ and the boundary conditions $h = 0$ on $y = \phi(x,t)$ and $\delta h = 0$ on $y = L$ have been used to eliminate the boundary integral. Substituting in the steady solution, $h = h_0(x,y)$ and $\phi = \phi_0(x)$, yields

$$\delta \mathcal{H}(h_0) = \iint_{\Omega_0} \delta h \left(\frac{1}{2}\nabla h_0 \cdot \nabla h_0 + h_0 \Delta h_0 + \Phi'(h_0)\right) dxdy, \tag{2.243}$$

where in this subsection

$$\Omega_0 \equiv \{(x,y) | \, \phi_0(x) < y < L \leq \infty, -x_0 < x < x_0\}, \tag{2.244}$$

since

$$\Omega \equiv \{(x,y) | \, \phi(x,t) < y < L \leq \infty, -x_0 < x < x_0\}. \tag{2.245}$$

The first order necessary condition for an extremum, that is, $\delta \mathcal{H}(h_0) = 0$, is satisfied if

$$\Phi'(h_0) = -\left(\frac{1}{2}\nabla h_0 \cdot \nabla h_0 + h_0 \Delta h_0\right) \tag{2.246}$$

$$= -F(h_0), \tag{2.247}$$

where F is the steady state function defined in equation (2.236). Consequently the Casimir density may be written in the form

$$\Phi(h) - \Phi(0) = -\int_0^h F(\xi)\, d\xi, \tag{2.248}$$

to ensure that $\delta \mathcal{H}(h_0) = 0$.

Finally, Swaters (1993) found the second variation of the constrained Hamiltonian $\mathcal{H}(h)$ evaluated at the steady state solution, $h_0(x,y)$, to be negative definite. The second variation of the constrained Hamiltonian, $\mathcal{H}(h)$, is given by

$$\delta^2 \mathcal{H}(h) = \iint_\Omega \delta^2 h \left(\frac{1}{2}\nabla h \cdot \nabla h + h\Delta h - F(h)\right) dxdy$$

$$+ \iint_\Omega \delta h \left(\nabla h \cdot \nabla \delta h + \delta h \Delta h + h \Delta \delta h - F'(h)\, \delta h\right) dxdy$$

$$- \int_{-x_0}^{x_0} \left[\left(\frac{1}{2}\nabla h \cdot \nabla h + h\Delta h - F(h)\right) \delta h\right]_{y=\phi(x,t)} \delta\phi dxdy. \tag{2.249}$$

Substituting in the steady state solution $h = h_0(x,y)$, gives

$$\delta^2 \mathcal{H}(h_0) = \iint_{\Omega_0} \delta h \left(\nabla h_0 \cdot \nabla \delta h + \delta h \Delta h_0 + h_0 \Delta \delta h - F'(h_0)\, \delta h\right) dxdy, \tag{2.250}$$

where equation (2.246) has been used to eliminate the first integral and the boundary integral. This integral can be rewritten, via Green's theorem, as

$$\delta^2 \mathcal{H}(h_0) = \iint_{\Omega_0} (\delta h)^2 \left(\Delta h_0 - F'(h_0)\right) + \delta h \left(\nabla h_0 \cdot \nabla \delta h + h_0 \Delta \delta h\right) dxdy$$

$$= \iint_{\Omega_0} \left((\delta h)^2 \left(\Delta h_0 - F'(h_0)\right) + \delta h \nabla h_0 \cdot \nabla \delta h - \nabla (h_0 \delta h) \cdot \nabla \delta h\right) dxdy$$

$$+ \int_{\partial\Omega_0} h_0 \delta h \left(\mathbf{n} \cdot \nabla \delta h\right) ds. \tag{2.251}$$

The last boundary integral is again zero by using the smooth periodicity of h and δh at $x = \pm x_0$, $h_0 = 0$ on $y = \phi_0(x)$ and $\delta h = 0$ on $y = L \leq \infty$. The remaining portion of equation (2.251) can be simplified to give

$$\delta^2\mathcal{H}(h_0) = \iint\limits_{\Omega_0} \left((\delta h)^2 (\Delta h_0 - F'(h_0)) + \delta h \nabla h_0 \cdot \nabla \delta h\right) dxdy$$

$$- \iint\limits_{\Omega_0} (h_0 \nabla \delta h \cdot \nabla \delta h + \delta h \nabla h_0 \cdot \nabla \delta h)\, dxdy$$

$$= \iint\limits_{\Omega_0} \left((\delta h)^2 (\Delta h_0 - F'(h_0)) - h_0 \nabla \delta h \cdot \nabla \delta h\right) dxdy. \tag{2.252}$$

It is now demonstrated that $\delta^2\mathcal{H}(h_0)$ is an invariant of the linear stability equation, that is,

$$\frac{d}{dt}\delta^2\mathcal{H}(h_0) = 0, \tag{2.253}$$

where δh evolves according to the linear stability equation (2.237). Using the second variation of the constrained Hamiltonian given in equation (2.252) its time derivative is given by

$$\frac{d}{dt}\delta^2\mathcal{H}(h_0) = \iint\limits_{\Omega_0} 2\delta h \delta h_t (\Delta h_0 - F'(h_0)) - 2h_0 \nabla \delta h \cdot \nabla \delta h_t \, dxdy$$

$$= 2\iint\limits_{\Omega_0} [\delta h_t \delta h (\Delta h_0 - F'(h_0)) - \nabla (h_0 \delta h_t) \cdot \nabla \delta h + \delta h_t \nabla h_0 \cdot \nabla \delta h]\, dxdy$$

$$= 2\iint\limits_{\Omega_0} [\delta h_t \delta h (\Delta h_0 - F'(h_0)) + h_0 \delta h_t \Delta \delta h + \delta h_t \nabla h_0 \cdot \nabla \delta h]\, dxdy$$

$$-2\int_{\partial\Omega_0} h_0 \delta h_t \mathbf{n} \cdot \nabla \delta h ds$$

$$= 2\iint\limits_{\Omega_0} \delta h_t [\delta h \Delta h_0 + h_0 \Delta \delta h + \nabla h_0 \cdot \nabla \delta h - F'(h_0)\delta h]\, dxdy$$

$$= -2\iint\limits_{\Omega_0} [J(h_0 \Delta \delta h + \delta h \Delta h_0 + \nabla h_0 \cdot \nabla \delta h - F'(h_0)\delta h, h_0)] \times$$

$$[\delta h \Delta h_0 + h_0 \Delta \delta h + \nabla h_0 \cdot \nabla \delta h - F'(h_0)\delta h]\, dxdy, \tag{2.254}$$

where Green's theorem is used and where the boundary integral is zero because of the periodicity of h_0 and δh at $x = \pm x_0$, and the boundary conditions, $h_0 = 0$ on $y = \phi_0(x)$ and $\delta h \equiv 0$ on $y = L$. Using the identities (2.120) and (2.122) followed by the divergence theorem, this can be rewritten as

$$\frac{d}{dt}\delta^2\mathcal{H}(h_0) = \iint_{\Omega_0} \left[J\left(\left[h_0\Delta\delta h + \delta h\Delta h_0 + \nabla h_0\cdot\nabla\delta h - F'(h_0)\,\delta h\right]^2, h_0\right)\right] dxdy$$

$$= \iint_{\Omega_0} \nabla\cdot\left[\left(h_0\Delta\delta h + \delta h\Delta h_0 + \nabla h_0\cdot\nabla\delta h - F'(h_0)\,\delta h\right)^2(\hat{\mathbf{e}}_3\times\nabla h_0)\right] dxdy$$

$$= \int_{\partial\Omega_0} \left[h_0\Delta\delta h + \delta h\Delta h_0 + \nabla h_0\cdot\nabla\delta h - F'(h_0)\,\delta h\right]^2(\hat{\mathbf{e}}_3\times\nabla h_0)\cdot\mathbf{n}ds = 0 \quad (2.255)$$

since $(\hat{\mathbf{e}}_3\times\nabla h_0)\cdot\mathbf{n} = 0$ on $y = \phi_0(x)$ and L and because of the periodicity of h_0 and δh at $x = \pm x_0$. This invariance will be used to establish linear stability in the sense of Liapunov.

In order to determine the sign of the second variation (2.252), it must be simplified somewhat first by eliminating the function $F'(h_0)$. This is done by taking the derivative with respect to y of equation (2.247) yielding

$$h_{0y}\Delta h_0 + h_0\Delta h_{0y} + \nabla h_{0y}\cdot\nabla h_0 = F'(h_0)\,h_{0y}, \quad (2.256)$$

and solving for $F'(h_0)$ in order to substitute into equation (2.252). Using this equation, the second variation (2.252) can be rewritten as

$$\delta^2\mathcal{H}(h_0) = -\iint_{\Omega_0} (\delta h)^2\left(\frac{h_0}{h_{0y}}\Delta h_{0y} + \frac{\nabla h_0\cdot\nabla h_{0y}}{h_{0y}}\right) + h_0\nabla\delta h\cdot\nabla\delta h\; dxdy, \quad (2.257)$$

where it is assumed that the slope of the steady state front, $h_0(y)$, is nonzero everywhere in the domain. Simplifying using Green's theorem on the first integral and then rearranging gives

$$\delta^2\mathcal{H}(h_0) = -\iint\limits_{\Omega_0} -\nabla\left(\frac{(\delta h)^2 h_0}{h_{0y}}\right)\cdot\nabla h_{0y} + \frac{(\delta h)^2 \nabla h_0\cdot\nabla h_{0y}}{h_{0y}} + h_0\nabla\delta h\cdot\nabla\delta h\,dxdy$$

$$+\int_{\partial\Omega_0}(\partial h)^2\frac{h_0}{h_{0y}}\mathbf{n}\cdot\nabla h_{0y}ds.$$

The boundary integral again vanishes because of the periodicity of h_0 and δh at $x=\pm x_0$ and the boundary conditions, $h_0=0$ on $y=\phi_0(x)$ and $\delta h=0$ on $y=L$. The remaining terms simplify as follows:

$$\delta^2\mathcal{H}(h_0) = \iint\limits_{\Omega}\left[\frac{(\delta h)^2}{h_{0y}}\nabla h_0\cdot\nabla h_{0y} - h_0\nabla\left(\frac{(\delta h)^2}{h_{0y}}\right)\cdot\nabla h_{0y}\right]dxdy$$

$$-\iint\limits_{\Omega}\left[(\delta h)^2\frac{\nabla h_0\cdot\nabla h_{0y}}{h_{0y}} + h_0\nabla\delta h\cdot\nabla\delta h\right]dxdy$$

$$=\iint\limits_{\Omega} h_0\left[-2\frac{\delta h}{h_{0y}}\nabla\delta h\cdot\nabla h_{0y} - \frac{(\delta h)^2}{(h_{0y})^2}\nabla h_{0y}\cdot\nabla h_{0y} - \nabla\delta h\cdot\nabla\delta h\right]dxdy$$

$$=-\iint\limits_{\Omega} h_0(h_{0y})^2\left[\frac{(\delta h)^2}{(h_{0y})^4}\nabla h_{0y}\cdot\nabla h_{0y} + 2\frac{\delta h}{(h_{0y})^3}\nabla\delta h\cdot\nabla h_{0y}\right]dxdy$$

$$-\iint\limits_{\Omega} h_0(h_{0y})^2\left[\frac{h_{0y}^2(\nabla\delta h\cdot\nabla\delta h)}{(h_{0y})^4}\right]dxdy$$

$$=-\iint\limits_{\Omega} h_0(h_{0y})^2\left(\frac{\delta h\nabla h_{0y} - h_{0y}\nabla\delta h}{(h_{0y})^2}\right)\left(\frac{\delta h\nabla h_{0y} - h_{0y}\nabla\delta h}{(h_{0y})^2}\right)dxdy$$

$$=-\iint\limits_{\Omega} h_0(h_{0y})^2\nabla\left(\frac{\delta h}{h_{0y}}\right)\cdot\nabla\left(\frac{\delta h}{h_{0y}}\right)dxdy. \qquad (2.258)$$

It is obvious that the second variation is negative definite since $h_0\geq 0$ with $h_0>0$ in some region. Formal stability, as defined by Holm *et al.* (1985), has thus been established.

These derivations can now be used to establish linear stability in the sense of

Liapunov since the second variation establishes a conserved norm (Swaters, 1993).

Theorem 3 *The steady state solution, $h_0(x,y)$, as defined by equation (2.232) and that satisfies the first order necessary condition for an extremum is linearly stable in the sense of Liapunov with respect to the disturbance norm given by* $\|h\| \equiv \left[-\delta^2\mathcal{H}(h_0)\right]^{\frac{1}{2}}$.

Proof.

$$\|h\|^2 \equiv \left[-\delta^2\mathcal{H}(h_0)\right]^{\frac{1}{2}} = \left[-\delta^2\mathcal{H}(h_0)\right]^{\frac{1}{2}}_{t=0} = \left\|\tilde{h}\right\|^2, \tag{2.259}$$

on using the invariance of $\delta^2\mathcal{H}(h_0)$ and where $\tilde{h}(x,y) \equiv h(x,y,t=0)$. Therefore, if $\left\|\tilde{h}\right\| < \delta = \epsilon$ then $\|h\| < \epsilon$ and linear stability in the sense of Liapunov is established. ∎

2.4.3 Linear stability of parallel shear flow

In this thesis, the class of steady state solutions known as parallel shear flows are of particular interest. Parallel shear flow solutions of this system are denoted by

$$h_0 = h_0(y), \tag{2.260}$$

$$\phi_0 = a, \tag{2.261}$$

which satisfies the governing equation trivially. The stability of this parallel shear flow is also examined by using the stability algorithm outlined by Holm *et al.* (1985) and introduced in Section 2.3.2. However in this subsection, a constrained linear momentum invariant is used instead of a constrained Hamiltonian. The first step in this algorithm involves forming the linear stability problem for the parallel shear flow. The linear stability problem is formed by assuming that the basic state, the parallel shear flow, $h_0(y)$, with outcropping, ϕ_0, is perturbed slightly such that the total flow and outcropping becomes

$$h\left(x,y,t\right)=h_0\left(y\right)+\delta h\left(x,y,t\right), \tag{2.262}$$

$$\phi\left(x,t\right)=a+\delta\phi\left(x,t\right), \tag{2.263}$$

where $h_0\left(y\right)$ is the parallel shear flow that satisfies the governing equation (2.87) identically, where a is a constant that satisfies the boundary condition (2.88) and where δh and $\delta\phi$ are small perturbations.

This total flow (2.262) is substituted into the governing equation (2.87). The perturbation terms associated with the frontal thickness and the outcropping, δh and $\delta\phi$, respectively, are assumed small enough that the nonlinear effects can be ignored. Thus quadratic and higher order terms of the perturbation can be neglected resulting in the linear equation,

$$\delta h_t+\left(h_{0y}\right)^2\delta h_{yx}+h_0h_{0y}\Delta\delta h_x-\delta h_xh_{0y}h_{0yy}-\delta h_xh_0h_{0yyy}=0, \tag{2.264}$$

which is identical to the linear stability equation (2.207) used for the normal mode problem.

The next step in the algorithm involves determining a conserved quantity of the present Hamiltonian system whose first variation vanishes at the steady state, $h_0\left(y\right)$. In contrast to the constrained Hamiltonian considered in the previous subsection, Karsten and Swaters (1995) found that the appropriate conserved quantity for this analysis should be defined as

$$\mathcal{M}\left(h\right)=M\left(h\right)+C\left(h\right), \tag{2.265}$$

where $M\left(h\right)$ is the invariant defined in equation (2.190) and $C\left(h\right)$ is the Casimir defined in equation (2.200). This functional, $\mathcal{M}\left(h\right)$, is also invariant in time as it is a linear combination of the two invariant functionals, $M\left(h\right)$ and $C\left(h\right)$ (by

Noether's Theorem (2) and equation (2.201)). The expression for $\mathcal{M}(h)$ explicitly becomes

$$\mathcal{M}(h) = \iint_{\Omega} (-yh + \Phi(h))\, dxdy, \tag{2.266}$$

by using equations (2.190) and (2.200).

A specific Casimir is now found to ensure that the first variation of this functional, $\mathcal{M}(h)$, vanishes at the steady state solution, $h_0(y)$. Karsten and Swaters (1995) showed that the first variation is given as

$$\delta\mathcal{M}(h) = \iint_{\Omega} (-y + \Phi'(h))\, \delta h \; dxdy. \tag{2.267}$$

The first variation evaluated at $h = h_0(y)$ and $\phi_0 = a$ is given by

$$\delta\mathcal{M}(h_0) = \iint_{\Omega_0} (-y + \Phi'(h_0))\, \delta h \; dxdy, \tag{2.268}$$

where in this subsection

$$\Omega_0 \equiv \{(x, y) |\, a \leq y < L \leq \infty, -x_0 < x < x_0\}. \tag{2.269}$$

It follows that $\delta\mathcal{M}(h_0) = 0$ if

$$\Phi'(h_0) = y. \tag{2.270}$$

Written in terms of the steady state this may be written in the form

$$h_0 = [\Phi']^{-1}(y), \tag{2.271}$$

where $[\Phi']^{-1}$ is the inverse function associated with Φ'.

The next step in the stability algorithm involves determining whether the second variation of this invariant (2.266) is positive definite or negative definite. The definiteness of the second variation of $\mathcal{M}(h)$ however cannot be determined without placing conditions on the Casimir function, $\Phi(h)$. The second variation is given as

$$\delta^2 \mathcal{M}(h) = \iint\limits_{\Omega} \left(-y + \Phi'(h)\right) \delta^2 h + \Phi''(h) (\delta h)^2 \, dxdy, \tag{2.272}$$

which when evaluated at the parallel shear flow solution gives

$$\delta^2 \mathcal{M}(h_0) = \iint\limits_{\Omega} \Phi''(h_0) (\delta h)^2 \, dxdy, \tag{2.273}$$

since $\Phi'(h_0) = y$.

This second variation is an invariant of the linear stability equation (2.264) which can be shown by first differentiating equation (2.270) to give

$$\Phi''(h_0) = \frac{1}{h_{0y}}. \tag{2.274}$$

If this equation (2.274) is substituted into the second variation (2.273) and the result is differentiated with respect to time, it follows that

$$\frac{d}{dt} \delta^2 \mathcal{M}(h_0) = \iint\limits_{\Omega_0} \frac{2}{h_{0y}} \delta h \delta h_t \, dxdy$$

$$= \iint\limits_{\Omega_0} \frac{2}{h_{0y}} \delta h \left(-(h_{0y})^2 \delta h_{yx} - h_0 h_{0y} \Delta \delta h_x + \delta h_x h_{0y} h_{0yy} + \delta h_x h_0 h_{0yyy} \right) dxdy$$

$$= 2 \iint\limits_{\Omega_0} \left(-h_{0y} \delta h \delta h_{yx} - h_0 \delta h \Delta \delta h_x + \frac{1}{2} \left[(\delta h)^2 \right]_x \left(h_{0yy} + \frac{h_0 h_{0yyy}}{h_{0y}} \right) \right) dxdy$$

$$= -2 \iint\limits_{\Omega} \left(h_{0y} \delta h \delta h_{yx} + h_0 \delta h \Delta \delta h_x \right) dxdy + \int_a^{\infty} \left(h_{0yy} + \frac{h_0 h_{0yyy}}{h_{0y}} \right) \delta h^2 \Big|_{-x_0}^{x_0} dy$$

$$= -2 \iint\limits_{\Omega_0} \left(h_{0y} \delta h \delta h_{yx} + h_0 \delta h \Delta \delta h_x \right) dxdy \tag{2.275}$$

where integration by parts and the smooth periodicity of $\delta h(x, y, t)$, and $h_0(y)$

on the boundaries $x = \pm x_0$ has been used. Using Green's theorem on the second term in this integral gives

$$\frac{d}{dt}\delta^2\mathcal{M}(h_0) = 2\iint\limits_{\Omega_0}\left(-h_{0_y}\delta h\delta h_{yx} + \nabla(h_0\delta h)\cdot\nabla\delta h_x\right)dxdy$$

$$-2\int_{\partial\Omega} h_0\delta h\mathbf{n}\cdot\nabla(\delta h_x)\,ds$$

$$= 2\iint\limits_{\Omega_0}\left(-h_{0_y}\delta h\delta h_{yx} + h_{0_y}\delta h\delta h_{yx} + h_0\nabla\delta h\cdot\nabla\delta h_x\right)dxdy$$

$$+2\int_a^\infty\left[h_0\delta h\delta h_{xx}\right]_{x=-x_0}dy - 2\int_a^\infty\left[h_0\delta h\delta h_{xx}\right]_{x=x_0}dy$$

$$-2\int_{-x_0}^{x_0}\left[h_0\delta h\left(\phi_x\delta h_{xx} - \delta h_{xy}\right)\right]_{y=\phi_0}dx - 2\int_{-x_0}^{x_0}\left[h_0\delta h\delta h_{xy}\right]_{y=L}dx$$

$$= \iint\limits_{\Omega_0} h_0\left(\nabla\delta h\cdot\nabla\delta h\right)_x dxdy = 0,$$

where $h_0(a) = 0$, $\delta h(x, L, t) = 0$ and the smooth periodicity of all fields at $x = \pm x_0$ has been used. Therefore the second variation of this constrained linear momentum invariant is invariant in time.

In order to establish linear stability of $h_0(y)$, the definiteness of the second variation must be established. It is easily seen from equation (2.273) that the definiteness of the second variation is dependent on the definiteness of the Casimir $\Phi''(h_0)$. In other words, the parallel shear flow $h = h_0(y)$ is stable if either

$$\inf_{\Omega_0}\ \Phi''(h_0) > 0, \tag{2.276}$$

or

$$\sup_{\Omega_C}\ \Phi''(h_0) < 0 \tag{2.277}$$

(Karsten and Swaters, 1995).

Theorem 4 *The parallel shear flow* $h = h_0(y)$ *that satisfies the first order nec-*

essary condition for an extremum is linearly stable in the sense of Liapunov with respect to the perturbation norm,

$$\|\delta h\|^2 = \iint_{\Omega} (\delta h)^2 \, dxdy, \tag{2.278}$$

if the Casimir density satisfies

$$\inf_{\Omega_0} \Phi''(h_0) > 0, \tag{2.279}$$

or

$$\sup_{\Omega_0} \Phi''(h_0) < 0. \tag{2.280}$$

Proof. Assume

$$\inf_{\Omega_0} \Phi''(h_0) > 0. \tag{2.281}$$

Using the expression for $\delta^2\mathcal{M}(h_0)$ (2.273) then gives,

$$\Gamma_1 \|\delta h\|^2 \leq \delta^2\mathcal{M}(h_0) \leq \Gamma_2 \|\delta h\|^2, \tag{2.282}$$

where

$$\Gamma_1 = \inf_{\Omega_0} \Phi''(h_0) > 0, \tag{2.283}$$

$$\Gamma_2 = \sup_{\Omega_0} \Phi''(h_0) > 0. \tag{2.284}$$

On using the invariance of $\delta^2\mathcal{M}(h_0)$ in the inequality (2.282) gives

$$\|\delta h\|^2 \leq (\Gamma_1)^{-1}\delta^2\mathcal{M}(h_0) \equiv \left[(\Gamma_1)^{-1}\delta^2\mathcal{M}(h_0)\right]_{t=0} \leq (\Gamma_1)^{-1}\Gamma_2 \left\|\delta\widetilde{h}\right\|^2, \tag{2.285}$$

where $\delta\widetilde{h}(x,y) = \delta h(x,y,0)$. Therefore, if

$$\left\|\delta\widetilde{h}\right\| < \delta = \left(\frac{\Gamma_2}{\Gamma_1}\right)^{1/2}\epsilon, \tag{2.286}$$

then

$$\|\delta h\| \le \left(\frac{\Gamma_1}{\Gamma_2}\right)^{1/2} \left\|\delta \tilde{h}\right\| < \left(\frac{\Gamma_1}{\Gamma_2}\right)^{1/2} \left(\frac{\Gamma_2}{\Gamma_1}\right)^{1/2} \epsilon = \epsilon, \tag{2.287}$$

and linear stability in the sense of Liapunov is proven. The proof for

$$\mathrm{s}\text{[illegible]}\;{}_0) < 0, \tag{2.288}$$

follows similarly. ■

2.5 A Linear Small Amplitude Solution

The steady solutions to the frontal model (2.87) were demonstrated to be linearly stable in the sense of Liapunov. However, in general, it is very difficult to analytically solve the linear stability problem for an arbitrary $h_0(x,y)$. One exact solution that can be obtained (Cushman–Roisin, 1986) is for the wedge–like front given by

$$h_0(y) = \alpha y, \tag{2.289}$$

where $\alpha > 0$ so that the thickness of the front increases in the positive y direction.

If this specific shear flow is substituted into the linear stability problem (2.207) the following equation results

$$h_t + \alpha^2 y h_{xxx} + \alpha^2 y h_{yyx} + \alpha^2 h_{yx} = 0, \tag{2.290}$$

where h is the perturbation thickness. A solution to this equation is sought where $h(x,0,t)$ is bounded, that is, where the perturbation at the unperturbed interface, $y = 0$, is a finite value and also where $h(x,y,t)$ remains bounded as $y \to \infty$.

Following the discussion given in Section 2.4.1, a normal mode solution to this equation (2.290) is constructed in the form

$$h\left(x,y,t\right)=\widetilde{h}\left(y\right)\exp\left(ik\left(x-ct\right)\right)+c.c., \tag{2.291}$$

where *c.c.* represents the complex conjugate, k is the along front wavenumber that is chosen to be positive and real and c is the phase speed whose value is dependent on k and is determined by the eigenvalue relation. Substituting this normal mode solution into the linearized equation (2.290) and simplifying yields

$$y\widetilde{h}''+\widetilde{h}'-\left(k^2y+\frac{c}{\alpha^2}\right)\widetilde{h}=0, \tag{2.292}$$

where primes denote differentiation with respect to y.

In further solving of this problem, Cushman-Roisin (1986) introduced the new variable,

$$\zeta=2ky, \tag{2.293}$$

into equation (2.292) resulting in the differential equation,

$$\zeta\widetilde{h}''+\widetilde{h}'-\left(\frac{\zeta}{4}+\frac{c}{2k\alpha^2}\right)\widetilde{h}=0, \tag{2.294}$$

where primes now denote differentiation with respect to ζ. The solution to equation (2.294) can be written in the form

$$\widetilde{h}\left(\zeta\right)=\overline{h}\left(\zeta\right)\exp\left(-\zeta/2\right). \tag{2.295}$$

Substituting this relation into equation (2.294) gives the differential equation,

$$\zeta\overline{h}''+\left(1-\zeta\right)\overline{h}'-\left(\frac{1}{2}+\frac{c}{2k\alpha^2}\right)\overline{h}=0 \tag{2.296}$$

According to differential equation theory (e.g., Nagle and Saff, 1993), $\zeta=0$ is a regular singular point of this equation (2.296). Therefore series solutions of this equation take the form

$$\overline{h}(\zeta) = \zeta^r \sum_{m=0}^{\infty} a_m \zeta^m, \tag{2.297}$$

where r is a solution of the indicial equation,

$$r(r-1) + p_0 r + q_0 = 0, \tag{2.298}$$

with

$$p_0 = \lim_{\zeta \to 0} \zeta \frac{1-\zeta}{\zeta} \text{ and } q_0 = \lim_{\zeta \to 0} \zeta^2 \frac{-\left(\frac{1}{2} + \frac{c}{2k\alpha^2}\right)}{\zeta}. \tag{2.299}$$

Taking these limits (2.299) and substituting into the indicial equation (2.298) shows that the root r satisfies the equation

$$r(r-1) + r = r^2 = 0. \tag{2.300}$$

Therefore the indicial equation (2.300) produces a double root of value $r = 0$ and two linear independent solutions will be given by the series (Nagle and Saff, 1993)

$$\overline{h}_1(\zeta) = \sum_{m=0}^{\infty} a_m \zeta^m, \tag{2.301}$$

$$\overline{h}_2(\zeta) = \sum_{m=1}^{\infty} b_m \zeta^m + \overline{h}_1(\zeta) \ln \zeta. \tag{2.302}$$

Since only solutions that are bounded at $\zeta = 0$ are physically realistic, the second solution must be discarded since $\ln \zeta \to -\infty$ as $y \to 0^+$. If the first series solution and its derivatives are substituted into the differential equation, the following recurrence relation for the coefficients holds:

$$a_m = \frac{\left(\frac{1}{2} + \frac{c}{2k\alpha^2}\right)\left(\frac{1}{2} + \frac{c}{2k\alpha^2} + 1\right)\left(\frac{1}{2} + \frac{c}{2k\alpha^2} + 2\right)\ldots\left(\frac{1}{2} + \frac{c}{2k\alpha^2} + m - 1\right)a_0}{(m!)^2}. \tag{2.303}$$

However only solutions where $\widetilde{h}(x,y,t) \to 0$ as $y \to \infty$ are being considered. This means that $\overline{h}_1(\zeta)$ must remain smaller than $\exp(\zeta/2)$ as $\zeta \to \infty$ (see equation (2.295)). However the series expansion, equation (2.301), with its coefficients given by equation (2.303) are only less than $\exp(\zeta/2)$ if the series terminates after a finite number of terms. The expression for the coefficients (2.303) shows that the power series terminates only if the following holds:

$$-\left(\frac{1}{2}+\frac{c}{2k\alpha^2}\right)=n, \tag{2.304}$$

where n is some positive integer.

Making this substitution into the differential equation (2.296) gives

$$\zeta\overline{h}''+(1-\zeta)\,\overline{h}'+n\overline{h}=0, \tag{2.305}$$

which is recognized as the Laguerre differential equation with the solution given by the Laguerre polynomials

$$\overline{h}_1(\zeta)=L_n(\zeta), \tag{2.306}$$

where

$$-\left(\frac{1}{2}+\frac{c}{2k\alpha^2}\right)=n,$$

and n is a non–negative integer.

Using the normal mode substitution (2.291) and the original variables (2.293) the complete solution for the perturbation is given as

$$h(x,y,t)=AL_n(2ky)\exp(-ky)\exp(ik(x-ct))+c.c., \tag{2.307}$$

where A is some free amplitude, $c.c.$ stands for complex conjugate and where equation (2.304) can be rearranged to show that the phase speed, c, satisfies the eigenvalue or dispersion relationship

$$c = -k\alpha^2 (2n + 1), \tag{2.308}$$

where n is a non–negative integer.

This solution describes waves that decay exponentially away from the frontal outcropping. They are dispersive waves since the phase speed, c, depends on the wavenumber, k. This relation also shows that waves with longer wavelengths propagate at slower speeds compared to waves with shorter wavelengths. In addition, since the wavenumber, k, is positive, c will be negative for all n meaning that the perturbation will propagate with the outcropping on its left. Most importantly, this is a neutrally stable solution since the phase speed, c, is real and therefore the perturbation, $h(x, y, t)$, merely oscillates in time.

In conclusion, the complete solution to the linear stability problem using the wedge profile as the basic state becomes

$$h(x, y, t) = \alpha y + AL_n(2ky)\exp(-ky)\exp(ik(x - ct)) + c.c. \tag{2.309}$$

with

$$c = -k\alpha^2 (2n + 1). \tag{2.310}$$

This solution serves as the basis for studying the nonlinear evolution of finite amplitude frontal waves in Chapter 3.

2.6 Nonlinear Stability

2.6.1 Introduction

In Section 2.4, the linear stability problem was formed by substituting the expression

$$h(x,y,t) = h_0 + h(x,y,t), \tag{2.311}$$

where h_0 was a steady state solution to the model (either a general steady solution or a parallel shear flow) and h was a small perturbation, into the governing equation (2.87). All quadratic and higher order terms of the perturbation thickness, h, were subsequently neglected. It was argued that since the perturbation thickness, h, was considered initially small that quadratic terms would be smaller yet and could justifiably be dropped. However even though the perturbations may initial[illegible] they may grow to some finite value. When this happens, linearized [illegible] no longer valid. Using this reasoning it can be argued that linearized theory may only be considered valid during some finite time interval (Eckhaus, 1965). If long time intervals are considered, nonlinear effects become important and must be considered.

The nonlinear stability problem, just as the linear stability problem, can be examined using the Hamiltonian formulation. Whereas, in finite dimensional space the definiteness of the second variation of some conserved quantity of the Hamiltonian system whose first variation vanishes at the steady state solution is sufficient to establish nonlinear stability, in infinite dimensions these conditions are not sufficient. This is a result of the topology of infinite dimensional vector space in which the definiteness of the second variation does not necessarily ensure that the norm used for nonlinear stability in the sense of Liapunov can be bounded away from zero due to the loss of compactness of the vector space (Ebin and Marsden, 1970).

The additional conditions that are required to prove nonlinear stability are referred to by Holm et al. (1985) as convexity conditions. These conditions are determined by examining invariants of the form

$$Q_1(\Delta u) + Q_2(\Delta u) \le B(u_e + \Delta u) - B(u_e) + A(u_e + \Delta u) - A(u_e), \quad (2.312)$$

where u_e is the steady state solution, A and B are conserved functionals of the Hamiltonian system that meet the first order necessary condition for an extremum and Δu is the deviation from the steady state solution. Once estimates on the magnitude of this invariant are determined thereby determining Q_1 and Q_2, a norm can be established such that

$$0 < Q_1(\Delta u) + Q_2(\Delta u) = \|\Delta u\|^2, \quad (2.313)$$

which can be used to prove nonlinear stability in the sense of Liapunov:

Definition 2 *The steady solution $\varphi_s(x, y)$ is said to be* ***nonlinearly stable in the sense of Liapunov*** *with respect to the norms $\|*\|_I$ and $\|*\|_{II}$, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\Psi_0 - \varphi_s\|_I < \delta$ then $\|\Psi - \varphi_s\|_{II} < \epsilon$ for all $t \ge 0$ where $\Psi_0(x, y) \equiv \Psi(x, y, t = 0)$ and where $\Psi(x, y, t)$ solves the nonlinear stability problem.*

2.6.2 The constrained Hamiltonian argument

There is a technical problem related to the appropriate domain to be used in the nonlinear stability problem. If the mean flow, $h_0(x, y)$, has any outcroppings associated with it, its domain of definition need not be the same as the total flow, $h_{Total}(x, y, t) = h_0(x, y) + h(x, y, t)$, where $h(x, y, t)$ is the finite amplitude perturbation due to dynamical oscillations of the outcropping. This leads to difficulties in the integration by parts that need to be done. The resolution to this difficulty is to assume that $h_0(x, y)$ and $h_{Total}(x, y, t)$ can be smoothly embedded in a fixed boundary domain denoted here as Ω. This domain is chosen so that the dynamical distortions of the perturbed outcropping are always contained in the interior of this domain and so that $h_0(x,y)$ can be smoothly extended to all of Ω.

The domain that is needed for this section then is given by

$$\Omega = \{(x,y)|\, B < \phi(x,t) < y < L \leq \infty, -x_0 < x < x_0\}, \tag{2.314}$$

in which all flow quantities are assumed to be smoothly periodic at $x = \pm x_0$ and, if there is an outcropping located at $y = \phi(x,t)$, $h_0(x,y) = h(x,y,t) = 0$ on $y = B$. If there is no outcropping then $h_0(x,y)$ extends to $y = B$ in any event and must satisfy $h_0(x,B) = h_B \equiv constant$ and $h(x,B,t) = 0$. On $y = L$ it is assumed that $h_0(x,L) = h_L \equiv constant$ and $h(x,L,t) = 0$.

In examining the nonlinear stability of the basic steady state flow, $h_0(x,y)$, Swaters (1993) introduces the functional $\mathcal{N}(h)$ given by

$$\mathcal{N}(h) \equiv H(h_0 + h) - H(h_0) + C(h_0 + h) - C(h_0), \tag{2.315}$$

where H is the Hamiltonian defined by equation (2.138) and C is the Casimir defined by equation (2.200) both of which are assumed to be defined in the domain Ω given by (2.314). The variable $h_0 = h_0(x,y)$ defines the steady solution and $h = h(x,y,t)$ represents the finite amplitude perturbation flow.

This functional, $\mathcal{N}(h)$, is invariant since $H(h_{Total})$ and $C(h_{Total})$ are invariant (see equation (2.148) and (2.201)) and since $H(h_0)$ and $C(h_0)$ are both independent of time.

Using our expressions for $H(h)$ and $C(h)$, this functional can be expressed explicitly as

$$\mathcal{N}(h) = -\frac{1}{2}\iint_{\Omega} (h_0 + h)\nabla(h_0 + h)\cdot\nabla(h_0 + h)\,dxdy$$

$$+\frac{1}{2}\iint_{\Omega} h_0 \nabla h_0 \cdot \nabla h_0 dxdy + \iint_{\Omega} \Phi\left(h_0 + h\right) dxdy - \iint_{\Omega} \Phi\left(h_0\right) dxdy \tag{2.316}$$

$$= -\frac{1}{2}\iint_{\Omega} \left(h_0 + h\right) \nabla h \cdot \nabla h dxdy$$

$$-\iint_{\Omega} h_0 \nabla h_0 \cdot \nabla h + \frac{1}{2} h \nabla h_0 \cdot \nabla h_0 + h \nabla h_0 \cdot \nabla h \; dxdy$$

$$+\iint_{\Omega} \Phi\left(h_0 + h\right) dxdy - \iint_{\Omega} \Phi\left(h_0\right) dxdy \tag{2.317}$$

$$= \frac{1}{2}\iint_{\Omega} h \nabla h_0 \cdot \nabla h_0 - \left(h_0 + h\right) \nabla h \cdot \nabla h \; dxdy$$

$$-\iint_{\Omega} \nabla\left(h_0 h\right) \cdot \nabla h_0 + \frac{1}{2} \nabla h^2 \cdot \nabla h_0 \; dxdy$$

$$+\iint_{\Omega} \Phi\left(h_0 + h\right) dxdy - \iint_{\Omega} \Phi\left(h_0\right) dxdy \tag{2.318}$$

$$= \frac{1}{2}\iint_{\Omega} h \nabla h_0 \cdot \nabla h_0 - \left(h_0 + h\right) \nabla h \cdot \nabla h \; dxdy$$

$$+\int_{\partial\Omega} h_0 h \left(\mathbf{n} \cdot \nabla h_0\right) ds + \iint_{\Omega} h_0 h \Delta h_0 + \frac{1}{2} h^2 \Delta h_0 \; dxdy$$

$$-\frac{1}{2}\int_{\partial\Omega} h^2 \left(\mathbf{n} \cdot \nabla h_0\right) ds$$

$$+\iint_{\Omega} \Phi\left(h_0 + h\right) dxdy - \iint_{\Omega} \Phi\left(h_0\right) dxdy$$

$$= \frac{1}{2}\iint_{\Omega} h \nabla h_0 \cdot \nabla h_0 - \left(h_0 + h\right) \nabla h \cdot \nabla h \; dxdy$$

$$+\iint_{\Omega} h_0 h \Delta h_0 + \frac{1}{2} h^2 \Delta h_0 \; dxdy$$

$$+\iint_{\Omega} \Phi\left(h_0 + h\right) dxdy - \iint_{\Omega} \Phi\left(h_0\right) dxdy, \tag{2.319}$$

where the boundary integrals,

$$\int_{\partial\Omega} h_0 h\,(\mathbf{n}\cdot\nabla h_0)\,ds \text{ and } \int_{\partial\Omega} h^2\,(\mathbf{n}\cdot\nabla h_0)\,ds,$$

vanish due to the periodicity of h and h_0 on $x = \pm x_0$ and the boundary conditions $h = 0$ on $y = B$ and L, respectively. Using equations (2.236) and the definition of the Casimir (2.248), Φ, to simplify gives

$$\mathcal{N}(h) = \frac{1}{2}\iint_{\Omega} h^2\Delta h_0 - (h+h_0)\,\nabla h\cdot\nabla h\;dxdy$$

$$-\iint_{\Omega}\left(\int_{h_0}^{h_0+h} F(\xi)\,d\xi - hF(h_0)\right)dxdy. \tag{2.320}$$

The second step involved in showing nonlinear stability is to determine appropriate conditions to ensure that $\mathcal{N}(h)$ is definite. It is possible to show that $\mathcal{N}(h)$ is negative definite but not without certain conditions (Swaters, 1993).

Theorem 5 *If the Casimir density satisfies*

$$\sup_{\Omega}(\Delta h_0) < \alpha < F'(\xi) < \beta < \infty, \tag{2.321}$$

then the functional $\mathcal{N}(h)$ is negative definite for all $h = h(x,y,t)$.

Proof. Suppose

$$\alpha < F'(\xi) < \beta. \tag{2.322}$$

Integrating this expression once with respect to ξ from h_0 to z gives

$$\alpha(z-h_0) < F(z) - F(h_0) < \beta(z-h_0). \tag{2.323}$$

Integrating once more with respect to the variable z and from h_0 to $h_0 + h$ gives

$$\frac{\alpha}{2}h^2 < \int_{h_0}^{h_0+h} F(\xi)\, d\xi - hF(h_0) < \frac{\beta}{2}h^2. \tag{2.324}$$

Substituting this into the expression for $\mathcal{N}(h)$ (2.320) gives the inequality,

$$\iint_\Omega \frac{1}{2}h^2(\Delta h_0 - \beta) - \frac{1}{2}(h + h_0)\nabla h \cdot \nabla h\, dxdy < \mathcal{N}(h)$$

$$< \iint_\Omega \frac{1}{2}h^2(\Delta h_0 - \alpha) - \frac{1}{2}(h + h_0)\nabla h \cdot \nabla h\, dxdy$$

$$< \iint_\Omega \frac{1}{2}h^2(\Delta h_0 - \alpha)\; dxdy, \tag{2.325}$$

since the total thickness of the front must be positive, i.e. $h_T = (h + h_0) > 0$. By supposition $\sup_\Omega(\Delta h_0) < \alpha$, so $\Delta h_0 - \alpha < 0$ and the functional satisfies

$$\mathcal{N}(h) < \iint_\Omega \frac{1}{2}h^2(\Delta h_0 - \alpha)\; dxdy < 0. \tag{2.326}$$

Therefore the functional $\mathcal{N}(h)$ is negative definite. ■

Now that the definiteness of $\mathcal{N}(h)$ has been established, nonlinear stability in the sense of Liapunov can be established. The same conditions to prove the definiteness of $\mathcal{N}(h)$ however need to be assumed.

Theorem 6 *The steady solution $h_0 = h_0(x,y)$ is nonlinearly stable in the sense of Liapunov with respect to the perturbation norms,*

$$\|h\|_I^2 = \iint_\Omega h^2 dxdy, \tag{2.327}$$

and

$$\|h\|_{II}^2 = \iint_\Omega \left(h^2 + (h + h_0)\nabla h \cdot \nabla h\right) dxdy, \tag{2.328}$$

if the Casimir density

$$\Phi(h) - \Phi(0) \equiv -\int_0^h F(\xi)\, d\xi, \tag{2.329}$$

satisfies

$$\sup_{\Omega}(\Delta h_0) < \alpha < F'(\xi) < \beta < \infty. \tag{2.330}$$

Proof. Suppose that the Casimir density meets the conditions stated above. As was shown in equation (2.326) when the Casimir meets these conditions, the functional $\mathcal{N}(h)$ satisfies the inequality

$$\mathcal{N}(h) < \iint_{\Omega} \frac{1}{2} h^2 (\Delta h_0 - \alpha)\ dxdy < 0. \tag{2.331}$$

Therefore if the constant Γ is introduced as

$$\Gamma = \frac{1}{2} \sup_{\Omega} (\Delta h_0 - \alpha) < 0, \tag{2.332}$$

then the functional satisfies

$$\mathcal{N}(h) < \Gamma \iint_{\Omega} h^2\ dxdy \text{ or } \iint_{\Omega} h^2\ dxdy < \Gamma^{-1}\mathcal{N}(h). \tag{2.333}$$

The perturbation norm becomes

$$\|h\|_I^2 = \iint_{\Omega} h^2 dxdy < \Gamma_1^{-1}\mathcal{N}(h) \equiv \Gamma_1^{-1}\mathcal{N}\left(\tilde{h}\right), \tag{2.334}$$

on using the invariance of the functional, $\mathcal{N}(h)$, and where $\tilde{h}(x,y) \equiv h(x,y,t=0)$. Using the inequality found in equation (2.325), this can now be written as

$$\|h\|_I^2 < \Gamma_1^{-1} \iint_{\Omega} \frac{1}{2}\tilde{h}^2 (\Delta h_0 - \beta) - \frac{1}{2}\left(\tilde{h} + h_0\right) \nabla\tilde{h} \cdot \nabla\tilde{h}\ dxdy$$

$$< \Gamma_1^{-1}\Gamma_2 \iint\limits_{\Omega} \left(\tilde{h}^2 + \left(\tilde{h} + h_0\right) \nabla\tilde{h} \cdot \nabla\tilde{h}\right) \, dxdy, \tag{2.335}$$

where

$$\Gamma_2 = \min\left\{\inf \frac{1}{2}\left(\Delta h_0 - \beta\right), -1\right\} < 0. \tag{2.336}$$

Therefore

$$\|h\|_I^2 < \Gamma_1^{-1}\Gamma_2 \iint\limits_{\Omega} \left(\tilde{h}^2 + \left(\tilde{h} + h_0\right) \nabla\tilde{h} \cdot \nabla\tilde{h}\right) \, dxdy = \Gamma_1^{-1}\Gamma_2 \left\|\tilde{h}\right\|_{II}^2 . \tag{2.337}$$

Liapunov stability can be proven by choosing

$$\delta = \left(\Gamma_1 \Gamma_2^{-1}\right)^{1/2} \epsilon, \tag{2.338}$$

which would give

$$\|h\|_I < \left(\Gamma_1^{-1}\Gamma_2\right)^{1/2} \left\|\tilde{h}\right\|_{II} < \left(\Gamma_1^{-1}\Gamma_2\right)^{1/2} \left(\Gamma_1\Gamma_2^{-1}\right)^{1/2} \epsilon = \epsilon, \tag{2.339}$$

and nonlinear stability in the sense of Liapunov is established. ∎

2.6.3 Nonlinear stability of parallel shear flow

The functional introduced here to examine the nonlinear stability problem for the parallel shear flow is composed of the invariants introduced to study the linear stability of parallel shear flows. Karsten and Swaters (1995) show that this functional takes the form.

$$\mathcal{L}(h) = M(h + h_0) - M(h_0) + C(h + h_0) - C(h_0), \tag{2.340}$$

where M is the defined by equation (2.190) and C is the related Casimir given by equation (2.200). The variables $h_0(y)$ and $h(x, y, t)$ are the steady flow solution and the finite amplitude perturbation flow, respectively. The functional $\mathcal{L}(h)$ is a conserved functional since $M(h_{Total})$ and $C(h_{Total})$ are invariant (by Noether's Theorem (2) and equation (2.201)) and $M(h_0)$ and $C(h_0)$ are both independent of time. This invariant can be written explicitly as

$$\mathcal{L}(h) = \iint_{\Omega} -y(h+h_0) + yh_0 + \Phi(h+h_0) - \Phi(h_0)\ dxdy$$

$$= \iint_{\Omega} -\Phi'(h_0)h + \Phi(h+h_0) - \Phi(h_0)\ dxdy, \qquad (2.341)$$

where the first order necessary condition for an extremum, equation (2.270) has been used to simplify.

Here, as in the stability argument developed using the constrained Hamiltonian, the working domain Ω is assumed to be a periodic channel in which the steady state flow $h_0(y)$ is extended to the boundaries of Ω. However, since the following derivations do not require any assumptions on the differentiability of $h_0(y)$, it is sufficient to assume that $h_0(y)$ is only continuously extended to all of Ω. In what follows the domain Ω is assumed to be given by

$$\Omega = \{(x,y)|\, B < \phi(x,t) < y < L \leq \infty, -x_0 < x < x_0\}. \qquad (2.342)$$

It is possible to show that $\mathcal{L}(h)$ is negative definite but not without certain conditions (Karsten and Swaters, 1995).

Theorem 7 *If the Casimir density* $\Phi(h)$ *satisfies*

$$-\infty < \alpha_1 < \Phi''(\xi) < \beta_1 < 0, \qquad (2.343)$$

or

$$0 < \alpha_2 < \Phi''(\xi) < \beta_2 < \infty, \qquad (2.344)$$

where α_1, α_2, β_1, β_2 *are real constants, then the functional* $\mathcal{L}(h)$ *will be positive* *or negative definite, respectively, for all* $h = h(x, y, t)$.

Proof. Suppose

$$\alpha_1 < \Phi''(\xi) < \beta_1 < 0. \quad (2.345)$$

Integrating this expression with respect to ξ from h_0 to z gives

$$\alpha_1 (z - h_0) < \Phi'(z) - \Phi'(h_0) < \beta_1 (z - h_0). \quad (2.346)$$

Integrating once more with respect to z from h_0 to $h_0 + h$ gives

$$\frac{\alpha_1}{2} h^2 < \Phi(h_0 + h) - \Phi(h_0) - h\Phi'(h_0) < \frac{\beta_1}{2} h^2. \quad (2.347)$$

Substituting this equation (2.347) into the expression for $\mathcal{L}(h)$ (2.341) results in

$$\iint_\Omega \frac{\alpha_1}{2} h^2 \, dxdy < \mathcal{L}(h) < \iint_\Omega \frac{\beta_1}{2} h^2 \, dxdy < 0. \quad (2.348)$$

Therefore $\mathcal{L}(h)$ is negative definite for all $h = h(x, y, t)$. The argument for positive definiteness follows similarly. ■

The conditions for the definiteness of the functional can now be used to establish nonlinear stability in the sense of Liapunov:

Theorem 8 *The parallel shear flow* $h_0 = h_0(y)$ *is nonlinearly stable in the sense of Liapunov with respect to the perturbation norm*

$$\|h\|^2 = \iint_\Omega h^2 dxdy, \quad (2.349)$$

if the Casimir density satisfies either

$$-\infty < \alpha_1 < \Phi''(\xi) < \beta_1 < 0, \quad (2.350)$$

or

$$0 < \alpha_2 < \Phi''(\xi) < \beta_2 < \infty, \tag{2.351}$$

where α_1, α_2, β_1, β_2 *are real constants (Karsten and Swaters, 1995).*

Proof. Suppose

$$-\infty < \alpha_1 < \Phi''(\xi) < \beta_1 < 0, \tag{2.352}$$

so that from equation (2.348),

$$\mathcal{L}(h) < \iint\limits_{\Omega} \frac{\beta_1}{2} h^2 \, dxdy \text{ or } \iint\limits_{\Omega} h^2 \, dxdy < \frac{2}{\beta_1} \mathcal{L}(h). \tag{2.353}$$

The perturbation norm then satisfies

$$\|h\|^2 = \iint\limits_{\Omega} h^2 \, dxdy < \frac{2}{\beta_1} \mathcal{L}(h) = \frac{2}{\beta_1} \mathcal{L}\left(\tilde{h}\right), \tag{2.354}$$

on using the invariance of the functional $\mathcal{L}(h)$ and where $\tilde{h}(x,y) = h(x,y,t=0)$. By using the second inequality for $\mathcal{L}(h)$ given in equation (2.348), this becomes

$$\|h\|^2 < \frac{2}{\beta_1} \mathcal{L}\left(\tilde{h}\right) < \frac{2}{\beta_1} \frac{\alpha_1}{2} \iint\limits_{\Omega} \tilde{h}^2 \, dxdy = \frac{\alpha_1}{\beta_1} \left\|\tilde{h}\right\|^2. \tag{2.355}$$

Therefore if

$$\left\|\tilde{h}\right\| < \delta = \left(\frac{\beta_1}{\alpha_1}\right)^{1/2} \epsilon, \tag{2.356}$$

then

$$\|h\| < \left(\frac{\alpha_1}{\beta_1}\right)^{1/2} \left\|\tilde{h}\right\| < \left(\frac{\alpha_1}{\beta_1}\right)^{1/2} \left(\frac{\beta_1}{\alpha_1}\right)^{1/2} \epsilon = \epsilon, \tag{2.357}$$

and nonlinear stability in the sense of Liapunov is proven. The proof assuming

$$0 < \alpha_2 < \Phi''(\xi) < \beta_2 < \infty, \tag{2.358}$$

follows similarly. ■

2.7 The wedge profile

In Theorem 8, it was shown that if the Casimir density satisfied

$$0 < \alpha_2 < \Phi''(\xi) < \beta_2 < \infty,$$

then the steady solution $h_0(y)$ was nonlinearly stable. In Section 2.5, the wedge profile

$$h_0 = \alpha y,$$

was considered and a linearly stable solution was found. If this wedge profile is used in Theorem 8, it can be shown that this steady solution is nonlinearly stable.

Theorem 9 *The wedge profile*

$$h_0(y) = \alpha y, \tag{2.359}$$

where $\alpha > 0$ is nonlinearly stable in the sense of Liapunov.

Proof. The Casimir density as defined in equation (2.274) is given by

$$\Phi''(h_0) = \frac{1}{h_{0y}}.$$

For the wedge profile this becomes

$$\Phi''(h_0) = \frac{1}{\alpha}. \tag{2.360}$$

Since

$$\alpha_2 < \frac{1}{\alpha} < \beta_2, \tag{2.361}$$

for some positive α_2 and β_2, then by Theorem 8, the wedge profile is nonlinearly stable in the sense of Liapunov. ■

Chapter 3

Weakly Nonlinear Evolution of Wedge-Front Modes

3.1 Introduction

In chapter 2, the stability characteristics of steady solutions of the frontal model (2.87) were determined. By using the Hamiltonian formulation of the model (Theorem 1) it was found that all steady state solutions were linearly stable and that certain steady state solutions meeting certain conditions (see Theorem 6 and Theorem 8) were also nonlinearly stable. In particular, it was found that if the basic state was the wedge front given by

$$h_0(y) = \alpha y, \tag{3.1}$$

then the flow was nonlinearly stable.

In Section 2.4, the wedge front was assumed as the basic state and a solution was found for the linear stability problem. However, if the wedge front is assumed as basic state for the nonlinear stability problem (see equation (2.206)) the problem is not analytically solvable. In order to examine the nonlinear evolution of this solution a multiple scales analysis for the perturbation thickness, $h(x, y, t)$, and interface, $\phi(x, y, t)$, around a small amplitude parameter ϵ is developed. The problems up to the $O(\epsilon^2)$ are solved and the free amplitude, A, introduced in equation (2.307) is found to satisfy the Nonlinear Schrödinger equation (Newell, 1985)

$$A_\tau = \frac{i\omega''}{2} A_{\xi\xi} + i\beta |A|^2 A, \tag{3.2}$$

where τ and ξ are slow time and space variables introduced in the multiple scales analysis.

The Nonlinear Schrödinger (NLS) equation appropriately describes the nonlinear dynamics of this system for two reasons. As Newell (1985) explains, the NLS equation describes dynamics that are weakly nonlinear but strongly dispersive. The linear stability problem whose solution describes the fundamental mode of the nonlinear stability problem yielded the dispersion relation,

$$c = -k\alpha^2 (2n + 1), \tag{3.3}$$

where c is the phase speed, k is the wavenumber and n is a non–negative integer. Since c is a function of k, waves of different wavenumbers will travel at different velocities or in other words these waves will disperse. In addition the group velocity which is the velocity of energy propagation is defined by

$$c_g = \frac{\partial \omega}{\partial k} = -2k\alpha^2 (2n + 1), \tag{3.4}$$

showing that the energy of the wave propagates at twice the phase velocity as the wave implying that there is a group amplitude different from the phase amplitude.

The wedge front is nonlinearly stable. Its nonlinear evolution, however, is not analytically solvable. Through a multiple scales analysis and an asymptotic expansion in the frontal thickness and interface, the nonlinear stability problem is divided into equations of different orders with respect to the ordering parameter ϵ which can be solved. Thus the finite amplitude evolution of the wedge front is determined.

3.2 The nonlinear problem

The complete nonlinear problem is derived by beginning with the governing equation,

$$\frac{\partial h}{\partial t} + J\left(h\Delta h + \frac{1}{2}\nabla h \cdot \nabla h, h\right) = 0, \tag{3.5}$$

and substituting in

$$h(x,y,t) = h_0(y) + \overline{h}(x,y,t), \tag{3.6}$$

where $h_0(y)$ is a general parallel shear flow and $\overline{h}(x,y,t)$ is a small amplitude perturbation, yielding

$$\begin{aligned} &h_t + h_0 h_{0y} h_{xxx} + h_0 h_y h_{xxx} + h h_{0y} h_{xxx} + h h_y h_{xxx} + h_0 h_{0y} h_{yyx} + h_0 h_y h_{yyx} \\ &+ h h_{0y} h_{yyx} + h h_y h_{yyx} + h_x h_{xx} h_{0y} + h_x h_{xx} h_y + (h_{0y})^2 h_{yx} + 2 h_{0y} h_y h_{yx} \\ &+ (h_y)^2 h_{yx} - h_0 h_{xxy} h_x - h h_{xxy} h_x - h_{0y} h_{0yy} h_x - h_{0y} h_{yy} h_x - h_y h_{0yy} h_x \\ &- h_y h_{yy} h_x - h_0 h_{0yyy} h_x - h_0 h_{yyy} h_x - h h_{0yyy} h_x - h h_{yyy} h_x - h_x h_{xy} h_x = 0, \end{aligned} \tag{3.7}$$

where the overbar has been omitted and where subscripts denote partial differentiation.

Nonlinear boundary conditions are also derived in the same manner by inserting the perturbed interface,

$$\phi(x,t) = a + \overline{\phi}(x,t) = \overline{\phi}(x,t), \tag{3.8}$$

where a is set equal to zero, along with the total flow (equation (3.6)) into the boundary condition (2.88) to give the nonlinear condition,

$$h_0 + h = 0, \text{ on } y = \phi(x,t), \tag{3.9}$$

where the overbar again has been omitted.

Initially the perturbation can be considered to be small, that is, of $O(\epsilon)$ where ϵ is introduced here as a small parameter and so the perturbation flow, $h(x,y,t)$, is rewritten as

$$h(x,y,t) = \epsilon \tilde{h}(x,y,t), \tag{3.10}$$

where $\tilde{h}(x,y,t) \simeq O(1)$ initially. Substituting (3.10) into the nonlinear problem (3.7) gives

$$\begin{aligned}
&h_t + h_0 h_{0y} h_{xxx} + \epsilon h_0 h_y h_{xxx} + \epsilon h h_{0y} h_{xxx} + \epsilon^2 h h_y h_{xxx} \\
&+ h_0 h_{0y} h_{yyx} + \epsilon h_0 h_y h_{yyx} + \epsilon h h_{0y} h_{yyx} + \epsilon^2 h h_y h_{yyx} \\
&+ \epsilon h_x h_{xx} h_{0y} + \epsilon^2 h_x h_{xx} h_y + (h_{0y})^2 h_{yx} + 2\epsilon h_{0y} h_y h_{yx} \\
&+ \epsilon^2 (h_y)^2 h_{yx} - \epsilon h_0 h_{xxy} h_x - \epsilon^2 h h_{xxy} h_x - h_{0y} h_{0yy} h_x \\
&- \epsilon h_{0y} h_{yy} h_x - \epsilon h_y h_{0yy} h_x - \epsilon^2 h_y h_{yy} h_x - h_0 h_{0yyy} h_x \\
&- \epsilon h_0 h_{yyy} h_x - \epsilon h h_{0yyy} h_x - \epsilon^2 h h_{yyy} h_x - \epsilon^2 h_x h_{xy} h_x = 0,
\end{aligned} \tag{3.11}$$

where the tildes have been omitted and the equation (3.11) has been divided through by ϵ. Rearranging, equation (3.11) can be written as

$$\begin{aligned}
&h_t + h_0 h_{0y} h_{xxx} + h_0 h_{0y} h_{yyx} + (h_{0y})^2 h_{yx} - h_{0y} h_{0yy} h_x - h_0 h_{0yyy} h_x = \\
&- \epsilon h_0 h_y h_{xxx} - \epsilon h h_{0y} h_{xxx} - \epsilon h_0 h_y h_{yyx} - \epsilon h h_{0y} h_{yyx} \\
&- \epsilon h_x h_{xx} h_{0y} - 2\epsilon h_{0y} h_y h_{yx} + \epsilon h_0 h_{xxy} h_x + \epsilon h_{0y} h_{yy} h_x
\end{aligned}$$

$$+\epsilon h_y h_{0yy} h_x + \epsilon h_0 h_{yyy} h_x + \epsilon h h_{0yyy} h_x - \epsilon^2 h h_y h_{xxx}$$

$$-\epsilon^2 h h_y h_{yyx} - \epsilon^2 h_x h_{xx} h_y - \epsilon^2 (h_y)^2 h_{yx} + \epsilon^2 h h_{xxy} h_x$$

$$+\epsilon^2 h_y h_{yy} h_x + \epsilon^2 h h_{yyy} h_x + \epsilon^2 h_x h_{xy} h_x. \tag{3.12}$$

This equation (3.12) shows that the nonlinear effects are all $O(\epsilon)$ or $O(\epsilon^2)$ compared to the linear terms. In other words, the substitution $h(x,y,t) = \epsilon\widetilde{h}(x,y,t)$ suggests that the nonlinear terms make slight but cumulative modifications on the linear underlying motion, that is, this motion is weakly nonlinear.

Since these nonlinear changes are of $O(\epsilon)$ or $O(\epsilon^2)$, they make noticeable changes only over long space and time periods. To emphasize these periods, Newell (1985) suggests introducing the auxiliary slow variables,

$$T = \epsilon t, \qquad X = \epsilon x, \tag{3.13}$$

into the nonlinear problem. Since an asymptotic expansion up to the order of ϵ^2 will be carried out, the additional slow time scale,

$$\tau = \epsilon^2 t, \tag{3.14}$$

is also introduced.

The various partial derivatives using these variables are given as follows:

$$\partial_t = \partial_t + \epsilon\partial_T + \epsilon^2\partial_\tau, \tag{3.15}$$

$$\partial_x = \partial_x + \epsilon\partial_X, \qquad \partial_{xx} = \partial_{xx} + 2\epsilon\partial_{xX} + \epsilon^2\partial_{XX}, \tag{3.16}$$

$$\partial_{xxx} = \partial_{xxx} + 3\epsilon\partial_{xxX} + 3\epsilon^2\partial_{xXX} + O\left(\epsilon^3\right), \tag{3.17}$$

$$\partial_{yx} = \partial_{yx} + \epsilon\partial_{yX}, \qquad \partial_{yyx} = \partial_{yyx} + \epsilon\partial_{yyX}, \tag{3.18}$$

$$\partial_{xxy} = \partial_{xxy} + 2\epsilon\partial_{xXy} + \epsilon^2\partial_{XXy}, \tag{3.19}$$

so that the nonlinear problem (3.11) becomes

$$h_t + \epsilon h_T + \epsilon^2 h_\tau + h_0 h_{0y}\left(h_{xxx} + 3\epsilon h_{xxX} + 3\epsilon^2 h_{xXX}\right)$$

$$+\epsilon h_0 h_y\left(h_{xxx} + 3\epsilon h_{xxX} + 3\epsilon^2 h_{xXX}\right) + \epsilon h_{0y} h\left(h_{xxx} + 3\epsilon h_{xxX} + 3\epsilon^2 h_{xXX}\right)$$

$$+\epsilon^2 h h_y\left(h_{xxx} + 3\epsilon h_{xxX} + 3\epsilon^2 h_{xXX}\right) + h_0 h_{0y}\left(h_{yyx} + \epsilon h_{yyX}\right)$$

$$+\epsilon h_0 h_y\left(h_{yyx} + \epsilon h_{yyX}\right) + \epsilon h_{0y} h\left(h_{yyx} + \epsilon h_{yyX}\right)$$

$$+\epsilon^2 h h_y\left(h_{yyx} + \epsilon h_{yyX}\right) + \epsilon h_{0y}\left(h_x + \epsilon h_X\right)\left(h_{xx} + 2\epsilon h_{xX} + \epsilon^2 h_{XX}\right)$$

$$+\epsilon^2 h_y\left(h_x + \epsilon h_X\right)\left(h_{xx} + 2\epsilon h_{xX} + \epsilon^2 h_{XX}\right) + \left(h_{0y}\right)^2\left(h_{yx} + \epsilon h_{yX}\right)$$

$$+2\epsilon h_{0y} h_y\left(h_{yx} + \epsilon h_{yX}\right) + \epsilon^2\left(h_y\right)^2\left(h_{yx} + \epsilon h_{yX}\right)$$

$$-\epsilon h_0\left(h_{xxy} + 2\epsilon h_{xXy} + \epsilon^2 h_{XXy}\right)\left(h_x + \epsilon h_X\right)$$

$$-\epsilon^2 h\left(h_{xxy} + 2\epsilon h_{xXy} + \epsilon^2 h_{XXy}\right)\left(h_x + \epsilon h_X\right) - h_{0y} h_{0yy}\left(h_x + \epsilon h_X\right)$$

$$-\epsilon h_{0y} h_{yy}\left(h_x + \epsilon h_X\right) - \epsilon h_{0yy} h_y\left(h_x + \epsilon h_X\right) - \epsilon^2 h_y h_{yy}\left(h_x + \epsilon h_X\right)$$

$$-h_0 h_{0yyy}\left(h_x + \epsilon h_X\right) - \epsilon h_0 h_{yyy}\left(h_x + \epsilon h_X\right) - \epsilon h_{0yyy} h\left(h_x + \epsilon h_X\right)$$

$$-\epsilon^2 h h_{yyy}\left(h_x + \epsilon h_X\right) - \epsilon^2\left(h_x + \epsilon h_X\right)^2\left(h_{xy} + \epsilon h_{Xy}\right) + O\left(\epsilon^3\right) = 0. \tag{3.20}$$

The corresponding boundary conditions are formed in the same manner. Since initially the perturbed interface is considered small, it is rewritten as

$$\phi\left(x,t\right) = \epsilon\tilde{\phi}\left(x,t\right). \tag{3.21}$$

Substituting this equation (3.21) along with equation (3.10) into the nonlinear boundary equation (3.9) and dividing through by ϵ gives the boundary conditions

$$\frac{1}{\epsilon}h_0 + h = 0, \text{ on } y = \epsilon\phi\left(x,t\right), \tag{3.22}$$

where the tildes have been omitted. The slow space and time variables given by equations (3.13) and (3.14) are introduced into this boundary condition (3.22) to give

$$\frac{1}{\epsilon}h_0 + h = 0, \text{ on } y = \epsilon\phi(x,t). \tag{3.23}$$

An additional simplification is made here by using Taylor expansion. The frontal thickness evaluated on $y = \epsilon\phi(x,t)$ can be Taylor expanded about $y = 0$ to give

$$(h_0(\epsilon\phi), h(x,\epsilon\phi,t)) \simeq (h_0(0), h(x,0,t)) + (h_0(0), h(x,0,t))_y \epsilon\phi$$

$$+ (h_0(0), h(x,0,t))_{yy} (\epsilon\phi)^2 + O\left(\epsilon^3\right). \tag{3.24}$$

Substituting this Taylor expansion (3.24) into the boundary condition (3.22) gives the boundary condition:

$$h_{0y}\phi + \epsilon h_{0yy}\phi^2 + \epsilon^2 h_{0yyy}\phi^3 + h + \epsilon h_y\phi + \epsilon^2 h_{yy}\phi^2 + O\left(\epsilon^3\right) = 0 \text{ on } y = 0, \tag{3.25}$$

where the boundary condition $h_0(0) = 0$ has been used.

Finally the perturbation flow, $h(x,y,t)$, and interface, $\phi(x,t)$, are asymptotically expanded in ϵ so that

$$h = \sum_{n=0}^{\infty} \epsilon^n h^{(n)}(x,y,t;X,T,\tau), \tag{3.26}$$

$$\phi = \sum_{n=0}^{\infty} \epsilon^n \phi^{(n)}(x,t;X,T,\tau). \tag{3.27}$$

In equation (2.55) an asymptotic expansion of the frontal thickness was also introduced in order to determine the leading order motion of the front given by the nonlinear equation being analyzed here (3.5). This present asymptotic expansion, (3.26) and (3.27), can be thought of as expanding the leading order motion yet further. However since the leading order motion describes dynamics that are greater then $O(R_0)$ (R_0 being defined by equation (2.38)), this asymptotic analysis is valid only as long as it remains greater than $O(R_0)$. Since an expansion up to $O(\epsilon^2)$ is going to be considered, it is assumed then that the parameter, ϵ, satisfies the relation

$$0 < R_0 \ll \epsilon^2 \ll \epsilon \ll 1, \tag{3.28}$$

in order to be asymptotically valid.

The asymptotic expansion (3.26) and (3.27) are introduced into the final form of the governing equation (3.20) and the final form of the boundary conditions (3.25). The next three sections will involve finding and solving the $O(1)$, $O(\epsilon)$, and $O(\epsilon^2)$ problem for the wedge front,

$$h_0(y) = \alpha y. \tag{3.29}$$

3.3 The $O(1)$ problem

After the asymptotic expansion of the perturbation thickness h has been included in equation (3.20), the $O(1)$ problem is formed by retaining only those terms that contain no powers of ϵ. The following equation results:

$$h_t^{(0)} + h_0 h_{0_y} h_{xxx}^{(0)} + h_0 h_{0_y} h_{yyx}^{(0)} + (h_{0_y})^2 h_{yx}^{(0)} - h_{0_y} h_{0_{yy}} h_x^{(0)} - h_0 h_{0_{yyy}} h_x^{(0)} = 0. \tag{3.30}$$

The steady state parallel shear flow will be considered to be $h_0(y) = \alpha y$, where $\alpha > 0$. Substituting this into the equation (3.30) yields

$$h_t^{(0)} + \alpha^2 y h_{xxx}^{(0)} + \alpha^2 y h_{yyx}^{(0)} + \alpha^2 h_{yx}^{(0)} = 0. \tag{3.31}$$

The $O(1)$ boundary conditions are formed in the same way by equating all terms in equation (3.25) that contain no powers of ϵ to zero after the asymptotic expansions (3.26) and (3.27) have been introduced. This gives the following boundary condition:

$$h_{0_y}\phi^{(0)} + h^{(0)} = 0 \text{ on } y = 0. \tag{3.32}$$

Substituting in the wedge–shaped solution for the basic state gives the boundary condition

$$\alpha\phi^{(0)} + h^{(0)} = 0 \text{ on } y = 0. \tag{3.33}$$

Since the coefficients of this $O(1)$ problem (3.31) and its boundary conditions (3.33) are independent of x and t a normal mode substitution is made such that

$$h^{(0)} = A(X, T, \tau)\,\psi(y)\exp(ik(x - ct)) + c.c., \tag{3.34}$$

where A is the envelope amplitude function which depends on only the slow space and time variables, k is the along–front wavenumber, c is the along–front phase speed and $c.c.$ is the complex conjugate, to give

$$\alpha^2 y\psi'' + \alpha^2\psi' - \left(k^2\alpha^2 y + c\right)\psi = 0, \tag{3.35}$$

where primes denote differentiation with respect to y. A solution to equation (3.35) can be constructed in the form

$$\psi(y) = \exp(-ky)\,F(y), \tag{3.36}$$

with derivatives,

$$\psi'(y) = -k\exp(-ky)\,F(y) + \exp(-ky)\,F'(y), \tag{3.37}$$

$$\psi'' = k^2\exp(-ky)\,F(y) - 2k\exp(-ky)\,F'(y) + \exp(-ky)\,F''(y). \tag{3.38}$$

Substituting these expressions (3.36)–(3.38) into equation (3.35) and simplifying

gives

$$yF'' + (1-2ky)\,F' - \left(k + \frac{c}{\alpha^2}\right)F = 0. \tag{3.39}$$

In order to simplify this equation somewhat, the change of variable $z = 2ky$ is made to give the equation

$$zF'' + (1-z)\,F' - \left(\frac{1}{2} + \frac{c}{2\alpha^2 k}\right)F = 0, \tag{3.40}$$

where primes now denote differentiation with respect to z. The only solution to this equation that is bounded at $z = 0$ and approaches a finite value as $z \to \infty$ (so that the total amplitude $\psi(y)$ decays exponentially) is given by the Laguerre polynomials (Birkhoff and Gian–Carlo, 1978),

$$F(y) = L_n(z) = L_n(2ky)\ , \tag{3.41}$$

with the eigenvalue or dispersion relationship

$$c = -\alpha^2 k\,(2n+1)\,, \tag{3.42}$$

where n is a any non–negative integer.

Therefore the complete solution to the $O(1)$ problem is given as

$$h^{(0)} = A\exp(-ky)\,L_n(2ky)\exp(i\Theta) + A^{*}\exp(-ky)\,L_n(2ky)\exp(-i\Theta)\,, \tag{3.43}$$

where $A = A(X, T, \tau)$ is the envelope amplitude; A^{*} is its complex conjugate; $\Theta = k(x - ct)$; and c is given by the dispersion relationship (3.42).

The $O(1)$ boundary, $\phi^{(0)}$, is determined by substituting this solution (3.43) into equation (3.33) to give

$$\phi^{(0)} = -\frac{A}{\alpha}\exp(ik(x-ct)) + c.c. \tag{3.44}$$

3.4 The $O(\epsilon)$ problem

The $O(\epsilon)$ problem is found by demanding that the coefficients of the terms proportional to ϵ^1 be zero in the nonlinear stability equation (3.20) after the asymptotic expansion of h (3.26) has been included. The following equation results:

$$h_t^{(1)} + h_T^{(0)} + h_0 h_{0y} h_{xxx}^{(1)} + 3h_0 h_{0y} h_{xxX}^{(0)} + h_0 h_y^{(0)} h_{xxx}^{(0)} + h_{0y} h^{(0)} h_{xxx}^{(0)}$$

$$+h_0 h_{0y} h_{yyx}^{(1)} + h_0 h_{0y} h_{yyX}^{(0)} + h_0 h_y^{(0)} h_{yyx}^{(0)} + h_{0y} h^{(0)} h_{yyx}^{(0)} + h_{0y} h_x^{(0)} h_{xx}^{(0)}$$

$$+ (h_{0y})^2 h_{yx}^{(1)} + (h_{0y})^2 h_{yX}^{(0)} + 2h_{0y} h_y^{(0)} h_{yx}^{(0)} - h_0 h_{xxy}^{(0)} h_x^{(0)}$$

$$-h_{0y} h_{0yy} h_x^{(1)} - h_{0y} h_{0yy} h_X^{(0)} - h_{0y} h_{yy}^{(0)} h_x^{(0)} - h_{0yy} h_y^{(0)} h_x^{(0)}$$

$$-h_0 h_{0yyy} h_x^{(1)} - h_0 h_{0yyy} h_X^{(0)} - h_0 h_{yyy}^{(0)} h_x^{(0)} - h_{0yyy} h^{(0)} h_x^{(0)} = 0. \qquad (3.45)$$

In order to examine the evolution of the wedge front, the parallel shear flow, $h_0 = \alpha y$, is substituted in and the equation is rearranged to give

$$h_t^{(1)} + \alpha^2 y h_{xxx}^{(1)} + +\alpha^2 y h_{yyx}^{(1)} + \alpha^2 h_{yx}^{(1)} = -h_T^{(0)} - 3\alpha^2 y h_{xxX}^{(0)} - \alpha y h_y^{(0)} h_{xxx}^{(0)}$$

$$-\alpha h^{(0)} h_{xxx}^{(0)} - \alpha^2 y h_{yyX}^{(0)} - \alpha y h_y^{(0)} h_{yyx}^{(0)} - \alpha h^{(0)} h_{yyx}^{(0)} - \alpha h_x^{(0)} h_{xx}^{(0)} - \alpha^2 h_{yX}^{(0)}$$

$$-2\alpha h_y^{(0)} h_{yx}^{(0)} + \alpha y h_{xxy}^{(0)} h_x^{(0)} + \alpha h_{yy}^{(0)} h_x^{(0)} + \alpha y h_{yyy}^{(0)} h_x^{(0)}. \qquad (3.46)$$

The expression for $h^{(0)}$, (3.43), is substituted in and, after a little algebra, yields the following nonhomogeneous differential equation:

$$h_t^{(1)} + \alpha^2 y h_{xxx}^{(1)} + \alpha^2 y h_{yyx}^{(1)} + \alpha^2 h_{yx}^{(1)} =$$

$$\left(-A_T \psi + 3\alpha^2 y k^2 A_X \psi - \alpha^2 y A_X \psi'' - \alpha^2 A_X \psi'\right) \exp(i\Theta)$$

$$+c.c. + \alpha A^2 ik \left(2k^2 \psi^2 - y\psi'\psi'' - 2(\psi')^2 + y\psi'''\psi\right) \exp(2i\Theta) + c.c., \qquad (3.47)$$

where $\psi = \exp(-ky) L_n(2ky)$, $\Theta = k(x - ct)$, and *c.c.* stands for complex conjugate.

The $O(\epsilon)$ boundary conditions are found by retaining all terms from equation (3.25) that contain ϵ^1 after the asymptotic expansions (3.26) and (3.27) have been included resulting in

$$h_{0y}\phi^{(1)} + h_{0yy}\phi^{(0)}\phi^{(0)} + h^{(1)} + h_y^{(0)}\phi^{(0)} = 0 \text{ on } y = 0. \tag{3.48}$$

Substituting in the wedge profile for the steady state solution gives

$$\alpha\phi^{(1)} + h^{(1)} + h_y^{(0)}\phi^{(0)} = 0 \text{ on } y = 0. \tag{3.49}$$

These can be simplified using the $O(1)$ boundary conditions (3.44) to give

$$\phi^{(1)} = -\frac{h^{(1)}}{\alpha} + \frac{h_y^{(0)}h^{(0)}}{\alpha^2} \text{ on } y = 0. \tag{3.50}$$

The solution to the $O(\epsilon)$ problem (3.47) is given as

$$h^{(1)} = h_e^{(1)} + h_p^{(1)}, \tag{3.51}$$

where $h_e^{(1)}$ is the solution to the homogeneous problem and $h_p^{(1)}$ is the particular solution to the nonhomogeneous equation. The homogeneous equation is identical to the $O(1)$ problem and its solution is given by

$$h_e^{(1)} = b(y; X, T, \tau). \tag{3.52}$$

This solution (3.52) is a real–valued mean flow which will be determined in the $O(\epsilon^2)$ problem. No terms proportional to the $O(1)$ solution are included in the homogeneous solution, $h_e^{(1)}$, since they can always be absorbed directly into the $O(1)$ solution (Newell, 1985).

An examination of the nonhomogeneous equation shows that the right hand side is composed of terms proportional to $\exp(\pm i\Theta)$ and terms proportional to $\exp(\pm 2i\Theta)$. Therefore, in principle, a particular solution associated with these terms can be constructed in the form

$$h_p^{(1)} = C(X,T,\tau)\, G(y) \exp(i\Theta) + A^2(X,T,\tau)\, H(y) \exp(2i\Theta) + c.c. \qquad (3.53)$$

Substituting into the $O(\epsilon)$ problem (3.47) gives the following two inhomogeneous ordinary differential equations:

$$yG'' + G' - \left(k^2 y - k(2n+1)\right) G$$

$$= \left(ik\alpha^2 C\right)^{-1} \left[-A_T \psi + \alpha^2 A_X \left(3yk^2\psi - y\psi'' - \psi'\right)\right], \qquad (3.54)$$

$$yH'' + H' - \left(4k^2 y - k(2n+1)\right) H$$

$$= \frac{1}{2\alpha}\left(2k^2\psi^2 - y\psi'\psi'' - 2(\psi')^2 + y\psi'''\psi\right), \qquad (3.55)$$

where $\psi = \exp(-ky)\, L_n(2ky)$ and primes denote differentiation with respect to y. The right hand side of these equations can be simplified somewhat by considering equation (3.35) and its first derivative given by

$$y\psi''' + 2\psi'' - \left(k^2 y - k(2n+1)\right)\psi' - k^2\psi = 0. \qquad (3.56)$$

Solving equation (3.35) for $y\psi''$ and equation (3.56) for $y\psi'''$ and substituting into the right hand side of equations (3.54) and (3.55) gives the simplified equations,

$$yG'' + G' - \left(k^2 y - k(2n+1)\right) G$$

$$= \left(ik\alpha^2 C\right)^{-1} \left[-A_T \psi + \alpha^2 A_X \left(2k^2 y\psi + k(2n+1)\psi\right)\right], \qquad (3.57)$$

$$yH'' + H' - \left(4k^2 y - k(2n+1)\right) H = \frac{1}{2\alpha}\left(3k^2\psi^2 - (\psi')^2 - 2\psi\psi''\right). \qquad (3.58)$$

The first differential equation (3.57) is initially considered. A solution to this nonhomogeneous equation exists only if (3.57) satisfies the conditions defined by the Fredholm Alternative Theorem (Zwillinger, 1989). The **Fredholm Alternative Theorem** states that an inhomogeneous ordinary differential equation of the form $\mathcal{L}G = \Phi$ has a solution if and only if $\langle \Phi, \phi \rangle = 0$ where $\phi \in Ker\left(\mathcal{L}^A\right)$, $\mathcal{L}^A$ being the adjoint operator of $\mathcal{L}$. The homogeneous part of the differential equation (3.57) given as

$$yG_e'' + G_e' - \left(k^2 y - k\left(2n+1\right)\right) G_e = 0, \tag{3.59}$$

can be written as

$$\left(yG_e'\right)' - \left(k^2 y - k\left(2n+1\right)\right) G_e = 0, \tag{3.60}$$

demonstrating that this equation is self–adjoint. Thus all solutions in the kernel of the adjoint operator are just solutions of (3.59) itself and the Fredholm Alternative Theorem states that

$$\langle \Phi, G_e \rangle = \int_\Omega \Phi\left(y\right) G_e\left(y\right) dy = 0, \tag{3.61}$$

where $G_e\left(y\right)$ is a solution to the homogeneous problem (3.59), $\Phi\left(y\right)$ is equal to the right hand side of (3.57), and Ω is the domain of y which in this case is $\{y|\, y \in [0, \infty)\}$. Examining (3.59) shows that this is the same equation solved in the $O\left(1\right)$ problem (equation (3.35)). Its solution is given by

$$G_e\left(y\right) = \exp\left(-ky\right) L_n\left(2ky\right) \equiv \psi\left(y\right). \tag{3.62}$$

Therefore a solution exists to the differential equation (3.57) only if

$$0 = \int_\Omega \Phi\left(y\right) \psi\left(y\right) dy$$

$$= -A_T \int_0^\infty \psi^2 dy + \alpha^2 A_X \int_0^\infty \psi^2 \left(2k^2 y + k\left(2n+1\right)\right) dy$$

$$= \int_0^\infty \exp(-2ky)\,[L_n(2ky)]^2\left[-A_T + \alpha^2 A_X\left(2k^2 y + k(2n+1)\right)\right] dy, \qquad (3.63)$$

where equation (3.62) has been used. This integral is easier to evaluate if the variable change, $z = 2ky$, is made giving

$$\int_0^\infty \exp(-z)\,[L_n(z)]^2\left[-A_T + \alpha^2 A_X\left(kz + k(2n+1)\right)\right] dz = 0. \qquad (3.64)$$

On using integration by parts and the identity (Gradshteyn and Ryzhik, 1980),

$$z\frac{d}{dz}L_n(z) = n\left(L_n - L_{n-1}\right), \qquad (3.65)$$

the third term can be rewritten giving

$$\int_0^\infty z\exp(-z)\,[L_n(z)]^2\,dz = \int_0^\infty \exp(-z)\,[L_n(z)]^2\,dz$$

$$+2n\int_0^\infty \exp(-z)\left([L_n(z)]^2 - L_n(z)\,L_{n-1}(z)\right) dz. \qquad (3.66)$$

The integral (3.64) then becomes

$$\left(-A_T + 2\alpha^2 A_X k\,[1+2n]\right)\int_0^\infty \exp(-z)\,[L_n(z)]^2\,dz \qquad (3.67)$$

$$-2n\alpha^2 A_X k\int_0^\infty \exp(-z)\,L_n(z)\,L_{n-1}(z)\,dz = 0.$$

Using the orthogonality condition for the Laguerre polynomials given by (Gradshteyn and Ryzhik, 1980),

$$\int_0^\infty \exp(-z)\,L_n(z)\,L_m(z)\,dz = \delta_{mn}, \qquad (3.68)$$

where δ_{mn} is the Kronecker delta defined by

$$\delta_{mn} = \begin{cases} 1 & \text{when} \quad m = n \\ 0 & \text{when} \quad m \neq n \end{cases}, \qquad (3.69)$$

makes the second integral zero and the first integral unity showing that the amplitude, A, must satisfy

$$A_T - 2\alpha^2 k\,(2n+1)\,A_X = 0. \tag{3.70}$$

Solutions to this equation are given as

$$A = A\,(X - c_g T, \tau)\,, \tag{3.71}$$

where c_g is the group velocity (3.4) given as

$$c_g = -2\alpha^2 k\,(2n+1)\,. \tag{3.72}$$

This leads to the introduction of a new variable, ξ, defined by

$$\xi = X - c_g T, \tag{3.73}$$

so that

$$A = A\,(\xi, \tau)\,. \tag{3.74}$$

Thus it is shown that the $O\,(1)$ envelope function is constant to $O\,(\epsilon)$ following the group velocity. Note that the partial derivatives associated with this new variable become

$$\partial_T\,(*) = -c_g \partial_\xi\,(*) = 2\alpha^2 k\,(2n+1)\,\partial_\xi\,(*)\,, \tag{3.75}$$

$$\partial_X\,(*) = \partial_\xi\,(*)\,. \tag{3.76}$$

It therefore follows that (3.57) can be written in the form

$$yG'' + G' - \left(k^2 y - k\,(2n+1)\right)G = \left(-2ky + (2n+1)\right)\psi, \tag{3.77}$$

where we have chosen $C\,(\xi, \tau) = iA_\xi$.

The solution to this equation can be found by using variation of parameters (Nagel and Saff, 1993). The homogeneous solution is given as

$$G_e = \exp\left(-ky\right) L_n\left(2ky\right) = \psi\left(y\right), \tag{3.78}$$

so that the particular solution can be assumed to be of the form

$$G_p = P\left(y\right)\psi\left(y\right). \tag{3.79}$$

Substituting this into this differential equation and multiplying the equation by the integrating factor $\psi\left(y\right)$ gives the integral relation

$$P'y\psi^2 = \frac{1}{2k}\int_0^{2ky} \exp\left(-z\right)\left[L_n\left(z\right)\right]^2\left(2n+1-z\right)dz. \tag{3.80}$$

Using integration by parts simplifies the integral to give

$$\begin{aligned}
P'y\phi^2 = \frac{1}{2k}\int_0^{2ky} &\Big(\exp\left(-z\right)\left[L_n\left(z\right)\right]^2\left(2n+1\right)\Big)\,dz \\
+y\exp\left(-z\right)\left[L_n\left(z\right)\right]^2 - \frac{1}{2k}\left(2n+1\right)&\int_0^{2ky}\left(\exp\left(-z\right)\left[L_n\left(z\right)\right]^2\right)dz \\
+\frac{n}{k}&\int_0^{2ky}\exp\left(-z\right)L_n\left(z\right)L_{n-1}\left(z\right)dz \\
=&\frac{1}{2k}\left[2ky\exp\left(-2ky\right)\left[L_n\left(2ky\right)\right]^2\right] \\
+\frac{1}{2k}&\left[2n\int_0^{2ky}\left(\exp\left(-z\right)L_n\left(z\right)L_{n-1}\left(z\right)\right)dz\right].
\end{aligned} \tag{3.81}$$

Using the relation (Gradshteyn and Ryzhik, 1980),

$$L_{n-1}\left(*\right) = L^1_{n-1}\left(*\right) - L^1_{n-2}\left(*\right), \tag{3.82}$$

gives

$$P'y\phi^2 = \frac{1}{2k}2ky\exp\left(-2ky\right)\left[L_n\left(2ky\right)\right]^2$$

$$+\frac{1}{k}n\int_0^{2ky}\exp\left(-z\right)L_n\left(z\right)\left(L^1_{n-1}\left(z\right)-L^1_{n-2}\left(z\right)\right)dz$$

$$=\frac{1}{2k}\left[2ky\exp\left(-2ky\right)\left[L_n\left(2ky\right)\right]^2-4kny\exp\left(-2ky\right)L_n\left(2ky\right)L^1_{n-2}\left(2ky\right)\right]$$

$$+\frac{1}{2k}\left[2n^2\int_0^{2ky}\exp\left(-z\right)\left(L_n\left(z\right)L^1_{n-1}\left(z\right)-L^1_{n-2}\left(z\right)L_{n-1}\left(z\right)\right)dz\right]$$

$$=\frac{1}{2k}\left[2ky\exp\left(-2ky\right)\left[L_n\left(2ky\right)\right]^2-4kny\exp\left(-2ky\right)L_n\left(2ky\right)L^1_{n-2}\left(2ky\right)\right]$$

$$+\frac{1}{2k}\left[2n^2\int_0^{2ky}\exp\left(-z\right)\left(-2L^1_{n-2}\left(z\right)L_{n-1}\left(z\right)+L_n\left(z\right)L^1_{n-1}\left(z\right)\right)dz\right]$$

$$+\frac{n^2}{k}\int_0^{2ky}\exp\left(-z\right)L^1_{n-2}\left(z\right)L_{n-1}\left(z\right)dz$$

$$=\frac{1}{2k}\left[2ky\exp\left(-2ky\right)\left[L_n\left(2ky\right)\right]^2-4kny\exp\left(-2ky\right)L_n\left(2ky\right)L^1_{n-2}\left(2ky\right)\right]$$

$$+\frac{1}{2k}2n^2\left[\exp\left(-2ky\right)\left[L_{n-1}\left(2ky\right)\right]^2-1\right]$$

$$+\frac{1}{2k}2n^2\int_0^{2ky}\exp\left(-z\right)\left(L^1_{n-2}\left(z\right)L_{n-1}\left(z\right)+L_n\left(z\right)L^1_{n\;1}\left(z\right)+\left[L_{n-1}\left(z\right)\right]^2\right)dz$$

$$=\frac{1}{2k}\left[2ky\exp\left(-2ky\right)\left[L_n\left(2ky\right)\right]^2-4kny\exp\left(-2ky\right)L_n\left(2ky\right)L^1_{n-2}\left(2ky\right)\right]$$

$$+\frac{1}{2k}2n^2\left[\exp\left(-2ky\right)\left[L_{n-1}\left(2ky\right)\right]^2-1+\int_0^{2ky}\exp\left(-z\right)L_{n-1}\left(z\right)L_n\left(z\right)dz\right]$$

$$+\frac{1}{2k}2n^2\int_0^{2ky}\exp\left(-z\right)\left(L^1_{n-2}\left(z\right)L_{n-1}\left(z\right)+L_n\left(z\right)L^1_{n-1}\left(z\right)\right)dz$$

$$+\frac{1}{2k}2n^2\int_0^{2ky}\exp\left(-z\right)\left(\left[L_{n-1}\left(z\right)\right]^2-L_{n-1}\left(z\right)L_n\left(z\right)\right)dz$$

$$=\frac{1}{2k}\left[2ky\exp\left(-2ky\right)\left[L_n\left(2ky\right)\right]^2-4kny\exp\left(-2ky\right)L_n\left(2ky\right)L^1_{n-2}\left(2ky\right)\right]$$

$$+\frac{1}{2k}\left[2n^2\exp\left(-2ky\right)\left[L_{n-1}\left(2ky\right)\right]^2+2n^2\exp\left(-2ky\right)L_{n-1}\left(2ky\right)L_n\left(2ky\right)\right]$$

$$+\frac{1}{2k}2n^2\int_0^{2ky}\exp(-z)\left(L_{n-1}(z)\left(L_{n-2}^1(z)-L_{n-1}^1(z)\right)+[L_{n-1}(z)]^2\right)dz$$

$$+\frac{1}{2k}2n^2\int_0^{2ky}\exp(-z)\left(L_n(z)\left(L_{n-1}^1(z)-L_{n-2}^2(z)\right)-L_{n-1}(z)L_n(z)\right)dz$$

$$=\frac{1}{2k}\left[2ky\exp(-2ky)[L_n(2ky)]^2-4kny\exp(-2ky)L_n(2ky)L_{n-2}^1(2ky)\right]$$

$$+\frac{1}{2k}2n\exp(-2ky)L_{n-1}(2ky)(nL_{n-1}(2ky)-nL_n(2ky))$$

$$+\frac{1}{2k}2n^2\int_0^{2ky}\exp(-z)\left(-[L_{n-1}(z)]^2+(L_{n-1}(z))^2\right)dz$$

$$+\frac{1}{2k}2n^2\int_0^{2ky}\exp(-z)[L_n(z)L_{n-1}(z)-L_{n-1}(z)L_n(z)]dz$$

$$=\frac{1}{2k}\left[2ky\exp(-2ky)[L_n(2ky)]^2-4kny\exp(-2ky)L_n(2ky)L_{n-2}^1(2ky)\right]$$

$$+\frac{1}{2k}4knyL_{n-1}^1(2ky)L_{n-1}(2ky)$$

$$=y\exp(-2ky)[L_n(2ky)]^2$$

$$+2ny\exp(-2ky)\left(L_{n-1}^1(2ky)L_{n-1}(2ky)-L_n(2ky)L_{n-2}^1(2ky)\right). \tag{3.83}$$

Therefore, solving equation (3.83) for $P'(y)$ gives

$$P'(y)=\left(1+2n\left(\frac{\left(L_{n-1}^1(2ky)L_{n-1}(2ky)-L_n(2ky)L_{n-2}^1(2ky)\right)}{[L_n(2ky)]^2}\right)\right)$$

$$=\left(1+\frac{n}{k}\left[\frac{L_{n-1}(2ky)}{L_n(2ky)}\right]'\right). \tag{3.84}$$

Integrating this equation from 0 to y gives

$$P(y)=\left(y+\frac{n}{k}\left[\frac{L_{n-1}(2ky)}{L_n(2ky)}\right]\right). \tag{3.85}$$

Substituting (3.85) into the particular solution (3.79) and then substituting the particular solution into (3.53) implies that $C(X,T,\tau)G(y)\exp(i\Theta)$ takes the form

$$iA_\xi \exp(-ky)\left(yL_n(2ky) + \frac{n}{k}L_{n-1}(2ky)\right). \tag{3.86}$$

The particular solution for the second differential equation (3.58) is not so easily obtained. First, the right hand side of (3.58) is simplified by making the substitution $\psi(y) = \exp(-ky)\, L_n(2ky)$ giving the equation

$$yH'' + H' - \left(4k^2 y - k(2n+1)\right)H = -\frac{2k^2}{\alpha}\exp(-2ky) \times$$

$$\left(3L_n(2ky)\, L_{n-1}^1(2ky) + \left[L_{n-1}^1(2ky)\right]^2 + 2L_n(2ky)\, L_{n-2}^2(2ky)\right). \tag{3.87}$$

In order to eliminate the exponential term on the right hand side of this equation (3.87) the substitution,

$$H(y) = H_p(y) = \frac{k}{\alpha}\exp(-2ky)\, \Psi(y), \tag{3.88}$$

is made giving

$$y\Psi'' + (1 - 4ky)\Psi' + k(2n-1)\Psi$$

$$= -2k\left(3L_n L_{n-1}^1 + \left(L_{n-1}^1\right)^2 + 2L_n L_{n-2}^2\right), \tag{3.89}$$

where the argument of the Laguerre polynomials is $2ky$. This argument suggests making the variable change, $z = 2ky$, giving the equation

$$z\Psi'' + (1 - 2z)\Psi' + \left(\frac{2n-1}{2}\right)\Psi = -\left(3L_n L_{n-1}^1 + \left(L_{n-1}^1\right)^2 + 2L_n L_{n-2}^2\right), \tag{3.90}$$

where primes now denote differentiation with respect to z, and the argument of the Laguerre polynomials is z.

This equation cannot be solved in general using variation of parameters or other known methods for solving inhomogeneous differential equations. Thus it must be solved for each individual value of n. This is done by substituting the

Laguerre polynomials for each value of n defined by the formula (Gradshteyn and Ryzhik, 1980)

$$L_n^{\alpha}(x) = \sum_{m=0}^{n} (-1)^m \begin{pmatrix} n+\alpha \\ n-m \end{pmatrix} \frac{x^m}{m!}, \tag{3.91}$$

into the right hand side of equation (3.90). The following list of equations results:

$$z\Psi_0'' + (1-2z)\,\Psi_0' + \left(-\tfrac{1}{2}\right)\Psi_0 = 0 \quad \text{when} \quad n = 0, \tag{3.92}$$

$$z\Psi_1'' + (1-2z)\,\Psi_1' + \left(\tfrac{1}{2}\right)\Psi_1 = (3z-4) \quad \text{when} \quad n = 1, \tag{3.93}$$

$$z\Psi_2'' + (1-2z)\,\Psi_2' + \left(\tfrac{3}{2}\right)\Psi_2 = \left(\tfrac{3}{2}z^3 - 11z^2 + 23z - 12\right) \quad \text{when} \quad n = 2, \tag{3.94}$$

$$\begin{aligned} &z\Psi_3'' + (1-2z)\,\Psi_3' + \left(\frac{5}{2}\right)\Psi_3 \\ &= \left(\frac{1}{4}z^5 - \frac{13}{3}z^4 + \frac{53}{2}z^3 - 69z^2 + 74z - 24\right) \text{ when } n = 3, \end{aligned} \tag{3.95}$$

$$\begin{aligned} z\Psi_4'' + (1-2z)\,\Psi_4' + \left(\frac{7}{2}\right)\Psi_4 &= \frac{1}{48}z^7 - \frac{47}{72}z^6 + \frac{95}{12}z^5 - \frac{142}{3}z^4 + \frac{887}{6}z^3 \\ &-235z^2 + 170z - 40 \text{ when } n = 4, \end{aligned} \tag{3.96}$$

$$\begin{aligned} z\Psi_5'' + (1-2z)\,\Psi_5' + \left(\frac{9}{2}\right)\Psi_5 &= \frac{1}{960}z^9 - \frac{37}{720}z^8 + \frac{149}{144}z^7 - \frac{265}{24}z^6 + 68z^5 \\ &-247z^4 + \frac{1039}{2}z^3 - 595z^2 + 325z - 60 \text{ when } n = 5, \end{aligned} \tag{3.97}$$

$$\begin{aligned} z\Psi_6'' + (1-2z)\,\Psi_6' + \left(\frac{11}{2}\right)\Psi_6 &= \frac{1}{28800}z^{11} - \frac{107}{43200}z^{10} + \frac{43}{576}z^9 \\ &-\frac{299}{240}z^8 + \frac{203}{16}z^7 - \frac{9841}{120}z^6 + \frac{1697}{5}z^5 - \frac{2660}{3}z^4 \\ &+1407z^3 - 1260z^2 + 553z - 84 \text{ when } n = 6, \end{aligned} \tag{3.98}$$

$$\begin{aligned} z\Psi_7'' + (1-2z)\,\Psi_7' + \left(\frac{13}{2}\right)\Psi_7 &= \frac{1}{1209600}z^{13} - \frac{73}{907200}z^{12} + \frac{293}{86400}z^{11} \\ &-\frac{3521}{43200}z^{10} + \frac{5353}{4320}z^9 - \frac{25}{2}z^8 + \frac{6833}{80}z^7 - \frac{142763}{360}z^6 + \frac{37111}{30}z^5 \end{aligned}$$

$$-\frac{7574}{3}z^4+\frac{9653}{3}z^3-2366z^2+868z-112 \text{ when } n=7, \tag{3.99}$$

$$z\Psi_8''+(1-2z)\Psi_8'+\left(\frac{15}{2}\right)\Psi_8=\frac{1}{677376600}z^{15}-\frac{191}{101606400}z^{14}$$
$$+\frac{383}{3628800}z^{13}-\frac{29}{8400}z^{12}+\frac{6329}{86400}z^{11}-\frac{45907}{43200}z^{10}$$
$$+\frac{5197}{480}z^9-\frac{39503}{504}z^8+\frac{2032871}{5040}z^7-\frac{58481}{40}z^6+\frac{36609}{10}z^5$$
$$-6132z^4+6531z^3-4074z^2+1284z-144 \text{ when } n=8, \tag{3.100}$$

$$z\Psi_9''+(1-2z)\Psi_9'+\left(\frac{17}{2}\right)\Psi_9=\frac{1}{4877107200}z^{17}-\frac{121}{3657830400}z^{16}$$
$$+\frac{97}{40642560}z^{15}-\frac{1151}{11289600}z^{14}+\frac{373}{129600}z^{13}$$
$$-\frac{3433}{60480}z^{12}+\frac{929}{1152}z^{11}-\frac{507361}{60480}z^{10}+\frac{1296901}{20160}z^9$$
$$-\frac{610469}{1680}z^8+\frac{1512373}{1008}z^7-\frac{35693}{8}z^6+9333z^5-13258z^4$$
$$+12129z^3-6570z^2+1815z-180 \text{ when } n=9, \tag{3.101}$$

and

$$z\Psi_{10}''+(1-2z)\Psi_{10}'+\left(\frac{19}{2}\right)\Psi_{10}=\frac{1}{438939648000}z^{19}-\frac{299}{658409472000}z^{18}$$
$$+\frac{599}{14631321600}z^{17}-\frac{1619}{731566080}z^{16}+\frac{48893}{609638400}z^{15}-\frac{209711}{101606400}z^{14}$$
$$+\frac{4421}{113400}z^{13}-\frac{4999}{9072}z^{12}+\frac{17809}{3024}z^{11}-\frac{288983}{6048}z^{10}+\frac{10664333}{36288}z^9$$
$$-\frac{1372283}{1008}z^8+\frac{4732123}{1008}z^7-\frac{284075}{24}z^6+21263z^5-26224z^4$$
$$+\frac{42053}{2}z^3-10065z^2+2475z-220 \text{ when } n=10. \tag{3.102}$$

Since the right hand side of these equations (3.92)–(3.102) is a polynomial of degree $2n-1$, the solution to each of these equations is assumed to be a series of the form,

$$\Psi_n = \sum_{k=0}^{2n-1} a_k z^k, \tag{3.103}$$

so that each solution is also a polynomial of degree $2n-1$. By substituting this series along with its derivatives into the differential equations, values for each of the coefficients can be found. This is done by using a computer program, *Mathematica*. The assumed series solution (3.103) is entered for the value of n that is being considered. *Mathematica* is then asked to find the derivatives of the series. Adding these together using the formulas given in equations (3.92)–(3.102) and equating to the polynomials on the right hand side of the appropriate equation in (3.92)–(3.102), *Mathematica* then solves for each of the coefficients, a_k. The results are summarized in the following tables:

n	0	1	2	3	4	5
a_0	0	-4	$\frac{28}{5}$	$-\frac{856}{77}$	$\frac{2024}{221}$	$-\frac{80772}{4807}$
a_1	-	-2	$-\frac{102}{5}$	$\frac{292}{77}$	$-\frac{15924}{221}$	$\frac{75054}{4807}$
a_2	-	-	$\frac{16}{5}$	$\frac{1388}{77}$	$\frac{15364}{221}$	$\frac{343660}{4807}$
a_3	-	-	$-\frac{1}{3}$	$-\frac{359}{77}$	$-\frac{4917}{221}$	$-\frac{1010665}{14421}$
a_4	-	-	-	$\frac{7}{11}$	$\frac{3821}{663}$	$\frac{124494}{48076}$
a_5	-	-	-	$-\frac{1}{30}$	$-\frac{1135}{1326}$	$-\frac{30064}{4807}$
a_6	-	-	-	-	$\frac{10}{153}$	$\frac{3671}{3933}$
a_7	-	-	-	-	$-\frac{1}{504}$	$-\frac{865}{10488}$
a_8	-	-	-	-	-	$\frac{13}{3312}$
a_9	-	-	-	-	-	$-\frac{1}{12960}$

Table 3.1 – Series coefficients

n	6	7	8	9	10
a_0	$\frac{442116}{32045}$	$-\frac{17416304}{745085}$	$\frac{859286096}{48612265}$	$-\frac{801925100}{27378487}$	$\frac{214918225916}{9711234785}$
a_1	$-\frac{5123418}{32045}$	$\frac{29756456}{745085}$	$-\frac{2688962376}{9722453}$	$\frac{1888235690}{27378487}$	$-\frac{4178194798902}{9711234785}$
a_2	$\frac{8913212}{32045}$	$\frac{128207432}{745085}$	$\frac{6818230680}{9722453}$	$\frac{9354605480}{27378487}$	$\frac{2768588354232}{1942246957}$
a_3	$-\frac{17915506}{96135}$	$-\frac{138892646}{447051}$	$-\frac{21157693634}{29167359}$	$-\frac{24663598250}{27378487}$	$-\frac{11591983856827}{5826740871}$
a_4	$\frac{7894012}{96135}$	$\frac{31414738}{149017}$	$\frac{4629761710}{9722453}$	$\frac{24608291528}{27378487}$	$\frac{1456142057348}{832391553}$
a_5	$-\frac{4366978}{160225}$	$-\frac{987288437}{11176275}$	$-\frac{57303200941}{243061325}$	$-\frac{75057625282}{136892435}$	$-\frac{8004283057298}{6936596275}$
a_6	$\frac{8682241}{1442025}$	$\frac{864159599}{33528825}$	$\frac{37328260117}{437510385}$	$\frac{97085888161}{410677305}$	$\frac{35872676266669}{62429366475}$
a_7	$-\frac{5435033}{6211800}$	$-\frac{295876829}{56897400}$	$-\frac{11979020955}{544457368}$	$-\frac{1705700025551}{22997929080}$	$-\frac{1038817447025699}{4894462331640}$
a_8	$\frac{59953}{730800}$	$\frac{2750701}{3793160}$	$\frac{7666325417}{1884660120}$	$\frac{2355380027423}{137987574480}$	$\frac{571959382207301}{9788924663280}$
a_9	$-\frac{991}{208800}$	$-\frac{2240299}{32341680}$	$-\frac{51387457}{95024880}$	$-\frac{1030977858271}{354825191520}$	$-\frac{2135294760025933}{176200643939040}$
a_{10}	$\frac{1}{6525}$	$\frac{155563}{35154000}$	$\frac{40772833}{791874000}$	$\frac{120029361427}{3268126764000}$	$\frac{247293495102877}{129559297014000}$
a_{11}	$-\frac{1}{475200}$	$-\frac{8431}{46872000}$	$-\frac{72400151}{20905473600}$	$-\frac{1956540391}{56836987200}$	$-\frac{7779147847517}{33932196837000}$
a_{12}	-	$\frac{19}{4536000}$	$\frac{115319}{720878400}$	$\frac{67435393}{28418493600}$	$\frac{85495674833}{4071863620440}$
a_{13}	-	$-\frac{1}{23587200}$	$-\frac{1471}{305827200}$	$-\frac{1409077}{119174432800}$	$-\frac{204867739}{140409090360}$
a_{14}	-	-	$\frac{11}{130183200}$	$\frac{32791}{7944955200}$	$\frac{1903323409}{25018347009600}$
a_{15}	-	-	$-\frac{1}{1524096000}$	$-\frac{3919}{41069306880}$	$-\frac{18940829}{6491246791680}$
a_{16}	-	-	-	$\frac{1}{764080128}$	$\frac{28013}{351829094400}$
a_{17}	-	-	-	$-\frac{1}{124366233600}$	$-\frac{5543}{3799754219520}$
a_{18}	-	-	-	-	$\frac{1}{62313753600}$
a_{19}	-	-	-	-	$-\frac{1}{12509779968000}$

Table 3. 2– Series coefficients cont.

These coefficients are substituted into the series solution (3.103) to form the particular solution to the second differential equation (3.89). In further derivations these solutions will be denoted as $\Psi_n\,(2ky.)$

In summary, the particular solution for the differential equation, (3.58), is given as

$$H_p(y) = \frac{k}{\alpha} \exp(-2ky)\, \Psi_n(2ky), \tag{3.104}$$

where $\Psi_n(2ky)$ is the series solution given in equation (3.103) with coefficients given by Table 3.1 and Table 3.2 and where solutions have been obtained for $n = 0$ to 10.

The total solution for the $O(\epsilon)$ can now be written as

$$h^{(1)} = h_e^{(1)} + h_p^{(1)}$$

$$= C(X,T,\tau)\, G(y) \exp(i\Theta) + D(X,T,\tau)\, H(y) \exp(2i\Theta) + c.c. + b(y; X, T, \tau)$$

$$= \frac{iA_\xi}{k} \exp(-ky) \left[kyL_n(2ky) + nL_{n-1}(2ky)\right] \exp(i\Theta)$$

$$+ \frac{A^2 k}{\alpha} \exp(-2ky)\, \Psi_n(2ky) \exp(2i\Theta) + c.c. + b(y; \xi, \tau), \tag{3.105}$$

where $\Theta = k(x - ct)$; $A = A(\xi, \tau)$; $\Psi_n(2ky)$ is the series (3.103) whose coefficients are defined by Tables 3.1 and 3.2; $\xi = X + 2\alpha^2 k(2n+1)T$; and $b(y; X, T, \tau)$ is the mean flow caused by the interaction of the fundamental mode with itself and which will be determined in the $O(\epsilon^2)$ problem.

The corresponding boundary condition can also now be determined by taking the $O(\epsilon)$ solution (3.105) and the $O(1)$ solution (3.43) and substituting into the boundary conditions (3.50) to give

$$\phi^{(1)} = -\frac{k\left[A\exp(i\Theta) + A^* \exp(-i\Theta)\right]^2 (1+2n)}{\alpha^2} - \frac{in\left[A_\xi \exp(i\Theta) - A_\xi^* \exp(-i\Theta)\right]}{k\alpha}$$

$$-\frac{k\left[A^2 \exp(2i\Theta) + A^{*2} \exp(-2i\Theta)\right]}{\alpha^2} \Psi_n(0) - \frac{b(0; X, T, \tau)}{\alpha}, \tag{3.106}$$

where $\Theta = k(x - ct)$; $A = A(\xi, \tau)$; $\Psi_n(0)$ is the series (3.103) evaluated at $y = 0$ whose coefficients are defined by Tables 3.1 and 3.2; $\xi = X + 2\alpha^2 k(2n+1)T$; and $b(0; X, T, \tau)$ is the mean flow evaluated at $y = 0$ caused by the interaction of the fundamental mode with itself.

To facilitate the next calculation this solution (3.105) is written in the form

$$h^{(1)} = \frac{iA_\xi}{2k}\zeta_1(y)\exp(i\Theta) + \frac{A^2k}{\alpha}\zeta_2(y)\exp(2i\Theta) + c.c. + b(y;X,T,\tau), \quad (3.107)$$

where

$$\zeta_1(y) = \exp(-ky)\left[2kyL_n(2ky) + 2nL_{n-1}(2ky)\right], \quad (3.108)$$

$$\zeta_2(y) = \exp(-2ky)\,\Psi_n(2ky), \quad (3.109)$$

with $\Psi_n(2ky)$ being the series (3.103) whose coefficients are defined by Tables 3.1 and 3.2.

3.5 The $O(\epsilon^2)$ Problem

The $O(\epsilon^2)$ problem is formed by demanding that the sum of the coefficients of the terms proportional to ϵ^2 be zero after the asymptotic expansion (3.26) is introduced into nonlinear stability equation (3.20). This results in the following equation:

$$\begin{aligned}
h_t^{(2)} &+ h_0h_{0y}h_{xxx}^{(2)} + h_0h_{0y}h_{yyx}^{(2)} + (h_{0y})^2 h_{yx}^{(2)} = -h_T^{(1)} - h_\tau^{(0)} - 3h_0h_{0y}h_{xxX}^{(1)} \\
&-3h_0h_{0y}h_{xXX}^{(0)} - h_0h_y^{(0)}h_{xxx}^{(1)} - h_0h_y^{(1)}h_{xxx}^{(0)} - 3h_0h_y^{(0)}h_{xxX}^{(0)} - h_{0y}h^{(0)}h_{xxx}^{(1)} \\
&-h_{0y}h^{(1)}h_{xxx}^{(0)} - 3h_{0y}h^{(0)}h_{xxX}^{(0)} - h^{(0)}h_y^{(0)}h_{xxx}^{(0)} - h_0h_{0y}h_{yyX}^{(1)} - h_0h_y^{(0)}h_{yyx}^{(1)} \\
&-h_0h_y^{(1)}h_{yyx}^{(0)} - h_0h_y^{(0)}h_{yyX}^{(0)} - h_{0y}h^{(0)}h_{yyx}^{(1)} - h_{0y}h^{(1)}h_{yyx}^{(0)} - h_{0y}h^{(0)}h_{yyX}^{(0)} \\
&-h^{(0)}h_y^{(0)}h_{yyx}^{(0)} - h_{0y}h_x^{(1)}h_{xx}^{(0)} - h_{0y}h_x^{(0)}h_{xx}^{(1)} - 2h_{0y}h_x^{(0)}h_{xX}^{(0)} - h_{0y}h_X^{(0)}h_{xx}^{(0)} \\
&-h_x^{(0)}h_{xx}^{(0)}h_y^{(0)} - (h_{oy})^2 h_{yX}^{(1)} - 2h_{0y}h_y^{(1)}h_{yx}^{(0)} - 2h_{0y}h_y^{(0)}h_{yx}^{(1)} - 2h_{0y}h_y^{(0)}h_{yX}^{(0)} \\
&-\left(h_y^{(0)}\right)^2 h_{yx}^{(0)} + h_0h_{xxy}^{(1)}h_x^{(0)} + h_0h_{xxy}^{(0)}h_x^{(1)} + 2h_0h_{xXy}^{(0)}h_x^{(0)} + h_0h_{xxy}^{(0)}h_X^{(0)}
\end{aligned}$$

$$+h^{(0)}h^{(0)}_{xxy}h^{(0)}_x + h_{0y}h^{(1)}_{yy}h^{(0)}_x + h_{0y}h^{(0)}_{yy}h^{(1)}_x + h_{0y}h^{(0)}_{yy}h^{(0)}_X + h^{(0)}_y h^{(0)}_{yy}h^{(0)}_x$$

$$+h_0 h^{(1)}_{yyy}h^{(0)}_x + h_0 h^{(0)}_{yyy}h^{(1)}_x + h_0 h^{(0)}_{yyy}h^{(0)}_X + h^{(0)}h^{(0)}_{yyy}h^{(0)}_x + \left(h^{(0)}_x\right)^2 h^{(0)}_{xy}. \tag{3.110}$$

This equation is not solved for $h^{(2)}$. Instead conditions for the solvability of this equation (namely the Fredholm Alternative Theorem) establish equations governing the evolution of the envelope amplitude, A, introduced in the $O(1)$ problem and the mean flow, $b(y; X, T, \tau)$, introduced in the $O(\epsilon)$ problem. In order to obtain these equations the expressions for $h^{(1)}(x, y, t; X, T, \tau)$ and $h^{(0)}(x, y, t; X, T, \tau)$ as given in equations (3.107) and (3.43), respectively, and the wedge profile, $h_0(y) = \alpha y$, for the steady state solution are first substituted into equation (3.110) resulting in the equation,

$$h^{(2)}_t + \alpha^2 y h^{(2)}_{xxx} + \alpha^2 y h^{(2)}_{yyx} + \alpha^2 h^{(2)}_{yx} = -\left(b_T + \alpha^2 y b_{yyX} + \alpha^2 b_{yX}\right)$$

$$+\alpha\left(A_X A^* + AA^*_X\right)\left(3k^2 y\psi'\psi + 2k^2\psi^2 + \frac{1}{2}y\psi'\zeta_1'' - \frac{1}{2}y\psi''\zeta_1'\right)$$

$$+\alpha\left(A_X A^* + AA^*_X\right)\left(-y\psi'\psi'' + \psi\zeta_1'' - \psi''\zeta_1 - 2\left(\psi'\right)^2\right)$$

$$+\alpha\left(A_X A^* + AA^*_X\right)\left(+k^2 y\psi\psi' + \frac{1}{2}y\psi\zeta_1''' - \frac{1}{2}y\psi'''\zeta_1 + y\psi\psi'''\right)$$

$$+\left(-\frac{i}{2k}A_{XT}\zeta_1 + \frac{3\alpha^2 ik}{2}A_{XX}y\zeta_1 - 3\alpha^2 ikA_{XX}y\psi\right)\exp(i\Theta)$$

$$-\left(\frac{\alpha^2 i}{2k}A_{XX}y\zeta_1'' + \frac{\alpha^2 i}{2k}A_{XX}\zeta_1'\right)\exp(i\Theta)$$

$$+iA^2A^*\left\{6k^4 y\psi'\zeta_2 + 3k^4 y\psi\zeta_2' + 6k^4\psi\zeta_2 - k^4\psi\zeta_2 - 2k^2 y\psi'\zeta_2''\right.$$

$$+k^2 y\psi''\zeta_2' - 3k^2\psi\zeta_2'' + 3k^2\psi''\zeta_2 - 2k^2\psi'\zeta_2' - k^2 y\psi\zeta_2'''$$

$$\left.+2k^2 y\psi'''\zeta_2 + k\psi^2\psi''' + 4k^3\psi^2\psi' - k\left(\psi'\right)^3\right\}\exp(i\Theta) - A_\tau\psi\exp(i\Theta)$$

$$+i\alpha A\left(k^3 y\psi b_y + k^3\psi b - ky\psi'' b_y - k\psi'' b - 2k\psi' b_y +\right)\exp(i\Theta)$$

$$+i\alpha A\left(k\psi b_{yy} + ky\psi b_{yyy}\right)\exp(i\Theta)$$

$$+\beta_2(y)\exp(2i\Theta)+\beta_3(y)\exp(3i\Theta)+c.c., \tag{3.111}$$

where $\beta_2(y)$ and $\beta_3(y)$ are the terms proportional to $\exp(2i\Theta)$ and $\exp(3i\Theta)$, respectively, which will not be needed in the following calculations.

An examination of the right hand side of the $O(\epsilon^2)$ problem (3.111) shows that there are four parts to the particular solution where only the first two will be discussed since they give the relevant information on the free amplitude that is needed. The first part corresponds to the terms on the right hand side of equation (3.111) which are independent of x and t. If a particular solution was found for these terms this solution would be secular, that is, the solution would be unbounded in either x or t. Thus in order to eliminate any such behavior associated with these terms, the collection must be set to zero to give

$$b_T+\alpha^2 y b_{yyX}+\alpha^2 b_{yX}=\alpha(AA^*)_X\left(4k^2y\psi'\psi+2k^2\psi^2\right)$$

$$+\alpha(AA^*)_X\left(\frac{1}{2}y\psi'\zeta_1''-\frac{1}{2}y\psi''\zeta_1'-y\psi'\psi''+\psi\zeta_1''-\psi''\zeta_1\right)$$

$$+\alpha(AA^*)_X\left(-2(\psi')^2+\frac{1}{2}y\psi\zeta_1'''-\frac{1}{2}y\psi'''\zeta_1+y\psi\psi'''\right). \tag{3.112}$$

The second part corresponds to the terms proportional to $\exp(\pm i\Theta)$. In principle, a particular solution associated with these terms can be constructed in the form,

$$h_p^{(2)}=N(X,T,\tau)\,\Gamma(y)\exp(i\Theta)+c.c. \tag{3.113}$$

When this form is substituted into the $O(\epsilon^2)$ problem (3.111), it implies that $\Gamma(y)$ is determined by

$$Nik\alpha^2\left[y\Gamma''+\Gamma'+k(2n+1-ky)\Gamma\right]=-\frac{i}{2k}A_{XT}\zeta_1+\frac{3\alpha^2ik}{2}A_{XX}y\zeta_1-A_\tau\psi$$

$$-3\alpha^2ikA_{XX}y\psi-\frac{\alpha^2 i}{2k}A_{XX}y\zeta_1''-\frac{\alpha^2 i}{2k}A_{XX}\zeta_1'+iA^2A^*\left(6k^4y\psi'\zeta_2+3k^4y\psi\zeta_2'\right)$$

$$+iA^2A^*\left(5k^4\psi\zeta_2 - 2k^2y\psi'\zeta_2'' + k^2y\psi''\zeta_2' - 3k^2\psi\zeta_2'' + 3k^2\psi''\zeta_2\right)$$

$$+iA^2A^*\left(-2k^2\psi'\zeta_2' - k^2y\psi\zeta_2''' + 2k^2y\psi'''\zeta_2 + k\psi^2\psi''' + 4k^3\psi^2\psi' - k\left(\psi'\right)^3\right)$$

$$+i\alpha A\left(k^3y\psi b_y + k^3\psi b - ky\psi''b_y - k\psi''b - 2k\psi'b_y + k\psi b_{yy} + ky\psi b_{yyy}\right). \quad (3.114)$$

These two equations can be solved by first solving equation (3.112), since the second equation (3.114) has mean flow, b, terms that need to be defined by the first equation (3.112). First, the variable, $z = 2ky$, is introduced into equation (3.112) to give

$$b_T + 2k\alpha^2\left(zb_{zzX} + b_{zX}\right) = \alpha k^2\left(AA^*\right)_X\left(4z\psi\psi' + 2\psi^2 + 2z\psi'\zeta_1'' - 2z\psi''\zeta_1' - 4z\psi\psi''\right)$$

$$+\alpha k^2\left(AA^*\right)_X\left(4\psi\zeta_1'' - 4\psi''\zeta_1 - 8\left(\psi'\right)^2 + 2z\psi\zeta_1''' - 2z\psi'''\zeta_1 + 4z\psi\psi'''\right), \quad (3.115)$$

where primes now denote differentiation with respect to z. This equation can be simplified further by using the relationship between X and T found in equation (3.73) given by the variable

$$\xi = X + 2k\alpha^2\left(2n+1\right)T, \quad (3.116)$$

which gives the partial derivatives,

$$\partial_T\left(*\right) = 2\alpha^2k\left(2n+1\right)\partial_\xi\left(*\right), \quad \partial_X = \partial_\xi. \quad (3.117)$$

Substituting these derivatives into equation (3.115) gives the equation

$$\left[\left(2n+1\right)b + \left(zb_{zz} + b_z\right)\right]_\xi = \frac{k}{2\alpha}\left(AA^*\right)_\xi\left(4z\psi\psi' + 2\psi^2 + 2z\psi'\zeta_1''\right)$$

$$+\frac{k}{2\alpha}\left(AA^*\right)_\xi\left(-2z\psi''\zeta_1' - 4z\psi\psi'' + 4\psi\zeta_1'' - 4\psi''\zeta_1\right)$$

$$+\frac{k}{2\alpha}\left(AA^*\right)_\xi\left(-8\left(\psi'\right)^2 + 2z\psi\zeta_1''' - 2z\psi'''\zeta_1 + 4z\psi\psi'''\right). \quad (3.118)$$

The right hand side of this equation (3.118) can be simplified by using the differential equations for ψ and ζ_1,

$$z\psi'' + \psi' + \left(\frac{2n+1}{2} - \frac{z}{4}\right)\psi = 0, \tag{3.119}$$

$$z\zeta_1'' + \zeta_1' + \left(\frac{2n+1}{2} - \frac{z}{4}\right)\zeta_1 = (2n+1-z)\,\psi, \tag{3.120}$$

(given in equations (3.35) and (3.77)) along with their derivatives taken with respect to z :

$$z\psi''' + 2\psi'' + \left(\frac{2n+1}{2} - \frac{z}{4}\right)\psi' - \frac{1}{4}\psi = 0, \tag{3.121}$$

$$z\zeta_1''' + 2\zeta_1'' + \left(\frac{2n+1}{2} - \frac{z}{4}\right)\zeta_1' - \frac{1}{4}\zeta_1 = (2n+1-z)\,\psi' - \psi. \tag{3.122}$$

These equations are solved for $z\psi''$, $z\zeta_1''$, $z\psi'''$ and $z\zeta_1'''$ and then substituted into the right hand side of equation (3.118). After some simplifications, the resulting equation is given by

$$\begin{aligned}&[zb_{zz} + b_z + (2n+1)\,b]_\xi \\ &= \frac{k}{2\alpha}\,(AA^*)_\xi\left(4\,(2n+1)\,\psi\psi' - 4\,(\psi')^2 - 8\psi\psi + \psi^2\right).\end{aligned} \tag{3.123}$$

Integrating this equation once with respect to ξ gives

$$\begin{aligned}&zb_{zz} + b_z + (2n+1)\,b \\ &= \frac{k}{2\alpha}\,(AA^*)\left(4\,(2n+1)\,\psi\psi' - 4\,(\psi')^2 - 8\psi\psi'' + \psi^2\right).\end{aligned} \tag{3.124}$$

The right hand side can be simplified further using $\psi = \exp(-z/2)\,L_n(z)$ resulting in the equation:

$$\begin{aligned}zb_{zz} + b_z + (2n+1)\,b = &-\frac{k}{2\alpha}\,(AA^*)\exp(-z)\left(4\,(n+1)\,[L_n]^2\right) \\ &-\frac{k}{2\alpha}\,(AA^*)\exp(-z)\left(8\,(n+2)\,L_nL_{n-1}^1 + 4\left(L_{n-1}^1\right)^2 + 8L_nL_{n-2}^2\right),\end{aligned} \tag{3.125}$$

where the argument of the Laguerre polynomials is z. The equation is now ready to be solved.

In order to solve this equation, first of all, the homogeneous solution to (3.125) denoted by $b_h(z)$ is found which is needed in solving the nonhomogeneous problem using Green's functions. In solving the homogeneous problem, the variable change

$$\nu = 2\left[(2n+1)z\right]^{1/2}, \tag{3.126}$$

is made to give

$$b_{h_{\nu\nu}} + \frac{1}{\nu} b_{h_\nu} + b_h = 0. \tag{3.127}$$

This is just Bessel's equation of order zero (Zwillinger, 1989) so that

$$b_h(z) = \begin{cases} J_0\left[2\sqrt{(2n+1)z}\right] \\ Y_0\left[2\sqrt{(2n+1)z}\right] \end{cases}. \tag{3.128}$$

Since our equation for b is of the Sturm - Liouville type, that is,

$$\frac{\partial}{\partial z}(zb_z) + (2n+1)b = f(z;\xi,\tau),$$

where $f(z;\xi,\tau)$ is the right hand of equation (3.125), the inhomogeneous solution can be found by considering Green's functions. The Green's functions corresponding to this problem are the functions, $K(z;\gamma)$, that satisfy the differential equation, (Zauderer, 1989)

$$-\frac{\partial}{\partial z}(-zK(z;\gamma)) + (2n+1)K(z;\gamma) = -\delta(z-\gamma), \tag{3.129}$$

with the boundary conditions:

$$K(0;\gamma) = 0, \qquad \lim_{z\to\infty} K(z;\gamma) = 0. \tag{3.130}$$

The differential equation (3.129) reduces to the problem

$$-\frac{\partial}{\partial z}\left(-zK\left(z;\gamma\right)\right)+\left(2n+1\right)K\left(z;\gamma\right)=0,\ z\neq\gamma, \tag{3.131}$$

with the jump conditions

$$K\left(z;\gamma\right) \text{ continuous at } z=\gamma, \tag{3.132}$$

$$\left[\frac{\partial K\left(z;\gamma\right)}{\partial z}\right]_{z=\gamma}=-\frac{1}{\gamma}. \tag{3.133}$$

The Green's solutions to this differential equation is identical to the homogeneous solutions of the original equation for b (3.127) multiplied by constants, R_1 and R_2, which will be determined by considering the boundary conditions (3.132) and (3.133). Thus the Green's solutions are

$$K\left(z,\gamma\right)=\begin{cases} R_1J_0\left[2\sqrt{\left(2n+1\right)z}\right], & 0\leq z<\gamma<\infty \\ R_2Y_0\left[2\sqrt{\left(2n+1\right)z}\right], & 0<\gamma<z<\infty \end{cases}, \tag{3.134}$$

since $J_0\left(0\right)=0$ and $\lim_{z\to\infty}Y_0\left[2\sqrt{\left(2n+1\right)z}\right]=0$. The first jump condition (3.132) is met when

$$R_1J_0\left[2\sqrt{\left(2n+1\right)\gamma}\right]=R_2Y_0\left[2\sqrt{\left(2n+1\right)\gamma}\right], \tag{3.135}$$

which means that

$$R_1=\frac{R_2Y_0\left[2\sqrt{\left(2n+1\right)\gamma}\right]}{J_0\left[2\sqrt{\left(2n+1\right)\gamma}\right]}. \tag{3.136}$$

The second jump condition (3.133) gives the equation

$$-\frac{R_2Y_0\left[2\sqrt{\left(2n+1\right)\gamma}\right]}{J_0\left[2\sqrt{\left(2n+1\right)\gamma}\right]}\left(2n+1\right)^{1/2}\gamma^{-1/2}J_0'\left[2\sqrt{\left(2n+1\right)\gamma}\right]$$

$$+R_2 (2n+1)^{1/2} \gamma^{-1/2} Y_0' \left[2\sqrt{(2n+1)\gamma}\right] = -\frac{1}{\gamma}. \tag{3.137}$$

Solving for R_2 and simplifying gives

$$R_2 = \frac{J_0}{(Y_0 J_0' - Y_0' J_0)(2n+1)\gamma^{1/2}}, \tag{3.138}$$

where the argument of the Bessel functions is assumed to be $2\sqrt{(2n+1)\gamma}$. Using the identity (Gradshteyn and Ryzhik, 1980),

$$J_0 Y_0' - Y_0 J_0' = \frac{1}{\pi (2n+1)^{1/2} \gamma^{1/2}}, \tag{3.139}$$

simplifies this equation to

$$R_2 = \pi J_0 \left[2\sqrt{(2n+1)\gamma}\right], \tag{3.140}$$

and gives the first coefficient as

$$R_1 = \pi Y_0 \left[2\sqrt{(2n+1)\gamma}\right]. \tag{3.141}$$

Therefore the Green's function is given as

$$K(z;\gamma) = \begin{cases} \pi Y_0 \left[2\sqrt{(2n+1)\gamma}\right] J_0 \left[2\sqrt{(2n+1)z}\right], & 0 \le z < \gamma < \infty \\ \pi J_0 \left[2\sqrt{(2n+1)\gamma}\right] Y_0 \left[2\sqrt{(2n+1)z}\right], & 0 < \gamma < z < \infty \end{cases}. \tag{3.142}$$

Using this Green's function to form the particular solution, b_p, of equation (3.125) gives

$$b_p = \pi Y_0 \left[2\sqrt{(2n+1)z}\right] \int_0^z J_0 \left[2\sqrt{(2n+1)\gamma}\right] f(\gamma;\xi,\tau)\, d\gamma$$
$$+\pi J_0 \left[2\sqrt{(2n+1)z}\right] \int_z^\infty Y_0 \left[2\sqrt{(2n+1)\gamma}\right] f(\gamma;\xi,\tau)\, d\gamma, \tag{3.143}$$

where

$$f(z;\xi,\tau) = -\frac{k}{2\alpha}(AA^*)\exp(-z)\left(4(n+1)[L_n(z)]^2 + 8(n+2)L_n(z)L^1_{n-1}(z)\right)$$

$$+\frac{k}{2\alpha}(AA^*)\exp(-z)\left(4\left[L^1_{n-1}(z)\right]^2 + 8L_n(z)L^2_{n-2}(z)\right). \qquad (3.144)$$

This forms the solution to the first differential equation (3.125) given in solving the $O(\epsilon^2)$ problem. Because of the dependence of this solution on the Laguerre polynomials and thus on n, the function, b, will be denoted as

$$b(y;\xi,\tau) = \frac{k}{2\alpha}|A|^2(\xi,\tau)\,b_n, \qquad (3.145)$$

where b_n is given by (3.143) and (3.144) with the coefficient $k|A|^2/2\alpha$ factored out.

The second differential equation (3.114) can now be considered. First of all, it is simplified in much the same manner as the first differential equation by making the variable change, $z = 2ky$, resulting in the equation

$$N2ik^2\alpha^2\left(z\Gamma'' + \Gamma' + \left(\frac{2n+1}{2} - \frac{z}{4}\right)\Gamma\right) = -\frac{i}{2k}A_{XT}\zeta_1 + \frac{3\alpha^2 i}{4}A_{XX}z\zeta_1 - A_\tau\psi$$

$$-\frac{3\alpha^2 i}{2}A_{XX}z\psi - \alpha^2 iA_{XX}z\zeta_1'' - \alpha^2 iA_{XX}\zeta_1' + iA^2A^*k^4(6z\psi'\zeta_2 + 3z\psi\zeta_2')$$

$$+iA^2A^*k^4(5\psi\zeta_2 - 8z\psi'\zeta_2'' + 4z\psi''\zeta_2' - 12\psi\zeta_2'' + 12\psi''\zeta_2)$$

$$+iA^2A^*k^4\left(-8\psi'\zeta_2' - 4z\psi\zeta_2''' + 8z\psi'''\zeta_2 + 8\psi^2\psi''' + 8\psi^2\psi' - 8(\psi')^3\right)$$

$$+i\alpha Ak^3(z\psi b_z + \psi b - 4z\psi''b_z - 4\psi''b - 8\psi'b_z + 4\psi b_{zz} + 4z\psi b_{zzz}). \qquad (3.146)$$

The right hand side of this equation can be simplified by using the relations given in equations (3.119)–(3.122) and solving them for $z\psi''$, $z\zeta_1''$, $z\psi'''$ and $z\zeta_1'''$. These relations are then substituted into the right hand side of equation (3.146) and the equation is simplified to give

$$N2ik^2\alpha^2\left(z\Gamma''+\Gamma'+\left(\frac{2n+1}{2}-\frac{z}{4}\right)\Gamma\right)$$

$$=\alpha^2 iA_{\xi\xi}\left(-\frac{2n+1}{2}\zeta_1+\frac{1}{2}z\zeta_1-\frac{1}{2}z\psi-(2n+1)\psi\right)-A_\tau\phi$$

$$+iA^2A^*k^4\left(8z\psi'\zeta_2+4z\psi\zeta_2'+7\psi\zeta_2-8z\psi'\zeta_2''-12\psi'\zeta_2'\right)$$

$$+iA^2A^*k^4\left(-2(2n+1)\psi\zeta_2'-12\psi\zeta_2''-4\psi''\zeta_2-4z\psi\zeta_2'''\right)$$

$$+iA^2A^*k^4\left(-4(2n+1)\psi'\zeta_2+8\psi^2\psi'''+8\psi^2\psi'-8(\psi')^3\right)$$

$$+i\alpha Ak^3\left(\psi b+4\psi'b_z+2(2n+1)\psi b_z-4\psi''b\right).$$

$$+i\alpha Ak^3\left(-8\psi'b_z+4\psi b_{zz}+4z\psi b_{zzz}\right). \tag{3.147}$$

In addition, the function ζ_2 and the mean flow, b, satisfy the equations,

$$z\zeta_2''+\zeta_2'+\left(\frac{2n+1}{2}-z\right)\zeta_2=\frac{3}{4}\psi^2-(\psi')^2-2\psi\psi'', \tag{3.148}$$

$$b=\frac{k}{2\alpha}(AA^*)b_n, \tag{3.149}$$

$$z(b_n)_{zz}+(b_n)_z+(1+2n)b_n=\left(4(2n+1)\psi\psi'-4(\psi')^2-8\psi\psi''+\psi^2\right), \tag{3.150}$$

(see equations (3.58) and (3.124)). These equations can be differentiated with respect to z showing that ζ_2 and b also satisfy the equations,

$$z\zeta_2'''+2\zeta_2''+\left(\frac{2n+1}{2}-z\right)\zeta_2'-\zeta_2=\frac{3}{2}\psi\psi'-4\psi'\psi''-2\psi\psi''', \tag{3.151}$$

$$z(b_n)_{zzz}+2(b_n)_{zz}+(1+2n)b_z$$

$$=4(2n+1)(\psi')^2+4(2n+1)\psi\psi''-16\psi'\psi''-8\psi\psi'''+2\psi\psi', \tag{3.152}$$

Solving these equations for zb_{zz}, $z\zeta_2''$, zb_{zzz} and $z\zeta_2'''$ and then substituting into equation (3.144) simplifies further to

$$
\begin{aligned}
&N\left(z\Gamma'' + \Gamma' + \left(\frac{2n+1}{2} - \frac{z}{4}\right)\Gamma\right) \\
&= \frac{1}{2k^2}A_{\xi\xi}\left(-\frac{2n+1}{2}\zeta_1 + \frac{1}{2}z\zeta_1 - \frac{1}{2}z\psi - (2n+1)\psi\right)\phi \\
&-\frac{A_\tau}{2i\alpha^2k^2}\psi + \frac{k^2}{2\alpha^2}A^2A^*\left(3\psi\zeta_2 - 4\psi'\zeta_2' - 4\psi\zeta_2'' - 4\psi''\zeta_2 +\right) \\
&+\frac{k^2}{2\alpha^2}A^2A^*\left(8\,(2n+1)\,\psi\,(\psi')^2 + 8\,(2n+1)\,\psi^2\psi'' + \frac{\psi b_n}{2}\right) \\
&+\frac{k^2}{2\alpha^2}A^2A^*\left(-2\psi'\,(b_n)_z - (2n+1)\,\psi\,(b_n)_z - 2\psi''b_n - 2\psi\,(b_n)_{zz}\right).
\end{aligned} \tag{3.153}
$$

This equation is now ready to be examined.

In Section 3.4, it was explained that by the Fredholm Alternative Theorem a solution to an inhomogeneous ordinary differential equation written in the form $\mathcal{L}F = \Phi$ exists if and only if

$$\langle\phi, \Phi\rangle = 0, \tag{3.154}$$

where $\phi \in Ker\left(\mathcal{L}^A\right)$, $\mathcal{L}^A$ being the adjoint operator to $\mathcal{L}$. Equation (3.153) can be rewritten as

$$(z\Gamma')' + \left(\frac{2n+1}{2} - \frac{z}{4}\right)\Gamma = \Phi\left(z;\xi,\tau\right), \tag{3.155}$$

which is a self adjoint equation. Any solutions then that are in the kernel of the adjoint of $\mathcal{L}$ are also in the kernel of $\mathcal{L}$, that is, they are solutions of the homogeneous equation.

$$(z\Gamma_e')' + \left(\frac{2n+1}{2} - \frac{z}{4}\right)\Gamma_e = 0. \tag{3.156}$$

This is again the same equation as that obtained in the linear stability problem and the $O(1)$ problem (see equation (3.40)) and its solution is given by

$$\Gamma_e(z) = \psi(z) = \exp(-z/2)\, L_n(z)\,, \tag{3.157}$$

where L_n are the Laguerre polynomials.

Therefore the Fredholm Alternative Theorem gives

$$\frac{1}{2k^2}A_{\xi\xi}\int_0^\infty \left(-\frac{2n+1}{2}\zeta_1 + \frac{1}{2}z\zeta_1 - \frac{1}{2}z\psi - (2n+1)\psi\right)\psi\, dz$$

$$-\frac{A_\tau}{2i\alpha^2k^2}\int_0^\infty \psi^2 dz + \frac{k^2}{2\alpha^2}A^2A^*\left[8(2n+1)\int_0^\infty \left(\psi^2(\psi')^2 + \psi^3\psi''\right)dz\right]$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\left[\int_0^\infty (3\psi\zeta_2 - 4\psi'\zeta_2' - 4\psi\zeta_2'' - 4\psi''\zeta_2)\,\psi\, dz\right]$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\int_0^\infty \left(\frac{\psi b_n}{2} - 2\psi'(b_n)_z - (2n+1)\psi(b_n)_z\right)\psi dz$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\int_0^\infty (-2\psi'' b_n - 2\psi(b_n)_{zz})\,\psi dz = 0. \tag{3.158}$$

In order to evaluate this integral, the substitutions,

$$\psi(z) = \exp(-z/2)\, L_n(z)\,,$$

$$\zeta_1(z) = \exp(-z/2)\,(zL_n(z) + 2nL_{n-1}(z))\,,$$

are made in the first two integrals where only quadratic terms of these functions are found. This gives the simplification

$$\frac{1}{2k^2}A_{\xi\xi}\int_0^\infty \exp(-z)\left[\left(\frac{1}{2}z^2 - (n+1)z - (2n+1)\right)[L_n(z)]^2\right]dz$$

$$\frac{1}{2k^2}A_{\xi\xi}\int_0^\infty \exp(-z)\,(nz - n(2n+1))\,L_n(z)\,L_{n-1}(z)\,dz$$

$$-\frac{A_\tau}{2i\alpha^2k^2}\int_0^\infty \exp(-z)\,[L_n(z)]^2\, dz$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\left[\int_0^\infty\left(3\psi\zeta_2-4\psi'\zeta_2'-4\psi\zeta_2''-4\psi''\zeta_2\right)\psi\,dz\right]$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\left[8\left(2n+1\right)\int_0^\infty\left(\psi^2\left(\psi'\right)^2+\psi^3\psi''\right)dz\right]$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\int_0^\infty\left(\frac{\psi b_n}{2}-2\psi'\left(b_n\right)_z-\left(2n+1\right)\psi\left(b_n\right)_z\right)\psi\,dz$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\int_0^\infty\left(-2\psi''b_n-2\psi\left(b_n\right)_{zz}\right)\psi dz=0. \tag{3.159}$$

In order to use the orthogonality conditions for the Laguerre polynomials (see equation (3.68)) the first two integrals are integrated by parts to give

$$-\frac{(2n+1)}{2k^2}A_{\xi\xi}\int_0^\infty\exp\left(-z\right)\left[\left[L_n\left(z\right)\right]^2+nL_n\left(z\right)L_{n-1}\left(z\right)\right]dz$$

$$-\frac{A_\tau}{2i\alpha^2k^2}\int_0^\infty\exp\left(-z\right)\left[L_n\left(z\right)\right]^2dz$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\left[\int_0^\infty\left(3\psi\zeta_2-4\psi'\zeta_2'-4\psi\zeta_2''-4\psi''\zeta_2\right)\psi\,dz\right]$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\left[8\left(2n+1\right)\int_0^\infty\left(\psi^2\left(\psi'\right)^2+\psi^3\psi''\right)dz\right]$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\int_0^\infty\left(\frac{\psi b_n}{2}-2\psi'\left(b_n\right)_z-\left(2n+1\right)\psi\left(b_n\right)_z\right)\psi\,dz$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\int_0^\infty\left(-2\psi''b_n-2\psi\left(b_n\right)_{zz}\right)\psi dz=0. \tag{3.160}$$

The orthogonality condition makes the first term of the first integral equal to one and the second term zero. The second integral is also equal to one resulting in the simplifications

$$-\frac{(2n+1)}{2k^2}A_{\xi\xi}-\frac{A_\tau}{2i\alpha^2k^2}+8\frac{k^2}{2\alpha^2}A^2A^*\left(2n+1\right)\int_0^\infty\left(\psi^2\left(\psi'\right)^2+\psi^3\psi''\right)dz$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\left[\int_0^\infty\left(3\psi\zeta_2-4\psi'\zeta_2'-4\psi\zeta_2''-4\psi''\zeta_2\right)\psi\,dz\right]$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\int_0^\infty\left(\frac{\psi b_n}{2}-2\psi'(b_n)_z-(2n+1)\psi(b_n)_z\right)\psi\,dz$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\int_0^\infty\left(-2\psi''b_n-2\psi(b_n)_{zz}\right)\psi dz=0. \qquad (3.161)$$

The rest of the integrals are harder to evaluate. However they first can be simplified somewhat using integration by parts resulting in

$$-\frac{(2n+1)}{2k^2}A_{\xi\xi}-\frac{A_\tau}{2i\alpha^2k^2}+\frac{k^2}{2\alpha^2}A^2A^*\left[\frac{4}{3}(2n+1)^2+\frac{16}{3}(2n+1)\int_0^\infty\psi^3\psi''dz\right]$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\int_0^\infty\left(3\psi^2-4(\psi')^2-8\psi\psi''\right)\zeta_2dz$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\int_0^\infty\left(\frac{1}{2}\psi^2-2(\psi')^2-4\psi\psi''+2(2n+1)\psi'\right)b_ndz$$

$$+\frac{k^2}{2\alpha^2}\left[A^2A^*-4\psi^2\zeta_2'+4\psi\psi'\zeta_2\right]_0^\infty$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\left[2\psi\psi'b_n-(2n+1)\psi^2b_n-2\psi^2(b_n)_z\right]_0^\infty=0. \qquad (3.162)$$

Substituting in ψ from equation (3.157) gives

$$-\frac{(2n+1)}{2k^2}A_{\xi\xi}-\frac{A_\tau}{2i\alpha^2k^2}+\frac{k^2}{2\alpha^2}A^2A^*\left[\frac{4}{3}(2n+1)^2+4\zeta_2'(0)+2(2n+1)\zeta_2(0)\right]$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\left[2(2n+1)b_n(0)+2\left[b_n(0)\right]_z\right]$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\left[(-4)\int_0^\infty\exp(-z)\left(3L_n(z)L_{n-1}^1(z)+\left[L_{n-1}^1(z)\right]^2\right)\zeta_2(z)\,dz\right]$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\left[(-8)\int_0^\infty\exp(-z)L_n(z)L_{n-2}^2(z)\zeta_2(z)\,dz\right]$$

$$-\frac{k^2}{\alpha^2}A^2A^*\left[\int_0^\infty\exp(-z)\left((n+1)\left[L_n(z)\right]^2+\right)b_n(z)\,dz\right]$$

$$-\frac{k^2}{\alpha^2}A^2A^*\left[\int_0^\infty\exp(-z)\left(2(n+2)L_n(z)L_{n-1}^1(z)\right)b_n(z)\,dz\right]$$

$$+\frac{k^2}{2\alpha^2}A^2A^*\left[(-2)\int_0^\infty\exp(-z)\left(\left[L_{n-1}^1(z)\right]^2+2L_n(z)L_{n-2}^2(z)\right)b_n(z)\,dz\right]$$

$$+\frac{4}{3}\frac{k^2}{2\alpha^2}A^2A^*(2n+1)\int_0^\infty [L_n(z)]^3\left(L_n(z)+L_{n-1}^1(z)\right)\exp(-2z)\,dz$$

$$+\frac{4}{3}\frac{k^2}{2\alpha^2}A^2A^*(2n+1)\int_0^\infty [L_n(z)]^3\,L_{n-2}^2(z)\exp(-2z)\,dz=0. \qquad (3.163)$$

The first remaining integral is evaluated by considering the Laguerre polynomials (see equation (3.91)) and the series solutions, $\zeta_2(z)$, (see equation (3.103) with the coefficients given in Tables 3.1 and 3.2) for different values of n. These polynomials are multiplied out using the computer program, *Mathematica*, which is also used to integrate the resulting polynomial. The last integral involves quartic powers of the Laguerre polynomials and thus the orthogonality condition cannot be used to simplify the terms. This integral, therefore, is also evaluated by considering the Laguerre polynomials for different values of n and using *Mathematica* to integrate the polynomials. The second and third integrals contain the mean flow b whose value involves calculating integrals of Bessel functions multiplied by Laguerre polynomials (see equation (3.143)). These two integrals could only be evaluated numerically (see Appendix A) for different values of n.

The results from the calculations for these four integrals are added and equation (3.163) becomes

$$-\frac{(2n+1)}{2k^2}A_{\xi\xi}-\frac{A_\tau}{2i\alpha^2k^2}+\frac{k^2}{2\alpha^2}A^2A^*\beta_n=0, \qquad (3.164)$$

where β_n is given by the following table of values:

β_0	3.3638623
β_1	98.575693
β_2	168.13063
β_3	539.89187
β_4	536.44862

Table 3.3 – Beta Values

Most importantly, it is noted that the sign of this parameter, β_n, is positive and seems to be growing with n. (Higher n values could not be obtained because of computer limitations.) Also in calculating these values it is noted that the computer calculations involving the mean flow, b, contribute a large positive portion of the values for β_n.

The equation governing the free amplitude, A, derived by using the Fredholm Alternative Theorem then is given by

$$-\frac{(2n+1)}{2k^2}A_{\xi\xi} - \frac{A_\tau}{2i\alpha^2k^2} + \frac{k^2}{2\alpha^2}A^2A^*\beta_n = 0, \tag{3.165}$$

which when rearranged gives the Nonlinear Schrödinger Equation

$$A_\tau = \frac{i\omega''}{2}A_{\xi\xi} + ik^4\beta_n A^2A^*, \tag{3.166}$$

where,

$$\omega'' = -2\alpha^2(2n+1), \tag{3.167}$$

and β_n is a positive parameter.

Chapter 4

Solutions to the Nonlinear Schrödinger Equation

4.1 Introduction

In chapter 2, a frontal model for buoyancy–driven flows was developed. The Hamiltonian formulation of this model was used to show that all steady–state solutions to this model are linearly stable and that certain steady state solutions meeting specific criteria are nonlinearly stable. Specifically, the steady state solution given by the wedge–front,

$$h_0(y) = \alpha y, \tag{4.1}$$

was found to satisfy these criteria and was therefore nonlinearly stable in the sense of Liapunov.

In chapter 3, the weakly nonlinear evolution of this wedge front was examined. The leading order solution, with respect to the amplitude expansion parameter ϵ, included a free complex valued envelope amplitude function, $A(\epsilon x, \epsilon t, \epsilon^2 t)$. The $O(\epsilon^2)$ problem, however, demonstrated that the free amplitude, A, satisfied the

Nonlinear Schrödinger (NLS) equation given as

$$A_\tau = \frac{i\omega''}{2} A_{\xi\xi} + ik^4\beta_n |A|^2 A, \tag{4.2}$$

where

$$\omega'' = -2\alpha^2 (2n+1), \tag{4.3}$$

where β_n is a positive parameter whose values dependent on n are given in Table 3.3, where τ is a slow time variable given by $\tau = \epsilon^2 t$, and where the variable ξ is defined by

$$\xi = \epsilon \left(x + 2\alpha^2 k (2n+1) t\right). \tag{4.4}$$

The evolution of this amplitude, A, is determined by the solutions to the NLS equation (4.2). Once these solutions are determined, the complete form of the nonlinear evolution of the perturbations to the wedge–front (4.1) is determined. Therefore, in this chapter several of the solutions to the NLS equation are examined.

4.2 The Stokes Wave Solution

The first solution considered is the Stokes wave solution (Newell, 1985). The derivation of this solution begins with first rewriting the complex amplitude in the form

$$A = a \exp(i\phi), \tag{4.5}$$

where $a = a(\xi, \tau)$ and $\phi = \phi(\xi, \tau)$ are both real functions. Substituting this form (4.5) into the NLS equation (4.2) gives, using the real and imaginary parts, respectively,

$$a_\tau = -\omega'' a_\xi \phi_\xi - \frac{\omega''}{2} a \phi_{\xi\xi}, \tag{4.6}$$

$$\phi_\tau = \frac{\omega''}{2}\frac{a_{\xi\xi}}{a} - \frac{\omega''}{2}(\phi_\xi)^2 + k^4\beta_n a^2. \tag{4.7}$$

Newell (1985) explains the physical meaning behind these equations by examining the underlying wave motion of the fundamental mode of the perturbation flow given by

$$h^{(0)} = A\psi(y)\exp(ik(x-ct)) + c.c.,$$
$$= a\psi(y)\exp(i(k(x-ct)+\phi)) + c.c.. \qquad (4.8)$$

Newell defines the *total* local wavenumber, $\widetilde{k}$, as the x derivative of the *total* phase given by,

$$\Phi = k(x-ct) + \phi, \qquad (4.9)$$

giving

$$\widetilde{k} = k + \epsilon\phi_\xi, \qquad (4.10)$$

and the *total* local frequency, $\widetilde{\omega}$, as the negative t derivative of the *total* phase giving

$$\widetilde{\omega} = kc + c_g\phi_\xi - \epsilon^2\phi_\tau. \qquad (4.11)$$

The imaginary part of the NLS equation, that is (4.7), then expresses the change in the local frequency at the $O(\epsilon^2)$. Also, if the conservation for the number of waves is considered as follows

$$0 = (\Phi_x)_t + (-\Phi_t)_x = \widetilde{k}_t + \widetilde{\omega}_x$$
$$= \epsilon^3\phi_{\xi\tau} - \epsilon^2 c_g\phi_{\xi\xi} + \epsilon^2 c_g\phi_{\xi\xi} - \epsilon^3\left(\frac{\omega''}{2}\left(\frac{a_{\xi\xi}}{a}\right)_\xi - \omega''\phi_\xi\phi_{\xi\xi} + 2k^4\beta_n a a_\xi\right)$$
$$= \epsilon^3\phi_{\xi\tau} - \epsilon^3\left(\frac{\omega''}{2}\left(\frac{a_{\xi\xi}}{a}\right)_\xi - \omega''\phi_\xi\phi_{\xi\xi} + 2k^4\beta_n a a_\xi\right), \qquad (4.12)$$

and the imaginary part (4.7) is differentiated with respect to ξ to give

$$\phi_{\tau\xi} = \frac{\omega''}{2}\left(\frac{a_{\xi\xi}}{a}\right)_\xi - \omega''\phi_\xi\phi_{\xi\xi} + 2k^4\beta_n a a_\xi, \qquad (4.13)$$

they are shown to be equivalent statements. Thus equation (4.13) expresses the conservation of the number of waves or crests as well. The real part of the NLS equation (4.6) expresses the conservation of wave action (Newell, 1985).

The Stokes wave solution to these equations is given by the monochromatic wave form (Newell, 1985),

$$a = a_0, \tag{4.14}$$

$$\phi = k^4 \beta_n a_0^2 \tau + const, \tag{4.15}$$

where a_0 is a constant. Using this wave form in the total phase (4.9) gives

$$\Phi = k\,(x - ct) + \phi\,(\tau)$$

$$= k\,(x - ct) + \epsilon^2 k^4 \beta_n a_0^2 t + const., \tag{4.16}$$

where the total local wavenumber (4.10) and total local frequency (4.11) are now given as

$$\tilde{k} = k, \tag{4.17}$$

$$\tilde{\omega} = kc - \epsilon^2 k^4 \beta_n a_0^2. \tag{4.18}$$

Examining the local wavenumber (4.17) and frequency (4.18) shows that there is no dependence on the variable ξ. Using the monochromatic waveform, (4.14) and (4.15) in the amplitude (4.5) gives the spatially independent amplitude as follows

$$A = a \exp\,(i\phi) = a_0 \exp\left(ik^4 \beta_n a_0^2 \tau + const.\right). \tag{4.19}$$

This amplitude is known as the Stokes wave.

The Stokes wave possesses an interesting property that could lead to instabilities. The phase speed of the Stokes wave is given as

$$\tilde{c} = \frac{\tilde{\omega}}{\tilde{k}} = \frac{kc - \epsilon^2 k^4 \beta_n a_0^2}{k} = c - \epsilon^2 k^3 \beta_n a_0^2, \tag{4.20}$$

where c is the phase speed of the underlying linear wave structure and equations (4.17) and (4.18) have been used. Examining this phase speed (4.20) shows that it is dependent on the amplitude, a_0, of the wave. This means that the wave motion near the peak or the trough of the envelope described by A, travels faster than the motion on either side of the peak. If the wave motion reinforces this effect, then instabilities known as Benjamin–Feir instabilities could develop (Newell, 1985).

4.3 Benjamin-Feir Stability

In Section 2.7, the wedge–front was found to be nonlinearly stable for the frontal model. Therefore all the solutions of the frontal model and thus of the NLS equation should be stable. Newell (1985) demonstrated how the specific Stokes wave solution can be examined for modulational or Benjamin–Feir instabilities.

Newell (1985) begins this stability analysis by perturbing the Stokes wave solution given by equations (4.14) and (4.15) in the form

$$a = a_0 + \tilde{a}, \qquad \phi = k^4 \beta_n a_0^2 \tau + \tilde{\phi}, \tag{4.21}$$

where $\tilde{a}$ and $\tilde{\phi}$ are assumed small perturbation variables. These perturbed variables are substituted into the real and imaginary parts of the NLS equation given by equations (4.6) and (4.7), respectively. Neglecting quadratic and higher order perturbation terms gives the linearized equations,

$$\tilde{a}_\tau = -\frac{\omega''}{2} a_0 \tilde{\phi}_{\xi\xi}, \tag{4.22}$$

$$k^4 \beta_n a_0^2 + \tilde{\phi}_\tau = \frac{\omega''}{2} \frac{\tilde{a}_{\xi\xi}}{a_0} + 2k^4 \beta_n a_0 \tilde{a}. \tag{4.23}$$

A single equation is formed by differentiating the first equation (4.22) once with respect to τ and the second equation (4.23) twice with respect to ξ. Adding the results gives the combined equation

$$\tilde{a}_{\tau\tau} = -\left(\frac{\omega''}{2}\right)^2 \tilde{a}_{\xi\xi\xi\xi} - \omega'' k^4 \beta_n a_0^2 \tilde{a}_{\xi\xi}. \tag{4.24}$$

Since the coefficients of this equation (4.24) are independent of ξ and τ, a normal mode solution of the form,

$$a = C \exp\left(ir\xi + \sigma\tau\right), \tag{4.25}$$

is substituted into equation (4.23), where r and σ are, respectively, a wavenumber and growth rate and where C is a free amplitude constant. This substitution gives the dispersion relation

$$\sigma^2 = -\left(\frac{\omega''}{2}\right)^2 r^4 + \omega'' k^4 \beta_n a_0^2 r^2. \tag{4.26}$$

Examining the normal mode solution (4.25) shows that if σ is purely imaginary then the solution merely oscillates in time. However if σ has a positive real component then the perturbation amplitude, a, grows exponentially with time. Therefore the sign of σ^2 must be determined by examining the right hand side of the dispersion relation (4.26). The relation

$$\omega'' = -2\alpha^2 \left(2n+1\right), \tag{4.27}$$

can be substituted into the dispersion relation to give

$$\sigma^2 = -\left[\left(\alpha^2 \left(2n+1\right)\right)^2 r^4 + 2\alpha^2 \left(2n+1\right) k^4 \beta_n a_0^2 r^2\right]. \tag{4.28}$$

Since β_n is a positive parameter (see Section 3.4), the right hand side of this equation (4.28) is negative definite. Therefore $\sigma^2 < 0$ and σ is purely imaginary meaning that the perturbation amplitude merely oscillates in time without increasing in size.

The Stokes wave solution,

$$A = a_0 \exp\left(ik^4\beta_n a_0^2\tau + const\right), \tag{4.29}$$

is therefore stable which is consistent with the conclusions obtained using the Hamiltonian formulation (see Section 2.7). This stability necessarily depends on the fact that the parameter β_n is positive. Since, as noted earlier (Section 3.5), the generated mean flow significantly contributes to making this parameter β_n positive, the mean flow is a stabilizing influence on the evolution of the wedge–front.

4.4 Plane Wave Solutions

4.4.1 The Derivations

Since the coefficients of the NLS equation (4.2) are independent of ξ and τ, normal mode solutions or plane wave solutions can also be formed in addition to the Stokes wave solution. Normal mode solutions for the amplitude, A, are assumed to be of the form

$$A = a_0 \exp\left[i\left(\lambda\xi + \Omega\tau\right)\right], \tag{4.30}$$

where a, λ, and Ω are assumed real. If equation (4.30) is substituted into the NLS equation (4.2) the corresponding dispersion relation is derived giving

$$\Omega = \alpha^2\lambda^2\left(2n+1\right) + k^4\beta_n\left|a_0\right|^2. \tag{4.31}$$

Thus the plane wave solution (4.30) satisfies the NLS equation provided the dispersion relation (4.31) is satisfied.

The complete nonlinear evolution of the wedge–front is given as

$$h = \alpha y + \epsilon h^{(0)}(x,y,t) + \epsilon^2 h^{(1)}(x,y,t) + O\left(\epsilon^2\right)$$

$$= \alpha y + \epsilon A \exp(i\Theta) L_n(2ky) \exp(-ky)$$

$$+\epsilon^2 \frac{i}{2k} A_\xi \exp(i\Theta)\left[2kyL_n(2ky) + 2nL_{n-1}(2ky)\right]\exp(-ky)$$

$$+\epsilon^2 \frac{k}{\alpha} A^2 \exp(2i\Theta)\Psi_n(2ky)\exp(-2ky) + c.c.$$

$$+\epsilon^2 \frac{k}{2\alpha}|A|^2 b_n(y;\xi,\tau), \tag{4.32}$$

where equations (3.43) and (3.105) have been used and where $\Theta = k(x-ct)$, $c = -k\alpha^2(2n+1)$, and $\xi = \epsilon\left(x + 2k\alpha^2(2n+1)t\right)$ and where $\Psi_n(2ky)$ is the series solution given in equation (3.103) with coefficients given in Tables 3.1 and 3.2 and where b_n is the mean flow given in equation (3.143).

The plane wave solutions can be substituted for A into this description (4.32) of the nonlinear evolution. Using the point generating programs given in Appendix B.1 and the graphing program *SpyGlass*, Figures (4.1)–(4.4) are constructed to give contour graphs of the perturbed front for these nonlinear plane wave solutions of the amplitude function, A.

In these contour plots, the slope of the wedge front, α, the wavenumber, k, the wavenumber of the plane wave solution, λ, and the amplitude of the plane wave solution, a_0, are all set equal to 1.0 for simplicity. The plots give a picture of the complete flow for $t = 0$. The value chosen for ϵ for all four plots is 0.25 which is quite large so that the effects of the perturbation can be seen on the total flow.

In each successive plot the value of the integer, n, is increased beginning with 0. As n is increased the complexity of the flow increases as the Laguerre polynomials describing the flow increase in degree. In the first plot for $n = 0$, there is simple wave–like interface (contour height of 0.0) and the effects of the perturbation quickly die out as y gets large ($\simeq 2$). In the second plot for $n = 1$, the flow also dies out quickly but the interface is more complicated and seems to be forming eddies. For $n = 2$ and $n = 3$, the interface gets more complex and yet the perturbation still dies out by $y \simeq 4$.

4.4.2 The Total Plane Front

Figure 4-1: Total Plane Front -- n=0

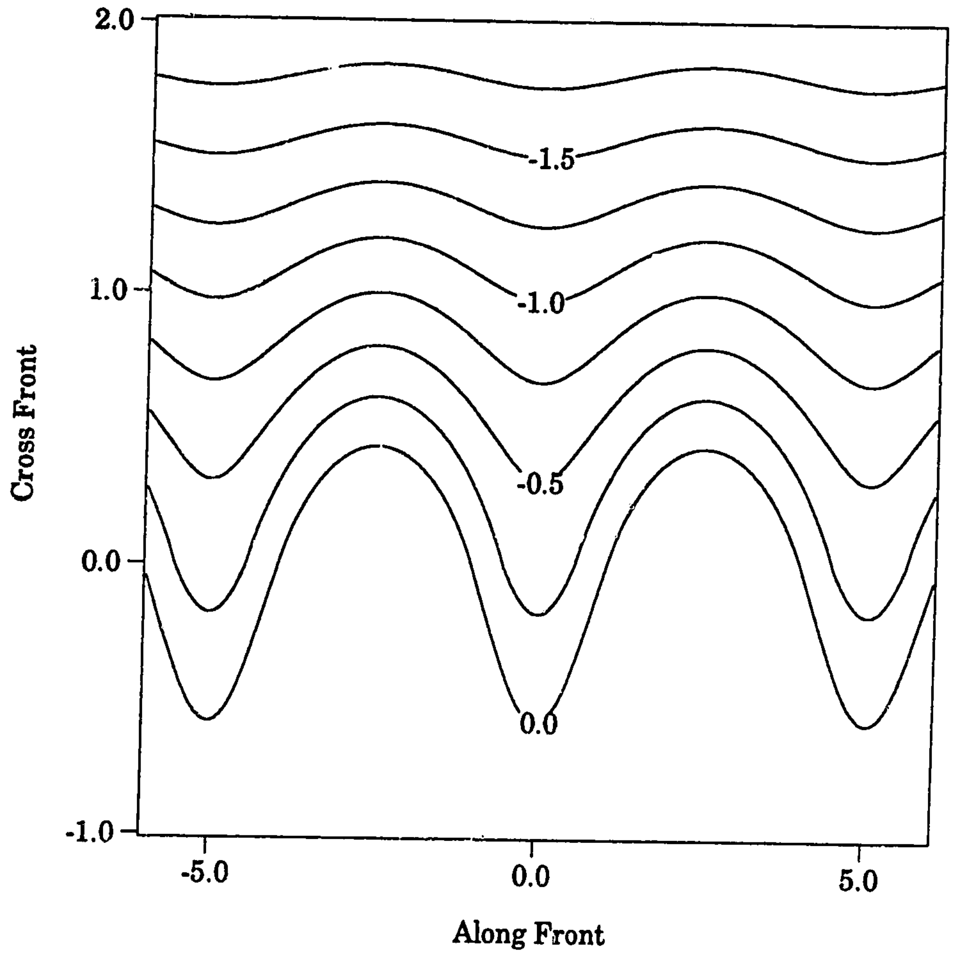


Figure 4-2: Total Plane Front -- n=1

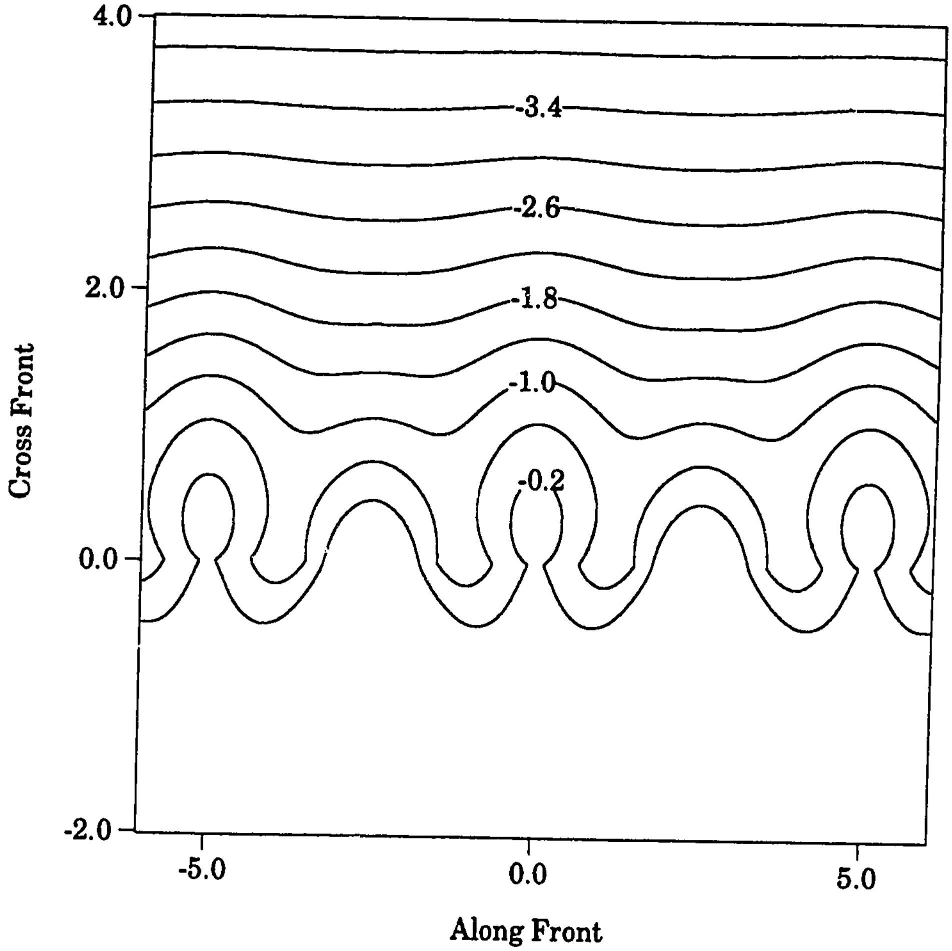


Figure 4-3: Total Plane Front -- n=2

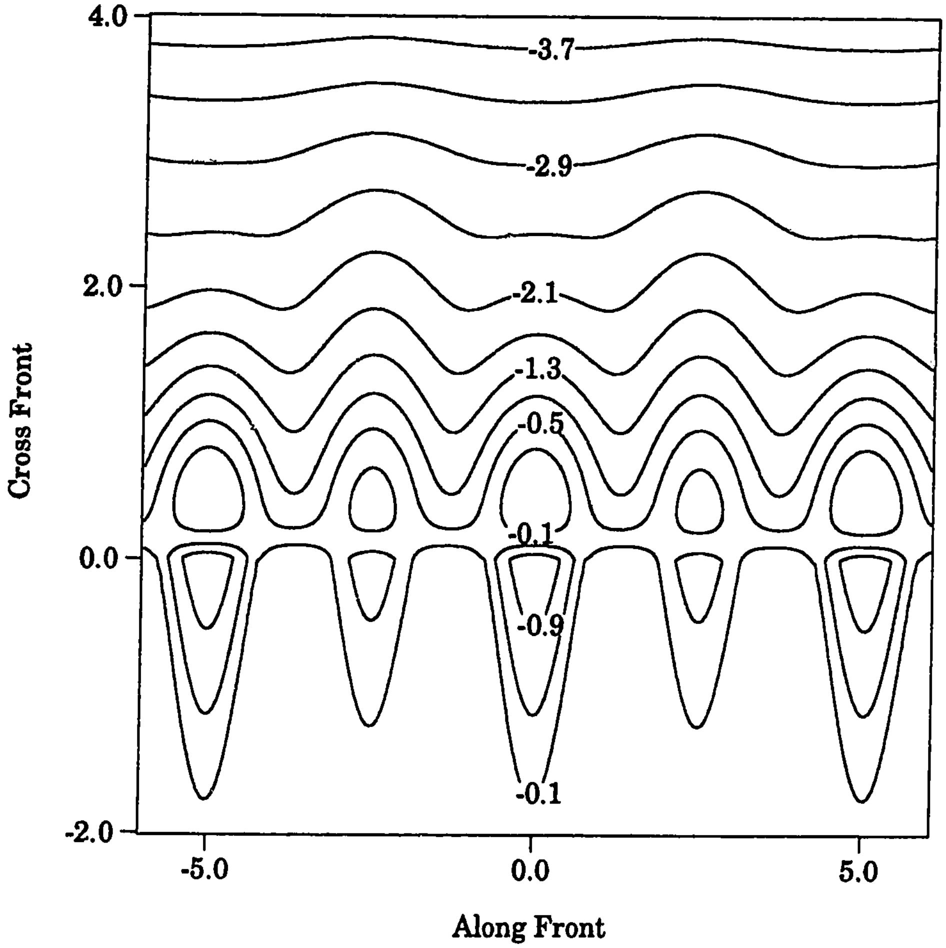
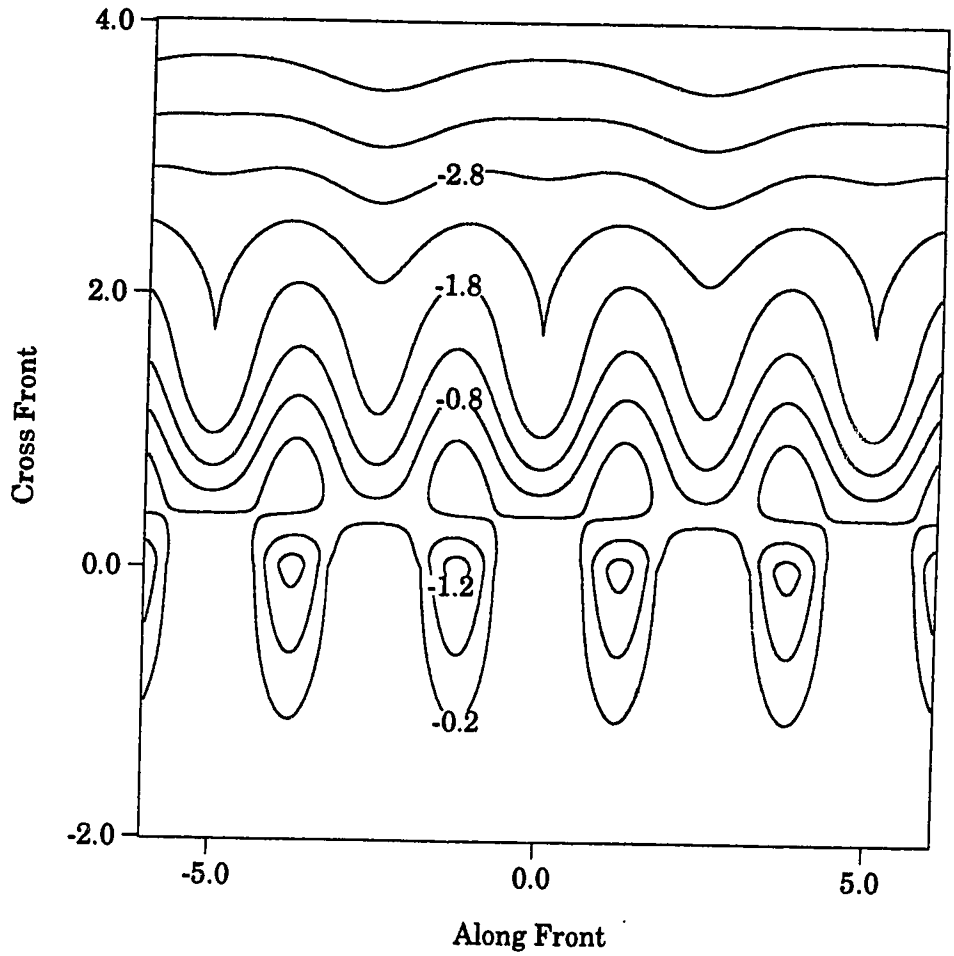


Figure 4-4: Total Plane Front -- n=3



4.5 The Snoidal Wave Solution

4.5.1 The Derivations

The third solution given to the Nonlinear Schrödinger equation describes a quasi-periodic solution for $A(\xi, \tau)$. In order to find this solution, the NLS equation is simplified somewhat by introducing the new independent variables

$$Y = \left(\frac{k^4 \beta_n}{\alpha^2 (2n+1)}\right)^{1/2} \xi, \tag{4.33}$$

$$q = -k^4 \beta_n \tau. \tag{4.34}$$

Substituting these variables (4.33) and (4.34) into the NLS equation (4.2) gives the standard form of the NLS equation

$$iA_q + A_{YY} - A^2 A^* = 0. \tag{4.35}$$

Following methods used by Drazin and Johnson (1989), the amplitude is now assumed to have a real and imaginary part in the form

$$A = a \exp\left[i\left(\phi - mq\right)\right], \tag{4.36}$$

where $a = a(Y, q)$ is a real quantity; $\phi = \phi(Y, q)$; and m is some constant value. Substituting (4.33) into the standard Nonlinear Schrödinger equation (4.35) and separating out the real and imaginary parts gives the following set of equations:

$$a_q + 2\phi_Y a_Y + a\phi_{YY} = 0, \tag{4.37}$$

$$ma - \phi_q a + a_{YY} - a\left(\phi_Y\right)^2 - a^3 = 0. \tag{4.38}$$

The snoidal wave solution is a travelling wave solution so a new variable defined as

$$\psi = Y - \kappa q, \tag{4.39}$$

where κ is the velocity of the travelling wave, is introduced into equations (4.37) and (4.38). This gives the equations

$$-\kappa a_\psi + 2\phi_\psi a_\psi + a\phi_{\psi\psi} = 0, \tag{4.40}$$

$$ma + \kappa\phi_\psi a + a_{\psi\psi} - a\left(\phi_\psi\right)^2 - a^3 = 0. \tag{4.41}$$

In order to solve these equations, the first equation (4.40) is initially simplified. Multiplying equation (4.40) by a and integrating once with respect to ψ gives

$$-\frac{\kappa}{2}a^2 + \phi_\psi a^2 = c_1, \tag{4.42}$$

where c_1 is a constant of integration, so that

$$\phi_\psi = \frac{1}{2}\left(\kappa + \frac{c_1}{a^2}\right). \tag{4.43}$$

Equation (4.43) can now be used to simplify the second equation (4.41) to give

$$a_{\psi\psi} = a^3 - ma - \frac{\kappa^2}{4}a + \frac{c_1^2}{4a^3}. \tag{4.44}$$

If this equation (4.44) is multiplied through by a_ψ, it can be integrated once to give

$$\left(a_\psi\right)^2 = \frac{1}{2}a^4 - \left(m + \frac{\kappa^2}{4}\right)a^2 + c_2 - \frac{c_1^2}{4a^2}, \tag{4.45}$$

where c_2 is another constant of integration. In order to simplify yet further, equation (4.45) is multiplied through by $4a^2$ to give

$$\left(2aa_\psi\right)^2 = 2a^6 - \left(4m + \kappa^2\right)a^4 + c_2a^2 - c_1^2. \tag{4.46}$$

The variable $r = a^2$ with $r_\psi = 2aa_\psi$ is introduced into equation (4.46) to give the simplified equation

$$(r_\psi)^2 = 2\left(r^3 - 2\left(m + \frac{\kappa^2}{4}\right)r^2 + c_2 r - \frac{c_1^2}{2}\right) \equiv R(r). \qquad (4.47)$$

Different solutions to the NLS equation can be formed depending on how many roots exist to $R(r)$. The roots of this equation however depend on the value of the integration constants c_1 and c_2. Therefore the behavior of the resulting snoidal wave will depend on the values chosen for c_1 and c_2.

Following Drazin and Johnson (1989), the first thing that is considered in this equation is the sign of $R(r)$. Since r_ψ is a real quantity, its square will be positive and thus $R(r)$ must necessarily be positive. Therefore in what follows it is assumed that only r values for which $R(r)$ is positive will be considered. Secondly, since $R(r)$ is a cubic polynomial with real coefficients, it follows that it possesses at least one real root. The largest real root and possibly only real root of $R(r)$ will be denoted by r_1. Since $\lim_{r\to\infty} R(r) = \infty$ it is important to note that $R(r) > 0$ for all $r > r_1$ and thus $(r_\psi)^2$ is positive and r_ψ real for all $r > r_1$.

Drazin and Johnson (1989) demonstrate that there are three different classifications of r_1 that need to be considered. The first classification is that r_1 is a simple zero. If r_1 is a simple zero $R(r)$ can be Taylor expanded around r_1 so that

$$\begin{aligned} R(r) &= R(r_1) + R'(r_1)(r - r_1) + O\left((r - r_1)^2\right) \\ &= R'(r_1)(r - r_1) + O\left((r - r_1)^2\right), \end{aligned} \qquad (4.48)$$

for $r \to r_1$. This gives the differential equation

$$(r_\psi)^2 = 2R'(r_1)(r - r_1) + O\left((r - r_1)^2\right). \qquad (4.49)$$

Solving equation (4.49) for r gives

$$r = r_1 + \frac{1}{2}R'(r_1)(\psi - \psi_1)^2 + O\left((\psi - \psi_1)^3\right), \tag{4.50}$$

where ψ_1 satisfies $r(\psi_1) \equiv r_1$. Examining equation (4.50) shows that since $r_\psi(\psi_1) = 0$ and $r_{\psi\psi}(\psi_1) = R'(r_1) \neq 0$ there must exist an extreme point at $\psi = \psi_1$. This extreme point will be a maximum if $r_{\psi\psi}(\psi_1) = R'(r_1) < 0$ and a minimum if $r_{\psi\psi}(\psi_1) = R'(r_1) > 0$. In either case, the sign of r_ψ changes across $r = r_1$ and thus for $r < r_1$, r is increasing with ψ and for $r > r_1$, r is decreasing with ψ or vice versa.

The second classification is that r_1 is a double root which would make $R'(r_1) = 0$ so that the Taylor expansion of $R(r)$ becomes

$$R(r) = \frac{R''(r_1)}{2}(r - r_1)^2 + O\left((r - r_1)^3\right), \tag{4.51}$$

with the resulting differential equation,

$$(r_\psi)^2 = R''(r_1)(r - r_1)^2 + O\left((r - r_1)^3\right), \tag{4.52}$$

as $r \to r_1$. Again in order that r_ψ be real, its square must be positive so $R''(r_1) > 0$. By neglecting the $O\left((r - r_1)^3\right)$ the solution for r can be found to be

$$r - r_1 \sim K \exp\left[\pm\left(R''(r_1)\right)^{1/2}\psi\right] \text{ as } \psi \to \mp\infty, \tag{4.53}$$

where the signs are vertically ordered throughout. This wave has only one peak at $\psi = 0$ and approaches the constant r_1 as $\psi \to \pm\infty$. As will be shown in the next section, equation (4.53) describes the asymptotic structure of the soliton solution to (4.47) as $|\psi| \to \infty$.

Finally the root r_1 could be a triple root. This would mean that

$$R(r) = r^3 - 2\left(m + \frac{\kappa^2}{4}\right) r^2 + c_2 r - \frac{c_1^2}{2} = (r - r_1)^3. \tag{4.54}$$

In order to make $R(r)$ a perfect cube in this way, values for the constants must be set at

$$c_2 = \frac{4}{3}\left(m + \frac{\kappa^2}{4}\right)^2, \tag{4.55}$$

$$c_1 = \frac{4}{3\sqrt{3}}\left(m + \frac{\kappa^2}{4}\right)^{3/2}, \tag{4.56}$$

with the root r_1 being given as

$$r_1 = \frac{2}{3}\left(m + \frac{\kappa^2}{4}\right). \tag{4.57}$$

When this occurs the differential equation becomes

$$r_\psi = \left(r - \frac{2}{3}\left(m + \frac{\kappa^2}{4}\right)\right)^{3/2}. \tag{4.58}$$

This is a simple first order differential equation that can be solved for r to give

$$r = \frac{4}{(\psi + c_3)^2} + \frac{2}{3}\left(m + \frac{\kappa^2}{4}\right), \tag{4.59}$$

where c_3 is another constant of integration. This equation, however, is unbounded at $\psi = -c_3$. Therefore this solution is discarded since it is not bounded for all ψ.

In summary, bounded solutions may exist to the Nonlinear Schrödinger equation if $r = r_1$ is a single root in which case r_ψ changes sign across $r = r_1$ or if $r = r_1$ is a double root in which case $r \to r_1$ only as $\psi \to \pm\infty$. However within these two classifications, bounded solutions only exist if the other roots of the cubic equation meet certain conditions. In the case where $r = r_1$ is a double root the only possible case is that the other root r_2 is a simple root. This solution is discussed in the next section.

In the case where $r = r_1$ is a single root, $r = r_1$ can, first of all, be the only root of $R(r)$. Therefore $r = r_1$ will be the only point at which r_ψ equals zero and as shown earlier the sign of r_ψ will change sign at $r = r_1$. This means that everywhere except at $r = r_1$, r_ψ is either positive or negative and r is either increasing or decreasing with ψ. As $\psi \to \infty$, then, $r \to \pm\infty$ and the solution will be unbounded. Thus the first case where r_1 is a simple root and the only root must be discarded.

Secondly, $r = r_1$ can be a simple root and a second root, $r = r_2$, can exist as a double root of $R(r)$. This can be written as

$$R(r) = (r - r_1)(r - r_2)^2, \text{ where } r_2 < r_1. \tag{4.60}$$

with the resulting differential equation,

$$r_\psi = \sqrt{2}\,(r - r_1)^{1/2}(r - r_2). \tag{4.61}$$

Solving this equation gives the value

$$r = (r_1 - r_2)\tan^2\left(\left(\frac{r_1 - r_2}{2}\right)^{1/2}\psi + c_4\right) + r_1. \tag{4.62}$$

This solution is not bounded for

$$\left(\frac{r_1 - r_2}{2}\right)^{1/2}\psi + c_4 = \frac{\pi n}{2}, \tag{4.63}$$

where n is any integer and thus must be discarded.

Finally, $r = r_1$ can be a simple root of $R(r)$ and two other roots, $r = r_2$ and $r = r_3$ can also exist which both necessarily must be simple roots. In this case,

$$R(r) = (r - r_1)(r - r_2)(r - r_3), \tag{4.64}$$

and

$$r_\psi = \sqrt{2}\,(r - r_1)^{1/2}\,(r - r_2)^{1/2}\,(r - r_3)^{1/2}. \tag{4.65}$$

This case gives the snoidal wave solution.

As established previously, $R(r) \geq 0$ in order to obtain a real solution for $r(\psi)$. Drazin and Johnson (1989) consider the situation where $R(r) \geq 0$ for $r \in [r_3, r_2]$ and $r \geq r_1$. The roots r_1, r_2, and r_3 are all simple roots and as found previously, r_ψ will change sign across these roots. Therefore in the region $r > r_1$, r_ψ will not change sign and so r is either increasing or decreasing with ψ. Thus for all $r > r_1$, $r \to \pm\infty$ as $\psi \to \infty$ or in other words the solutions are unbounded and must be discarded. As a result, the values for r must be restricted to the region $[r_3, r_2]$. Since r_2 and r_3 are both simple roots, the behavior of r around these points is approximated by

$$r = r_{2,3} + \frac{1}{2}R'(r_{2,3})\,(\psi - \psi_{2,3}) + O\left((\psi - \psi_{2,3})^2\right), \tag{4.66}$$

as $r \to r_{2,3}$. This shows that r is algebraic around r_2 and r_3 and therefore these points will be separated by only a finite distance. Also, as a result of the fact that they are simple roots, the sign of r_ψ will change at these points. Thus, r will increase from r_3 to r_2 but then since the sign of r_ψ changes at r_2, r will start decreasing again towards r_3. Once it reaches r_3, again the sign of r_ψ changes and once again r increases towards r_2. Thus r will oscillate between r_2 and r_3 and it will have a periodic solution. The period of this solution is given as

$$2\int_{r_3}^{r_2} \frac{dr}{r_\psi} = 2\int_{r_3}^{r_2} \frac{dr}{[2R(r)]^{1/2}}. \tag{4.67}$$

The form of the periodic solution can be obtained by examining the differential equation

$$r_\psi = \sqrt{2}\,(r - r_1)^{1/2}\,(r - r_2)^{1/2}\,(r - r_3)^{1/2}. \tag{4.68}$$

Solutions to this equation (4.68) are given implicitly by

$$\psi = \psi_3 + \frac{1}{\sqrt{2}} \int_{r_3}^{r} \frac{ds}{\left[(s-r_1)(s-r_2)(s-r_3)\right]^{1/2}}, \text{ where } r_1 > r_2 > r_3. \tag{4.69}$$

In order to solve this implicit equation (4.69), Jacobian elliptic functions must be considered (Milne–Thomson, 1950). First of all a change of variables in equation (4.69) is made by using the variable,

$$\Phi = \left(\frac{s-r_3}{r_2-r_3}\right)^{1/2}, \tag{4.70}$$

to give the new integral equation,

$$\psi = \psi_3 + \left(\frac{2}{r_1-r_3}\right)^{1/2} \int_0^B \frac{d\Phi}{(1-M\Phi^2)^{1/2}(1-\Phi^2)^{1/2}}, \tag{4.71}$$

where $B = (r-r_3)^{1/2}(r_2-r_3)^{-1/2}$ and $M = (r_2-r_3)/(r_1-r_3)$. This is one form of an integral describing the Jacobian elliptic function, sn(∗). In order to solve for this function, a second change of variable is made so that,

$$\Phi = \mathrm{sn}\,(v|M). \tag{4.72}$$

With $B' = \mathrm{sn}^{-1}(B|M)$, this variable (4.72) changes the integral equation (4.71) again to

$$\begin{aligned} \psi &= \psi_3 + \left(\frac{2}{r_1-r_3}\right)^{1/2} \int_0^{B'} \frac{\mathrm{cn}\,(v|M)\,\mathrm{dn}\,(v|M)}{(1-M\mathrm{sn}^2(v|M))^{1/2}(1-\mathrm{sn}^2(v|M))^{1/2}} dv, \\ &= \psi_3 + \left(\frac{2}{r_1-r_3}\right)^{1/2} \int_0^{B'} \frac{\mathrm{cn}\,(v|M)\,\mathrm{dn}\,(v|M)}{\mathrm{dn}\,(v|M)\,\mathrm{cn}\,(v|M)} dv, \\ &= \psi_3 + \left(\frac{2}{r_1-r_3}\right)^{1/2} \int_0^{B'} dv. \end{aligned} \tag{4.73}$$

This equation (4.73) simplifies to

$$\psi = \psi_3 + \left(\frac{2}{r_1 - r_3}\right)^{1/2} \mathrm{sn}^{-1}\left(\left(\frac{r - r_3}{r_2 - r_3}\right)^{1/2} \middle| \frac{r_2 - r_3}{r_1 - r_3}\right), \tag{4.74}$$

which can be solved for the variable r to give the solution,

$$r = (r_2 - r_3)\,\mathrm{sn}^2\left[\left(\frac{r_1 - r_3}{2}\right)^{1/2} (\psi - \psi_3) \middle| \frac{r_2 - r_3}{r_1 - r_3}\right] + r_3, \tag{4.75}$$

where $r_1 > r_2 > r_3$. This snoidal solution (4.75) is bounded and periodic with period,

$$2\int_{r_3}^{r_2} \frac{ds}{\left[(s - r_1)(s - r_2)(s - r_3)\right]^{1/2}}. \tag{4.76}$$

A special limit of this solution can be found by using the identity (Milne-Thomson, 1950),

$$\mathrm{sn}\,(u|1) = \tanh u. \tag{4.77}$$

In order to find this limit, the variable M is equated to one or

$$\frac{r_2 - r_3}{r_1 - r_3} = 1, \tag{4.78}$$

which is equivalent to stating that $r_1 = r_2$. Therefore this reduces to the case where $r_1 = r_2 = r$ is a double root and $r_3 = r$ is the other smaller simple root. Substituting the identity (4.77) into the solution (4.75) gives the solution

$$r = (r_1 - r_3)\tanh^2\left[\left(\frac{r_1 - r_3}{2}\right)^{1/2} (\psi - \psi_3)\right] + r_3. \tag{4.79}$$

This is the solution that will be discussed in the next section.

An exact solution for r can be found by finding values for r_1, r_2 and r_3 so that they are simple roots of the cubic equation $R(r)$. The values of r_1, r_2 and r_3 will be simple roots of $R(r)$ if

$$R(r) = (r - r_1)(r - r_2)(r - r_3) \text{ with } r_1 > r_2 > r_3, \tag{4.80}$$

or in other words,

$$\begin{aligned} &r^3 - 2\left(m + \frac{\kappa^2}{4}\right) r^2 + c_2 r - \frac{c_1^2}{2} \\ &= r^3 - (r_1 + r_2 + r_3) r^2 + (r_1 r_2 + r_2 r_3 + r_1 r_3) r - r_1 r_2 r_3. \end{aligned} \tag{4.81}$$

Because $r = a^2$, the variable r must be greater than or equal to zero. This means that the minimum value of r, that is, r_3, should be greater than or equal to zero. In order to simplify equation (4.81) somewhat r_3 is set equal to zero so that

$$r^3 - 2\left(m + \frac{\kappa^2}{4}\right) r^2 + c_2 r - \frac{c_1^2}{2} = r^3 - (r_1 + r_2) r^2 + (r_1 r_2) r. \tag{4.82}$$

It is obvious that $c_1 = 0$ and thus the resulting cubic equations on both sides can be factored to give

$$r\left(r - R^-\right)\left(r - R^+\right) = r(r - r_1)(r - r_2), \tag{4.83}$$

where

$$r_{1,2} = R^{\pm} = \left(m + \frac{\kappa^2}{4}\right) \pm \sqrt{\left(m + \frac{\kappa^2}{4}\right)^2 - c_2}, \tag{4.84}$$

and where the value of $c_2 \neq 0$ then will depend on the boundary conditions. The exact solution for r is found by substituting these chosen values for r_1 and r_2 given in equation (4.84) and the assumption $r_3 = 0$ into equation (4.75) to give

$$r = R^- \mathrm{sn}^2 \left[\left(\frac{R^+}{2}\right)^{1/2} (\psi - \psi_3) \middle| \frac{R^-}{R^+} \right], \tag{4.85}$$

where $R^{\pm}$ is given by equation (4.84).

Going back to the original variable a, the real part of the amplitude solution,

gives

$$\sqrt{r} = a = \left(R^-\right)^{1/2} \text{sn}\left[\left(\frac{R^+}{2}\right)^{1/2}(\psi - \psi_3)\middle|\frac{R^-}{R^+}\right]. \tag{4.86}$$

The second function of the NLS equation, ϕ, can be found by solving the resulting imaginary part of the NLS equation (4.43). This equation is simplified by substituting equation (4.86) into equation (4.43) giving the differential equation,

$$\phi_\psi = \frac{1}{2}\left(\kappa + \frac{c_1}{a^2}\right) = \frac{\kappa}{2}, \tag{4.87}$$

since $c_1 = 0$ for this specific solution. The solution to this equation is simply

$$\phi = \frac{\kappa}{2}\psi, \tag{4.88}$$

where the constant of integration is assumed to be zero. Thus the complete snoidal solution using equation (4.86) and (4.88) in equation (4.36) is given by

$$A = \left(R^-\right)^{1/2} \exp\left(i\left(\frac{\kappa}{2}\psi - mq\right)\right) \text{sn}\left[\left(\frac{R^+}{2}\right)^{1/2}(\psi - \psi_3)\middle|\frac{R^-}{R^+}\right]$$

$$= \left(R^-\right)^{1/2} \exp\left(i\left(\frac{\kappa}{2}\psi + mk^4\beta_n\tau\right)\right) \text{sn}\left[\left(\frac{R^+}{2}\right)^{1/2}(\psi - \psi_3)\middle|\frac{R^-}{R^+}\right], \tag{4.89}$$

where

$$\psi = \left(\frac{k^4\beta_n}{\alpha^2(2n+1)}\right)^{1/2}\xi + \kappa k^4\beta_n\tau; \tag{4.90}$$

and $R^\pm = \left(m + \frac{\kappa^2}{4}\right) \pm \sqrt{\left(m + \frac{\kappa^2}{4}\right)^2 - c_2}$ with $c_2 \neq 0$ as a constant of integration whose value depends on the boundary conditions.

In the Figures 4–5 and 4–6, the amplitude function, A, given by (4.89) and (4.90) is depicted alone. The first figure (4–5) shows the solution when $R^- = R^+$, that is, in the limiting case of when the snoidal function becomes the hyperbolic tangent function. In order to achieve this the parameters are set so that $m = 0$,

$\kappa = 1$, and $c_1 = 0.25$. The rest of the parameters are chosen arbitrarily so that $k = 1$, $\alpha = 1$ and $\epsilon = 0.025$. The simplest value for n, that is, $n = 0$, is also chosen. The hyperbolic tangent structure can be easily seen in this figure.

Figure 4–6 shows the amplitude where the constants have been chosen arbitrarily as $m = 4$, $\kappa = 1$, $k = 1$, $\alpha = 1$ and $c_2 = .25$ so that $R^{\pm} = \frac{17}{4} \pm 3\sqrt{2}$. The simplest value of n, $n = 0$, has again been chosen. These parameters show that this amplitude function is bounded and quasiperiodic. Even though the snoidal function is periodic, the amplitude function involves multiplying this solution by an exponential and thus true periodicity is lost.

Similar to the plane wave solutions presented last section this snoidal wave solution (4.89) is substituted into the complete nonlinear evolution of the wedge-front given by equation (4.32). The point generating programs for this solution is given in Appendix B.2. Passing the generated coordinates through *SpyGlass* yielded Figures 4.7–4.15.

Figures 4–7 to 4–11, however, illustrate the perturbation thickness alone. This is achieved by running the programs with the basic state, αy, omitted in the function. Figure 4–7 is the total perturbation thickness with the parameters set equal to the parameters used in graphing 4-5, the hyperbolic tangent limit. The dotted lines show negative perturbation thicknesses whereas the solid lines show positive perturbation thicknesses. As will be observed, this perturbation flow has many of the same characteristics as the solitary perturbation wave flow. Figure 4–8 is the total thickness with the same parameter values as those chosen for Figure 4–6. The quasiperiodicity is again evident in this figure. In Figure 4–9, the domain from Figure 4–8 is simply extended to show that this perturbation flow does in fact exhibit definite quasiperiodicity.

Figures 4–10 and 4–11 are identical to Figures 4–7 and 4–8, respectively, except that the value of ϵ has been increased by tenfold so that the magnitude of the perturbation thickness has increased. The periodicity is again evident. These are the perturbations to which the wedge–front is added in creating Figure 4–12 and 4–13.

The total snoidal front is shown in Figures 4–12 to 4–15. The parameters set for Figure 4–12 are again identical to the parameters chosen for the hyperbolic tangent limit shown in Figure 4–5 except that the larger $\epsilon = 0.25$ is chosen so that the effects of the perturbation can be seen on the flow. Figure 4–12 shows that the hyperbolic tangent amplitude gives the flow a basic periodicity except for a small nodal point at $x \simeq -2$. Figure 4–13 is the total snoidal front with the parameters chosen to be the same as for Figure 4–6 except that again ϵ is increased to $\epsilon = 0.25$ so that the effects of the perturbation can be seen. This flow has a definite periodicity.

In Figure 4–14 and 4–15 the same parameters as for Figure 4–12 and Figure 4–13, respectively, are used except that this time the value for n has been increased to 2. The basic structure of Figure 4–12 and Figure 4–13 is repeated in Figure 4–14 and Figure 4–15, respectively, except for some increased complexity. In both of these flows also, a nodal line exists at $y \simeq 2$. This occurs because of the increased complexity of the Laguerre polynomials with n.

4.5.2 The Snoidal Amplitude

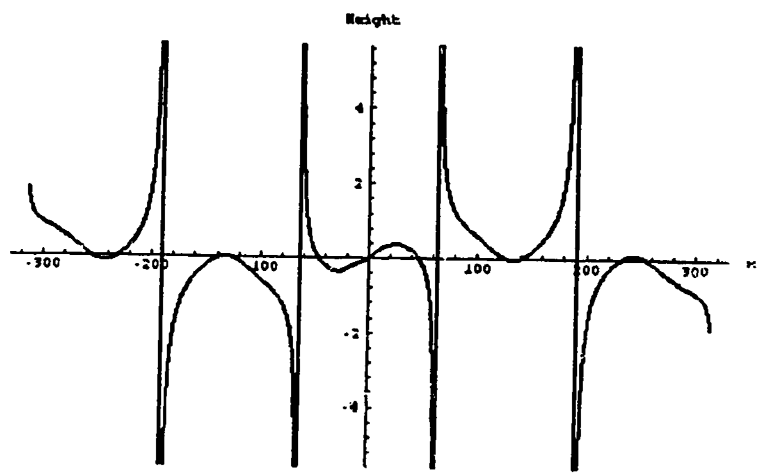


Figure 4-5: The Snoidal Amplitude - Hyperbolic Tangent Limit

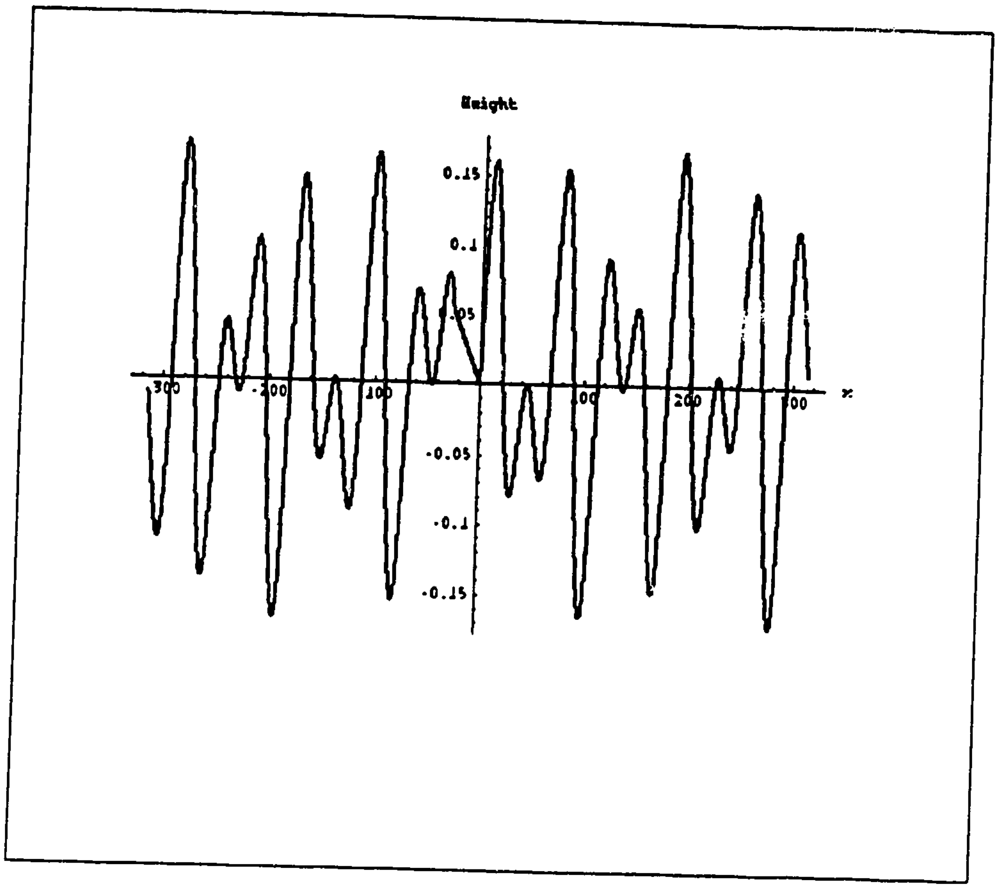


Figure 4-6: The Snoidal Amplitude

4.5.3 The Snoidal Perturbation Thickness

Figure 4-7: Gravest Mode -- m=0 -- Small Epsilon

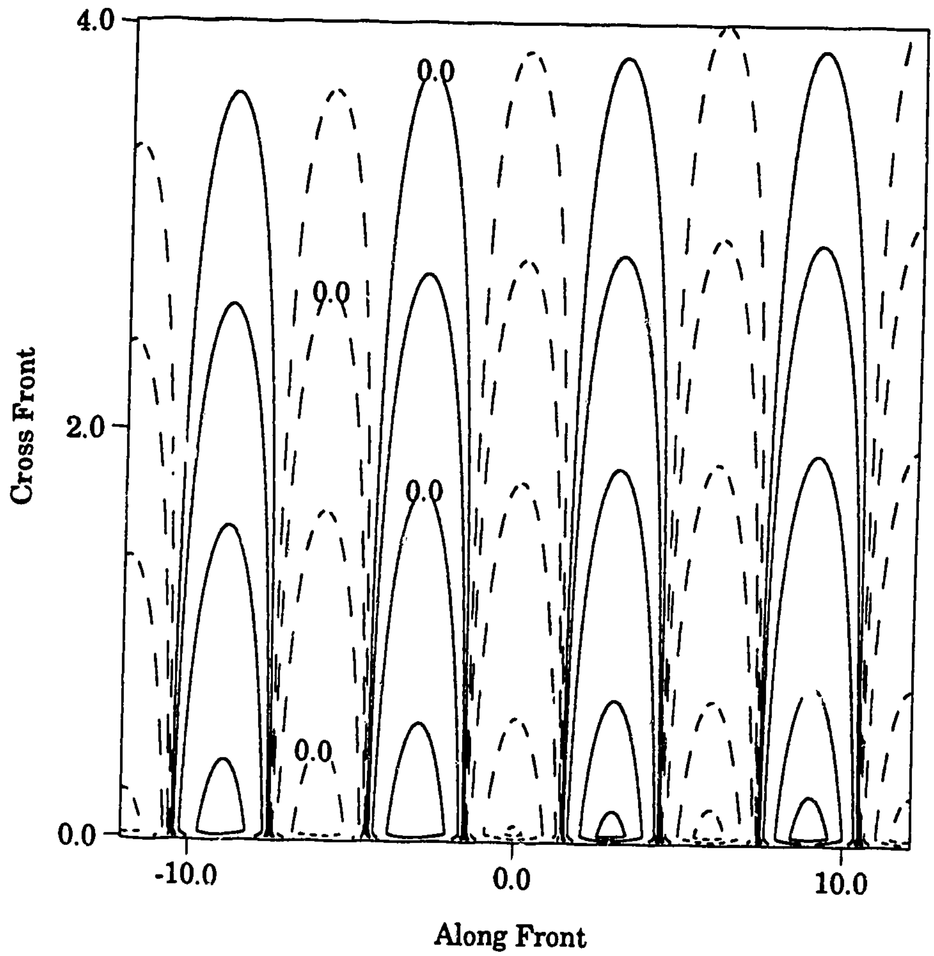


Figure 4-8: Gravest Mode -- m=4 -- Small Epsilon

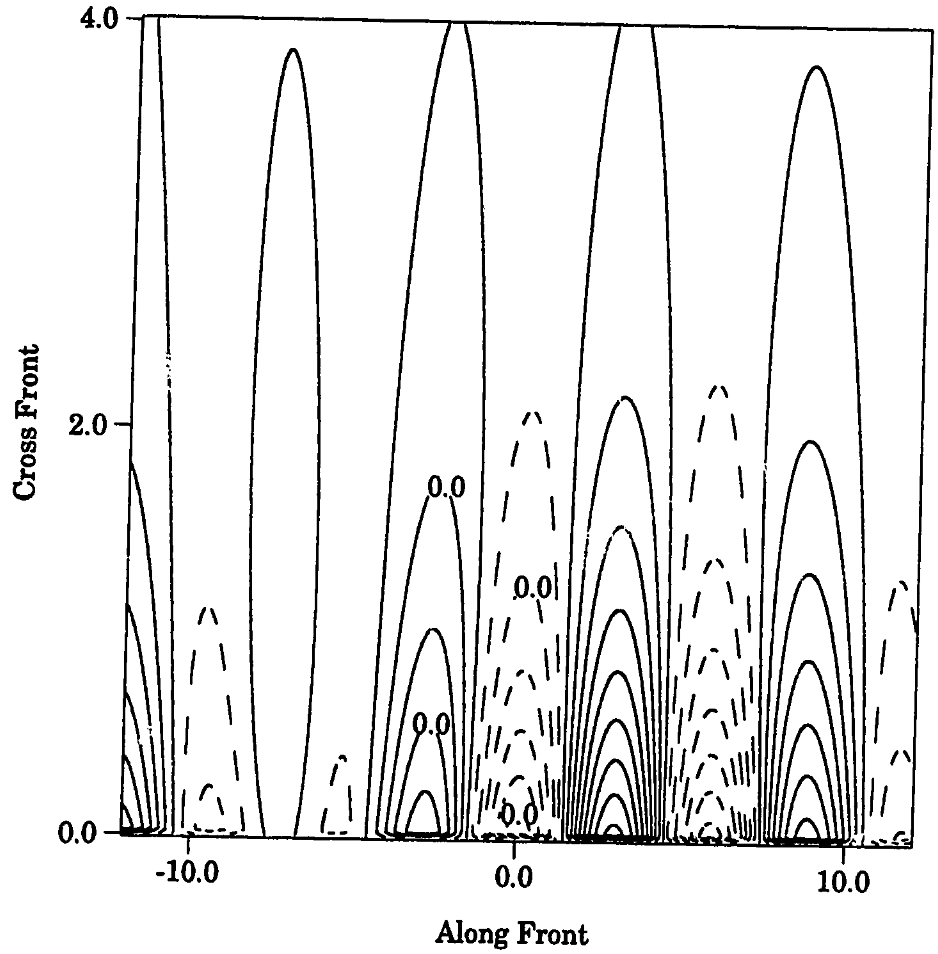


Figure 4-9: Gravest Mode -- m=4 -- Small Epsilon

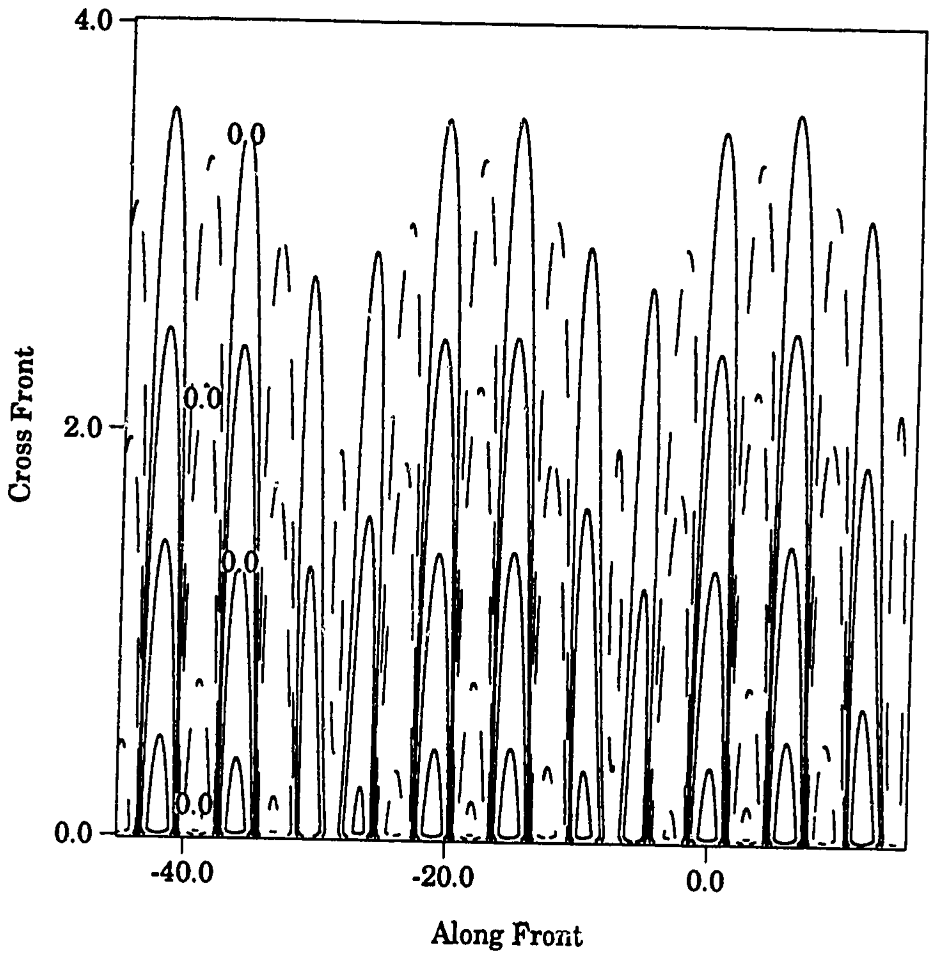


Figure 4-10: Gravest Mode -- Hyperbolic Tangent Limit

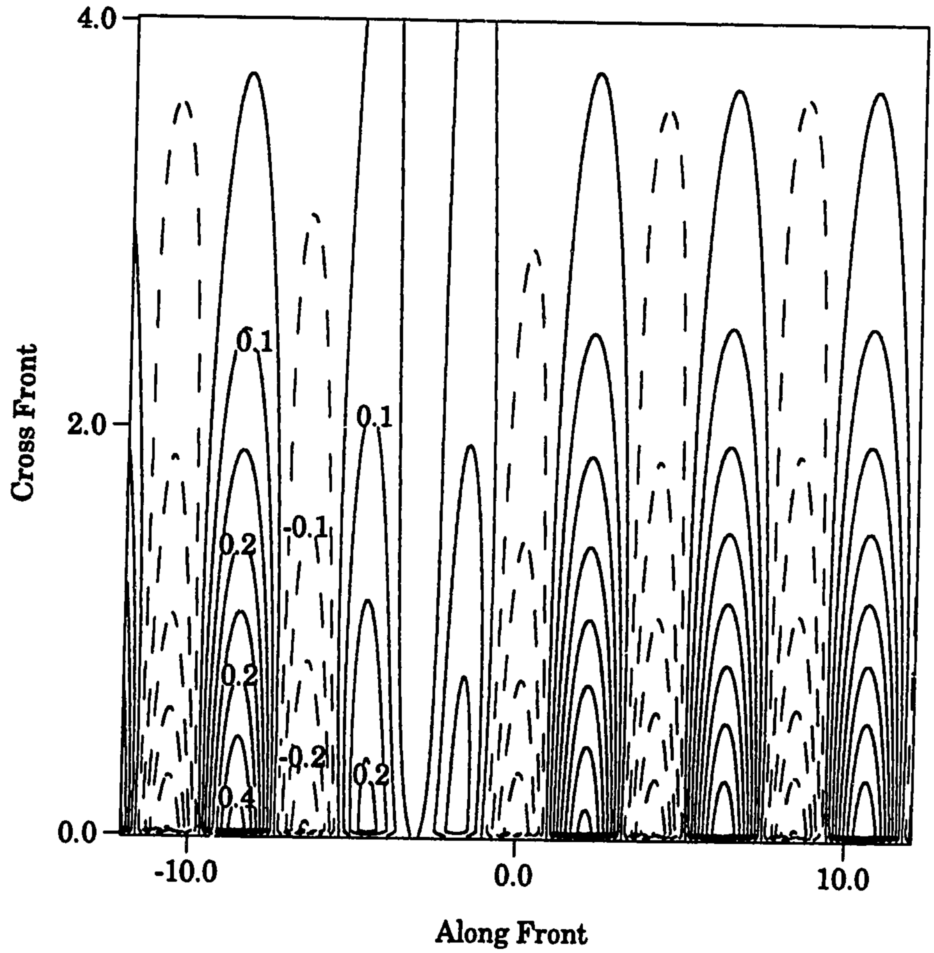
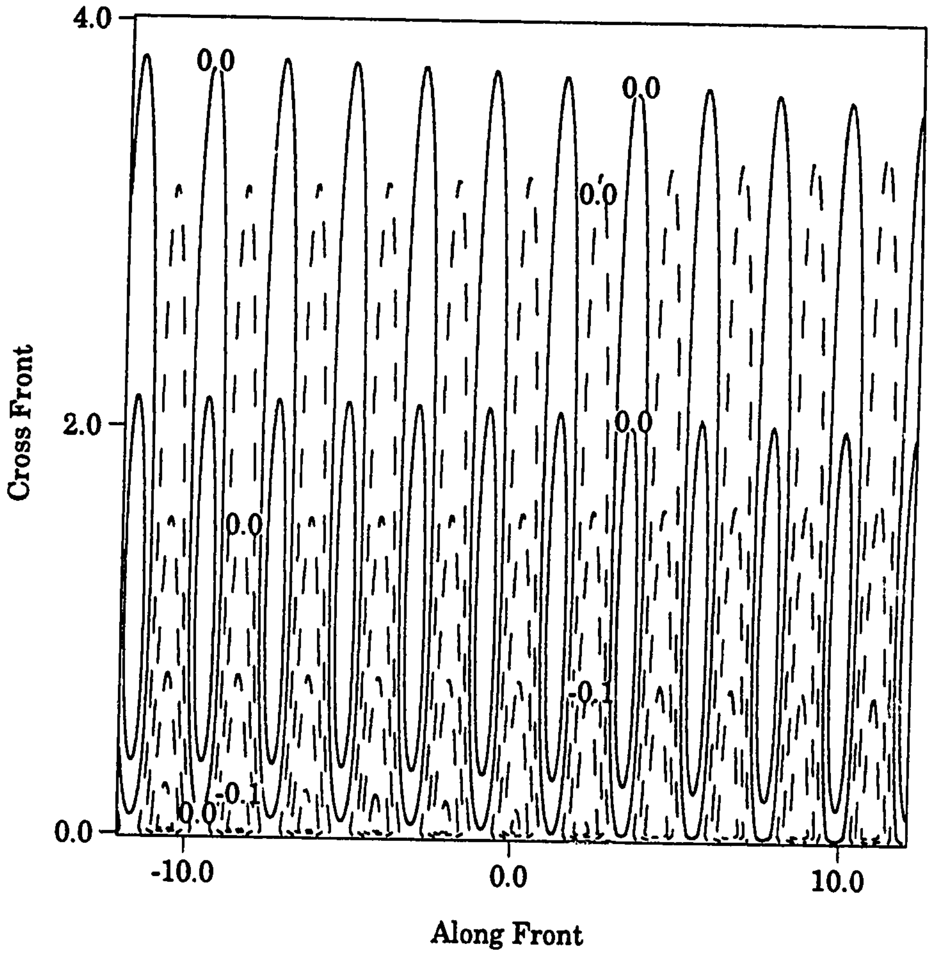


Figure 4-11: Gravest Mode -- m=4



4.5.4 The Total Snoidal Front

Figure 4-12: Gravest Mode -- Hyperbolic Tangent Limit

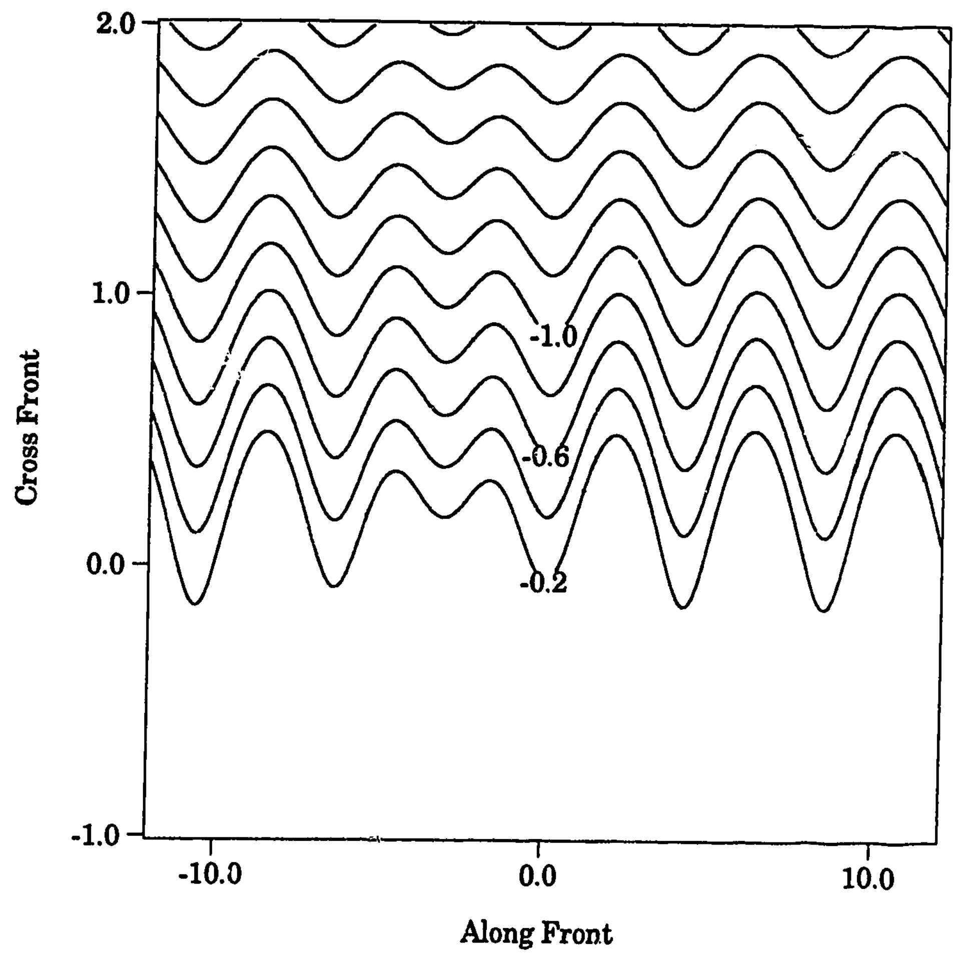


Figure 4-13: Gravest mode -- m=4

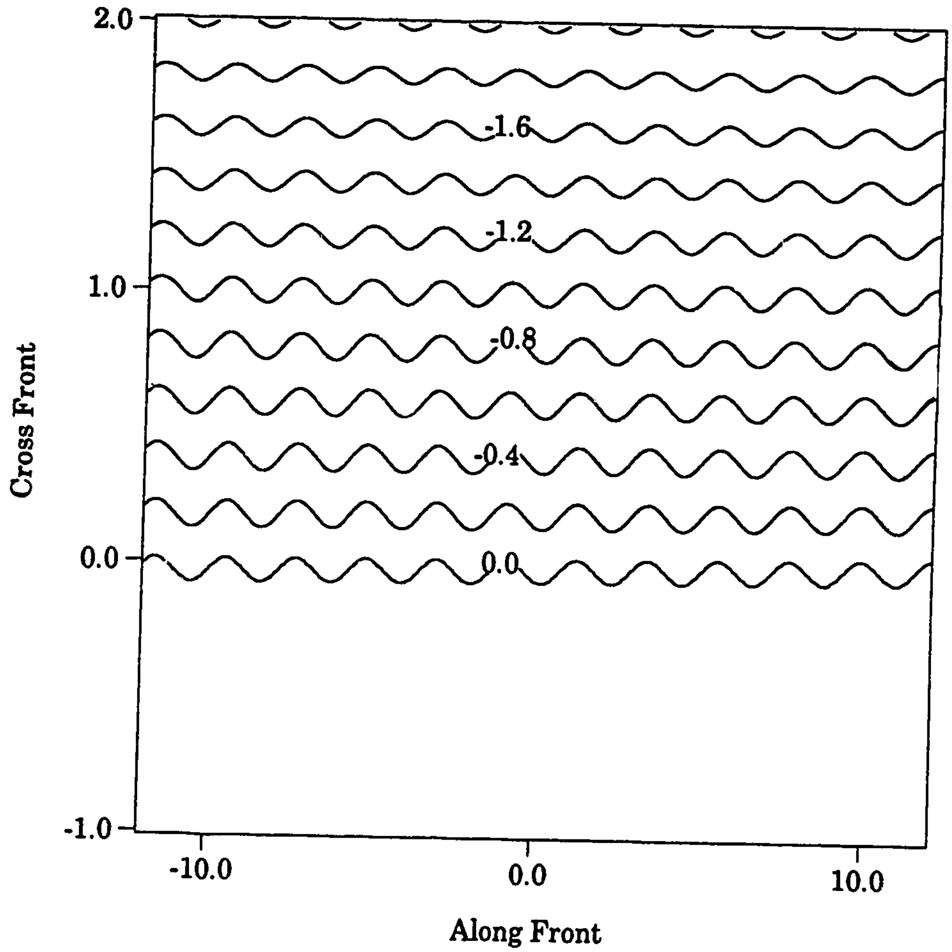


Figure 4-14: n=2 -- Hyperbolic Tangent Limit

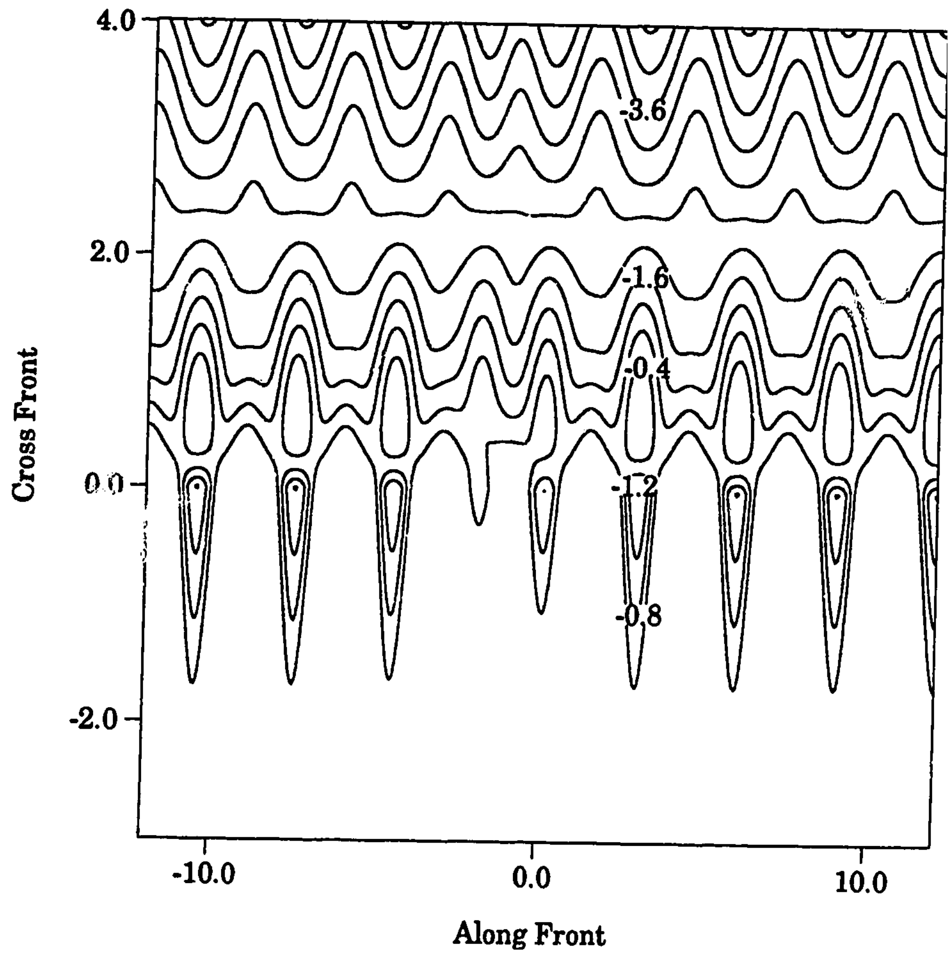
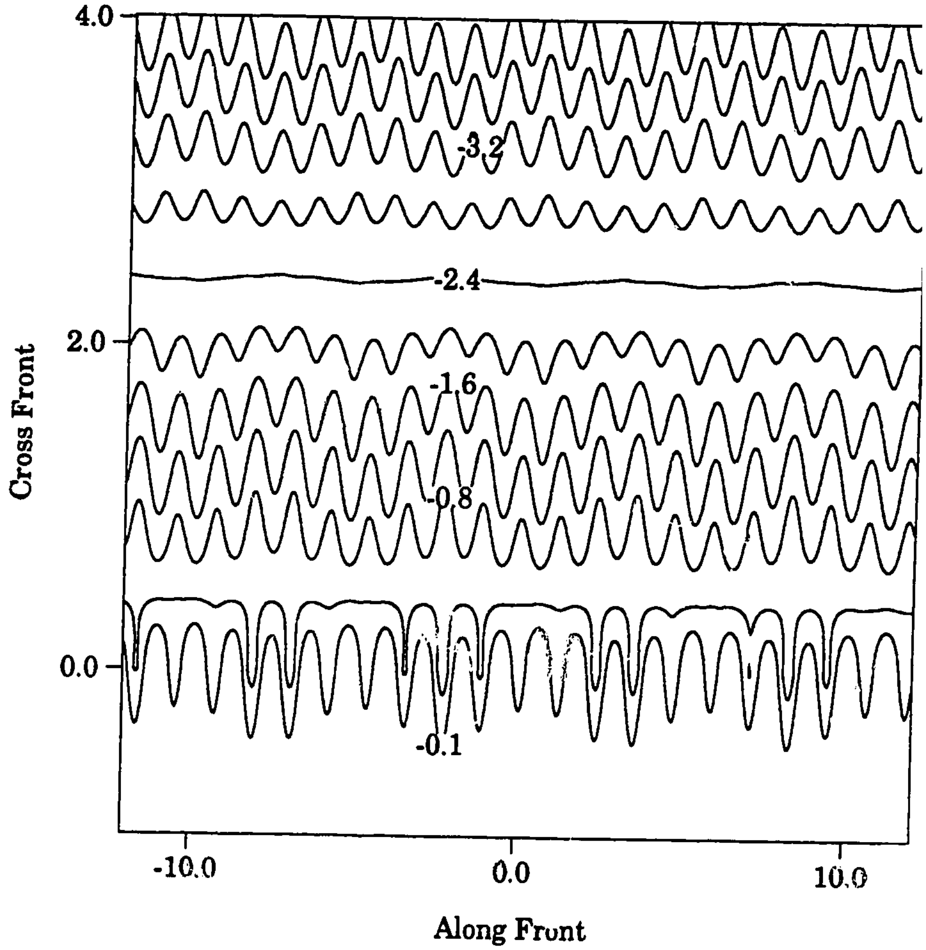


Figure 4-15: n=2 -- m=4



4.6 The Solitary Wave Solution

4.6.1 The Derivations

The final solution to the NLS equation given in this thesis describes a solitary wave. Ablowitz and Clarkson (1991) explain that a solitary wave is a solution of a nonlinear equation which represents a wave of permanent form in that it does not change its shape and propagates at a constant velocity. Ablowitz and Clarkson (1991) also explain that a solitary wave solution is a travelling wave solution of the form

$$u(\xi, \tau) = u(\xi - \kappa\tau) = u(z), \tag{4.91}$$

where $z = \xi - \kappa\tau$, whose transition is from one constant asymptotic state as $z \to -\infty$ to possibly another or the same constant asymptotic state as $z \to +\infty$. Solitary waves are formed when the effects from the nonlinear steepening or breaking represented by the cubic term of the NLS equation and the effects from the dispersive spreading balance to create a stable soliton.

This solitary wave solution results when there is a double root, r_1, and a simple root, r_2, with $r_1 > r_2$ to the cubic equation considered in equation (4.46). Briefly reviewing how this cubic equation is derived, recall that the amplitude A is assumed to be of the form

$$A = a \exp\left[i(\phi - mq)\right]. \tag{4.92}$$

Substituting this form (4.92) into the standard NLS equation (4.35), equating real and imaginary parts and introducing the variable $\psi = Y - \kappa q$ results in the two differential equations

$$\phi_\psi = \frac{1}{2}\left(\kappa + \frac{c_1}{a^2}\right), \tag{4.93}$$

$$(a_\psi)^2 = \frac{1}{2}a^4 - \left(m + \frac{\kappa^2}{4}\right)a^2 + c_2 - \frac{c_1^2}{4a^2}, \tag{4.94}$$

where this last equation (4.94) has been integrated twice with respect to ψ and c_1 and c_2 are introduced as the constants of integration. The variable $r = a^2$ is introduced into the second differential equation (4.94) resulting in the cubic equation

$$(r_\psi)^2 = 2\left(r^3 - 2\left(m + \frac{\kappa^2}{4}\right) r^2 + c_2 r - \frac{c_1^2}{2}\right) = R\left(r\right). \tag{4.95}$$

As was shown in the last section, bounded solutions for r exist only if $R(r)$ has three simple roots which gave the cnoidal wave solution or if $R(r)$ has one double root, r_1, and a simple root, r_2, with $r_1 > r_2$. This latter case is the one considered here and gives the solitary wave solution.

Suppose $R(r)$ has one double root, $r = r_1$, and a simple root, $r = r_2$ with $r_1 > r_2$, so that

$$R\left(r\right) = 2\left(r - r_1\right)^2\left(r - r_2\right), \tag{4.96}$$

so that

$$r_\psi = \sqrt{2}\left(r - r_1\right)\left(r - r_2\right)^{1/2}, \tag{4.97}$$

where $r_1 > r_2$. Solutions to this equation are given as

$$r = r_2 + \left(r_1 - r_2\right)\tanh^2\left(\left(\frac{r_1 - r_2}{2}\right)^{1/2}\psi + c_5\right), \tag{4.98}$$

which is a bounded solution for all ψ and where c_5 is another constant of integration. This solution is identical to the limiting case of the snoidal wave solution given in equation (4.79).

The qualitative nature of this solution can be examined using the fact that as $\psi \to \infty$, $\tanh^2(*) \to 1$ and thus

$$r \to r_2 + \left(r_1 - r_2\right) = r_1, \tag{4.99}$$

a constant value. This is the same result that was found previously when examining r_1 as a double root. This solution also dips to $r = r_2$ when $\psi =$

$-\sqrt{2}c_5/\left(r_1 - r_2\right)^{1/2}$ but quickly approaches the value of r_1 as ψ gets slightly bigger. This describes a upside-down soliton shape that is known as a dark soliton (Drazin and Johnson, 1989). Therefore when the largest root of $R(r)$ is a double root and the second root is a simple root, the solution to the differential equation forms a solitary wave.

Now that the conditions for forming a soliton solution are known they can be used to form a specific solution by determining values for the integration constants c_1 and c_2. Methods similar to the ones used to find the specific snoidal wave solution are used. The conditions are that $R(r)$ must have one double root that is larger than the remaining single root. These are met when

$$R(r) = (r - r_1)^2 (r - r_2) \text{ where } r_1 > r_2, \tag{4.100}$$

or in other words,

$$r^3 - 2\left(m + \frac{\kappa^2}{4}\right) r^2 + c_2 r - \frac{c_1^2}{2} = r^3 - (2r_1 + r_2) r^2 + \left(r_1^2 + 2r_1 r_2\right) r - r_1^2 r_2. \tag{4.101}$$

Equating coefficients of the powers of r is equation (4.101) shows that r_1, r_2, c_1 and c_2 must satisfy the relations

$$2\left(m + \frac{\kappa^2}{4}\right) = 2r_1 + r_2, \tag{4.102}$$

$$c_2 = r_1 (r_1 + 2r_2), \tag{4.103}$$

$$c_1 = 2r_1^2 r_2, \tag{4.104}$$

$$r_1 > r_2. \tag{4.105}$$

The values

$$r_1 = m, \tag{4.106}$$

and

$$r_2 = \kappa^2/2 \text{ with } m > \kappa^2/2, \tag{4.107}$$

satisfy these relations (4.102)–(4.105) and when substituted into (4.97) results in the differential equation,

$$r_\psi = \sqrt{2}\,(r-m)\left(r-\frac{\kappa^2}{2}\right)^{1/2}. \tag{4.108}$$

It can be shown that these values, (4.102) to (4.105), correspond to the boundary conditions

$$r \longrightarrow m \text{ as } \psi \longrightarrow \infty, \tag{4.109}$$

$$r_\psi, r_{\psi\psi} \longrightarrow 0 \text{ as } \psi \longrightarrow \infty. \tag{4.110}$$

Using these values and the solution obtained previously (equation 4.98), r is given as

$$r = \frac{1}{2}\left(\kappa^2 + \left(2m-\kappa^2\right)\tanh^2\left[\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right]\right), \tag{4.111}$$

or, in the alternate form,

$$r = m - \frac{1}{2}\left(2m-\kappa^2\right)\operatorname{sech}^2\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right), \tag{4.112}$$

where the constant of integration is assumed zero and $m > \kappa^2/2$. Recalling that $a^2 = r$, the solution for the real part of the Nonlinear Schrödinger equation is then given by

$$a = \left(m - \frac{1}{2}\left(2m-\kappa^2\right)\operatorname{sech}^2\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right)\right)^{1/2}, \quad m > \frac{\kappa^2}{2}. \tag{4.113}$$

The imaginary part of the Nonlinear Schrödinger equation in this instance will be the solution of the differential equation (see equation (4.93)),

$$\phi_\psi = \frac{1}{2}\left(\kappa + \frac{c_1}{a^2}\right)$$

$$= -\frac{\kappa}{2}\left(2m - \kappa^2\right)\left(\frac{\operatorname{sech}^2\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right)}{\kappa^2 + \left(2m-\kappa^2\right)\tanh^2\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right)}\right). \tag{4.114}$$

Making the substitution

$$\left(2m-\kappa^2\right)^{1/2}\tanh\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right) = \kappa\tan\theta, \tag{4.115}$$

transforms this equation (4.115) into

$$d\phi = -d\theta,$$

so that

$$\phi = -\theta = -\tan^{-1}\left(\frac{1}{\kappa}\left(2m-\kappa^2\right)^{1/2}\tanh\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right)\right). \tag{4.116}$$

Thus the complete solution to the Nonlinear Schrödinger equation corresponding to the boundary conditions

$$a^2 \rightarrow m \text{ as } \psi \rightarrow \infty, \tag{4.117}$$

$$a_\psi, a_{\psi\psi} \rightarrow 0 \text{ as } \psi \rightarrow \infty, \tag{4.118}$$

is given as

$$A = \exp\left(i\left(mq + \phi\right)\right) a\left(\psi\right)$$

$$= \exp\left(imq\right)\exp\left(-i\tan^{-1}\left(\frac{1}{\kappa}\left(2m-\kappa^2\right)^{1/2}\tanh\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right)\right)\right)\times$$

$$\left(m - \frac{1}{2}\left(2m-\kappa^2\right)\operatorname{sech}^2\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right)\right)^{1/2}, \; m > \frac{\kappa^2}{2}. \tag{4.119}$$

This can be simplified on using the identity

$$\exp(i\theta) = \cos\theta + i\sin\theta, \tag{4.120}$$

so that the second exponential in equation (4.119) can be rewritten as

$$\exp\left(-i\tan^{-1}\left(\frac{1}{\kappa}\left(2m-\kappa^2\right)^{1/2}\tanh\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right)\right)\right) =$$

$$\cos\left(\tan^{-1}\left(\frac{1}{\kappa}\left(2m-\kappa^2\right)^{1/2}\tanh\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right)\right)\right)$$

$$-i\sin\left(\tan^{-1}\left(\frac{1}{\kappa}\left(2m-\kappa^2\right)^{1/2}\tanh\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right)\right)\right)$$

$$= \left[\kappa - i\left(2m-\kappa^2\right)^{1/2}\tanh\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right)\right] \times$$

$$\left(2m - \left(2m-\kappa^2\right)\operatorname{sech}^2\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right)\right)^{-1/2}. \tag{4.121}$$

Substituting this into the soliton solution (4.119) and simplifying gives

$$A = \frac{1}{\sqrt{2}}\exp(imq)\left[\kappa - i\left(2m-\kappa^2\right)^{1/2}\tanh\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right)\right], \tag{4.122}$$

where $m > \kappa^2/2$ and $\psi = Y - \kappa q$.

The final form for this particular soliton solution of the Nonlinear Schrödinger equation is found by substituting the original variables in to give

$$A = \frac{1}{\sqrt{2}}\exp\left(-mk^4\beta_n\tau\right) \times$$

$$\left[\kappa - i\left(2m-\kappa^2\right)^{1/2}\tanh\left(\frac{1}{2}\left(2m-\kappa^2\right)^{1/2}\psi\right)\right], \tag{4.123}$$

where $m > \kappa^2/2$ and

$$\psi = \left(\frac{k^4\beta_n}{\alpha^2(2n+1)}\right)^{1/2}\xi + \kappa k^4\beta_n\tau. \tag{4.124}$$

This solution is graphed in Figure 4–16. The dark soliton structure is clearly evident as the graph dips down at $\psi = 0$ and asymptotically approaches a constant ($\simeq 2$) as $\psi \to \pm\infty$. The parameters chosen for this graph are as follows: $\epsilon = 0.025$, $\kappa = 1$, $m = 5$, $n = 0$, $k = 1$, and $\alpha = 1$. These parameters are chosen because they showed the most obvious dark soliton structure.

This solution can also be substituted into the complete nonlinear evolution of the wedge-front given by equation (4.32). The point generating programs for this solution are found in Appendix B.3 and by using *SpyGlass* results in the pictures shown in Figures 4.16–4.26.

Figures 4–17 to 4–21 give only the perturbation flow. The parameters for these graphs are the same as those chosen for Figure 4–16, however the value of m has been changed so that $m = 1$ in Figures 4–17 and 4–19 and the value of n has been increased to 3 for Figures 4–19 to 4–21. In Figure 4–17 where $m = 1$ the dark soliton structure is not as evident as it is in Figure 4–18. Figure 4–17 does show more per[illegible]an is evident in Figure 4–18. In Figure 4–18 the perturbation also see[illegible] more far-reaching effects as it is still evident at $y \simeq 6$. Figure 4–19 and [illegible] show how the flow increases in complexity with increased n. The choice of $m = 1$ in Figure 4–19 again shows somewhat more periodicity whereas the choice of $m = 5$ in Figure 4–20 shows more of the dark soliton structure. Figure 4–21 is simply Figure 4–20 but with an extended x–domain. The dark soliton structure is clearly evident in this plot.

Figures 4–22 and 4–23 give simply the perturbation flow with the same parameters as used in Figure 4–16 except that ϵ is increased tenfold so that $\epsilon = 0.25$. The value for n has also been increased to 3 in Figure 4–23. Figure 4–22 looks identical to Figure 4–17 except that the perturbation flow has increased in magnitude somewhat. Figure 4–23 shows the same 'far–reaching' effects as Figure 4–18 in that the perturbation is still observed at $y \simeq 4$.

The total solitary wave front is found in Figures 4–24 to 4–26. The same parameters used for Figures 4–22 and 4–23 are once again used here except that in Figure 4–24, $m = 5$, in Figure 4–25, $m = 1$ with $n = 2$ and Figure 4–26, $m = 1$ with $n = 3$. Figure 4–24 shows the effects of choosing a relatively large value for ϵ as the perturbed interface extends quite a distance from the front. The small hump in the contours at $x = 0$ is a result of the dark soliton structure. Increasing n in Figure 4–25 once again increases the complexity of the flow and the choice of $m = 1$ does not make the soliton structure as evident. Figure 4–26 shows a bit more periodicity with the somewhat flattened contour at $x = 0$ due to the soliton structure.

4.6.2 The Solitary Wave Amplitude

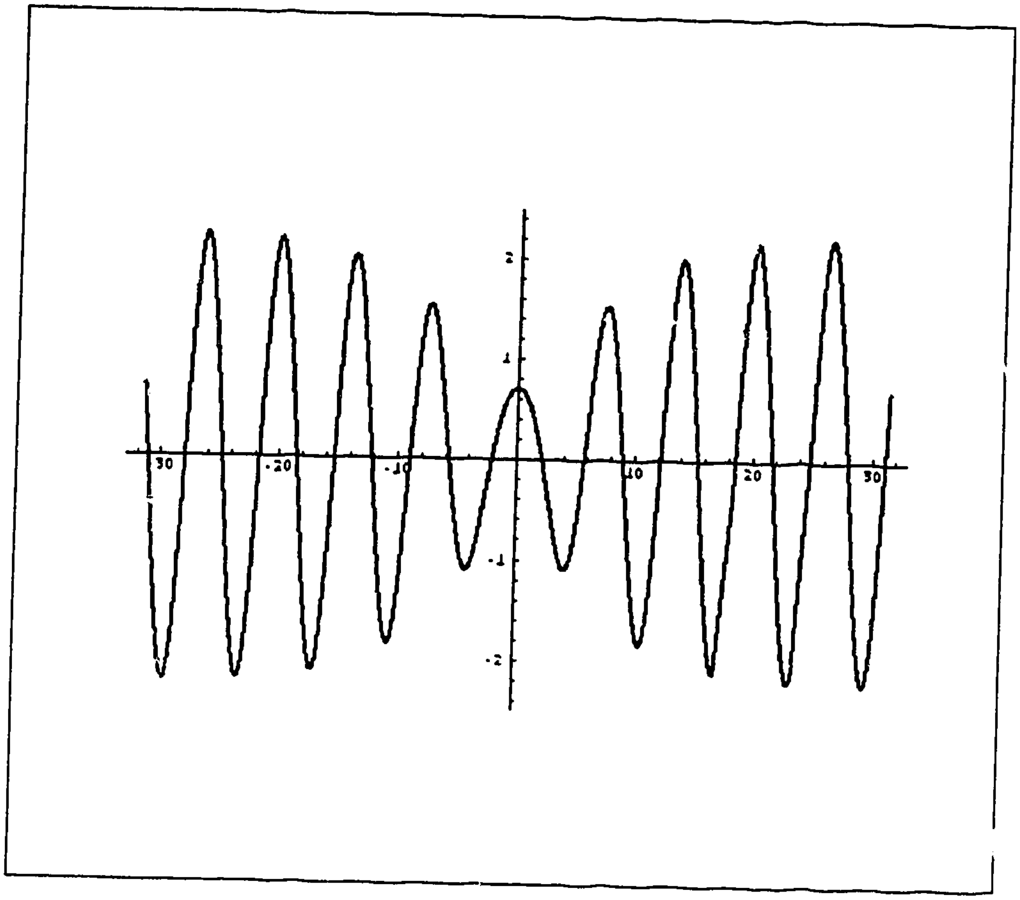


Figure 4-16: The Soliton Amplitude

4.6.3 The Solitary Wave Perturbation Thickness

Figure 4-17: Gravest Mode -- m=1 -- Small Epsilon

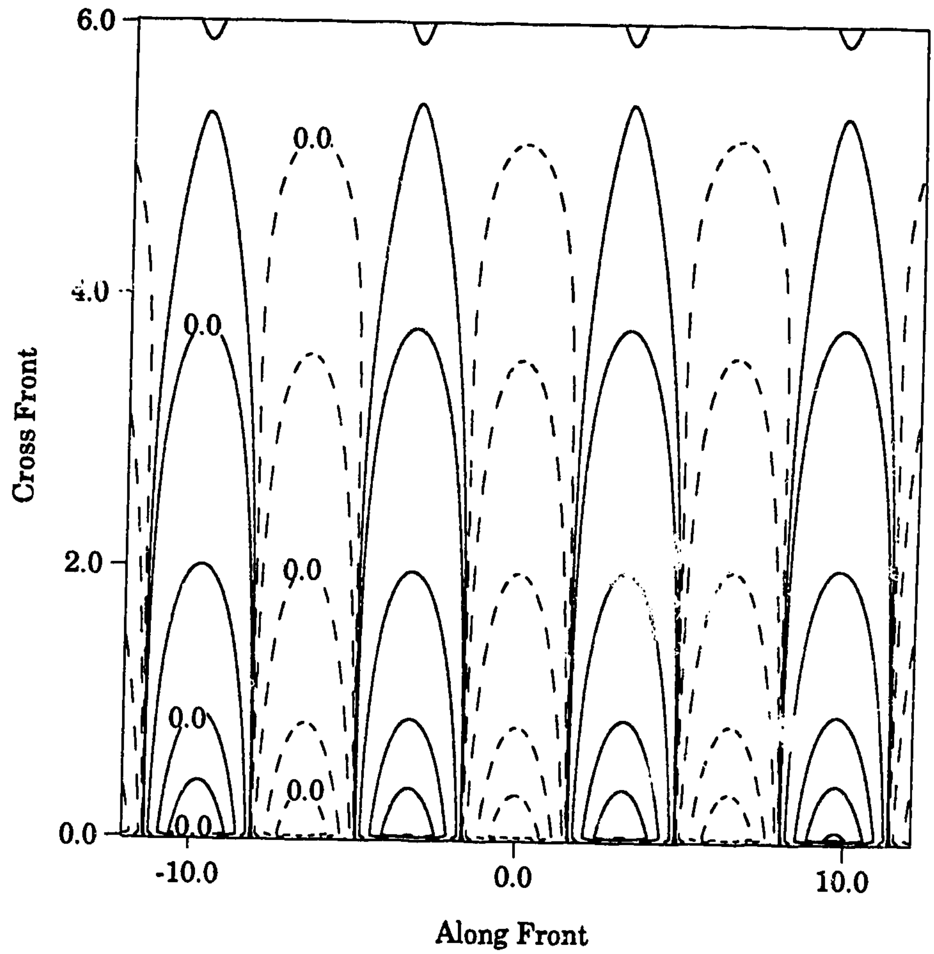


Figure 4-18: Gravest Mode -- m=5 -- Small Epsilon

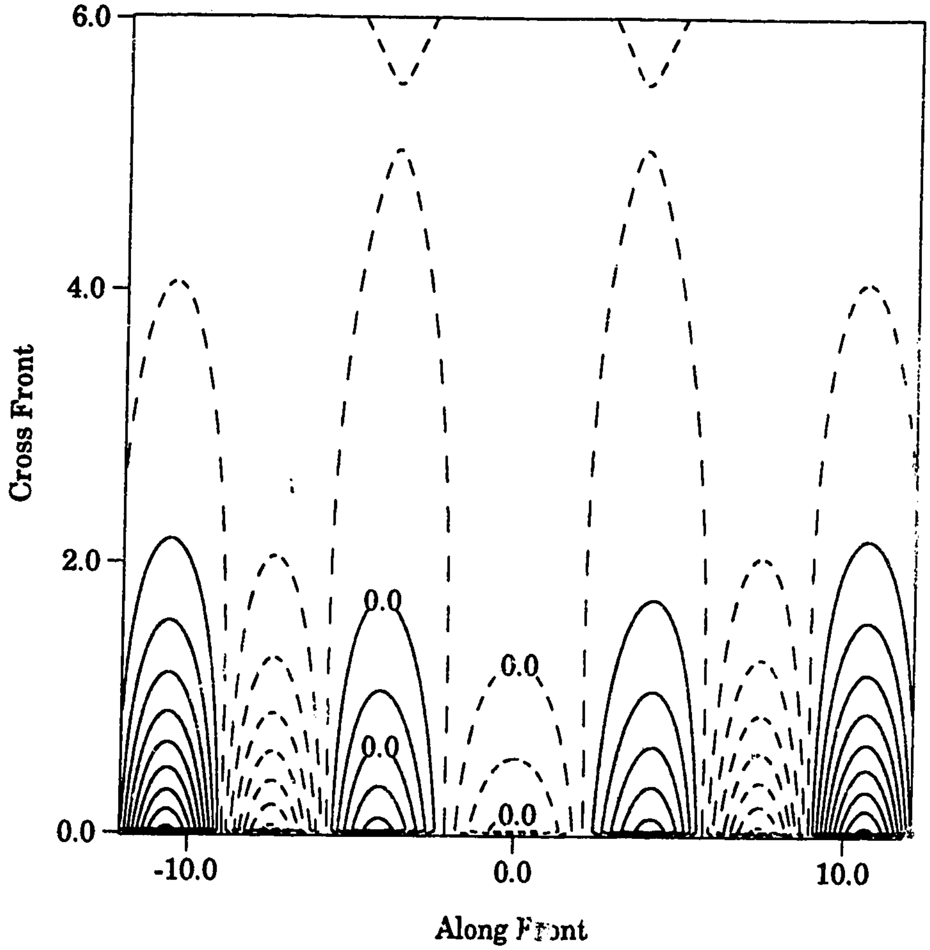


Figure 4-19: n=3 -- m=1 -- Small Epsilon

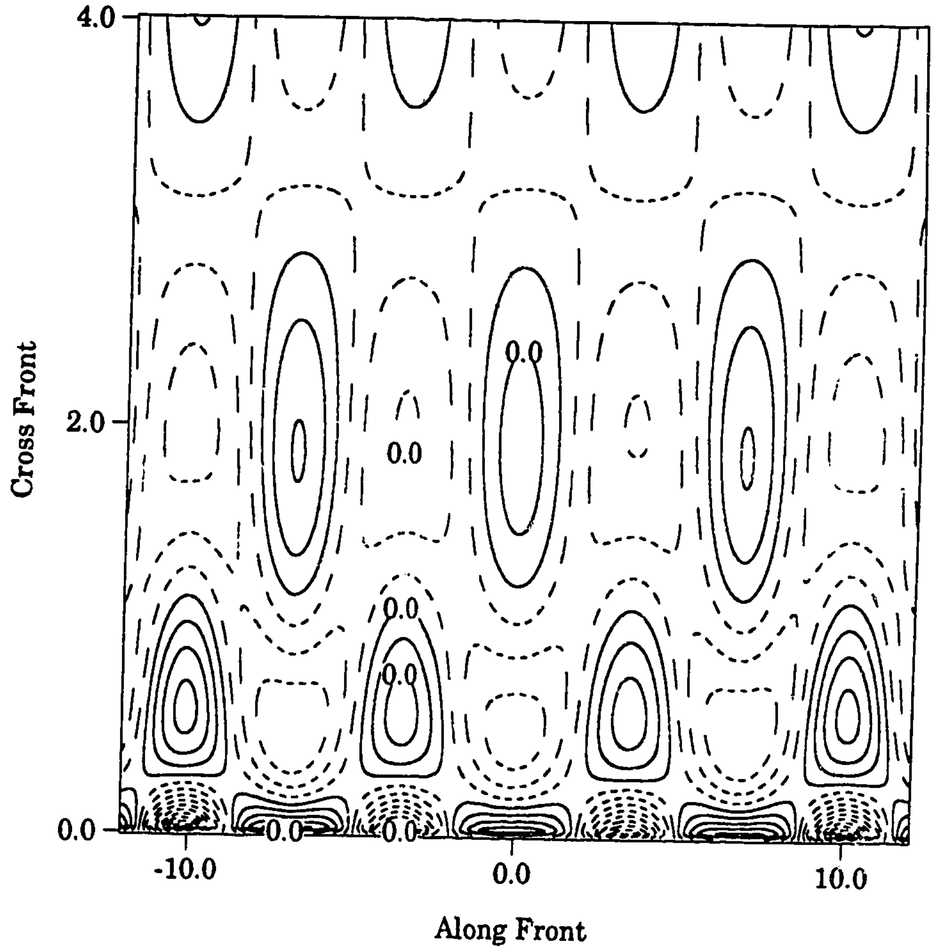


Figure 4-20: n=3 -- m=5 -- Small Epsilon

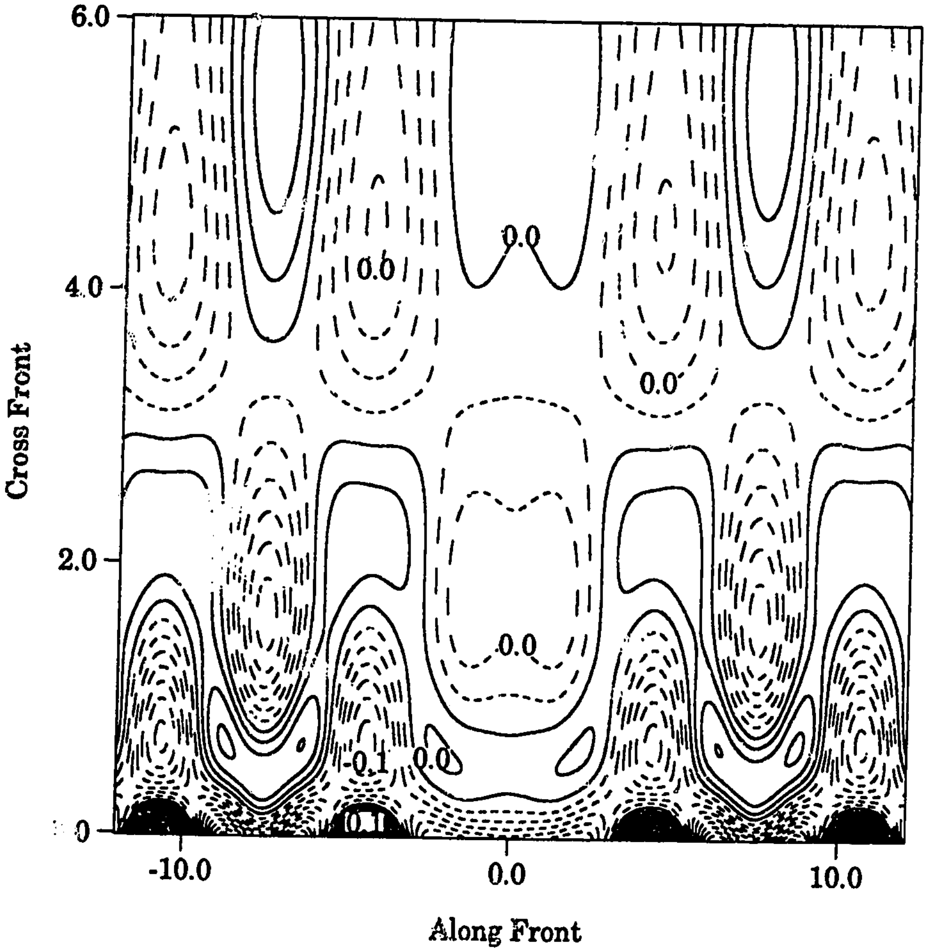


Figure 4-21: Extended Domain -- n=3 -- m=5 -- Small Epsilon

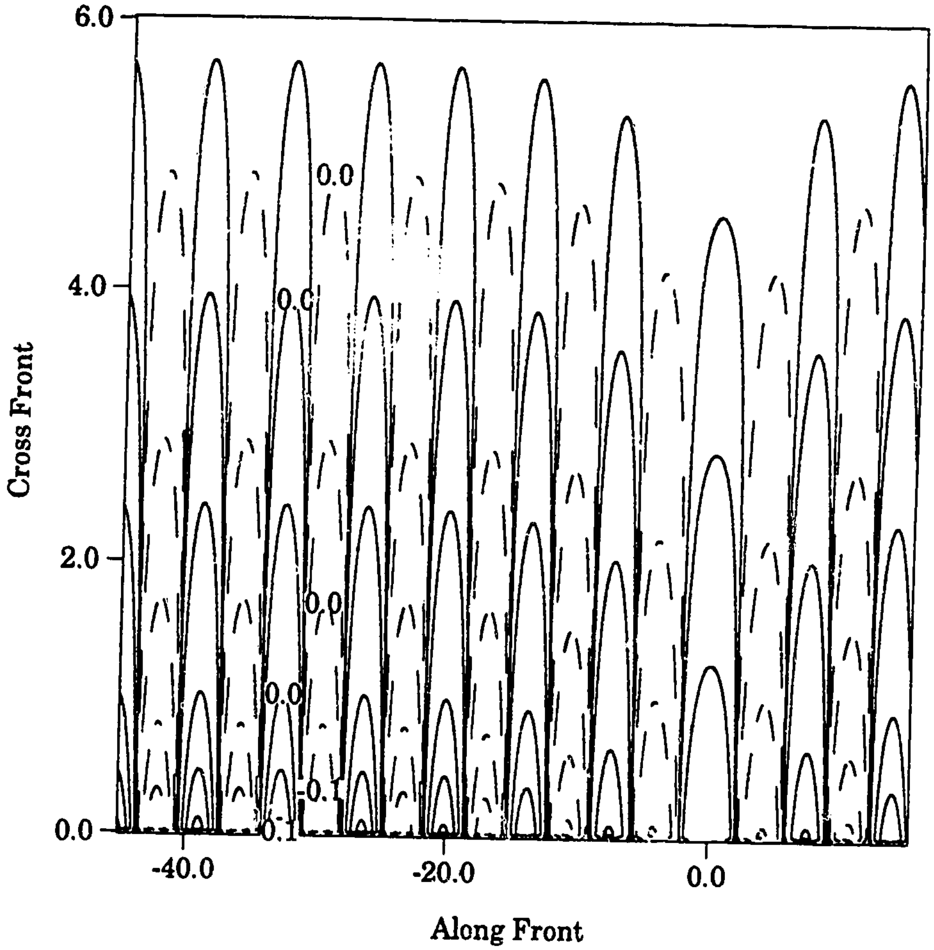


Figure 4-22: Gravest Mode -- m=1

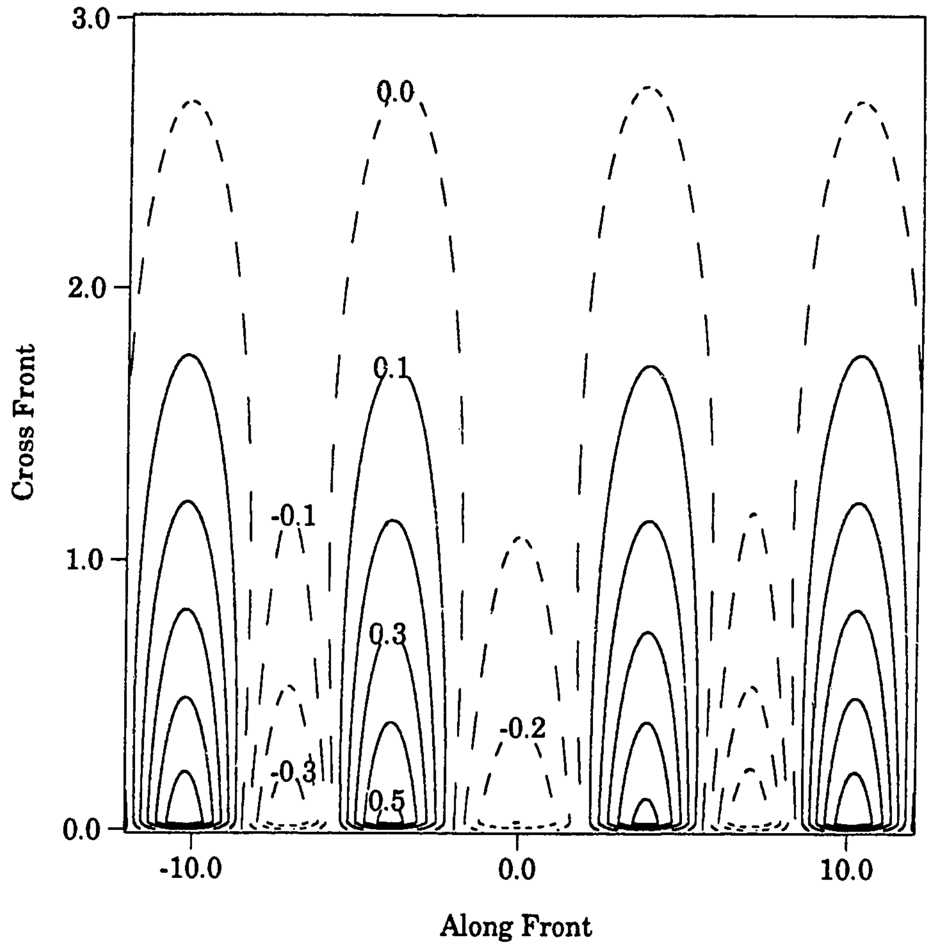
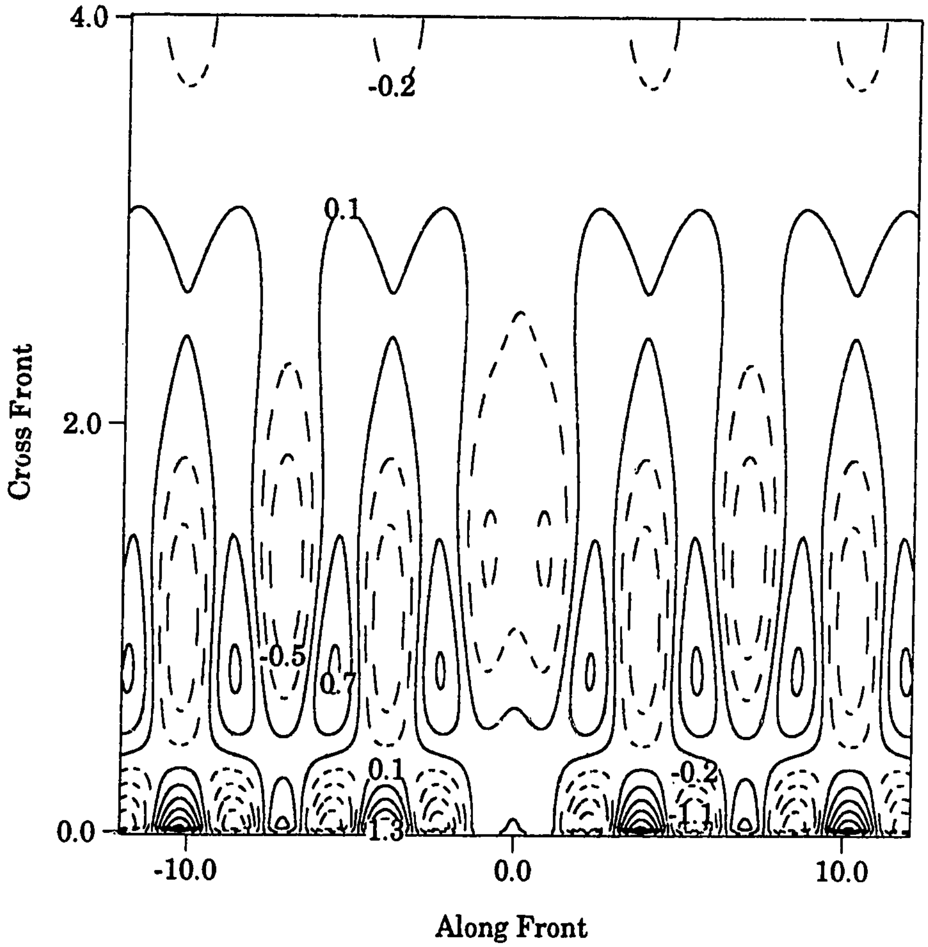


Figure 4-23: n=3 -- m=1



4.6.4 The Total Solitary Wave Front

Figure 4-24: Gravest Mode -- m=5

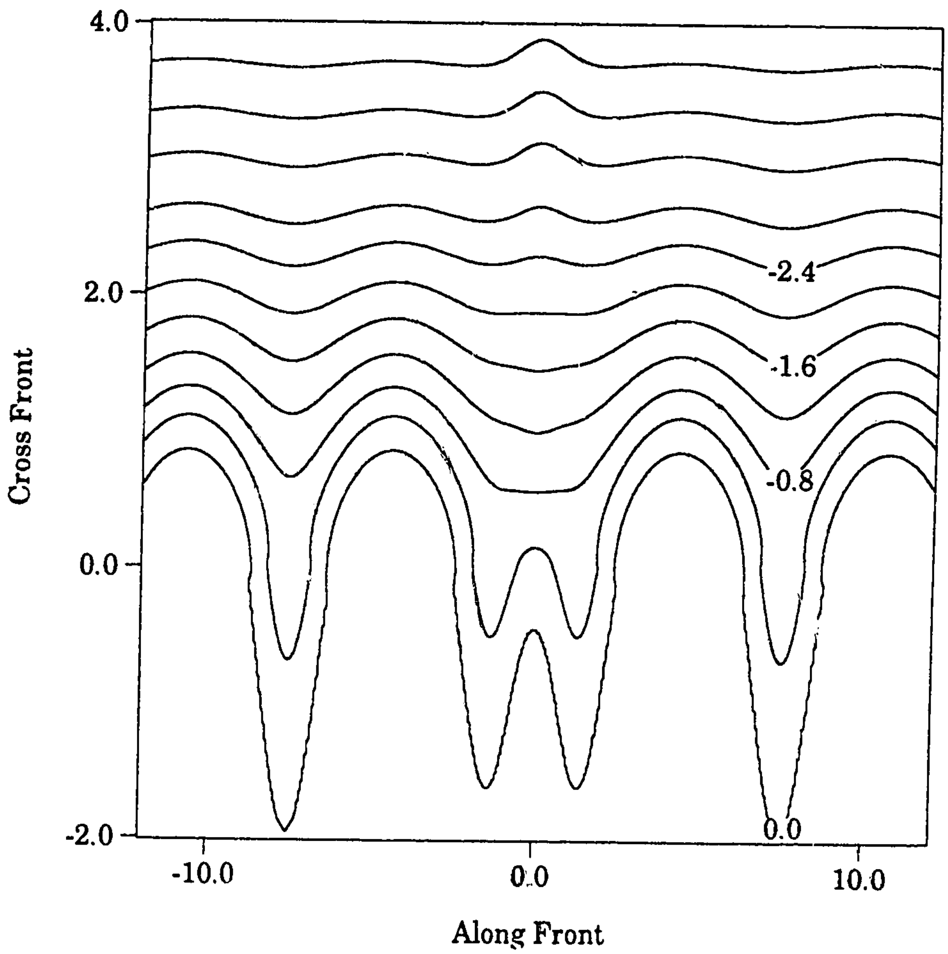


Figure 4-25: n=2 -- m=1

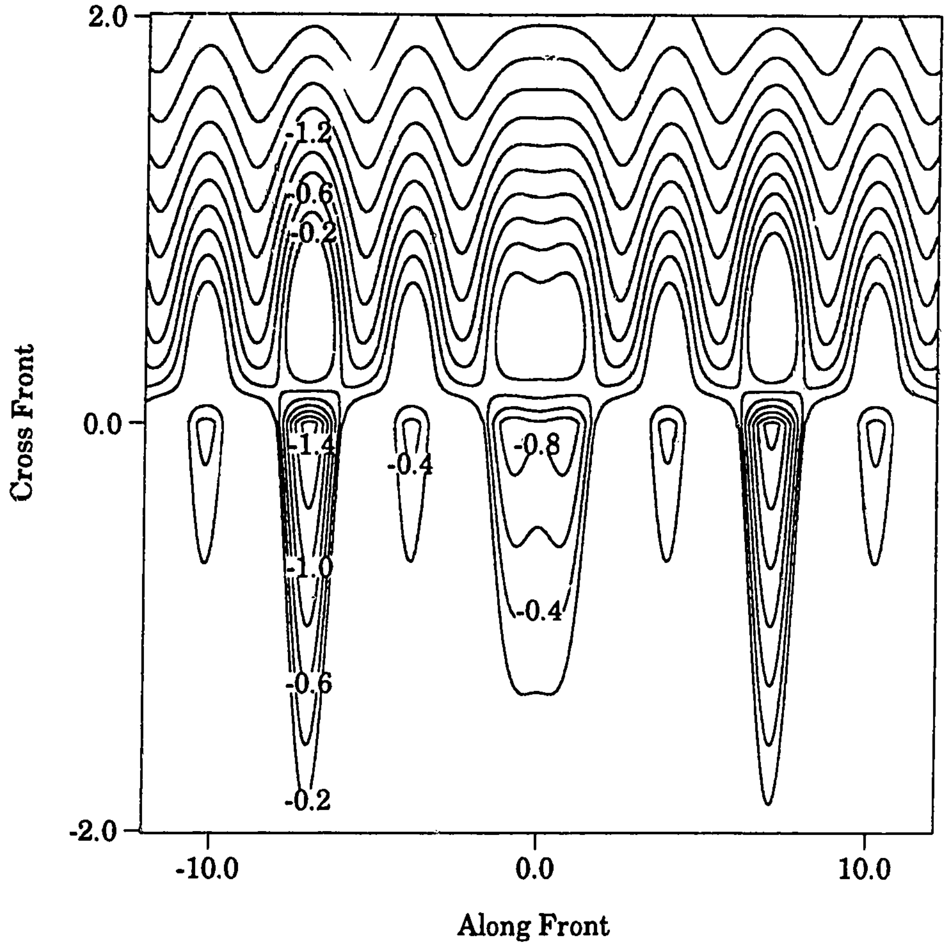
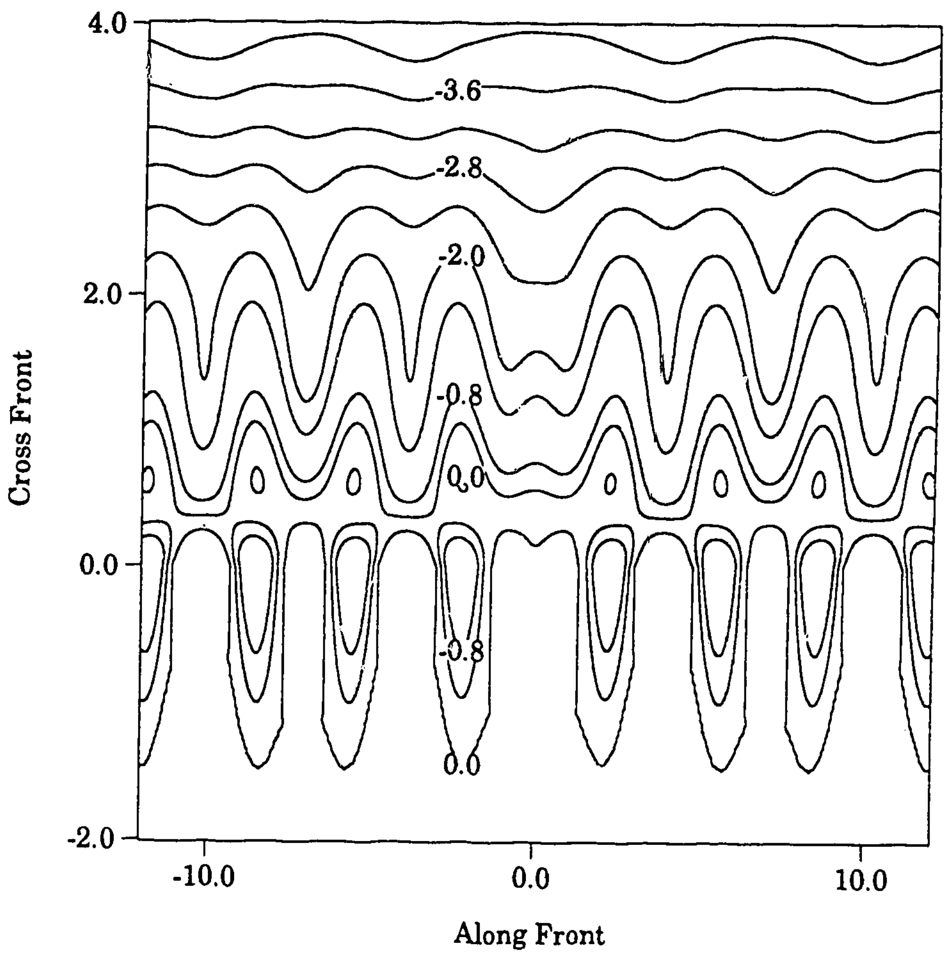


Figure 4-26: n=3 - m=1



Chapter 5

Conclusions

An analytic asymptotic theory examining the finite–amplitude evolution of a wedge–shaped front has been derived by using a barotropic reduced gravity model describing a two layer non–viscous and incompressible water system where only the uppermost layer was considered active. The derivation of the model began using the shallow water equations on a f–plane to describe the motion of the active layer. The nondimensionalization of the model was achieved so that the leading order motion was in geostrophic balance and so that the nonlinear terms in the horizontal momentum equations were somewhat weaker. This model was shown to possess a Hamiltonian structure. Using this Hamiltonian structure it was shown that all steady state solutions to this model were linearly stable. Nonlinear stability could not be shown unconditionally. However the wedge–shaped front given by $h_0(y) = \alpha y$ was found to satisfy these conditions necessary for stability and thus was found to be nonlinearly stable.

The nonlinear evolution of the perturbed wedge–shaped front was examined using a multiple–scales analysis. The frontal height was asymptotically expanded around a small amplitude parameter, ϵ, and problems up to the $O(\epsilon^2)$ were examined. Conditions for the solvability of the $O(\epsilon^2)$ problem showed that the amplitude of the perturbation waves was governed by the Nonlinear Schrödinger equation.

In order to determine the complete form of the finite amplitude evolution of the wedge front, solutions for the Nonlinear Schrödinger equation were derived. In addition to the Stokes wave solution and a plane wave solution, a snoidal wave solution was also found. The final solution to the Nonlinear Schrödinger equation described a solitary waveform. This solution described a dark soliton which went out to a constant at infinity.

In conclusion, this analysis revealed that:

1. This model possesses a Hamiltonian structure that demonstrated that **any** front described by this model is linearly stable.

2. The Hamiltonian structure also showed nonlinear stability but only under certain conditions.

3. Evolution of a wedge–shaped front governed by this reduced–gravity barotropic model is nonlinearly stable.

4. The amplitude of the perturbation waves is described by the Nonlinear Schrödinger equation.

5. Several solutions exist to this Nonlinear Schrödinger equation including a snoidal wave solution and a solitary wave solution.

There are several ways in which the theory presented could be modified. First of all, a different method could be used to examine the leading order equation. Multiple scales analysis, an analytical analysis, was used in this instance. However, numerical methods could have also been used to examine the stability of the wedge–shaped front.

Secondly, the shape of the front considered in the finite amplitude evolution could be changed. Some other interesting and realistic fronts might take a parabolic form or maybe a triangular form. The stability of these different shapes could also be examined.

Finally the model itself could have been modified. The theory presented here considered a barotropic model where the lower layer was considered motionless. However Killworth *et al.* (1984) demonstrated that motion in the second layer can significantly change the stability properties of a front. In other words the baroclinic stability mechanism is an important consideration. Swaters (1993) developed a model to describe a baroclinic two–layer system where both layers evolved over a sloping bottom. His analysis using the Hamiltonian structure of the model also revealed that the baroclinic model was nonlinearly stable but only under certain conditions. The same multiple scales analysis used here could be used in examining this baroclinic model for the evolution of the wedge–shaped front. Even though this analysis would be difficult, if this model shows instability then one can deduce that this instability is due to the baroclinic effects since in the barotropic limit the wedge–like front was stable.

Another concept that could be added to the model is the interaction between the layers. This model assumes that the two layers do not exchange mass or momentum even at the boundary between the layers. The effects of the interaction between the layers on the stability of the front could be examined also. Mixing across fronts is a very important consideration for those involved in waste disposal. A final variation of the model might be to consider the evolution on the beta–plane instead of the f–plane as was done here. This would result in considering a different lengthscale in which the beta effects become important. Cushman–Roisin (1986) derived a leading order governing equation for this instance as well.

Realistically this model is quite simplified but different components could be added one–by–one to examine the effects of each. This would be at the cost of complicating the mathematics and numerical work may have to be used. However, it would also give a better understanding of the stability of geostrophic fronts.

Bibliography

[1] Ablowitz, M. J., and P. A. Clarkson, 1991: *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press.

[2] Arnold, V. I., 1978: *Mathematical Methods of Classical Mechanics*, Springer–Verlag, Inc.

[3] Arnold, V. I., 1969: On an *a priori* estimate in the theory of hydrodynamical stability. *Am. Math. Soc. Transl.* Ser. 2, **79**, 267–269.

[4] Arnold, V.I., 1965: Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid. *Sov. Math.*, **6**, 773–777.

[5] Bender, C. M., and S. A. Orszag, 1978: *Advanced Mathematical Methods for Scientists and Engineers*, McGraw–Hill, Inc.

[6] Birkhoff, G., and R. Gian–Carlo, 1978: *Ordinary Differential Equations*, John Wiley and Sons, Inc.

[7] Bowman, M. J., and W. E. Esaias, 1978: *Oceanic Fronts in Coastal Processes*, Springer–Verlag.

[8] Courant, R. and D. Hilbert, 1953: *Methods of Mathematical Physics, Volume 1*, Interscience Publishers, Inc.

[9] Cushman–Roisin, B., 1994: *Introduction to Geophysical Fluid Dynamics*, Prentice–Hall, Inc.

[10] Cushman-Roisin, B., G. G. Sutyrin and B. Tang, 1992: Two-layer geostrophic dynamics. *J. Phys. Oceanogr.*, **22**, 117–127.

[11] Cushman-Roisin, B., 1986: Frontal geostrophic dynamics. *J. Phys. Oceanogr.*, **16**, 132-143.

[12] Drazin, P. G., and R. S. Johnson, 1989: *Solitons: An Introduction*, Cambridge University Press.

[13] Drazin, P. G., and W. H. Reid, 1981: *Hydrodynamic Stability*, Cambridge University Press.

[14] Ebin, D., and J. E. Marsden, 1970: Groups of diffeomorphisms and motion of an incompressible fluid. *Ann. Math.*, **92**, 102–163.

[15] Eckhaus,W., 1965: *Studies in Nonlinear Stability Theory*, Springer-Verlag.

[16] Federov, K. N., 1986: *The Physical Nature and Structure of Oceanic Fronts*, Springer–Verlag.

[17] Flierl, G. R., 1984: Rossby wave radiation from a strongly nonlinear warm eddy. *J. Phys. Oceanogr.*, **11**, 47-58.

[18] Gradshteyn, I. S., and I. M. Ryzhik, 1980: *Table of Integrals, Series and Products*, Academic Press.

[19] Griffiths, R. W., P. D. Killworth and M. E. Stern, 1982: Ageostrophic instability of ocean currents. *J. Fluid Mech.*, **117**, 343–377.

[20] Griffiths, R. W. and P. F. Linden, 1981: The stability of buoyancy–driven coastal currents. *Dyn. Atmos. Oceans*, **5**, 281–306.

[21] Holm, D. D., J. E. Marsden, T. Ratiu and A. Weinstein, 1985: Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.*, **123**, 1–116.

[22] Karsten, R. H., and G. E. Swaters, A note on the stability theory of buoyancy–driven ocean currents on a sloping bottom, *to appear.*

[23] Killworth, P. D., N. Paldor, and M. E. Stern, 1984: Wave propagation and growth on a surface front in a two–layer geostrophic current. *J. Mar. Res.*, **42**, 761–785.

[24] Killworth, P. D., and M. E. Stern, 1982: Instabilities on density–driven boundary currents and fronts. *Geophys. Astrophys. Fluid Dynamics*, **22**, 1–28.

[25] Kundu, P. K., 1990: *Fluid Mechanics*, Academic Press, Inc.

[26] LeBlond, P. H., and L. A. Mysak, 1978: *Waves in the Ocean*, Elsevier Scientific Publishing Company.

[27] Milne–Thomson, L. M., 1950: *Jacobian Elliptic Function Tables*, Dover Publications, Inc.

[28] Morrison, P. J., 1982: Poisson brackets for fluids and plasmas. *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems*, American Institute of Physics, 13–46.

[29] Nagle, R. K., and E. B. Saff, 1993: *Fundamentals of Differential Equations and Boundary Value Problems*, Addison–Wesley Publishing Company, Inc.

[30] Newell, A. C., 1985: *Solitons in Mathematics and Physics*, Society for Industrial and Applied Mathematics.

[31] Olver, P. J., 1982: A nonlinear Hamiltonian structure for the Euler equations. *J. math. Analyt. Appl.*, **89**, 233–250.

[32] Orlanski, I., 1968: Instability of frontal waves. *J. Atmos. Sci.*, **25**, 178–200.

[33] Paldor, N., 1983: Linear stability and stable modes of geostrophic motion. *Geophys. Astrophys. Fluid Dyn.*, **24**, 299–326.

[34] Pedlosky, J., 1979: *Geophysical Fluid Dynamics*, Springer-Verlag.

[35] Salmon, R., 1988: Hamiltonian Fluid Mechanics. *Ann. Rev. Fluid Mech.*, **20**, 225–256.

[36] Shepherd, T. G., 1990: Symmetries, Conservation Laws and Hamiltonian Structure in Geophysical Fluid Dynamics. *Adv. Geophys.*, **32**, 287-338.

[37] Smith, P. C., 1976: Baroclinic instability in the Denmark Strait overflow. *J. Phys. Oceanogr.*, **6**, 355–371.

[38] Swaters, G. E., 1993: On the baroclinic dynamics, Hamiltonian formulation and general stability characteristics of density driven surface currents and fronts over a sloping continental shelf. *Phil. Trans. R. Soc. Lond. A.*, **345**, 295-325.

[39] Swaters, G. E., 1991: Dynamics of ventilated coherent cold eddies on a sloping bottom. *J. Fluid Mech.*, **224**, 361-383.

[40] Swaters, G. E., and G. R. Flierl, 1991: Dynamics of ventilated coherent cold eddies on a sloping bottom. *J. Fluid Mech.*, **223**. 565-587.

[41] Zauderer, E., 1989: *Partial Differential Equations of Applied Mathematics*, John Wiley and Sons, Inc.

[42] Zwillinger, D., 1989: *Handbook of Differential Equations*, Academic Press, Inc.

Appendix A

Computer Calculations for Beta

```
C
C       *******************************************************
C       * PROGRAM BETA                                        *
C       *                                                     *
C       * PURPOSE:  THIS PROGRAM WILL FIND A VALUE FOR BETA   *
C       *                USING NUMERICAL INTEGRATION.         *
C       * WRITTEN:  AUGUST, 1994                              *
C       * BY:  CAROL SLOMP                                    *
C       *******************************************************
C
C
C
C
        REAL    F,A,AERR,RERR,INT, WK(1000),ERR,INT1,INT2
        REAL    WK2(1000),JO,YO,ERR1,ERR2,WK1(1000)
        INTEGER   MO, NUM, IERR, L, M, NM, IWK(250),NUM1
        INTEGER   IWK2(250),IWK1(250),N,IERR2,NUM2,IERR1
        EXTERNAL F,YO,JO
        COMMON/CAR/N
C
        N = 0
        L = 250
        M = 1000
        A = 0.0000001
        MO = 1
        AERR = .00001E0
        RERR = .00001E0
C
C
        CALL QAGI (F,A,MO,AERR,RERR,INT,ERR, NUM, IERR,
     &                   L,M,NM,IWK,WK)
```

```
      WRITE (7,5)
5       FORMAT ('THESE ARE THE RESULTS FOR N = 0.')
      WRITE (7,10) ERR
10      FORMAT ('THE VALUE OF THE ABSOLUTE ERROR IS:', F11.8)
      WRITE (7,15) IERR
15      FORMAT ('THE VALUE OF IERR IS:', I4)
      WRITE (7,20) NUM
20      FORMAT ('THE NUMBER OF POINTS AT WHICH F WAS
     &            EVALUATED', I7)
      WRITE (7,25) N
25      FORMAT ('THE NUMBER OF SUBINTERVALS IN THE PARTITION
     &            IS', I4)
      WRITE (7,30) INT
30      FORMAT ('THE VALUE OF THE INTEGRAL IS', F13.9)
C
C
      CALL QAGI (JO,A,MO,AERR,RERR,INT1,ERR1,NUM1,IERR1,
     &                     L,M,NM,IWK1,WK1)
      WRITE (7,35) INT1
35      FORMAT ('THE VALUE OF THE INFINITE INTEGRAL OF JO
     &                     IS:',F10.6)
C
      CALL QAGI (YO,A,MO,AERR,RERR,INT2,ERR2,NUM2,IERR2,
     &                     L,M,NM,IWK2,WK2)
      WRITE (7,40) INT2
40      FORMAT ('THE VALUE OF THE INFINITE INTEGRAL OF YO
     &                     IS:',F10.6)
C
C
      STOP
      END
C
C
      REAL FUNCTION F(Z)
C     THIS FUNCTION WILL CALCULATE THE INDEFINITE INTEGRAL
C     OF BESSEL FUNCTIONS AND MULTIPLY THEM BY THE ALTERNATE
C     BESSEL FUNCTION TO SEND TO THE MAIN PROGRAM.
C
      REAL Z,A,ABSE,RELE,IJ,IY,ERR1,ERR2,WRK1(1000)
      REAL F1,F2,F3,JO,YO,WRK2(1000)
      INTEGER NB1,NB2,IERR1,IERR2,P,Q,R,S,IWRK1(250)
      INTEGER IWRK2(250),MO2,N
      EXTERNAL JO,YO
      COMMON/CAR/N
C
      A = 0.0E0
      ABSE = .00001E0
      RELE = .00001E0
      P = 250
      Q = 1000
      MO2 = 1
C
C
```

```
      CALL QAGS (JO,A,Z,ABSE,RELE,IJ,ERR1,NB1,IERR1,
     &                        P,Q,R,IWRK1,WRK1)
      CALL QAGI (YO,Z,MO2,ABSE,RELE,IY,ERR2,NB2,IERR2,
     &                        P,Q,S,IWRK2,WRK2)
C
C
      F1 = YO(Z) * IJ
      F2 = JO(Z) * IY
      F= (F1 + F2)
C
C     WRITE (7,30) F
C30          FORMAT ('THE VALUE OF F IS:', F15.7)
C     WRITE (7,35) IERR1
C35         FORMAT ('THE VALUE OF IERR1 IS:', I4)
C     WRITE (7,40) IERR2
C40        FORMAT ('THE VALUE OF IERR2 IS:', I4)
      RETURN
      END
C
C
      REAL FUNCTION JO(Z)
C     THIS FUNCTION GIVES THE FUNCTION REPRESENTATION FOR
C     BESSELJ WITH ARGUMENT 2*SQRT((2*N+1)*Z).
C
      INTEGER N
      REAL Z
      COMPLEX ZC,W
      COMMON/CAR/N
C
      ZC = CMPLX (2.0E0*SQRT((2.0E0*FLOAT(N)+1.0E0)*Z),0.0E0)
      CALL BSSLJ (ZC,0,W)
      IF (N.EQ.0) THEN
      JO = REAL (W)*EXP(-Z)
      ENDIF
      IF (N.EQ.1) THEN
      JO = REAL (W)*EXP(-Z)*(9.0E0-10.0E0*Z+2.0E0*Z**2)
      ENDIF
      IF (N.EQ.2) THEN
      JO = REAL (W)*EXP(-Z)*(3.0E0*Z**4/4.0E0-10.0E0*Z**3
     & +41.0E0*Z**2-60.0E0*Z+25.0E0)
      ENDIF
      IF (N.EQ.3) THEN
      JO = REAL (W)*EXP(-Z)*(49.0E0-182.0E0*Z+215.0E0*Z**2
     & -328.0E0*Z**3/3.0E0+313.0E0*Z**4/12.0E0-17.0E0*Z**5/6.0E(
     & +Z**6/9.0E0)
      ENDIF
      IF (N.EQ.4) THEN
      JO = REAL (W)*EXP(-Z)*(81.0E0-408.0E0*Z+687.0E0*Z**2
     & -534.0E0*Z**3+2603.0E0*Z**4/12.0E0-145.0E0*Z**5/3.0E0
     & +47.0E0*Z**6/8.0E0-13.0E0*Z**7/36.0E0+5.0E0*Z**8/576.0E0)
      ENDIF
C
      RETURN
      END
```

```
      REAL FUNCTION YO(Z)
C     THIS FUNCTION GIVES THE REPRESENTATION FOR BESSELY
C     WITH ARGUMENT 2*SQRT((2*N+1)*Z).
C
      INTEGER N
      REAL Z
      COMPLEX ZC,T
      COMMON/CAR/N
C
      ZC = CMPLX (2.0E0*SQRT((2.0E0*FLOAT(N)+1.0E0)*Z),0.0E0)
      CALL BSSLY (ZC,0,T)
      IF (N.EQ.0) THEN
      YO = REAL (T)*EXP(-Z)
      ENDIF
      IF (N.EQ.1) THEN
      YO = REAL (T)*EXP(-Z)*(9.0E0-10.0E0*Z+2.0E0*Z**2)
      ENDIF
      IF (N.EQ.2) THEN
      YO = REAL (T)*EXP(-Z)*(3.0E0*Z**4/4.0E0-10.0E0*Z**3
     &   +41.0E0*Z**2-60.0E0*Z+25.0E0)
      ENDIF
      IF (N.EQ.3) THEN
      YO = REAL (T)*EXP(-Z)*(49.0E0-182.0E0*Z+215.0E0*Z**2
     & -328.0E0*Z**3/3.0E0+313.0E0*Z**4/12.0E0-17.0E0*Z**5/6.0E0
     & +Z**6/9.0E0)
      ENDIF
      IF (N.EQ.4) THEN
      YO = REAL (T)*EXP(-Z)*(81.0E0-408.0E0*Z+687.0E0*Z**2
     & -534.0E0*Z**3+2603.0E0*Z**4/12.0E0-145.0E0*Z**5/3.0E0
     & +47.0E0*Z**6/8.0E0-13.0E0*Z**7/36.0E0+5.0E0*Z**8/576.0E0)
      ENDIF
C
      RETURN
      END
C
```

Appendix B

Graphing Programs

B.1 Plane Wave Solution

```
C
C       ***********************************************************
C       * PROGRAM FRONTGRAPH                                      *
C       *                                                         *
C       * PURPOSE:  THIS PROGRAM WILL CALCULATE A TABLE OF        *
C       *           VALUES DESCRIBING THE EVOLUTION OF            *
C       *                THE FRONT WITH THE PLANE SOLUTION        *
C       *           DESCRIBING ITS AMPLITUDE.  THESE              *
C       *           RESULTS CAN BE USED BY A GRAPHING             *
C       *                PROGRAM TO PRODUCE A 3-D GRAPH OF        *
C       *           THE MOTION.                                   *
C       * WRITTEN:  MARCH, 1995                                   *
C       * BY:  CAROL SLOMP                                        *
C       ***********************************************************
C
C        THE N=0 RESULTS ARE IN: N=0 FORT 25, N=1 FORT 26,
C        N=2 FORT 27, N=3 FORT 28, N=4 FORT 29
C
C        VALUES FOR JUST THE PERTURBATION ARE FOUND IN: N=0
C        FORT 85, N=1 FORT 86, N=2 FORT 87, N=3 FORT 88, N=4
C        FORT 89.
C
C
         INTEGER L,R,J,N,I,F
         REAL Z(500,500),P,E,K,M,C,D,PI,H,LL,GG,G,LAG
         REAL X0,Y0,HNULL(500,500),ZI
         COMPLEX A
         EXTERNAL H,LAG,G
```

```
          P=1.0E0
C         P GIVES THE SLOPE OF THE WEDGE-SHAPED FRONT
          E=0.25E0
C         E IS THE SMALL PARAMETER EPSILON
          K=1.0E0
C         K IS THE WAVENUMBER
          N=4
C         N IS THE MODE BEING CONSIDERED
          M=1.0E0
C         M IS THE WAVENUMBER FOR THE GROUP OF WAVES WITH
C                 AMPLITUDE A
          A=CMPLX (1.0E0,0.0E0)
C         A IS THE AMPLITUDE OF THE SLOW MOVING GROUP OF WAVES
          L =100
          PI=3.0E0
C
          DO 50 I=0,L
                  DO 25 F=0,240
                  X0=-2.0E0*PI+FLOAT(I)*4.0E0*PI/FLOAT(L)
                  Y0=-12.0E0+12.0E0*FLOAT(F)/240.0E0
                  HNULL(I+1,F+1)=0.0E0
                  WRITE (29,12) X0,Y0,HNULL(I+1,F+1)
12                FORMAT (' ',F14.3,'  ',F14.3,'  ',F14.9)
25                CONTINUE
50        CONTINUE
C
              DO 120 R=0,L
                  DO 110 J=0,240
                          C=-2.0E0*PI+FLOAT(R)*4.0E0*PI/FLOAT(L)
                          D=12.0E0*FLOAT(J)/240.0E0
                          Z(R+1,J+1)=H(P,E,A,K,N,M,C,D)
C                         LL=LAG(N,2*D)
C                         GG=G(N,2*D)
                          WRITE (29,75)  C, D, Z(R+1,J+1)
75                        FORMAT (' ',F14.3,'  ',F14.3,'  ',F14.9)
110                       CONTINUE
120               CONTINUE
          STOP
          END
C
C
          REAL FUNCTION H(P,E,A,K,N,M,X,Y)
C         THIS FUNCTION WILL CALCULATE THE HEIGHT OF THE FRONT.
C
          INTEGER N
          REAL  E1,E2,P,E,K,M,X,Y,G,LAG
          COMPLEX  CE1,CE2,A,AA,HC,SCE1,SCE2,NCE1,NCE2
          COMPLEX  NSCE1,NSCE2
          EXTERNAL G,LAG
C
          CE1=CMPLX(0.0E0,K*X)
          NCE1=CMPLX(0.0E0,-K*X)
          CE2=CMPLX(0.0E0,2.0E0*K*X)
          NCE2=CMPLX(0.0E0,-2.0E0*K*X)
```

```
      E1=-K*Y
      E2=-2.0E0*K*Y
      SCE1=CMPLX(0.0E0,E*M*X)
      NSCE1=CMPLX(0.0E0,-E*M*X)
      SCE2=CMPLX(0.0E0,2.0E0*E*M*X)
      NSCE2=CMPLX(0.0E0,-2.0E0*E*M*X)
      AA=CMPLX(REAL(A),-IMAG(A))
C
      HC=E*(A*CEXP(CE1)*CEXP(SCE1)+AA*CEXP(NCE1)*CEXP(NSCE1))
     & *EXP(E1)*LAG(N,-E2)
     & -E**2*M/(2.0E0*K)*(A*CEXP(CE1)*CEXP(SCE1)+
     & AA*CEXP(NCE1)*CEXP(NSCE1))*EXP(E1)
     & *(-E2*LAG(N,-E2)+2.0E0*FLOAT(N)*LAG(N-1,-E2))
     & +E**2*K/P*G(N,-E2)*EXP(E2)
     & *(A**2*CEXP(CE2)*CEXP(SCE2)+AA**2*CEXP(NCE2)*CEXP(NSCE2))
     & +P*Y
C
      H=REAL(HC)
C
      RETURN
      END
C
C
      REAL FUNCTION G(N,E3)
      INTEGER N
      REAL E3,GS
C
      IF (N.EQ.0) THEN
      GS=0.0E0
      ENDIF
      IF (N.EQ.1) THEN
      GS=-4.0E0-2.0E0*E3
      ENDIF
      IF (N.EQ.2) THEN
      GS=28.0E0/5.0E0-102.0E0*E3/5.0E0
     & +16.0E0*E3**2/5.0E0-E3**3/3.0E0
      ENDIF
      IF (N.EQ.3) THEN
      GS=-856.0E0/77.0E0+292.0E0*E3/77.0E0+
     & 1388.0E0*E3**2/77.0E0-359.0E0*E3**3/77.0E0+
     & 7.0E0*E3**4/11.0E0-E3**5/30.0E0
      ENDIF
      IF (N.EQ.4) THEN
      GS=2024.0E0/221.0E0-15924.0E0*E3/221.0E0
     & +15364.0E0*E3**2/221.0E0-4917.0E0*E3**3/221.0E0+
     & 3821.0E0*E3**4/663.0E0-1135.0E0*E3**5/1326.0E0
      ENDIF
      IF (N.EQ.5) THEN
      GS=-80772.0E0/4807.0E0+75054.0E0*E3/4807.0E0
     & +343660.0E0*E3**2/4807.0E0-1010665.0E0*E3**3/14421.0E0+
     & 124494.0E0*E3**4/48076.0E0-30064.0E0*E3**5/4807.0E0+
     & 3671.0E0*E3**6/3933.0E0-865.0E0*E3**7/10488.0E0
     & +13.0E0*E3**8/3312.0E0-E3**9/12960.0E0
      ENDIF
```

```
      G=GS
      RETURN
      END
C
      REAL FUNCTION LAG(N,E4)
C     THIS FUNCTION CALCULATES THE VALUES OF THE LAGUERRE
C     POLYNOMIALS
C
      INTEGER N,T
      REAL E4,LN,FACTORIAL
      EXTERNAL FACTORIAL
C
      LN=1.0E0
      DO 150 T=1,N
      LN=LN+(-1.0E0)**T*FACTORIAL(N)/(FACTORIAL(N-T)
     & *(FACTORIAL(T))**2)*E4**T
150   CONTINUE
      LAG=LN
      RETURN
      END
C
C
      REAL FUNCTION FACTORIAL(Q)
C     THIS FUNCTION CALCULATES THE FACTORIAL OF A NUMBER.
      INTEGER Q,S,F1
C
      F1=1
      DO 200 S=1,Q
      F1=F1*S
200   CONTINUE
      FACTORIAL=FLOAT(F1)
C
      RETURN
      END
```

```
C
C      ****************************************************
C      * PROGRAM MEANFLOW                                 *
C      *                                                  *
C      * PURPOSE:  THIS PROGRAM WILL CALCULATE A TABLE OF *
C      *           VALUES DESCRIBING THE MEANFLOW OF THE  *
C      *                FRONT WITH THE PLANE SOLUTION     *
C      *           DESCRIBING ITS AMPLITUDE.              *
C      *           RESULTS CAN BE USED BY A GRAPHING      *
C      *           PROGRAM TO PRODUCE A 3-D GRAPH OF THE  *
C      *           MOTION.                                *
C      *                                                  *
C      * WRITTEN:  MARCH, 1995                            *
C      * BY:  CAROL SLOMP                                 *
C      ****************************************************
C
C
C       THE RESULTS FOR N=0 AREIN FORT 30, N=1 FORT 31, N=2
C       FORT 32, N=3 FORT 33, N=4 FORT 34.
C
C       VALUES FOR SMALL EPSILON ARE FOUND IN: N=0 FORT 90,
C       N=1 FORT 91, N=2 FORT 92, N=3 FORT 93, N=4 FORT 94.
C
        INTEGER N,L,R,I,S,J
        REAL K,P,E,PI,U(500), TOTAL,Y,Z,F,LL,X,LAG,LAG1
        REAL X0,Y0,LAG2,G,HNULL(500,500)
        COMPLEX AA,A
        EXTERNAL TOTAL, Y,LAG,LAG1,LAG2
        COMMON/CAR/N
C
        K=1.0E0
C       K IS THE WAVENUMBER
        N=4
C       N IS THE MODE BEING CONSIDERED
        P=1.0E0
C       P IS THE SLOPE OF THE WEDGE-SHAPED FRONT
        E=0.25E0
C       E IS THE SMALL PARAMETER EPSILON
        A=CMPLX(1.0E0,0.0E0)
C       A IS THE AMPLITUDE OF THE SLOW MOVING GROUP OF WAVES
        AA=CMPLX(REAL(A),-IMAG(A))
C       AA IS THE COMPLEX CONJUGATE OF A
        L=100
        PI=3.0E0
C
        DO 50 J=0,L
                DO 25 S=0,240
                X0=-2.0E0*PI+FLOAT(J)*4.0E0*PI/FLOAT(L)
                Y0=-12.0E0+12.0E0*FLOAT(S)/240.0E0
                HNULL(J+1,S+1)=0.0E0
                WRITE (34,12) X0,Y0,HNULL(J+1,S+1)
12              FORMAT ('  ',F14.3,'  ',F14.3,'  ',F14.9)
25              CONTINUE
50      CONTINUE
```

```
        DO 125 I=0,L
                DO 100 R=0,240
                        X=-2.0E0*PI+FLOAT(I)*4.0E0*PI/FLOAT(L)
                        G=12.0E0*FLOAT(R)/240.0E0
                        Z=2.0E0*K*G
                        U(R+1)=TOTAL(K,A,AA,P,E,Z)
                        WRITE (34,75) X,G,U(R+1)
 75             FORMAT ('  ',F14.3,'  ',F14.3, '  ',F14.9)
 100            CONTINUE
 125    CONTINUE
        STOP
        END
C
C
        REAL FUNCTION TOTAL(K,A,AA,P,E,Z)
C       THIS FUNCTION CALCULATES THE TOTAL VALUE OF THE MEAN FLOW
C
        INTEGER N
        REAL ZI1
        REAL PI,P,E,K,TT,F,Y,Z
        COMPLEX ZC1,Y8,J8
        COMPLEX A,AA
        EXTERNAL F,Y
        COMMON/CAR/N
C
        ZI1=(2.0E0*FLOAT(N)+1.0E0)*Z
        ZC1=CMPLX(2.0E0*SQRT(ZI1),0.0E0)
        CALL BSSLJ (ZC1,0,J8)
        IF (ZC1.EQ.(0.0E0,0.0E0)) THEN
        Y8=(1.0E0,0.0E0)
        ELSE
        CALL BSSLY (ZC1,0,Y8)
        ENDIF
        TT=-E**2*K*ACOS(-1.0D0)*REAL(A)*REAL(AA)/(2.0E0*P)*
     &  (REAL(Y8)*F(Z)+REAL(J8)*Y(Z))
        TOTAL=TT
C
        RETURN
        END
C
C
        REAL FUNCTION F(Z)
C       THIS FUNCTION WILL CALCULATE THE FIRST INTEGRAL IN
C       THE FORMULA FOR b, THAT IS, THE INTEGRAL FROM 0 TO 2*Y.
C
        INTEGER NB1,IERR1, P,Q,R,IWRK1(250)
        REAL JO, A,ABSE,RELE,IJ,ERR1,WRK1(1000),Z
        EXTERNAL JO
        COMMON/CAR/N
C
        A=0.0E0
        ABSE=.00001E0
        RELE=.00001E0
```

```
      P=250
      Q=1000
C
      CALL QAGS (JO,A,Z,ABSE,RELE,IJ,ERR1,NB1,IERR1,P,Q,R,IWRK1,WRK1)
      F=IJ
C
      RETURN
      END
C
C
      REAL FUNCTION Y(Z)
C     THIS FUNCTION WILL CALCULATE THE SECOND INTEGRAL
C     IN THE FORMULA FOR b, THAT IS, THE INTEGRAL FROM
C     2*Y TO INFINITY.
C
      INTEGER C,D,S, IWRK2(250), NB2,MO1,IERR2
      REAL Z, AERR, IERR, IY, ERR2, WRK2(1000),YO
      EXTERNAL YO
      COMMON/CAR/N
C
      MO1=1
      AERR=0.00001E0
      IERR=0.00001E0
      C=250
      D=1000
C
      CALL QAGI(YO,Z,MO1,AERR,IERR,IY,ERR2,NB2,IERR2,
     &                    C,D,S,IWRK2,WRK2)
      Y=IY
C
      RETURN
      END
C
C
      REAL FUNCTION JO(Z)
C     THIS FUNCTION CALCULATES THE INTEGRAND FOR THE FIRST
C     INTEGRAL FOR THE VALUE OF b.
C
      INTEGER N
      REAL ZI
      REAL Z,LAG,LAG1,LAG2
      COMPLEX ZC,T
      EXTERNAL LAG, LAG1, LAG2
      COMMON/CAR/N
C
      ZI=(2.0E0*FLOAT(N)+1.0E0)*Z
      ZC=CMPLX(2.0E0*SQRT(ZI),0.0E0)
      CALL BSSLJ (ZC,0,T)
      JO=REAL(T)*EXP(-Z)*(4.0E0*FLOAT(N+1)*(LAG(Z))**2+8.0E0*
     & FLOAT(N+2)*LAG(Z)*LAG1(Z)+4.0E0*(LAG1(Z))**2+8.0E0*
     & LAG(Z)*LAG2(Z))
C
      RETURN
      END
```

```
      REAL FUNCTION YO(Z)
C     THIS FUNCTION CALCULATES THE INTEGRAND FOR THE SECOND
C     INTEGRAL FOR THE VALUE OF b.
C
      INTEGER N
      REAL ZI2
      REAL Z,LAG,LAG1,LAG2,WR
      COMPLEX ZC2,W
      EXTERNAL LAG,LAG1,LAG2
      COMMON/CAR/N
C
      ZI2=(2.0E0*FLOAT(N)+1.0E0)*Z
      ZC2=CMPLX(2.0E0*SQRT(ZI2),0.0E0)
      CALL BSSLY(ZC2,0,W)
      YO=REAL(W)*EXP(-Z)*(4.0E0*FLOAT(N+1)*(LAG(Z))**2
     & +8.0E0*FLOAT(N+2)*LAG(Z)*LAG1(Z)+4.0E0*(LAG1(Z))**2
     & +8.0E0*LAG(Z)*LAG2(Z))
C     WRITE (30,*) YO
C
      RETURN
      END
C
C
      REAL FUNCTION LAG(E4)
C     THIS FUNCTION CALCULATES THE VALUES OF THE LAGUERRE
C     POLYNOMIALS
C
      INTEGER N,T
      REAL E4,LN,FACTORIAL
      EXTERNAL FACTORIAL
      COMMON/CAR/N
C
      LN=1.0E0
      DO 150 T=1,N
      LN=LN+(-1.0E0)**T*FACTORIAL(N)/(FACTORIAL(N-T)
     & *(FACTORIAL(T))**2)*E4**T
150   CONTINUE
      LAG=LN
      RETURN
      END
C
C
      REAL FUNCTION LAG1(E4)
C     THIS FUNCTION CALCULATES THE FIRST DERIVATIVE OF THE
C     LAGUERRE POLYNOMIALS.
C
      INTEGER N,H
      REAL LN1,FACTORIAL,E4
      EXTERNAL FACTORIAL
      COMMON/CAR/N
C
```

```
      LN1=0.0E0
      DO 200 H=1,N
      LN1=LN1+(-1.0E0)**H*FACTORIAL(N)*FLOAT(H)/
     & (FACTORIAL(N-H)*(FACTORIAL(H))**2)*E4**(H-1)
 200  CONTINUE
      LAG1=LN1
C
      RETURN
      END
C
C
      REAL FUNCTION LAG2(E4)
C     THIS FUNCTION CALCULATES THE SECOND DERIVATIVE OF
C     THE LAGUERRE POLYNOMIALS.
C
      INTEGER N,I
      REAL LN2, FACTORIAL,E4
      EXTERNAL FACTORIAL
      COMMON/CAR/N
C
      LN2=0
      DO 250 I=2,N
      LN2=LN2+(-1.0E0)**I*FACTORIAL(N)*FLOAT(I)*FLOAT(I-1)/
     & (FACTORIAL(N-I)*(FACTORIAL(I))**2)*E4**(I-2)
250   CONTINUE
      LAG2=LN2
C
      RETURN
      END
C
C
      REAL FUNCTION FACTORIAL(Q)
C     THIS FUNCTION CALCULATES THE FACTORIAL OF A NUMBER.
      INTEGER Q,S,F1
C
      F1=1
      DO 400 S=1,Q
      F1=F1*S
400   CONTINUE
      FACTORIAL=FLOAT(F1)
C
      RETURN
      END
```

```
C
C       ******************************************************
C       * PROGRAM HADDITION                                  *
C       *                                                    *
C       * PURPOSE:  THIS PROGRAM WILL CALCULATE A TABLE OF   *
C       *           VALUES DESCRIBING THE BOUNDARY OF THE    *
C       *                FRONT WITH THE PLANE SOLUTION       *
C       *           DESCRIBING ITS AMPLITUDE.                *
C       *           THESE RESULTS CAN BE USED BY A           *
C       *           GRAPHING PROGRAM TO PRODUCE A 3-D        *
C       *           GRAPH OF THE MOTION.                     *
C       *                                                    *
C       * WRITTEN:  MARCH, 1995                              *
C       * BY:  CAROL SLOMP                                   *
C       ******************************************************
C
C
C        THE RESULTS FOR N=0 ARE IN FORT 35, N=1 FORT 36, N=2
C        FORT 37, N=3 FORT 38, N=4 FORT 39.
C
         INTEGER N,L,R,F,I,J
         REAL K,M,P,E,PI,C,D,Z,HADD(500,500),HATY0,PHI
         REAL X0,Y0,ZI, HNULL (500,500)
         COMPLEX A
         EXTERNAL HATY0,PHI
C
         N=0
C        N IS THE MODE NUMBER
         A=CMPLX(1.0E0,0.0E0)
C        A IS THE AMPLITUDE OF THE WAVE
         K=1.0E0
C        K IS THE WAVENUMBER OF THE WAVE
         M=1.0E0
C        M IS THE SLOW TIME WAVENUMBER.
         P=1.0E0
C        P IS THE SLOPE OF THE WEDGE FRONT
         E=0.25E0
C        E IS THE SMALL PARAMETER EPSILON.
         PI =3.0E0
         L=100
C
C
         DO 200 R=0,L
                DO 150 F=0,240
                C=-2.0E0*PI+4*PI*FLOAT(R)/FLOAT(L)
                D=-12.0E0+12.0E0*FLOAT(F)/240.0E0
                Z=PHI(N,A,K,M,P,E,C)
                ZI=HATY0(N,A,K,M,P,E,C)
                IF (Z.GE.0.0E0) THEN
                HADD(R+1,F+1)=0
```

```
                ELSEIF (D.LT.Z) THEN
                HADD (R+1,F+1)=0
                ELSE
                HADD (R+1,F+1)=HATY0 (N,A,K,M,P,E,C)*(D-Z)/Z
                ENDIF
                WRITE (39,25) C,D,HADD(R+1,F+1)
25              FORMAT ('  ',F14.3,'    ',F14.3,'    ',F14.9)
150             CONTINUE
200         CONTINUE
        DO 300 J=0,L
                DO 250 I=0,240
                X0=-2.0E0*PI+4*PI*FLOAT(J)/FLOAT(L)
                Y0=12.0E0*FLOAT(I)/240.0E0
                HNULL (J+1,I+1)=0.0E0
                WRITE (39,225) X0,Y0,HNULL(J+1,I+1)
225             FORMAT ('  ',F14.3,'    ',F14.3,'    ',F14.9)
250             CONTINUE
300     CONTINUE
C
        STOP
        END
C
C
        REAL FUNCTION PHI(N,A,K,M,P,E,XC)
C       THIS FUNCTION CALCULATES THE DEFORMED BOUNDARY.
C
        INTEGER N
        REAL PER1,K,M,P,E,XC,G,B
        COMPLEX A, CPHI
        EXTERNAL G,B
C
        PER1=K*XC+M*E*XC
C
        CPHI=-2.0E0*E*A/P*COS(PER1)
     &  -2.0E0*E**2*A**2/P**2*COS(2.0E0*PER1)
     &  *(K+2.0E0*K*FLOAT(N))-E**2*A**2/P**2*
     &  (K+2.0E0*K*FLOAT(N))*2.0E0
     &  +E**2*2.0E0*FLOAT(N)*M*A/(K*P)*COS(PER1)
     &  -2.0E0*E**2*A**2*K/P**2*COS(2.0E0*PER1)*G(N)
     &  +B(N)/P
C
        PHI=REAL(CPHI)
        RETURN
        END
C
C
        REAL FUNCTION HATY0(N,A,K,M,P,E,X)
C       THIS FUNCTION CALCULATES THE HEIGHT OF THE FRONT AT Y=0.
C
        INTEGER N
        REAL E,K,X,M,P,G,B
        COMPLEX CE1,NCE1,CE2,NCE2,SCE1,NSCE1,SCE2,NSCE2,AA,A
        COMPLEX CHATY0
        EXTERNAL B, G
```

```
      CE1=CMPLX(0.0E0,K*X)
      NCE1=CMPLX(0.0E0,-K*X)
      CE2=CMPLX(0.0E0,2.0E0*K*X)
      NCE2=CMPLX(0.0E0,-2.0E0*K*X)
      SCE1=CMPLX(0.0E0,E*M*X)
      NSCE1=CMPLX(0.0E0,-E*M*X)
      SCE2=CMPLX(0.0E0,2.0E0*E*M*X)
      NSCE2=CMPLX(0.0E0,-2.0E0*E*M*X)
      AA=CMPLX(REAL(A),-IMAG(A))
C
      CHATY0=E*(A*CEXP(CE1)*CEXP(SCE1)+AA*CEXP(NCE1)
     & *CEXP(NSCE1))
     & -E**2*M/K*(A*CEXP(CE1)*CEXP(SCE1)+AA*CEXP(NCE1)
     & *CEXP(NSCE1))*FLOAT(N)
     & +E**2*K/P*G(N)*(A**2*CEXP(CE2)*
     & CEXP(SCE2)+AA**2*CEXP(NCE2)*CEXP(NSCE2))
     & +B(N)
C
      HATY0=REAL(CHATY0)
C
      RETURN
      END
C
C
      REAL FUNCTION G(N)
C     THIS FUNCTION REPRESENTS THE POLYNOMIAL FUNCTION FOUND
C     IN THE SOLUTION FOR THE HEIGHT OF THE FRONT EVALUATED
C     AT Y=0.
C
      INTEGER N
C
      IF (N.EQ.0) THEN
      G=0.0E0
      ENDIF
      IF (N.EQ.1) THEN
      G=-4.0E0
      ENDIF
      IF (N.EQ.2) THEN
      G=28.0E0/5.0E0
      ENDIF
      IF (N.EQ.3)THEN
      G=-856.0E0/77.0E0
      ENDIF
      IF (N.EQ.4) THEN
      G=2024.0E0/221.0E0
      ENDIF
C
      RETURN
      END
C
C
      REAL FUNCTION B(N)
C     THIS FUNCTION REPRESENTS THE VALUE OF THE MEAN FLOW
C     AT Y=0 FOR EPSILON EQUAL TO 0.25.
```

```
      IF (N.EQ.0) THEN
      B=-.087146886E0
      ENDIF
      IF (N.EQ.1) THEN
      B=0.306398422E0
      ENDIF
      IF (N.EQ.2) THEN
      B=0.285505891E0
      ENDIF
      IF (N.EQ.3) THEN
      B=0.025252942E0
      ENDIF
      IF (N.EQ.4) THEN
      B=-0.243944064E0
      ENDIF
C
      RETURN
      END
```

B.2 Snoidal Wave Solution

```
C       *****************************************************
C       * PROGRAM CFRONTGRAPH                               *
C       *                                                   *
C       * PURPOSE:  THIS PROGRAM WILL CALCULATE A TABLE OF  *
C       *           VALUES DESCRIBING THE EVOLUTION OF THE  *
C       *           FRONT WITH THE SNOIDAL SOLUTION         *
C       *           DESCRIBING ITS AMPLITUDE.               *
C       *           THESE RESULTS CAN BE USED BY A          *
C       *           GRAPHING PROGRAM TO PRODUCE A 3-D GRAPH *
C       *           OF THE MOTION.                          *
C       *                                                   *
C       * WRITTEN:  MARCH, 1995                             *
C       * BY:  CAROL SLOMP                                  *
C       *****************************************************
C
C       THE VALUES FOR N=0, M=0, ARE IN FORT7, N=0 M=4 ARE IN
C       FORT 8, N=2, M=0 ARE IN FORT9, N=2,M=4 ARE IN FORT10,
C       N=4 ARE IN FORT11.
C
C       THE VALUES FOR JUST THE PERTURBATION ARE AS FOLLOWS:
C       N=0 M=0 FORT30; N=0 M=4 FORT 31; N=2 M=0 FORT 32;
C       N=2 M=4 FORT 33; N=4 KAY=1 FORT 34.
C
C       THE VALUES FOR JUST THE PERTURBATION BUT SMALL EPSILON
C       ARE AS FOLLOWS: N=0 M=0 FORT 50, N=0 M=4 FORT51,
C       N=2 M=0 FORT52, N=2 M=4 FORT 53, N=4 FORT 54
C
C       THE VALUES FOR THE EXTENDED X N=0 M=4 FORT 60
C
        INTEGER L,R,J,N,I,F
        REAL Z(1000,1000),P,E,K,CE,DE,PI,H,LL,GG,G,LAG
        REAL X0,Y0,HNULL(1000,1000),TA
        COMPLEX A
        EXTERNAL H,LAG,G,A
C
        P=1.0E0
C       P GIVES THE SLOPE OF THE WEDGE-SHAPED FRONT
        E=0.25E0
C       E IS THE SMALL PARAMETER EPSILON
        K=1.0E0
C       K IS THE WAVENUMBER
        N=0
C       N IS THE MODE BEING CONSIDERED
        L =500
        PI=3.0E0
```

```
      DO 50 I=0,L
              DO 25 F=0,40
              X0=-4.0E0*PI+FLOAT(I)*8.0E0*PI/FLOAT(L)
              Y0=-1.0E0+1.0E0*FLOAT(F)/40.0E0
              HNULL(I+1,F+1)=0.0E0
              WRITE (7,12) X0,Y0,HNULL(I+1,F+1)
12            FORMAT (' ',F14.3,'  ',F14.3,'  ',F14.9)
25            CONTINUE
50    CONTINUE
C
C
      DO 120 R=0,L
              DO 110 J=0,160
                      CE=-4.0E0*PI+FLOAT(R)*8.0E0*PI/FLOAT(L)
                      DE=4.0E0*FLOAT(J)/160.0E0
                      Z(R+1,J+1)=H(P,E,K,N,CE,DE)
                      WRITE (7,75)  CE, DE, Z(R+1,J+1)
75                    FORMAT (' ',F14.3,'  ',F14.3,'  ',F14.9)
110           CONTINUE
120   CONTINUE
      STOP
      END
C
C
      REAL FUNCTION H(P,E,K,N,X,Y)
C     THIS FUNCTION WILL CALCULATE THE HEIGHT OF THE FRONT.
C
C
      INTEGER N
      REAL  E1,E2,P,E,K,M,X,Y,G,LAG
      COMPLEX  CE1,CE2,IA,AA,HC,NCE1,NCE2
      COMPLEX IDA,DAA,A,DA,I
      EXTERNAL G,LAG,A,DA
C
C
      CE1=CMPLX(0.0E0,K*X)
      NCE1=CMPLX(0.0E0,-K*X)
      CE2=CMPLX(0.0E0,2.0E0*K*X)
      NCE2=CMPLX(0.0E0,-2.0E0*K*X)
      E1=-K*Y
      E2=-2.0E0*K*Y
      IA=A(E,K,N,P,X)
      IDA=DA(E,K,N,P,X)
C     WRITE (7,*) IDA
      AA=CMPLX(REAL(IA),-IMAG(IA))
      DAA=CMPLX(REAL(IDA),-IMAG(IDA))
      I=CMPLX(0.0E0,1.0E0)
C
C
      HC=E*(IA*CEXP(CE1)+AA*CEXP(NCE1))*EXP(E1)*LAG(N,-E2)
     & -E**2*I/(2*K)*(IDA*CEXP(CE1)-DAA*CEXP(NCE1))*
     & EXP(E1)*(-E2*LAG(N,-E2)+2.0E0*FLOAT(N)*LAG(N-1,-E2))
     & +E**2*K/P*G(N,-E2)*EXP(E2)*(IA**2*CEXP(CE2)
     & +AA**2*CEXP(NCE2))+P*Y
```

```
      H=REAL(HC)
C
      RETURN
      END
C
C
      REAL FUNCTION G(N,E3)
C
      INTEGER N
      REAL E3,GS
C
      IF (N.EQ.0) THEN
      GS=0.0E0
      ENDIF
      IF (N.EQ.1) THEN
      GS=-4.0E0-2.0E0*E3
      ENDIF
      IF (N.EQ.2) THEN
      GS=28.0E0/5.0E0-102.0E0*E3/5.0E0
     & +16.0E0*E3**2/5.0E0-E3**3/3.0E0
      ENDIF
      IF (N.EQ.3) THEN
      GS=-856.0E0/77.0E0+292.0E0*E3/77.0E0+
     & 1388.0E0*E3**2/77.0E0-359.0E0*E3**3/77.0E0+
     & 7.0E0*E3**4/11.0E0-E3**5/30.0E0
      ENDIF
      IF (N.EQ.4) THEN
      GS=2024.0E0/221.0E0-15924.0E0*E3/221.0E0
     & +15364.0E0*E3**2/221.0E0-4917.0E0*E3**3/221.0E0+
     & 3821.0E0*E3**4/663.0E0-1135.0E0*E3**5/1326.0E0
      ENDIF
      IF (N.EQ.5) THEN
      GS=-80772.0E0/4807.0E0+75054.0E0*E3/4807.0E0
     & +343660.0E0*E3**2/4807.0E0-1010665.0E0*E3**3/14421.0E0+
     & 124494.0E0*E3**4/48076.0E0-30064.0E0*E3**5/4807.0E0+
     & 3671.0E0*E3**6/3933.0E0-865.0E0*E3**7/10488.0E0
     & +13.0E0*E3**8/3312.0E0-E3**9/12960.0E0
      ENDIF
C
      G=GS
      RETURN
      END
C
C
      REAL FUNCTION LAG(N,E4)
C     THIS FUNCTION CALCULATES THE VALUES OF THE LAGUERRE
C     POLYNOMIALS
C
      INTEGER N,T
      REAL E4,LN,FACTORIAL
      EXTERNAL FACTORIAL
C
      LN=1.0E0
      DO 150 T=1,N
```

```
      LN=LN+(-1.0E0)**T*FACTORIAL(N)/(FACTORIAL(N-T)*
     &     (FACTORIAL(T))**2)*E4**T
150   CONTINUE
C
      LAG=LN
C
      RETURN
      END
C
C
      REAL FUNCTION FACTORIAL(Q)
C     THIS FUNCTION CALCULATES THE FACTORIAL OF A NUMBER.
C
      INTEGER Q,S,F1
C
      F1=1
      DO 200 S=1,Q
      F1=F1*S
200   CONTINUE
      FACTORIAL=FLOAT(F1)
C
      RETURN
      END
C
C
      COMPLEX FUNCTION A(E,K,N,P,X)
C     THIS FUNCTION CALCULATES THE VALUE OF THE AMPLITUDE
C     THAT SATISFIES THE NONLINEAR SCHROEDINGER EQUATION.
C
      INTEGER IERR,N
      REAL S,C,D,KAY,LE,AK,OM,Y,Q,NR,R,RPSI,CONST,BETA
      REAL K,P,E,X,B,AY
      COMPLEX SCE1
      EXTERNAL BETA
C
      OM=1.0E0
      AK=1.0E0/SQRT(2.0E0)
      B=0.25E0
C     B IS A CONSTANT THAT MUST BE LESS THAN (OM-AK**2)**2
      CONST=1.0E0
      R=(OM-AK**2)+SQRT((OM-AK**2)**2-B)
      NR=(OM-AK**2)-SQRT((OM-AK**2)**2-B)
      KAY=NR/R
      LE=SQRT(1-KAY**2)
      AY=SQRT(K**4*BETA(N)/(P**2*(2.0E0*FLOAT(N)+1.0E0)))*E*X
      RPSI=SQRT(R/2)*AY+CONST
C
      CALL ELLPF (RPSI,KAY,LE,S,C,D,IERR)
C
C     FOR THIS FUNCTION KAY^2+L^2=1, S=SN(RPSI),C=CN(RPSI),
C     D=DN(RPSI).  IF IERR=0 THEN ELLIPTIC FUNCTIONS WERE
C     COMPUTED, IERR=1, KAY^2+LE^2=1 IS NOT SATISFIED,
C     IERR=2, RPSI IS TOO LARGE FOR KAY.
C
```

```
      SCE1=CMPLX(0.0E0,AK*AY)
      A=CEXP(SCE1)*SQRT(NR)*S
C
      RETURN
      END
C
C
      COMPLEX FUNCTION DA(E,K,N,P,X)
C     THIS FUNCTION CALCULATES THE DERIVATIVE OF THE
C     AMPLITUDE GIVEN BY THE FUNCTION A(E,K,N,X)
C
      INTEGER N,IERR
      REAL E,K,X,OM,B,D,CONST,NR,R,KAY,LE,AY,RPSI,S,C
      REAL AK,BETA,PARA,P
      COMPLEX I,SCE2
      EXTERNAL BETA
C
      OM=1.0E0
      AK=1.0E0/SQRT(2.0E0)
      B=0.25E0
C     B IS A CONSTANT THAT MUST BE LESS THAN (OM-AK**2)**2
      CONST=1.0E0
      R=(OM-AK**2)+SQRT((OM-AK**2)**2-B)
      NR=(OM-AK**2)-SQRT((OM-AK**2)**2-B)
      KAY=NR/R
      LE=SQRT(1-KAY**2)
      PARA=SQRT(K**4*BETA(N)/(P**2*(2.0E0*FLOAT(N)+1.0E0)))
      AY=PARA*E*X
      RPSI=SQRT(R/2)*AY+CONST
      I=CMPLX(0.0E0,1.0E0)
C
      CALL ELLPF (RPSI,KAY,LE,S,C,D,IERR)
C
C     FOR THIS FUNCTION KAY^2+LE^2=1, S=SN(RPSI),C=CN(RPSI),
C     D=DN(RPSI).  IF IERR=0 THEN ELLIPTIC FUNCTIONS WERE
C     COMPUTED, IERR=1, KAY^2+LE^2=1 IS NOT SATISFIED,
C     IERR=2, RPSI IS TOO LARGE FOR KAY.
C
      SCE2=CMPLX(0.0E0,AK*AY)
      DA=SQRT(NR)*CEXP(SCE2)*PARA*(I*AK*S+C*D*SQRT(R/2))
C
      RETURN
      END
C
C
      REAL FUNCTION BETA(N)
C
      INTEGER N
      REAL BT
C
      IF (N.EQ.0) THEN
      BT=7.3638623E0
      ENDIF
```

```
      IF (N.EQ.1) THEN
      BT=47.828495E0
      ENDIF
      IF (N.EQ.2) THEN
      BT =190.530625E0
      ENDIF
      IF (N.EQ.3) THEN
      BT=495.425275E0
      ENDIF
      IF (N.EQ.4) THEN
      BT=531.2647044E0
      ENDIF
C
      BETA=BT
C
      RETURN
      END
```

```
C
C      *****************************************************************
C      * PROGRAM CMEANFLOW                                             *
C      *                                                               *
C      * PURPOSE:  THIS PROGRAM WILL CALCULATE A TABLE OF              *
C      *           VALUES DESCRIBING THE MEANFLOW OF THE               *
C      *           FRONT WITH THE SNOIDAL SOLUTION                     *
C      *           DESCRIBING ITS AMPLITUDE.                           *
C      *           THESE RESULTS CAN BE USED BY A GRAPHING             *
C      *           PROGRAM TO PRODUCE A 3-D GRAPH OF                   *
C      *           THE MOTION.                                         *
C      *                                                               *
C      * WRITTEN:  MARCH, 1995                                         *
C      * BY:  CAROL SLOMP                                              *
C      *****************************************************************
C
C      THE RESULTS FOR N=0, M=0 ARE IN FORT 12, N=0,M=4
C      FORT 13, N=2 FORT 14, N=2, M=4 FORT 15, N=4 FORT 16.
C
C      THE VALUES FOR JUST THE PERTURBATION BUT FOR SMALL
C      EPSILON ARE AS FOLLLOWS:  FOR N=0,M=0,FORT55,
C      N=0, M=4, FORT 56, N=2 M=0 FORT 57, N=2 M=4 FORT 58,
C      N=4 FORT 59
C
       INTEGER N,L,T,I,S,J
       REAL K,P,E,PI,U(500), TOTAL,Y,Z,F,LL,X,LAG,LAG1,LAG2
       REAL X0,Y0,G,HNULL(500,500)
       EXTERNAL TOTAL, Y,LAG,LAG1,LAG2
       COMMON/CAR/N
C
       K=1.0E0
C      K IS THE WAVENUMBER
       N=0
C      N IS THE MODE BEING CONSIDERED
       P=1.0E0
C      P IS THE SLOPE OF THE WEDGE-SHAPED FRONT
       E=0.25E0
C      E IS THE SMALL PARAMETER EPSILON
       L=400
       PI=3.0E0
C
       DO 50 J=0,L
              DO 25 S=0,40
              X0=-4.0E0*PI+FLOAT(J)*8.0E0*PI/FLOAT(L)
              Y0=-1.0E0+1.0E0*FLOAT(S)/40.0E0
              HNULL(J+1,S+1)=0.0E0
              WRITE (12,12) X0,Y0,HNULL(J+1,S+1)
12            FORMAT ('  ',F14.3,'  ',F14.3,'  ',F14.9)
25            CONTINUE
50     CONTINUE
C
```

```
      DO 125 I=0,L
            DO 100 T=0,160
                  X=-4.0E0*PI+FLOAT(I)*8.0E0*PI/FLOAT(L)
                  G=4.0E0*FLOAT(T)/160.0E0
                  Z=2.0E0*K*G
                  U(R+1)=TOTAL(K,P,E,X,Z)
                  WRITE (12,75) X,G,U(R+1)
75          FORMAT ('  ',F14.3,'  ',F14.3, '  ',F14.9)
100         CONTINUE
125   CONTINUE
      STOP
      END
C
C
      REAL FUNCTION TOTAL(K,P,E,X,Z)
C     THIS FUNCTION CALCULATES THE TOTAL VALUE OF THE MEAN FLOW
C
      INTEGER N
      REAL ZI1
      REAL PI,P,E,K,TT,F,Y,Z,X
      COMPLEX ZC1,Y8,J8
      COMPLEX A,AA,AI
      EXTERNAL F,Y,A
      COMMON/CAR/N
C
      ZI1=(2.0E0*FLOAT(N)+1.0E0)*Z
      ZC1=CMPLX(2.0E0*SQRT(ZI1),0.0E0)
      CALL BSSLJ (ZC1,0,J8)
      IF (ZC1.EQ.(0.0E0,0.0E0)) THEN
      Y8=(1.0E0,0.0E0)
      ELSE
      CALL BSSLY (ZC1,0,Y8)
      ENDIF
      AI=A(E,K,P,X)
      AA=CMPLX(REAL(AI),-IMAG(AI))
      TT=-E**2*K*ACOS(-1.0E0)*AI*AA/(2.0E0*P)*
     & (REAL(Y8)*F(Z)+REAL(J8)*Y(Z))
      TOTAL=REAL(TT)
C
      RETURN
      END
C
      REAL FUNCTION F(Z)
C     THIS FUNCTION WILL CALCULATE THE FIRST INTEGRAL IN THE
C     FORMULA FOR b, THAT IS, THE INTEGRAL FROM 0 TO 2*Y.
C
      INTEGER NB1,IERR1, P,Q,R,IWRK1(250)
      REAL JO, A,ABSE,RELE,IJ,ERR1,WRK1(1000),Z
      EXTERNAL JO
      COMMON/CAR/N
C
      A=0.0E0
      ABSE=.00001E0
      RELE=.00001E0
```

```
      P=250
      Q=1000
C
      CALL QAGS (JO,A,Z,ABSE,RELE,IJ,ERR1,NB1,IERR1,
     &                    P,Q,R,IWRK1,WRK1)
C
      F=IJ
C
      RETURN
      END
C
C
      REAL FUNCTION Y(Z)
C     THIS FUNCTION WILL CALCULATE THE SECOND INTEGRAL IN
C     THE FORMULA FOR b, THAT IS, THE INTEGRAL FROM 2*Y TO
C     INFINITY.
C
      INTEGER C,D,S, IWRK2(250), NB2,MO1,IERR2
      REAL Z, AERR, IERR, IY, ERR2, WRK2(1000),YO
      EXTERNAL YO
      COMMON/CAR/N
C
      MO1=1
      AERR=0.00001E0
      IERR=0.00001E0
      C=250
      D=1000
C
      CALL QAGI(YO,Z,MO1,AERR,IERR,IY,ERR2,NB2,IERR2,
     &                    C,D,S,IWRK2,WRK2)
C
      Y=IY
C
      RETURN
      END
C
C
      REAL FUNCTION JO(Z)
C     THIS FUNCTION CALCULATES THE INTEGRAND FOR THE FIRST
C     INTEGRAL FOR THE VALUE OF b.
C
      INTEGER N
      REAL ZI
      REAL Z,LAG,LAG1,LAG2
      COMPLEX ZC,T
      EXTERNAL LAG, LAG1, LAG2
      COMMON/CAR/N
C
      ZI=(2.0E0*FLOAT(N)+1.0E0)*Z
      ZC=CMPLX(2.0E0*SQRT(ZI),0.0E0)
      CALL BSSLJ (ZC,0,T)
      JO=REAL(T)*EXP(-Z)*(4.0E0*FLOAT(N+1)*(LAG(Z))**2+8.0E0*
     &   FLOAT(N+2)*LAG(Z)*LAG1(Z)+4.0E0*(LAG1(Z))**2+8.0E0*
     &   LAG(Z)*LAG2(Z))
```

```
      RETURN
      END
C
C
      REAL FUNCTION YO(Z)
C     THIS FUNCTION CALCULATES THE INTEGRAND FOR THE
C     SECOND INTEGRAL FOR THE VALUE OF b.
C
      INTEGER N
      REAL ZI2
      REAL Z,LAG,LAG1,LAG2,WR
      COMPLEX ZC2,W
      EXTERNAL LAG,LAG1,LAG2
      COMMON/CAR/N
C
      ZI2=(2.0E0*FLOAT(N)+1.0E0)*Z
      ZC2=CMPLX(2.0E0*SQRT(ZI2),0.0E0)
      CALL BSSLY(ZC2,0,W)
      YO=REAL(W)*EXP(-Z)*(4.0E0*FLOAT(N+1)*(LAG(Z))**2
     &   +8.0E0*FLOAT(N+2)*LAG(Z)*LAG1(Z)+4.0E0*(LAG1(Z))**2
     &   +8.0E0*LAG(Z)*LAG2(Z))
C
      RETURN
      END
C
C
      REAL FUNCTION LAG(E4)
C     THIS FUNCTION CALCULATES THE VALUES OF THE LAGUERRE
C     POLYNOMIALS
C
      INTEGER N,T
      REAL E4,LN,FACTORIAL
      EXTERNAL FACTORIAL
      COMMON/CAR/N
C
      LN=1.0E0
      DO 150 T=1,N
      LN=LN+(-1.0E0)**T*FACTORIAL(N)/(FACTORIAL(N-T)*
     &   (FACTORIAL(T))**2)*E4**T
150   CONTINUE
C
      LAG=LN
      RETURN
      END
C
C
      REAL FUNCTION LAG1(E4)
C     THIS FUNCTION CALCULATES THE FIRST DERIVATIVE OF THE
C     LAGUERRE POLYNOMIALS.
C
      INTEGER N,H
      REAL LN1,FACTORIAL,E4
      EXTERNAL FACTORIAL
      COMMON/CAR/N
```

```
      LN1=0.0E0
      DO 200 H=1,N
      LN1=LN1+(-1.0E0)**H*FACTORIAL(N)*FLOAT(H)/
     & (FACTORIAL(N-H)*(FACTORIAL(H))**2)*E4**(H-1)
200   CONTINUE
C
      LAG1=LN1
C
      RETURN
      END
C
C
      REAL FUNCTION LAG2(E4)
C     THIS FUNCTION CALCULATES THE SECOND DERIVATIVE OF
C     THE LAGUERRE POLYNOMIALS.
C
      INTEGER N,I
      REAL LN2, FACTORIAL,E4
      EXTERNAL FACTORIAL
      COMMON/CAR/N
C
      LN2=0
      DO 250 I=2,N
      LN2=LN2+(-1.0E0)**I*FACTORIAL(N)*FLOAT(I)*FLOAT(I-1)/
     & (FACTORIAL(N-I)*(FACTORIAL(I))**2)*E4**(I-2)
250   CONTINUE
C
      LAG2=LN2
C
      RETURN
      END
C
C
      REAL FUNCTION FACTORIAL(Q)
C     THIS FUNCTION CALCULATES THE FACTORIAL OF A NUMBER.
      INTEGER Q,S,F1
C
      F1=1
      DO 400 S=1,Q
      F1=F1*S
400   CONTINUE
      FACTORIAL=FLOAT(F1)
C
      RETURN
      END
C
C
      COMPLEX FUNCTION A(E,K,P,X)
C     THIS FUNCTION CALCULATES THE VALUE OF THE AMPLITUDE
C     THAT SATISFIES THE NONLINEAR SCHROEDINGER EQUATION.
C
      INTEGER IERR,N
      REAL S,C,D,KAY,LE,AK,OM,Y,Q,NR,R,RPSI,CONST,BETA
      REAL K,P,E,X,B,AY
```

```
      COMPLEX SCE1
      EXTERNAL BETA
      COMMON/CAR/N
C
      OM=1.0E0
      AK=1/SQRT(2.0E0)
      B=0.25E0
C     B IS A CONSTANT THAT MUST BE LESS THAN (OM-AK**2)**2
      CONST=1.0E0
      R=(OM-AK**2)+SQRT((OM-AK**2)**2-B)
      NR=(OM-AK**2)-SQRT((OM-AK**2)**2-B)
      KAY=NR/R
      LE=SQRT(1-KAY**2)
      AY=SQRT(K**4*BETA(N)/(P**2*(2.0E0*FLOAT(N)+1.0E0)))*E*X
      RPSI=SQRT(R/2)*AY+CONST
      CALL ELLPF (RPSI,KAY,LE,S,C,D,IERR)
C
C     FOR THIS FUNCTION KAY**2+L**2=1, S=SN(RPSI),C=CN(RPSI),
C     D=DN(RPSI). IF IERR=0 THEN ELLIPTIC FUNCTIONS WERE
C     COMPUTED, IERR=1,KAY**2+LE**2=1 IS NOT SATISFIED,
C     IERP=2, RPSI IS TOO LARGE FOR KAY.
C
      SCE1=CMPLX(0.0E0,AK*AY)
      A=CEXP(SCE1)*SQRT(NR)*S
C
      RETURN
      END
C
C
      REAL FUNCTION BETA(N)
C
      INTEGER N
      REAL BT
C
      IF (N.EQ.0) THEN
      BT=7.3638623E0
      ENDIF
      IF (N.EQ.1) THEN
      BT=47.828495E0
      ENDIF
      IF (N.EQ.2) THEN
      BT=190.530625E0
      ENDIF
      IF (N.EQ.3) THEN
      BT=495.425275E0
      ENDIF
      IF (N.EQ.4) THEN
      BT=531.2647044E0
      ENDIF
C
      BETA=BT
C
      RETURN
      END
```

```
C
C      *****************************************************
C      * PROGRAM CHADDITION                                *
C      *                                                   *
C      * PURPOSE:  THIS PROGRAM WILL CALCULATE A TABLE OF  *
C      *           VALUES DESCRIBING THE BOUNDARY OF THE   *
C      *           FRONT WITH THE SNOIDAL SOLUTION         *
C      *           DESCRIBING ITS AMPLITUDE.  THESE        *
C      *           RESULTS CAN BE USED BY A GRAPHING       *
C      *           PROGRAM TO PRODUCE A 3-D GRAPH OF       *
C      *           THE MOTION.                             *
C      *                                                   *
C      * WRITTEN:  MARCH, 1995                             *
C      * BY:  CAROL SLOMP                                  *
C      *****************************************************
C
C      THE RESULTS FOR N=0,M=0 ARE IN FORT 17, N=0, M=4 FORT18,
C      N=2 FORT 19, N=2, M=4 FORT 20, N=4 FORT21.
C
C      THE VALUES FOR THE PERTURBATION CAN BE FOUND AS FOLLOWS:
C      N=0 KAY=1 FORT 40; N=0 M=4 FORT 41;
C
       INTEGER N,L,R,F,I,J
       REAL K,M,P,E,PI,CE,DE,HADD(500,500),HATY0,PHI
       REAL HNULL(500,500),X0,Y0,ZR,Z,TA
       COMPLEX ZC,A
       EXTERNAL HATY0,PHI,A
C
       N=0
C      N IS THE MODE NUMBER
       K=1.0E0
C      K IS THE WAVENUMBER OF THE WAVE
       P=1.0E0
C      P IS THE SLOPE OF THE WEDGE FRONT
       E=0.25E0
C      E IS THE SMALL PARAMETER EPSILON.
       PI =3.0E0
       L=400
C
       DO 200 R=0,L
             DO 150 F=0,40
             CE=-4.0E0*PI+8.0E0*PI*FLOAT(R)/FLOAT(L)
             DE=-1.0E0+1.0E0*FLOAT(F)/40.0E0
             Z=PHI(N,K,P,E,CE)
             IF (Z.GE.0) THEN
             HADD(R+1,F+1)=0
             ELSEIF (DE.LT.Z) THEN
             HADD (R+1,F+1)=0
             ELSE
             HADD (R+1,F+1)=HATY0 (N,K,P,E,CE)*(DE-Z)/Z
             ENDIF
```

```
                WRITE (17,25) CE,DE,HADD(R+1,F+1)
25              FORMAT ('  ',F14.3,'    ',F14.3,'   ',F14.9)
150             CONTINUE
200         CONTINUE
      DO 300 J=0,L
            DO 250 I=0,160
            X0=-4.0E0*PI+8.0E0*PI*FLOAT(J)/FLOAT(L)
            Y0=4.0E0*FLOAT(I)/160.0E0
            HNULL (J+1,I+1)=0.0E0
            WRITE (17,225) X0,Y0,HNULL(J+1,I+1)
225         FORMAT ('  ',F14.3,'    ',F14.3,'   ',F14.9)
250         CONTINUE
300   CONTINUE
C
      STOP
      END
C
C
      REAL FUNCTION PHI(N,K,P,E,XC)
C     THIS FUNCTION CALCULATES THE DEFORMED BOUNDARY.
C
      INTEGER N
      REAL K,P,E,XC,G
      COMPLEX CE1,CE2,NCE1,NCE2,CI,AA,DAA,A,DA,AI,IDA,CPHI,B
      EXTERNAL G,B,A,DA
C
      CE1=CMPLX(0.0E0,K*XC)
      CE2=CMPLX(0.0E0,2.0E0*K*XC)
      NCE1=CMPLX(0.0E0,-K*XC)
      NCE2=CMPLX(0.0E0,-2.0E0*K*XC)
      AI=A(E,K,N,P,XC)
      AA=CMPLX(REAL(AI),-IMAG(AI))
      IDA=DA(E,K,N,P,XC)
      DAA=CMPLX(REAL(IDA),-IMAG(IDA))
      CI=CMPLX(0.0E0,1.0E0)
C
      CPHI=-E/P*(AI*CEXP(CE1)+AA*CEXP(NCE1))
     & -E**2/P**2*(AI*CEXP(CE1)+AA*CEXP(NCE1))**2
     & *(1.0E0+2.0E0*FLOAT(N))*K
     & +E**2/(P**2*K)*CI*(IDA*CEXP(CE1)-DAA*CEXP(NCE1))
     & *FLOAT(N)
     & -E**2*K/P**2*(AI**2*CEXP(CE2)+AA**2*CEXP(NCE2))*G(N)
     & -B(E,K,P,N,XC)/P
C
      PHI=REAL(CPHI)
      RETURN
      END
C
C
      REAL FUNCTION HATY0(N,K,P,E,X)
C     THIS FUNCTION CALCULATES THE HEIGHT OF THE FRONT AT Y=0.
C
```

```
      INTEGER N
      REAL E,K,X,P,G,RHATY0
      COMPLEX CE1,CE2,NCE1,NCE2,CI,AA,DAA,A,DA,AI,IDA,CHATY0,B
      EXTERNAL B, G,A,DA
C
      CE1=CMPLX(0.0E0,K*X)
      CE2=CMPLX(0.0E0,2.0E0*K*X)
      NCE1=CMPLX(0.0E0,-K*X)
      NCE2=CMPLX(0.0E0,-2.0E0*K*X)
      AI=A(E,K,N,P,X)
      AA=CMPLX(REAL(AI),-IMAG(AI))
      IDA=DA(E,K,N,P,X)
      DAA=CMPLX(REAL(IDA),-IMAG(IDA))
      CI=CMPLX(0.0E0,1.0E0)
C
      CHATY0=E*(AI*CEXP(CE1)+AA*CEXP(NCE1))
     & -E**2*CI/K*(IDA*CEXP(CE1)-DAA*CEXP(NCE1))*FLOAT(N)
     & +E**2*K/P*(AI**2*CEXP(CE2)+AA**2*CEXP(NCE2))*G(N)
     & +B(E,K,P,N,X)
C
      RHATY0=REAL(CHATY0)
      HATY0=RHATY0
      RETURN
      END
C
C
      REAL FUNCTION G(N)
C     THIS FUNCTION REPRESENTS THE POLYNOMIAL FUNCTION FOUND
C     IN THE SOLUTION FOR THE HEIGHT OF THE FRONT EVALUATED
C     AT Y=0.
C
      INTEGER N
C
      IF (N.EQ.0) THEN
      G=0.0E0
      ENDIF
      IF (N.EQ.1) THEN
      G=-4.0E0
      ENDIF
      IF (N.EQ.2) THEN
      G=28.0E0/5.0E0
      ENDIF
      IF (N.EQ.3)THEN
      G=-856.0E0/77.0E0
      ENDIF
      IF (N.EQ.4) THEN
      G=2024.0E0/221.0E0
      ENDIF
C
      RETURN
      END
```

```
      COMPLEX FUNCTION B(E,K,P,N,X)
C     THIS FUNCTION REPRESENTS THE VALUE OF THE MEAN FLOW
C     AT Y=0 FOR EPSILON EQUAL TO 0.5.
C
      INTEGER N
      REAL E,K,P,X
      COMPLEX O,J8,AI,AA,A
      EXTERNAL A
C
      AI=A(E,K,N,P,X)
      AA=CMPLX(REAL(AI),-IMAG(AI))
      O=CMPLX(0.0E0,0.0E0)
      CALL BSSLJ (O,0,J8)
      IF (N.EQ.0) THEN
      B=-E**2*K*ACOS(-1.0E0)*AI*AA/(2.0E0*P)*REAL(J8)
     &         *0.887670875E0
      ENDIF
      IF (N.EQ.1) THEN
      B=-E**2*K*ACOS(-1.0E0)*AI*AA/(2.0E0*P)*REAL(J8)
     &            *-3.120948553E0
      ENDIF
      IF (N.EQ.2) THEN
      B=-E**2*K*ACOS(-1.0E0)*AI*AA/(2.0E0*P)*REAL(J8)
     &            *-2.908138990E0
      ENDIF
      IF (N.EQ.3) THEN
      B=-E**2*K*ACOS(-1.0E0)*AI*AA/(2.0E0*P)*REAL(J8)
     &            *1.0E0
      ENDIF
      IF (N.EQ.4) THEN
      B=-E**2*K*ACOS(-1.0E0)*AI*AA/(2.0E0*P)*REAL(J8)
     &            *2.484793901E0
      ENDIF
C
      RETURN
      END
C
C
      COMPLEX FUNCTION A(E,K,N,P,X)
C     THIS FUNCTION CALCULATES THE VALUE OF THE AMPLITUDE
C     THAT SATISFIES THE NONLINEAR SCHROEDINGER EQUATION.
C
      INTEGER IERR,N
      REAL S,D,C,KAY,LE,AK,OM,Q,NR,R,RPSI,CONST,BETA
      REAL K,P,E,X,BO,AY,BI
      COMPLEX SCE1
      EXTERNAL BETA
C
      OM=1.0E0
      AK=1.0E0/SQRT(2.0E0)
      BO=0.25E0
C     BO IS A CONSTANT THAT MUST BE LESS THAN (OM-AK**2)**2
      CONST=1.0E0
```

```
      R=(OM-AK**2)+SQRT((OM-AK**2)**2-BO)
      NR=(OM-AK**2)-SQRT((OM-AK**2)**2-BO)
      KAY=NR/R
      LE=SQRT(1.0E0-KAY**2)
      BI=BETA(N)
      AY=SQRT(K**4*BETA(N)/(P**2*(2.0E0*FLOAT(N)+1.0E0)))*E*X
      RPSI=SQRT(R/2.0E0)*AY+CONST
      CALL ELLPF (RPSI,KAY,LE,S,C,D,IERR)
C
C     FOR THIS FUNCITON KAY**2+LE**2=1, S=SN(RPSI),C=CN(RPSI),
C     D=DN(RPSI).  IF IERR=0 THEN THE ELLIPTIC FUNCITON WERE
C     COMPUTED, IERR=1,KAY*82+LE**2=1 IS NOT SATISFIED,
C     IERR=2, RPSI IS TOO LARGE FOR KAY.
C
      SCE1=CMPLX(0.0E0,AK*AY)
      A=CEXP(SCE1)*SQRT(NR)*S
C
      RETURN
      END
C
C
      COMPLEX FUNCTION DA(E,K,N,P,X)
C     THIS FUNCTION CALCULATES THE DERIVATIVE OF THE AMPLITUDE
C     GIVEN BY THE FUNCTION A(E,K,N,P,X)
C
      INTEGER N,IERR
      REAL E,K,X,OM,BO,D,CONST,NR,R,KAY,LE,AY,RPSI,S,C
      REAL AK,BETA,PARA,P
      COMPLEX CI,SCE2
      EXTERNAL BETA
C
C
      OM=1.0E0
      AK=1.0E0/SQRT(2.0E0)
      BO=0.25E0
C     BO IS A CONSTANT THAT MUST BE LESS THAN (OM-AK**2)**2
      CONST=1.0E0
      R=(OM-AK**2)+SQRT((OM-AK**2)**2-BO)
      NR=(OM-AK**2)-SQRT((OM-AK**2)**2-BO)
      KAY=NR/R
      LE=SQRT(1.0E0-KAY**2)
      PARA=SQRT(K**4*BETA(N)/(P**2*(2.0E0*FLOAT(N)+1.0E0)))
      AY=PARA*E*X
      RPSI=SQRT(R/2.0E0)*AY+CONST
      CI=CMPLX(0.0E0,1.0E0)
      CALL ELLPF (RPSI,KAY,LE,S,C,D,IERR)
C
C     FOR THIS FUNCTION KAY**2+LE**2=1, S=SN(RPSI),C=CN(RPSI),
C     D=DN(RPSI). IF IERR=0 THEN THE ELLIPTIC FUNCITON WERE
C     COMPUTED, IERR=1,KAY**2+LE**2=1 IS NOT SATISFIED,
C     IERR=2, RPSI IS TOO LARGE FOR KAY.
C
      SCE2=CMPLX(0.0E0,AK*AY)
      DA=SQRT(NR)*CEXP(SCE2)*PARA*(CI*AK*S+C*D*SQRT(R/2.0E0))
```

```
      RETURN
      END
C
C
      REAL FUNCTION BETA(N)
C
      INTEGER N
      REAL BT
C
      IF (N.EQ.0) THEN
      BT=7.3638623E0
      ENDIF
      IF (N.EQ.1) THEN
      BT=47.828495E0
      ENDIF
      IF (N.EQ.2) THEN
      BT=190.530625E0
      ENDIF
      IF (N.EQ.3) THEN
      BT=495.425275E0
      ENDIF
      IF (N.EQ.4) THEN
      BT=531.2647044E0
      ENDIF
C
      BETA=BT
C
      RETURN
      END
```

B.3 Solitary Wave Solution

```
C
C
C      ****************************************************
C      * PROGRAM SFRONT                                   *
C      *                                                  *
C      * PURPOSE:  THIS PROGRAM WILL CALCULATE A TABLE    *
C      *           VALUES DESCRIBING THE EVOLUTION OF THE *
C      *               FRONT WITH THE SOLITON SOLUTION    *
C      *           DESCRIBING ITS AMPLITUDE.              *
C      *           THESE RESULTS CAN BE USED BY A         *
C      *           GRAPHING PROGRAM TO PRODUCE A 3-D      *
C      *           GRAPH OF THE MOTION.                   *
C      *                                                  *
C      * WRITTEN:  MARCH, 1995                            *
C      * BY:  CAROL SLOMP                                 *
C      ****************************************************
C
C
C       THE RESULTS FOR N=0 ARE IN FORT 10 N=0 M=5
C       FORT 11,N=2 FORT 12, N=2 M=5 FORT 13 AND N=4 FORT14;
C       N=3 M=1 FORT 30; N=3 M=5 FORT 31;
C
C       THE VALUES FOR JUST THE PERTURBATION ARE FOUND AS ;
C       N=0 M=1 FORT 50; N=0 M=5 FORT 51; N=2 M=1 FORT 52;
C       N=2 M=5 FORT 53; N=3 M=1 FORT 60; N=3 M=5 FORT 61.
C
C       THE VALUES FOR SMALL E ARE AS FOLLOWS: N=3 M=5 FORT 70;
C       N=3 M=1 FORT 71; N=0 M=1 FORT 72; N=0 M=5 FORT 73 ;
C       (WITH CHANGED X-VALUES FORT 74;SECOND FORT 75)
C
        INTEGER L,R,J,N,I,F
        REAL Z(1000,1000),P,E,K,M,C,D,PI,H,LL,GG,G,LAG,KAP
        REAL X0,Y0,HNULL(1000,1000),HATY0
        EXTERNAL H,LAG,G
        P=1.0E0
C       P GIVES THE SLOPE OF THE WEDGE-SHAPED FRONT
        E=0.25E0
C       E IS THE SMALL PARAMETER EPSILON
        K=1.0E0
C       K IS THE WAVENUMBER
        N=0
C       N IS THE MODE BEING CONSIDERED
        L =500
        PI=3.0E0
```

```
      DO 50 I=0,L
            DO 25 F=0,80
            X0=-4.0E0*PI+FLOAT(I)*8.0E0*PI/FLOAT(L)
            Y0=-2.0E0+2.0E0*FLOAT(F)/80.0E0
            HATY0=H(P,E,K,N,X0,0)
            HNULL(I+1,F+1)=0.0E0
            WRITE (11,12) X0,Y0,HNULL(I+1,F+1)
12          FORMAT (' ',F14.3,'  ',F14.3,'  ',F14.9)
25          CONTINUE
50    CONTINUE
C
         DO 120 R=0,L
            DO 110 J=0,240
                  C=-4.0E0*PI+FLOAT(R)*8.0E0*PI/FLOAT(L)
                  D=6.0E0*FLOAT(J)/240.0E0
                  Z(R+1,J+1)=H(P,E,K,N,C,D)
                  WRITE (11,75)  C, D, Z(R+1,J+1)
75                FORMAT (' ',F14.3,'  ',F14.3,'  ',F14.9)
110         CONTINUE
120      CONTINUE
      STOP
      END
C
C
      REAL FUNCTION H(P,E,K,N,X,Y)
C     THIS FUNCTION WILL CALCULATE THE HEIGHT OF THE FRONT.
C
      INTEGER N
      REAL  E1,E2,P,E,K,X,Y,G,LAG
      COMPLEX  CE1,CE2,HC,NCE1,NCE2,I,A,AI,AA,DA,IDA,DAA
      EXTERNAL G,LAG,A,DA
C
      CE1=CMPLX(0.0E0,K*X)
      NCE1=CMPLX(0.0E0,-K*X)
      CE2=CMPLX(0.0E0,2.0E0*K*X)
      NCE2=CMPLX(0.0E0,-2.0E0*K*X)
      E1=-K*Y
      E2=-2.0E0*K*Y
      I=CMPLX(0.0E0,1.0E0)
      AI=A(E,N,K,P,X)
      AA=CMPLX(REAL(AI),-IMAG(AI))
      IDA=DA(E,N,K,P,X)
      DAA=CMPLX(REAL(IDA),-IMAG(IDA))
C
      HC=E*(AI*CEXP(CE1)+AA*CEXP(NCE1))*EXP(E1)*LAG(N,-E2)
    & +E**2*I/(2.0E0*K)*(CEXP(CE1)*IDA-DAA*CEXP(NCE1))*
    & (-E2*LAG(N,-E2)+2.0E0*FLOAT(N)*LAG(N-1,-E2))*EXP(E1)
    & +E**2*K/P*EXP(E2)*G(N,-E2)*(AI**2*CEXP(CE2)
    & +AA**2*CEXP(NCE2))+P*Y
C
      H=REAL(HC)
C
      RETURN
      END
```

```
      COMPLEX FUNCTION A(E,N,K,P,XE)
C     THIS FUNCTION WILL CALCULATE THE AMPLITUDE OF THE FLOW
C
      INTEGER N
      REAL KAP,CE,PSI,M,PARA,K,B,P,XE,E,BETA
      COMPLEX I
      EXTERNAL BETA
C
      KAP=1.0E0
      M=5.0E0
      CE=SQRT(2.0E0*M-KAP**2)/2.0E0
C     M MUST BE GREATE THAN KAP**2/2
      B=BETA(N)
      PARA=SQRT((K**4*B)/(P**2*(2.0E0*FLOAT(N)+1.0E0)))
      PSI=PARA*E*XE
      I=CMPLX(0.0E0,1.0E0)
C
      A=(KAP-2.0E0*I*CE*TANH(CE*PSI))/SQRT(2.0E0)
C
      RETURN
      END
C
C
      COMPLEX FUNCTION DA(E,N,K,P,XR)
C     THIS FUNCTION CALCULATES THE FIRST DERIVATIVE
C     OF THE AMPLITUDE
C
      INTEGER N
      REAL E,K,P,XR,B,KAP,CE,M,PARA,PSI,BETA
      COMPLEX I
      EXTERNAL BETA
C
      KAP=1.0E0
      M=5.0E0
      CE=SQRT(2.0E0*M-KAP**2)/2.0E0
C     M MUST BE GREATE THAN KAP**2/2
      B=BETA(N)
      PARA=SQRT((K**4*B)/(P**2*(2.0E0*FLOAT(N)+1.0E0)))
      PSI=PARA*E*XR
      I=CMPLX(0.0E0,1.0E0)
C
      DA=-SQRT(2.0E0)*I*CE**2/(COSH(CE*PSI))**2*PARA*E
C
      RETURN
      END
C
C
      REAL FUNCTION G(N,E3)
      INTEGER N
      REAL E3,GS
C
```

```
      IF (N.EQ.0) THEN
      GS=0.0E0
      ENDIF
      IF (N.EQ.1) THEN
      GS=-4.0E0-2.0E0*E3
      ENDIF
      IF (N.EQ.2) THEN
      GS=28.0E0/5.0E0-102.0E0*E3/5.0E0
     & +16.0E0*E3**2/5.0E0-E3**3/3.0E0
      ENDIF
      IF (N.EQ.3) THEN
      GS=-856.0E0/77.0E0+292.0E0*E3/77.0E0+
     & 1388.0E0*E3**2/77.0E0-359.0E0*E3**3/77.0E0+
     & 7.0E0*E3**4/11.0E0-E3**5/30.0E0
      ENDIF
      IF (N.EQ.4) THEN
      GS=2024.0E0/221.0E0-15924.0E0*E3/221.0E0
     & +15364.0E0*E3**2/221.0E0-4917.0E0*E3**3/221.0E0+
     & 3821.0E0*E3**4/663.0E0-1135.0E0*E3**5/1326.0E0
      ENDIF
      IF (N.EQ.5) THEN
      GS=-80772.0E0/4807.0E0+75054.0E0*E3/4807.0E0
     & +343660.0E0*E3**2/4807.0E0-1010665.0E0*E3**3/14421.0E0+
     & 124494.0E0*E3**4/48076.0E0-30064.0E0*E3**5/4807.0E0+
     & 3671.0E0*E3**6/3933.0E0-865.0E0*E3**7/10488.0E0
     & +13.0E0*E3**8/3312.0E0-E3**9/12960.0E0
      ENDIF
C
      G=GS
      RETURN
      END
C
C
      REAL FUNCTION LAG(N,E4)
C     THIS FUNCTION CALCULATES THE VALUES OF THE LAGUERRE
C     POLYNOMIALS
C
      INTEGER N,T
      REAL E4,LN,FACTORIAL
      EXTERNAL FACTORIAL
C
      LN=1.0E0
      DO 150 T=1,N
      LN=LN+(-1.0E0)**T*FACTORIAL(N)/(FACTORIAL(N-T)
     & *(FACTORIAL(T))**2)*E4**T
150   CONTINUE
      LAG=LN
      RETURN
      END
C
C
      REAL FUNCTION FACTORIAL(Q)
C     THIS FUNCTION CALCULATES THE FACTORIAL OF A NUMBER.
```

```
      INTEGER Q,S,F1
C
      F1=1
      DO 200 S=1,Q
      F1=F1*S
200   CONTINUE
      FACTORIAL=FLOAT(F1)
C
      RETURN
      END
C
C
      REAL FUNCTION BETA(N)
C     THIS FUNCTION SETS THE VALUE FOR BETA, THE POSITIVE
C     PARAMETER IN THE NONLINEAR SCHROEDINGER EQUATION.
C
      INTEGER N
      REAL B
C
      IF (N.EQ.0) THEN
      B=7.3638623E0
      ENDIF
      IF (N.EQ.1) THEN
      B=47.828495E0
      ENDIF
      IF (N.EQ.2) THEN
      B=190.530625E0
      ENDIF
      IF (N.EQ.3) THEN
      B=495.425275E0
      ENDIF
      IF (N.EQ.4) THEN
      B=531.2647044E0
      ENDIF
C
      BETA =B
C
      RETURN
      END
```

```
C
C
C       *******************************************************
C       * PROGRAM SMEANFLOW                                   *
C       *                                                     *
C       * PURPOSE:  THIS PROGRAM WILL CALCULATE A TABLE OF    *
C       *           VALUES DESCRIBING THE MEANFLOW OF THE     *
C       *               FRONT WITH THE SOLITON SOLUTION       *
C       *           DESCRIBING ITS AMPLITUDE.                 *
C       *           THESE RESULTS CAN BE USED BY A            *
C       *           GRAPHING PROGRAM TO PRODUCE A 3-D         *
C       *           TO PRODUCE A 3-D GRAPH OF THE MOTION.     *
C       *                                                     *
C       * WRITTEN:  MARCH, 1995                               *
C       * BY:  CAROL SLOMP                                    *
C       *******************************************************
C
C
C        THE VALUES FOR N=0 ARE IN FORT 15, N=0 M=5 FORT 16;
C        N=2 FORT 17; N=2 M=5 FORT 18 AND N=4 FORT 19.
C
C
         INTEGER N,L,R,I,S,J
         REAL K,P,E,PI,U(500,500), TOTAL,Y,Z,F,X,LAG,LAG1,LAG2,G
         REAL X0,Y0,KAP,M,C,HNULL(500,500)
         COMPLEX LL,A
         EXTERNAL TOTAL, Y,LAG,LAG1,LAG2,A
         COMMON/CAR/N
C
         K=1.0E0
C        K IS THE WAVENUMBER
         N=0
C        N IS THE MODE BEING CONSIDERED
         P=1.0E0
C        P IS THE SLOPE OF THE WEDGE-SHAPED FRONT
         E=0.25E0
C        E IS THE SMALL PARAMETER EPSILON
         L=400
         PI=3.0E0
C
         DO 50 J=0,L
                 DO 25 S=0,80
                 X0=-4.0E0*PI+FLOAT(J)*8.0E0*PI/FLOAT(L)
                 Y0=-2.0E0+2.0E0*FLOAT(S)/80.0E0
                 HNULL(J+1,S+1)=0.0E0
                 WRITE (15,12) X0,Y0,HNULL(J+1,S+1)
12               FORMAT ('  ',F14.3,'  ',F14.3,'  ',F14.9)
25               CONTINUE
50       CONTINUE
C
```

```
        DO 125 I=0,L
               DO 100 R=0,240
                      X=-4.0E0*PI+FLOAT(I)*8.0E0*PI/FLOAT(L)
                      G=6.0E0*FLOAT(R)/240.0E0
                      Z=2.0E0*K*G
C                     LL=A(E,K,P,X)
                      U(I+1,R+1)=TOTAL(K,P,E,Z,X)
                      WRITE (15,75) X,G,U(I+1,R+1)
75                    FORMAT (' ',F14.3,' ',F14.3, ' ',F14.9)
100            CONTINUE
125     CONTINUE
        STOP
        END
C
C
        REAL FUNCTION TOTAL(K,P,E,Z,X)
C       THIS FUNCTION CALCULATES THE TOTAL VALUE OF THE
C       MEAN FLOW
C
        INTEGER N
        REAL ZI1
        REAL P,E,K,F,Y,Z,X
        COMPLEX ZC1,Y8,J8,TT,IA,AA,A
        EXTERNAL F,Y,BETA,A
        COMMON/CAR/N
C
        IA=A(E,K,P,X)
        AA=CMPLX(REAL(IA),-IMAG(IA))
        ZI1=(2.0E0*FLOAT(N)+1.0E0)*Z
        ZC1=CMPLX(2.0E0*SQRT(ZI1),0.0E0)
        CALL BSSLJ (ZC1,0,J8)
        IF (ZC1.EQ.(0.0E0,0.0E0)) THEN
        Y8=(1.0E0,0.0E0)
        ELSE
        CALL BSSLY (ZC1,0,Y8)
        ENDIF
        TT=-E**2*K*ACOS(-1.0E0)*IA*AA/(2.0E0*P)*
     &  (Y8*F(Z)+J8*Y(Z))
C
        TOTAL=REAL(TT)
C
        RETURN
        END
C
C
        COMPLEX FUNCTION A(E,K,P,XE)
C       THIS FUNCTION CALCULATES THE VALUE OF THE
C       AMPLITUDE FUNCTION
C
        INTEGER N
        REAL BI,K,P,KAP,PSI,M,PARA,XE,E,BETA
        COMPLEX I
        EXTERNAL BETA
        COMMON/CAR/N
```

```
      BI=BETA(N)
      KAP=1.0E0
      M=1.0E0
      CE=SQRT(2.0E0*M-KAP**2)/2.0E0
C     M MUST BE GREATER THAN KAP**2/2
      PARA=SQRT(K**4*BI/(P**2*(2.0E0*FLOAT(N)+1.0E0)))
      PSI=PARA*E*XE
      I=CMPLX(0.0E0,1.0E0)
C
      A=(KAP-2.0E0*I*CE*TANH(CE*PSI))/SQRT(2.0E0)
C
      RETURN
      END
C
C
      REAL FUNCTION F(Z)
C     THIS FUNCTION WILL CALCULATE THE FIRST INTEGRAL IN
C     THE FORMULA FOR b, THAT IS, THE INTEGRAL FROM 0 TO 2*Y.
C
      INTEGER NB1,IERR1, P,Q,R,IWRK1(250)
      REAL JO, A,ABSE,RELE,IJ,ERR1,WRK1(1000),Z
      EXTERNAL JO
      COMMON/CAR/N
C
      A=0.0E0
      ABSE=.00001E0
      RELE=.00001E0
      P=250
      Q=1000
C
      CALL QAGS (JO,A,Z,ABSE,RELE,IJ,ERR1,NB1,IERR1,
     &                P,Q,R,IWRK1,WRK1)
C
      F=IJ
C
      RETURN
      END
C
C
      REAL FUNCTION Y(Z)
C     THIS FUNCTION WILL CALCULATE THE SECOND INTEGRAL IN
C     THE FORMULA FOR b, THAT IS, THE INTEGRAL FROM 2*Y TO
C     INFINITY.
C
      INTEGER C,D,S, IWRK2(250), NB2,MO1,IERR2
      REAL Z, AERR, IERR, IY, ERR2, WRK2(1000),YO
      EXTERNAL YO
      COMMON/CAR/N
C
      MO1=1
      AERR=0.00001E0
      IERR=0.00001E0
      C=250
      D=1000
```

```
      CALL QAGI(YO,Z,MO1,AERR,IERR,IY,ERR2,NB2,IERR2,
     &                     C,D,S,IWRK2,WRK2)
C
      Y=IY
C
      RETURN
      END
C
C
      REAL FUNCTION JO(Z)
C     THIS FUNCTION CALCULATES THE INTEGRAND FOR THE FIRST
C     INTEGRAL FOR THE VALUE OF b.
C
      INTEGER N
      REAL ZI
      REAL Z,LAG,LAG1,LAG2
      COMPLEX ZC,T
      EXTERNAL LAG, LAG1, LAG2
      COMMON/CAR/N
C
      ZI=(2.0E0*FLOAT(N)+1.0E0)*Z
      ZC=CMPLX(2.0E0*SQRT(ZI),0.0E0)
      CALL BSSLJ (ZC,0,T)
      JO=REAL(T)*EXP(-Z)*(4.0E0*FLOAT(N+1)*(LAG(Z))**2+8.0E0*
     & FLOAT(N+2)*LAG(Z)*LAG1(Z)+4.0E0*(LAG1(Z))**2+8.0E0*
     & LAG(Z)*LAG2(Z))
C
      RETURN
      END
C
C
      REAL FUNCTION YO(Z)
C     THIS FUNCTION CALCULATES THE INTEGRAND FOR THE
C     SECOND INTEGRAL FOR THE VALUE OF b.
C
      INTEGER N
      REAL ZI2
      REAL Z,LAG,LAG1,LAG2,WR
      COMPLEX ZC2,W
      EXTERNAL LAG,LAG1,LAG2
      COMMON/CAR/N
C
      ZI2=(2.0E0*FLOAT(N)+1.0E0)*Z
      ZC2=CMPLX(2.0E0*SQRT(ZI2),0.0E0)
      CALL BSSLY(ZC2,0,W)
      YO=REAL(W)*EXP(-Z)*(4.0E0*FLOAT(N+1)*(LAG(Z))**2
     & +8.0E0*FLOAT(N+2)*LAG(Z)*LAG1(Z)+4.0E0*(LAG1(Z))**2
     & +8.0E0*LAG(Z)*LAG2(Z))
C     WRITE (30,*) YO
C
      RETURN
      END
```

```
      REAL FUNCTION LAG(E4)
C     THIS FUNCTION CALCULATES THE VALUES OF THE LAGUERRE
C     POLYNOMIALS
      INTEGER N,T
      REAL E4,LN,FACTORIAL
      EXTERNAL FACTORIAL
      COMMON/CAR/N
C
      LN=1.0E0
      DO 150 T=1,N
      LN=LN+(-1.0E0)**T*FACTORIAL(N)/(FACTORIAL(N-T)*
     & (FACTORIAL(T))**2)*E4**T
150   CONTINUE
      LAG=LN
      RETURN
      END
C
C
      REAL FUNCTION LAG1(E4)
C     THIS FUNCTION CALCULATES THE FIRST DERIVATIVE OF THE
C     LAGUERRE POLYNOMIALS.
C
      INTEGER N,H
      REAL LN1,FACTORIAL,E4
      EXTERNAL FACTORIAL
      COMMON/CAR/N
C
      LN1=0.0E0
      DO 200 H=1,N
      LN1=LN1+(-1.0E0)**H*FACTORIAL(N)*FLOAT(H)/(FACTORIAL(N-H)*
     & (FACTORIAL(H))**2)*E4**(H-1)
200   CONTINUE
C
      LAG1=LN1
C
      RETURN
      END
C
C
      REAL FUNCTION LAG2(E4)
C     THIS FUNCTION CALCULATES THE SECOND DERIVATIVE OF THE
C     LAGUERRE POLYNOMIALS.
C
      INTEGER N,I
      REAL LN2, FACTORIAL,E4
      EXTERNAL FACTORIAL
      COMMON/CAR/N
C
      LN2=0
      DO 250 I=2,N
      LN2=LN2+(-1.0E0)**I*FACTORIAL(N)*FLOAT(I)*FLOAT(I-1)/
     & (FACTORIAL(N-I)*(FACTORIAL(I))**2)*E4**(I-2)
250   CONTINUE
      LAG2=LN2
```

```
      RETURN
      END
C
C
      REAL FUNCTION FACTORIAL(Q)
C     THIS FUNCTION CALCULATES THE FACTORIAL OF A NUMBER.
C
      INTEGER Q,S,F1
C
      F1=1
      DO 400 S=1,Q
      F1=F1*S
400   CONTINUE
      FACTORIAL=FLOAT(F1)
C
      RETURN
      END
C
C
      REAL FUNCTION BETA(N)
C     THIS FUNCTION SETS THE VALUE FOR BETA, THE POSITIVE
C     PARAMETER IN THE NONLINEAR SCHROEDINGER EQUATION.
C
      INTEGER N
      REAL BT
C
      IF (N.EQ.0) THEN
      BT=7.3638623E0
      ENDIF
      IF (N.EQ.1) THEN
      BT=47.828495E0
      ENDIF
      IF (N.EQ.2) THEN
      BT=190.530625E0
      ENDIF
      IF (N.EQ.3) THEN
      BT=495.425275E0
      ENDIF
      IF (N.EQ.4) THEN
      BT=531.2647044E0
      ENDIF
C
      BETA=BT
      RETURN
      END
```

```
C
C       ***********************************************************
C       * PROGRAM SHADDITION                                      *
C       *                                                         *
C       * PURPOSE:  THIS PROGRAM WILL CALCULATE A TABLE OF        *
C       *           VALUES DESCRIBING THE BOUNDARY OF THE         *
C       *           FRONT WITH THE SOLITON SOLUTION               *
C       *           DESCRIBING ITS AMPLITUDE.                     *
C       *           THESE RESULTS CAN BE USED BY A                *
C       *           GRAPHING PROGRAM TO PRODUCE A 3-D GRAPH       *
C       *           OF THE MOTION.                                *
C       *                                                         *
C       * WRITTEN:  MARCH, 1995                                   *
C       * BY:  CAROL SLOMP                                        *
C       ***********************************************************
C
C       THE DATA FOR N=0 IS IN FORT 20, N=0 M=5 FORT 21 N=2
C       FORT 22, N=2 M=9 FORT 23 AND N=4 FORT 24, N=3 M=1
C       FORT 40, N=3 M=5 FORT 41
C
        INTEGER N,L,R,F,I,J
        REAL K,P,E,PI,C,D,Z,HADD(500,500),HATY0,PHI,H
        REAL HNULL(500,500),X0,Y0,AA
        COMPLEX BI,B
        EXTERNAL HATY0,PHI,B
C
        N=0
C       N IS THE MODE NUMBER
        K=1.0E0
C       K IS THE WAVENUMBER OF THE WAVE
        P=1.0E0
C       P IS THE SLOPE OF THE WEDGE FRONT
        E=0.25E0
C       E IS THE SMALL PARAMETER EPSILON.
        PI =3.0E0
        L=400
C
        DO 200 R=0,L
                DO 150 F=0,80
                C=-4.0E0*PI+8.0E0*PI*FLOAT(R)/FLOAT(L)
                D=-2.0E0+2.0E0*FLOAT(F)/80.0E0
                Z=PHI(N,K,P,E,C)
                IF (Z.GE.0.0E0) THEN
                HADD(R+1,F+1)=0.0E0
                ELSEIF (D.LT.Z) THEN
                HADD (R+1,F+1)=0.0E0
                ELSE
                HADD (R+1,F+1)=HATY0 (N,K,P,E,C)*(D-Z)/Z
                ENDIF
                WRITE (20,25) C,D,HADD(R+1,F+1)
```

```
25          FORMAT ('  ',F14.3,'  ',F14.3,'  ',F14.9)
150         CONTINUE
200    CONTINUE
       DO 300 J=0,L
            DO 250 I=0,160
            X0=4.0E0*PI*(-1.0E0 + 2.0E0*FLOAT(J)/FLOAT(L))
            Y0=4.0E0*FLOAT(I)/160.0E0
            HNULL (J+1,I+1)=0.0E0
            WRITE (20,225) X0,Y0,HNULL(J+1,I+1)
225         FORMAT ('  ',F14.3,'  ',F14.3,'  ',F14.9)
250         CONTINUE
300    CONTINUE
C
       STOP
       END
C
C
       REAL FUNCTION PHI(N,K,P,E,X)
C      THIS FUNCTION CALCULATES THE DEFORMED BOUNDARY.
C
       INTEGER N
       REAL K,P,E,X,G
       COMPLEX CE1,CE2,NCE1,NCE2
       COMPLEX I,CPHI,AI,AA,A,DA,DAI,DAA,B
       EXTERNAL G,B,A
C
       AI=A(E,N,K,P,X)
       AA=CMPLX(REAL(AI),-IMAG(AI))
       DAI=DA(E,N,K,P,X)
       DAA=CMPLX(REAL(DAI),-IMAG(DAI))
       CE1=CMPLX(0.0E0,K*X)
       NCE1=CMPLX(0.0E0,-K*X)
       CE2=CMPLX(0.0E0,2.0E0*K*X)
       NCE2=CMPLX(0.0E0,-2.0E0*K*X)
       I=CMPLX(0.0E0,1.0E0)
C
       CPHI=-E*(AI*CEXP(CE1)+AA*CEXP(NCE1))/P
     & -E**2*K/P**2*(CEXP(CE1)*AI+CEXP(NCE1)*AA)**2*
     & (1.0E0+2.0E0*FLOAT(N))
     & -E**2*I/(P*K)*DFLOAT(N)*(DAI*CEXP(CE1)-DAA*CEXP(NCE1))
     & -E**2*K/P**2*G(N)*(AI**2*CEXP(CE2)+AA**2*CEXP(NCE2))
     & -B(N,E,K,P,X)/P
C
       PHI=REAL(CPHI)
       RETURN
       END
C
C
       REAL FUNCTION HATY0(N,K,P,E,X)
C      THIS FUNCTION CALCULATES THE HEIGHT OF THE FRONT AT Y=0.
C
       INTEGER N
       REAL K,P,E,X,G
       COMPLEX CE1,CE2,NCE1,NCE2
```

```
      COMPLEX I,CHATY0,AI,A,AA,DAI,DA,DAA,B
      EXTERNAL G,B,A,DA
C
      AI=A(E,N,K,P,X)
      AA=CMPLX(REAL(AI),-IMAG(AI))
      DAI=DA(E,N,K,P,X)
      DAA=CMPLX(REAL(DAI),-IMAG(DAI))
      CE1=CMPLX(0.0E0,K*X)
      NCE1=CMPLX(0.0E0,-K*X)
      CE2=CMPLX(0.0E0,2.0E0*K*X)
      NCE2=CMPLX(0.0E0,-2.0E0*K*X)
      I=CMPLX(0.0E0,1.0E0)
C
      CHATY0=E*(CEXP(CE1)*AI+CEXP(NCE1)*AA)
     & +E**2*I/K*FLOAT(N)*(CEXP(CE1)*DAI-CEXP(NCE1)*DAA)
     & +E**2*K/P*(CEXP(CE2)*AI**2+CEXP(NCE2)*AA**2)*G(N)
     & +B(N,E,K,P,X)/P
C
      HATY0=REAL(CHATY0)
C
      RETURN
      END
C
C
      REAL FUNCTION G(N)
C     THIS FUNCTION REPRESENTS THE POLYNOMIAL FUNCTION FOUND
C     IN THE SOLUTION FOR THE HEIGHT OF THE FRONT EVALUATED
C     AT Y=0.
C
      INTEGER N
C
      IF (N.EQ.0) THEN
      G=0.0E0
      ENDIF
      IF (N.EQ.1) THEN
      G=-4.0E0
      ENDIF
      IF (N.EQ.2) THEN
      G=28.0E0/5.0E0
      ENDIF
      IF (N.EQ.3)THEN
      G=-856.0E0/77.0E0
      ENDIF
      IF (N.EQ.4) THEN
      G=2024.0E0/221.0E0
      ENDIF
C
      RETURN
      END
C
C
      COMPLEX FUNCTION B(N,E,K,P,X)
C     THIS FUNCTION REPRESENTS THE VALUE OF THE MEAN FLOW
C     AT Y=0 FOR EPSILON EQUAL TO 0.25.
```

```
      INTEGER N
      REAL E,K,X,P
      COMPLEX J8,S,AI,AA,BN,A
      EXTERNAL A
C
      S=CMPLX(0.0E0,0.0E0)
      CALL BSSLJ (S,0,J8)
      AI=A(E,N,K,P,X)
      AA=CMPLX(REAL(AI),-IMAG(AI))
      IF (N.EQ.0) THEN
      BN=-E**2*K*ACOS(-1.0E0)*AI*AA/(2.0E0*P)
     & *J8*0.887670875E0
      ENDIF
      IF (N.EQ.1) THEN
      BN=-E**2*K*ACOS(-1.0E0)*AI*AA
     & *J8*-3.120948553E0/(2.0E0*P)
      ENDIF
      IF (N.EQ.2) THEN
      BN=-E**2*K*ACOS(-1.0E0)*AI*AA
     & *J8*-2.908138990E0/(2.0E0*P)
      ENDIF
      IF (N.EQ.3) THEN
      BN=-E**2*K*ACOS(-1.0E0)*AI*AA
     & *J8*-0.257224351E0/(2.0E0*P)
      ENDIF
      IF (N.EQ.4) THEN
      BN=-E**2*K*ACOS(-1.0E0)*AI*AA
     & *J8*2.484793901E0/(2.0E0*P)
      ENDIF
C
      B=BN
      RETURN
      END
C
C
      COMPLEX FUNCTION A(E,N,K,P,XE)
C     THIS FUNCTION COMPUTES THE AMPLITUDE OF THE GROUP
C     VELOCITY
C
      INTEGER N
      REAL KAP,CE,PSI,M,PARA,K,B,P,XE,E,BETA
      COMPLEX I
      EXTERNAL BETA
C
      KAP=1.0E0
      M=1.0E0
      CE=SQRT(2.0E0*M-KAP**2)/2.0E0
C     M MUST BE GREATER THAN KAP**2/2
      B=BETA(N)
      PARA=SQRT((K**4*B)/(P**2*(2.0E0*FLOAT(N)+1.0E0)))
      PSI=PARA*E*XE
      I=CMPLX(0.0E0,1.0E0)
C
      A=(KAP-2.0E0*I*CE*TANH(CE*PSI))/SQRT(2.0E0)
```

```
      RETURN
      END
C
C
      COMPLEX FUNCTION DA(E,N,K,P,XR)
C     THIS FUNCTION CALCULATES THE DERIVATIVE OF THE
C     AMPLITUDE FUNCTION
C
      INTEGER N
      REAL E,K,P,XR,B,KAP,CE,M,PARA,PSI,BETA
      COMPLEX I
      EXTERNAL BETA
C
      KAP=1.0E0
      M=1.0E0
      CE=SQRT(2.0E0*M-KAP**2)/2.0E0
C     M MUST BE GREATER THAN KAP*82/2
      B=BETA(N)
      PARA=SQRT((K**4*B)/(P**2*(2.0E0*FLOAT(N)+1.0E0)))
      PSI=PARA*E*XR
      I=CMPLX(0.0E0,1.0E0)
C
      DA=-SQRT(2.0E0)*I*CE**2/(COSH(CE*PSI))**2*PARA*E
C
      RETURN
      END
C
C
      REAL FUNCTION BETA(N)
C     THIS FUNCTION SETS THE VALUE FOR BETA, THE POSITIVE
C     PARAMETER IN THE NONLINEAR SCHROEDINGER EQUATION
C
      INTEGER N
      REAL BT
C
      IF (N.EQ.0) THEN
      BT=7.3638623E0
      ENDIF
      IF (N.EQ.1) THEN
      BT=47.828495E0
      ENDIF
      IF (N.EQ.2) THEN
      BT=190.530625E0
      ENDIF
      IF (N.EQ.3) THEN
      BT=495.425275E0
      ENDIF
      IF (N.EQ.4) THEN
      BT=531.2647044E0
      ENDIF
C
      BETA=BT
C
      RETURN
      END
```