CONJUGACY PROBLEMS FOR “CARTAN” SUBALGEBRAS IN INFINITE DIMENSIONAL LIE ALGEBRAS

by

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Abstract

Chevalley's theorem on the conjugacy of split Cartan subalgebras is one of the cornerstones of the theory of simple finite dimensional Lie algebras over a field of characteristic 0. Indeed, this theorem affords the most elegant proof that the root system is an invariant of the Lie algebra.

The analogous result for symmetrizable Kac-Moody Lie algebras is the celebrated theorem of Peterson and Kac. However, the methods they used are not suitable for attacking the problem of conjugacy in "higher nullity", i.e. for extended affine Lie algebras (EALA). In the thesis we develop a new cohomological approach which we use to prove

1) conjugacy of Cartan subalgebras in affine Kac-Moody Lie algebras;

2) conjugacy of maximal abelian *ad*-diagonalizable subalgebras (MADs) of EALA of finite type, coming as a part of the structure, where me assume that the centreless core is not isomorphc to $sl_2(R)$, R is a ring of Laurent polynomials in more then 1 variables.

We give a counterexample to conjugacy of arbitrary MADs in EALA.

Some relevant problems on the lifting of automorphisms are discussed as well.

Preface

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Contents

Chapter 1

Introduction

Throughout the dissertation, k denotes an algebraically closed field of characteristic 0.

Infinite dimensional Lie algebras became a useful tool in physics in the 1960s. They were the right mathematical apparatus to describe the supersymmetric phenomena. Currently, amongst the others infinite dimensional Lie algebras ones of particular interest to physicists are affine Kac-Moody Lie algebras and extended affine Lie algebras (EALAs for short).

Kac-Moody Lie algebras appeared in mathematics as a generalization of finite dimensional simple Lie algebras over a field of characteristic 0. While a general Kac-Moody Lie algebra is defined by generators and relations, this definition is quite elusive (often it is hard to say how the algebra looks), an affine Kac-Moody Lie algebra has a nice description: its derived subalgebra modulo its centre is a twisted loop algebra. Therefore, it can be considered not only as a Lie algebra over a base field k, but as a simple Lie algebra over a Laurent polynomial ring $k[t^{\pm 1}]$ in the sense of [SGA3] and it is a twisted form of a loop algebra $\mathfrak{g} \otimes k[t^{\pm 1}]$.

Let us recall the description of an affine Kac-Moody Lie algebra.

Split case. Let $\mathfrak{g}$ be a split simple finite dimensional Lie algebra over a field k and let $\mathbf{Aut}(\mathfrak{g})$ be its automorphism group. If $x, y \in \mathfrak{g}$, we denote their product in $\mathfrak{g}$ by $[x, y]$. We also let $R = k[t^{\pm 1}]$ and $L(\mathfrak{g}) = \mathfrak{g} \otimes_k R$. We again denote the Lie product in $L(\mathfrak{g})$ by $[x, y]$, where $x, y \in L(\mathfrak{g})$.

The main object under consideration in Chapter 4 is the affine (split or twisted)

Kac-Moody Lie algebra $\widehat{L}$ corresponding to $\mathfrak{g}$. Any split affine Kac-Moody Lie algebra is of the form (see [Kac])

$$\widehat{L} = \mathfrak{g} \otimes_k R \oplus k\, c \oplus k\, d.$$

The element c is central and d is a degree derivation for a natural grading of $L(\mathfrak{g})$: if $x \in \mathfrak{g}$ and $p \in \mathbb{Z}$ then

$$[d, x \otimes t^p]_{\widehat{L}} = p\, x \otimes t^p.$$

If $l_1 = x \otimes t^p,\ l_2 = y \otimes t^q \in L(\mathfrak{g})$ are viewed as elements in $\widehat{L}$, their Lie product is given by

$$[x \otimes t^p, y \otimes t^q]_{\widehat{L}} = [x, y] \otimes t^{p+q} + p\, \kappa(x, y)\, \delta_{0,p+q} \cdot c,$$

where κ is the Killing form on $\mathfrak{g}$ and $\delta_{0,p+q}$ is Kronecker's delta.

Twisted case. Let m be a positive integer and let $S = k[t^{\pm\frac{1}{m}}]$ be the ring of Laurent polynomials in the variable $s = t^{\frac{1}{m}}$ with coefficients in k. Let

$$L(\mathfrak{g})_S = L(\mathfrak{g}) \otimes_R S$$

be the Lie algebra obtained from the R-Lie algebra $L(\mathfrak{g})$ by the base change $R \to S$. Similarly, we define Lie algebras

$$\widetilde{L}(\mathfrak{g})_S = L(\mathfrak{g})_S \oplus kc \quad \text{and} \quad \widehat{L}(\mathfrak{g})_S = L(\mathfrak{g})_S \oplus kc \oplus kd.$$ [1]

Fix a primitive root of unity $\zeta \in k$ of degree m. The R-automorphism

$$\zeta^{\times} : S \to S \quad s \mapsto \zeta s$$

generates the Galois group $\Gamma = \mathrm{Gal}(S/R)$, which we may identify with the abstract group $\mathbb{Z}/m\mathbb{Z}$ by means of $\zeta^{\times}$. Note that Γ acts naturally on $\mathbf{Aut}(\mathfrak{g})(S) = \mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S)$ and on $L(\mathfrak{g})_S = L(\mathfrak{g}) \otimes_R S$ through the second factor.

Next, let σ be an automorphism of $\mathfrak{g}$ of order m. This gives rise to an S-automorphism of $L(\mathfrak{g})_S$ via $x \otimes s \mapsto \sigma(x) \otimes s$, for $x \in \mathfrak{g}$, $s \in S$. It then easily

[1] Unlike $L(\mathfrak{g})_S$, these object exist over k but not over S.

follows that the assignment

$$\overline{1} \mapsto z_{\overline{1}} = \sigma^{-1} \in \mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S)$$

gives rise to a cocycle $z = (z_{\overline{i}}) \in Z^1(\Gamma, \mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S))$. This cocycle, in turn, gives rise to a twisted action of Γ on $L(\mathfrak{g})_S$. Applying Galois descent formalism, we then obtain the Γ-invariant subalgebra

$$L(\mathfrak{g}, \sigma) := (L(\mathfrak{g})_S)^{\Gamma} = (L(\mathfrak{g}) \otimes_R S)^{\Gamma}.$$

This is a "simple Lie algebra over R" in the sense of [SGA3], which is a twisted form of the "split simple" R-Lie algebra $L(\mathfrak{g}) = \mathfrak{g} \otimes_k R$. Indeed, S/R is an étale extension and from properties of Galois descent we have

$$L(\mathfrak{g}, \sigma) \otimes_R S \simeq L(\mathfrak{g})_S = (\mathfrak{g} \otimes_k R) \otimes_R S.$$

Note that $L(\mathfrak{g}, id) = L(\mathfrak{g})$.

For $\overline{i} \in \mathbb{Z}/m\mathbb{Z}$, consider the eigenspace

$$\mathfrak{g}_{\overline{i}} = \{x \in \mathfrak{g} : \sigma(x) = \zeta^i x\}.$$

Simple computations show that

$$L(\mathfrak{g}, \sigma) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\overline{i}} \otimes k[t^{\pm 1}] s^i.$$

Let

$$\widetilde{L}(\mathfrak{g}, \sigma) := L(\mathfrak{g}, \sigma) \oplus kc \quad \text{and} \quad \widehat{L}(\mathfrak{g}, \sigma) := L(\mathfrak{g}, \sigma) \oplus kc \oplus kd.$$

We give $\widehat{L}(\mathfrak{g}, \sigma)$ a Lie algebra structure such that c is a central element, d is the degree derivation, i.e. if $x \in \mathfrak{g}_{\overline{i}}$ and $p \in \mathbb{Z}$ then

$$[d, x \otimes t^{\frac{p}{m}}] := px \otimes t^{\frac{p}{m}} \tag{1.0.0.1}$$

and if $y \otimes t^{\frac{q}{m}} \in L(\mathfrak{g}, \sigma)$ we get

$$[x \otimes t^{\frac{p}{m}}, y \otimes t^{\frac{q}{m}}]_{\widehat{L}(\mathfrak{g},\sigma)} = [x, y] \otimes t^{\frac{p+q}{m}} + p\,\kappa(\,x, y\,)\,\delta_{0,p+q} \cdot c,$$

where, as before, κ is the Killing form on $\mathfrak{g}$ and $\delta_{0,p+q}$ is Kronecker's delta.

1.0.1 Remark. Note that the Lie algebra structure on $\widehat{L}(\mathfrak{g},\sigma)$ is induced by that of $\widehat{L}(\mathfrak{g})_S$ if we view $\widehat{L}(\mathfrak{g},\sigma)$ as a subset of $\widehat{L}(\mathfrak{g})_S$.

1.0.2 Remark. Let $\widehat{\sigma}$ be an automorphism of $\widehat{L}(\mathfrak{g})_S$ such that $\widehat{\sigma}|_{L(\mathfrak{g})_S} = \sigma$, *$\widehat{\sigma}(c) = c$ and $\widehat{\sigma}(d) = d$.* Then $\widehat{L}(\mathfrak{g},\sigma) = (\widehat{L}(\mathfrak{g})_S)^{\widehat{\sigma}}$.

Realization Theorem. (a) *The Lie algebra $\widehat{L}(\mathfrak{g},\sigma)$ is an affine Kac-Moody Lie algebra, and every affine Kac-Moody Lie algebra is isomorphic to some $\widehat{L}(\mathfrak{g},\sigma)$.*

(b) *$\widehat{L}(\mathfrak{g},\sigma) \simeq \widehat{L}(\mathfrak{g},\sigma')$, where σ' is a diagram automorphism with respect to some Cartan subalgebra of $\mathfrak{g}$.*

Proof. See [Kac, Theorems 7.4, 8.3 and 8.5]. $\square$

Although there were some precursors (papers by Saito and Slodowy for nullity 2), it was in the paper [HT] by the physicists Hoegh-Krohn and Torrésani that the class of discrete extended affine Lie algebras was introduced, however not under this name. Rather, they were called “irreducible quasi-simple Lie algebras” and later ([BGK],[BGKN]) “elliptic quasi-simple Lie algebras.” The stated goal of the paper [HT] was applications in quantum gauge theory. The theory developed there did not, however, stand up to the scrutiny of mathematicians. The errors of [HT] were corrected in the AMS memoir [AABGP] by Allison, Azam, Berman, Gao and Pianzola, and it was here that the name “extended affine Lie algebra” first appeared, but not in the sense we use it in this work. Rather, the authors develop the basic theory of what is currently called discrete EALAs. Nevertheless, [AABGP] has become the standard reference for the more general EALAs, since many of the results presented there for discrete EALAs easily extend to the more general setting. The definition of an EALA that we will use in this work (see Section 2.1) is due to E. Neher in [Ne2].

Extended affine Lie algebras form a category of Lie algebras which contains the categories of finite-dimensional simple and affine Kac-Moody Lie algebras. An EALA is defined by a set of axioms prescribing its internal structure, rather than by a potentially elusive presentation. In particular, one of the axioms is that an EALA is a pair (E, H), where E is a Lie algebra over a field k and H is its maximal abelian *ad*-diagonalizable subalgebra. It is known that the set of weights of the adjoint representation of H on E form a so-called *extended* *affine root system*. The structure of an EALA is now well understood and is quite similar to that of an affine Kac-Moody Lie algebra. It is obtained from an invariant Lie torus by taking a central extension and adding some derivations. This invarant Lie torus is called the *centreless core* E_{cc} of an EALA (E, H), its central extension is called the *core* E_c of the EALA (E, H), and this really is the core of the matter. As Lie algebras invariant Lie tori have been classified: they are either multiloop Lie algebras or isomorphic to $sl_n(q)$, where q is a quantum torus.

In this work we will be primarily interested in studying extended affine Lie algebras with centreless cores isomorphic to some multiloop Lie algebra. As in the case of affine Kac-Moody Lie algebras, derived modulo centre, they can be viewed as simple finite dimensional Lie algebras not only over k, but also over a Laurent polynomial ring in several variables $R = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$; they are the twisted forms of the split Lie R-algebra $\mathfrak{g} \otimes R$.

The method of studying the structure and representation theory of a twisted (multi)loop Lie algebra by applying non-abelian Galois cohomology and descent theory to the corresponding untwisted (multi)loop Lie algebra has been successfully used in the research on the following topics:

- the central extensions of twisted forms of Lie algebras in [PPS], [Sun],
- the derivations of the twisted forms of Lie algebras in [P3],
- the conjugacy theorem of maximal abelian diagonalizable subalgebras (analogues of Cartan subalgebras) of twisted loop Lie algebras in [P1],[CGP].
- the finite-dimensional irreducible representations of twisted forms of Lie algebras in [Lau], [LP].

The primary goal of this dissertation is to obtain conjugacy theorems of maximal abelian adjoint-diagonalizable subalgebras (MADs) of affine Kac-Moody Lie algebras and EALAs analogous to the classical theorem of Chevalley, which says that Cartan subalgebras of a finite dimensional simple Lie algebra over an algebraically closed field of characteristic 0 are conjugate. The above observation on the shape of the centreless core of EALA (or affine Kac-Moody Lie algebra) will enable us to employ non-abelian Galois cohomology techniques, which serve as an important tool in getting the result. Even though the conjugacy theorem for affine Kac-Moody Lie algebras was known (due to

[PK]), it is relevant to point out that the cohomological methods that we are putting forward do have their advantages. The group under which conjugacy is achieved in this work has a very transparent structure (given in terms of Laurent polynomial points of simply connected group scheme.) This is in contrast to the Kac-Moody groups used in [PK] which, in the twisted case, are quite difficult to "see".

It is known that EALAs of nullity 0 are precisely finite dimensional simple Lie algebras and EALAs of nullity 1 are precisely affine Kac-Moody Lie algebras. Therefore, one can think of EALAs as generalizations of finite dimensional simple Lie algebras and affine Kac-Moody Lie algebras to higher nullity.

It may be possible to give a Lie algebra E multiple structures of an EALA. Therefore the fundamental question to ask is the following:

Is an extended affine root system corresponding to some EALA structure on E an invariant of E, i.e. does not it depend on a choice of an EALA structure on E?

Of course, this question has an affirmative answer, if all the structure MADs of E are conjugate:

1.0.3 Theorem. (Conjugacy theorem for EALAs.) *Let (E, H) be an extended affine Lie algebra with centreless core a multiloop Lie algebra, which is not isomorphic to $sl_2(R)$, where R is a ring of Laurent polynomials in more then 1 variable. Assume E admits the second structure (E, H') of an extended affine Lie algebra. Then H and H' are conjugate, i.e. there exists an automorphism $\phi \in \mathrm{Aut}_{k-Lie}(E)$ such that $\phi(H) = H'$.*

As a corollary, we have that an extended affine root system of an EALA (E, H) is an invariant of E, i.e. it does not depend on the choice of an EALA structure on E, if the centreless core of E is a multiloop Lie algebra.

Further, the following natural question arises:

Are all MADs of EALAs conjugate?

The answer to this question is "No!": we have constructed a counterexample, i.e. an EALA and its two MADs which are not conjugate, (see 8.2.9).

The structure of the dissertation is as follows. Chapters 2 and 3 will serve as preliminaries required for our subsequent discussions. In Chapter 2, we will provide a review of the general theory of Affine Kac-Moody Lie algebras, Lie tori and EALAs. In Chapter 3, we will review the basic terminology and facts from non-abelian Galois cohomology theory. Chapter 4 is devoted to the proof of the conjugacy theorem for affine Kac-Moody Lie algebras:

1.0.4 Theorem. (Conjugacy theorem for affine Kac-Moody.) *Let* $\widehat{L}(\mathfrak{g},\sigma)$ *be an affine Kac-Moody Lie algebra and* ${}_z\widetilde{\mathbf{G}}_R$ *be a simple simply connected group scheme over* R *corresponding to* $L(\mathfrak{g},\sigma)$ *(in the sense that we will make precise in Chapter 4). Let* ${}_z\widehat{\mathbf{G}}_R(R)$ *be the preimage of* $\{Ad(g) : g \in {}_z\widetilde{\mathbf{G}}_R(R)\}$ *under the canonical map* $\mathrm{Aut}_k(\widehat{L}(\mathfrak{g},\sigma)) \to \mathrm{Aut}_k(L(\mathfrak{g},\sigma))$*. Then all MADs of* $\widehat{L}(\mathfrak{g},\sigma)$ *are conjugate under* ${}_z\widehat{\mathbf{G}}_R(R)$*.*

The rest of the dissertation consists of Chapters 5-8, where our focus is on the conjugacy problem of MADs in EALAs. In Chapter 5, we deduce two important properties of the core of an EALA, namely, that it does not depend on the choice of the EALA structure and is automorphism-invariant. These allow us to consider a natural restriction map $res_{cc} : \mathrm{Aut}_{k-Lie}(E) \to \mathrm{Aut}_{k-Lie}(E_{cc})$, whose image and kernel are studied in Chapter 6. There we prove that the group $\mathrm{Aut}_{R-Lie}(E_{cc})$ of R-automorphisms of the centreless core E_{cc} is in the image of the restriction map res_{cc} (6.4.1), if E_{cc} is not isomorphic to $sl_2(R)$, where R is a ring of Laurent polynomials in more then 1 variable. Results of this chapter play a crucial role in Chapter 7, where we prove the main theorem of the dissertation (1.0.3), that is, the conjugacy theorem of *structure* MADs for EALAs with multiloop centreless core, which is not isomorphic to $sl_2(R)$, where R is a ring of Laurent polynomials in more then 1 variable. In Chapter 8, we construct an extended affine Lie algebra E and its two MADs H and H' such that they are not conjugate, i.e. there is no $\phi \in \mathrm{Aut}_{k-Lie}(E)$ such that $\phi(H') = H$ (see 8.2.9). In particular, this implies that not all MADs of EALA are structure MADs.

Chapter 2

Lie tori and EALAs

This chapter contains a review of the theory of Lie tori and extended affine Lie algebras. We will often refer to it in the following chapters of this work.

2.1 Definition of an extended affine Lie algebra

2.1.1 Definition. *An extended affine Lie algebra, or EALA for short, is a pair (E, H) consisting of a Lie algebra E over k and a subalgebra H satisfying the following axioms (EA1) - (EA6).*

(EA1): *E has an invariant nondegenerate symmetric bilinear form $(-,-)$.*

(EA2): *H is a nontrivial finite-dimensional* ad-*diagonalizable and self-centralizing subalgebra of E.*

Recall, that a bilinear form $(-,-)$ is called *invariant* if $([u,v],w) = (u,[v,w])$ for any $u, v, w \in E$; a subalgebra H is called ad-*diagonalizable* if there is a decomposition of E into a sum of eigenspaces for H with respect to the adjoint action, i.e.

$$E = \oplus_{\alpha \in H^*} E_\alpha,$$

where

$$E_\alpha = \{x \in E | [h,x] = \alpha(h)x \text{ for all } h \in H\}.$$

Before we state the other four axioms, we have to define the notion of a root (null or anisotropic) of E and a core of E. Since the form $(-,-)$ is invariant we get that

$$(E_\alpha, E_\beta) = 0 \text{ if } \alpha + \beta \neq 0.$$

Since the form is nondegenerate this implies that its restriction to $H = E_0$ is still nondegenerate. Therefore we can transfer the form to a nondegenerate symmetric bilinear form $(-,-)$ on H^* in an obvious way. We can now define

$\Phi = \{\alpha \in H^* | E_\alpha \neq 0\}$ (set of *roots* of (E, H)),

$\Phi^0 = \{\alpha \in \Phi | (\alpha, \alpha) = 0\}$ (set of *null roots*),

$\Phi^{an} = \{\alpha \in \Phi | (\alpha, \alpha) \neq 0\}$ (set of *anisotropic roots*).

We call Φ the set of roots and not a root system because we reserve the latter for the root system in its usual meaning.

We define the *core* of E as the subalgebra E_c generated by its anisotropic root spaces, i.e.

$$E_c = \langle \cup_{\alpha \in \Phi^{an}} E_\alpha \rangle_{subalg}.$$

Now we can state the remaining four axioms.

(EA3): For every $\alpha \in \Phi^{an}$ and any $x_\alpha \in E_\alpha$ the operator $\mathrm{ad}(x_\alpha)$ is locally nilpotent on E.

(EA4): Φ^{an} is connected in the sense that for any decomposition $\Phi^{an} = \Phi_1 \cup \Phi_2$ with $(\Phi_1, \Phi_2) = 0$ we have $\Phi_1 = \emptyset$ or $\Phi_2 = \emptyset$.

(EA5): The centralizer of the core E_c of E is contained in E_c.

(EA6): $\Lambda = span_{\mathbb{Z}}(\Phi^0) \subset H^*$ is a free abelian group of finite rank.

The rank of the free abelian group Λ in the axiom (EA6) is called the *nullity* of (E, H).

2.1.2 Remark. The set Φ of roots of an EALA E has special properties: it is a so-called *extended affine root system* in the sense of [AABGP, Ch. I]. More precisely, let V be a finite dimensional vector space over k equipped with a symmetric bilinear form $(-,-)$ and let Φ be a subset of V. A triple $(\Phi, V, (-,-))$ is called an extended affine root system, if the following axioms $(EARS1) - (EARS7)$ are fulfilled:

(EARS1): $0 \in \Phi$ and Φ spans V.

(EARS2): Φ has *unbroken finite root strings*, i.e., for every $\alpha \in \Phi^{an}$ and $\beta \in \Phi$ there exist $d, u \in \mathbb{N} = \{0, 1, 2, \dots\}$ such that

$$\{\beta + n\alpha | n \in \mathbb{Z}\} \cap \Phi = \{\beta - d\alpha, \dots, \beta + u\alpha\} \ \ and \ \ d - u = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}.$$

(EARS3): $\Phi^0 = \Phi \cap rad(V)$.

(EARS4): Φ is reduced: for every $\alpha \in \Phi^{an}$ we have $k\alpha \cap \Phi^{an} = \{\pm\alpha\}$.

(EARS5): Φ is connected: whenever $\Phi^{an} = \Phi_1 \cup \Phi_2$ with $(\Phi_1, \Phi_2) = 0$, then $\Phi_1 = \emptyset$ or $\Phi_2 = \emptyset$.

(EARS6): Φ is tame, i.e. $\Phi^0 \subset \Phi^{an} + \Phi^{an}$.

(EARS7): The abelian group $span_{\mathbb{Z}}(\Phi^0)$ is free of finite rank.

We will next present some examples of EALAs.

2.2 Examples of EALAs

2.2.1 EALAs of nullity 0.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over k with Cartan subalgebra $\mathfrak{h}$. Then $(\mathfrak{g}, \mathfrak{h})$ is an EALA of nullity 0.

(EA1) Up to a scalar multiple, there exists unique invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$, namely, the Killing form κ. Therefore we take $(-|-) = \kappa$.

(EA2) By definition of Cartan subalgebra, the Lie algebra $\mathfrak{g}$ has a root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha, \quad \mathfrak{g}_0 = \mathfrak{h},$$

where Σ is the root system of $(\mathfrak{g}, \mathfrak{h})$.

Hence the set of roots of $(\mathfrak{g}, \mathfrak{h})$ is $\Phi = \Sigma \cup \{0\}$. It is a standard fact that $\kappa(t_\alpha, t_\alpha) \neq 0$ for $t_\alpha \in \mathfrak{h}$ such that $\alpha(x) = \kappa(t_\alpha, x)$ for all $x \in \mathfrak{h}$. Therefore, $\Phi^{an} = \Sigma$ and $\Phi^0 = \{0\}$.

(EA3) Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and $\mathfrak{g}$ is finite dimensional, $\mathrm{ad}x_\alpha$ is nilpotent for any element $x_\alpha \in \mathfrak{g}_\alpha$.

(EA4) Since $\mathfrak{g}$ is simple Σ is irreducible and hence connected.

(EA5) We defined the core $\mathfrak{g}_c$ of $\mathfrak{g}$ to be a subalgebra generated by the root spaces $\mathfrak{g}_\alpha$, $\alpha \in \Sigma$. But $\mathfrak{h} = \sum_{\alpha\in\Sigma}[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. This implies that $\mathfrak{g}_c = \mathfrak{g}$ and therefore the axiom (EA5) holds.

(EA6) $\Lambda = span_{\mathbb{Z}}(\Phi^0) = \{0\}$.

It worth to be pointed out that this example exhaust all the EALAs of nullity 0.

2.2.2 EALAs of nullity 1.

We wil show that any affine Kac-Moody Lie algebra is an extended affine Lie algebra of nullity 1.

We will use some basic facts about affine Kac-Moody Lie algebras all of which can be found in Kac's book [Kac]. This reference uses the field of complex numbers $\mathbb{C}$ for the base field, but everything we say here is true for an arbitrary algebraically closed field k of characteristic 0.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over k and σ is a diagram automorphism of $\mathfrak{g}$. We let m be the order of σ, and denote the canonical map $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ by $n \mapsto \overline{n}$. Then

$$\widehat{L} = \widehat{L}(\mathfrak{g}, \sigma) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\bar{i}} \otimes k[t^{\pm 1}]s^i \oplus kc \oplus kd$$

is an affine Kac-Moody Lie algebra corresponding to $\mathfrak{g}$ and σ (see Chapter 1). We now verify the axioms (EA1)-(EA6).

(EA1) We define a bilinear form on $\widehat{L}$ by

$$\begin{aligned}&(u_{\overline{\lambda}} \otimes t^\lambda + s_1 c + s_1' d | v_{\overline{\mu}} \otimes t^\mu + s_2 c + s_2' d)\\ &= \kappa(u_{\overline{\lambda}}, v_{\overline{\mu}})\delta_{\lambda,-\mu} + s_1 s_2' + s_2 s_1'.\end{aligned}$$

One can check that this form is invariant symmetric nondegenerate.

(EA2) To construct a subalgebra H as required in axiom (EA2) we start with a

Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Since σ is a diagram automorphism, it leaves $\mathfrak{h}$ invariant. We let

$$\mathfrak{h}_{\overline{0}} = \mathfrak{h} \cap \mathfrak{g}_{\overline{0}} = \{h \in \mathfrak{h} : \sigma(h) = h\}$$

and put

$$H = \mathfrak{h}_{\overline{0}} \oplus kc \oplus kd.$$

It is known that $\mathfrak{g}_{\overline{0}}$ is a simple Lie algebra with Cartan subalgebra $h_{\overline{0}}$ ([Kac, Proposition 7.9]). The grading property implies that $[\mathfrak{g}_{\overline{0}}, \mathfrak{g}_{\overline{n}}] \subset \mathfrak{g}_{\overline{n}}$ for $n \in \mathbb{Z}$. Hence $\mathfrak{g}_{\overline{0}}$ acts on $\mathfrak{g}_{\overline{n}}$ by the adjoint action. Let $\Delta_{\overline{n}}$ be the set of weights of the $\mathfrak{g}_{\overline{0}}$-module $\mathfrak{g}_{\overline{n}}$ with respect to $\mathfrak{h}_{\overline{0}}$:

$$\begin{aligned} &\mathfrak{g}_{\overline{n}} = \textstyle\bigoplus_{\gamma \in \Delta_{\overline{n}}} \mathfrak{g}_{\overline{n},\gamma} \\ &\mathfrak{g}_{\overline{n},\gamma} = \{x \in \mathfrak{g}_{\overline{n}} |\ [h_{\overline{0}}, x] = \gamma(h_{\overline{0}})x \text{ for all } h_{\overline{0}} \in \mathfrak{h}_{\overline{0}}\}. \end{aligned}$$

In particular, $\Delta_{\overline{0}} \backslash \{0\}$ is the root system of $\mathfrak{g}_{\overline{0}}$ with respect to $\mathfrak{h}_{\overline{0}}$ and $\mathfrak{h}_{\overline{0}} = \mathfrak{g}_{\overline{0},0}$.

We extend $\Delta_{\overline{n}} \subset \mathfrak{h}_{\overline{0}^*}$ to a linear form on H by zero, i.e., for $\gamma \in \Delta_{\overline{n}}$ we put

$$\gamma(\mathfrak{h}_{\overline{0}} + sc + s'd) = \gamma(h_{\overline{0}})$$

and define a linear form δ on H by

$$\delta(h_{\overline{0}} + sc + s'd) = s'.$$

Then for $\gamma \in \Delta_{\overline{n}}, n \in \mathbb{Z}$, we have

$$\widehat{L}_{\gamma \oplus n\delta} = \{u \subset \widehat{L} |\ [h, u] = (\gamma \oplus n\delta)(h)u \text{ for all } h \in H\}.$$

That is $\widehat{L}_{\gamma \oplus n\delta} = \mathfrak{g}_{\overline{n},\gamma} \otimes t^n$, if $\gamma \oplus n\delta \neq 0$, and $\widehat{L}_0 = H$.

This implies that H is an ad-diagonalizable subalgebra of $\widehat{L}$ with the set of roots

$$\Phi = \{\gamma + n\delta |\ \gamma \in \Delta_{\overline{n}}\}.$$

This establishes (EA2).

One checks that

$$\Phi^{an} = \{\gamma + n\delta \in \Phi |\ \gamma \neq 0\} \ \text{ and } \ \Phi^0 = \mathbb{Z}\delta,$$

which in the theory of affine Kac-Moody Lie algeras are usually called *real* and *imaginary roots*. Now we can verify the remaining axioms.

(EA3) From the description of the root spaces we see that this axiom holds in a stronger form: $\mathrm{ad}(x_\alpha)$ is nilpotent for any $x_\alpha \in \widehat{L}_\alpha, \alpha \in \Phi^{an}$.

(EA4) This is easy.

(EA5) It follows from the description of the root spaces $\widehat{L}_{\gamma+n\delta}$ that the core $\widehat{L}_c$ of $\widehat{L}$ coincides with the derived subalgebra $\widetilde{L} = [\widehat{L}, \widehat{L}]$. It easy to check that the centralizer of $\widehat{L}_c$ in $\widehat{L}$ is $kc \subset \widehat{L}_c$.

(EA6) $span_{\mathbb{Z}}(\Phi^0) = \mathbb{Z}\delta$. In particular, the nullity of $\widehat{L}$ is 1.

2.2.3 EALAs of higher nullity

Let $R = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be a ring of Laurent polynomials in n variables with coefficients in field k. Let $L = L(\mathfrak{g}) = \mathfrak{g} \otimes R$ be an associated *untwisted multiloop algebra*. L has a 2-cocycle $\sigma : L \times L \to \mathcal{C} = k^n$, given by

$$\sigma(u \otimes t^\lambda, v \otimes t^\mu) = \delta_{\lambda,-\mu}\kappa(u, v)\lambda.$$

We can therefore define a central extension

$$K = L \oplus \mathcal{C}$$

with a product

$$[l_1 + c_1, l_2 + c_2]_K = [l_1, l_2]_L + \sigma(l_1, l_2).$$

Define the i-th *degree derivation* ∂_i of K by

$$\partial_i(u \otimes t^\lambda + c) = \lambda_i u \otimes t^\lambda \text{ for } \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$$

and put

$$\mathcal{D} = span_k\{\partial_i | 1 \leq i \leq n\},$$

the space of *degree derivations*.

Define a Lie algebra E as a semi-direct product

$$E = K \rtimes \mathcal{D}.$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and put

$$H = \mathfrak{h} \oplus \mathcal{C} \oplus \mathcal{D}.$$

We claim that (E, H) is an extended affine Lie algebra of nullity n.

(EA1) We will mimic the construction of an invariant nondegenerate symmetric bilinear form in 2.2.2. Thus, we require

- $(L(\mathfrak{g}), \mathcal{C} \oplus \mathcal{D}) = 0$.
- $\mathcal{C} \oplus \mathcal{D}$ is a hyperbolic space with $(\mathcal{C}, \mathcal{C}) = 0 = (\mathcal{D}, \mathcal{D})$ and

  $$(\sum_i s_i c_i, \sum_i s_i' d_i) = \sum_i s_i s_i',$$

  where $c_1, \ldots, c_n$ is the canonical basis of k^n. In other words, $\mathcal{C} \oplus \mathcal{D}$ is an orthgonal sum of n hyperbolic planes $kc_i \oplus kd_i$.
- On $L(\mathfrak{g})$ the form is a tensor product form of the Killing form κ on $\mathfrak{g}$ and the natural invariant bilinear form on R.

Putting all these requirements together, we get a bilinear form on E, given by

$$\begin{aligned}&(u \otimes t^\lambda + \textstyle\sum_i s_i c_i + \sum_j s_j' d_j, v \otimes t^\mu + \sum_i t_i c_i + \sum_j t_j' d_j) \\ &= \kappa(u, v)\delta_{\lambda,-\mu} + \textstyle\sum_i (s_i t_i' + t_i s_i').\end{aligned}$$

Let $\mathfrak{h}$ be a splitting Cartan subalgebra and let Σ be the root system of $(\mathfrak{g}, \mathfrak{h})$. We put $\Delta = \{0\} \cup \Sigma$ and hence have a weight space decomposition $\mathfrak{g} = \bigoplus_{\gamma \in \Delta} \mathfrak{g}_\gamma$ with $\mathfrak{g}_0 = \mathfrak{h}$.

We embed $\Delta \hookrightarrow H^*$ by requiring $\gamma | \mathcal{C} \oplus \mathcal{D} = 0$ for $\gamma \in \Delta$. Also we embed $\Lambda = \mathbb{Z}^n \hookrightarrow H^*$ by $\lambda(\mathfrak{h} \oplus \mathcal{C}) = 0$ and $\lambda(d_i) = \lambda_i$ for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda$. Then

E has a root space decomposition $E = \bigoplus_{\alpha\in\Phi} E_\alpha$ with root spaces

$$E_{\gamma\oplus\lambda} = \mathfrak{g}_\gamma \otimes t^\lambda \ (\gamma \oplus \lambda \neq 0),\ E_0 = H.$$

Moreover, $\Phi^{an} = \Sigma \times \Lambda$ and $\Phi^0 = \Lambda$. It is easy now to verify axioms (EA2)-(EA6).

2.2.4 Remark. There are many more EALAs in nullity $n \geq 2$. Other examples can be found in [AABGP, Chapter III], some of them involving nonassociative algebras, like octonion algebras and Jordan algebras over Laurent polynomial rings.

2.3 Lie tori

2.3.1 Definition of a Lie torus

Here the term "root system" means a finite, not necessarily reduced root system Δ in the usual sense, except that we will assume $0 \in \Delta$, as for example in [AABGP]. In other words, a root system Δ is a finite subset of k-vector space V such that:

(RS1): $0 \in \Delta$ and Δ spans V.

(RS2): For every nonzero $\alpha \in \Delta$ there exists a linear form $\alpha^\vee$ such that $\alpha^\vee(\alpha) = 2$ and $s_\alpha(\Delta) = \Delta$, where s_α is the reflection of V defined by $s_\alpha(y) = y - \alpha^\vee(y)\alpha$.

(RS)3: For every nonzero $\alpha \in \Delta$ the set $\alpha^\vee(\Delta)$ is contained in $\mathbb{Z}$.

We denote by

$$\Delta_{\text{ind}} = \{0\} \cup \{\xi \in \Delta : \xi/2 \notin \Delta\}$$

the subsystem of indivisible roots and by $\mathcal{Q}(\Delta) = \text{span}_{\mathbb{Z}}(\Delta)$ the root lattice of Δ. To avoid some degeneracies we will always assume that $\Delta \neq \{0\}$.

Let Δ be a finite irreducible root system, and let Λ be an abelian group. A *Lie torus of type* (Δ, Λ) is a Lie algebra L satisfying the following conditions (LT1) – (LT4).

(LT1) (a) L is graded by $\mathcal{Q}(\Delta)\oplus\Lambda$. We write this grading as $L = \bigoplus_{\xi\in\mathcal{Q}(\Delta),\lambda\in\Lambda} L_\xi^\lambda$ and thus have $[L_\xi^\lambda, L_\zeta^\mu] \subset L_{\xi+\zeta}^{\lambda+\mu}$. It is convenient to define

$$L_\xi = \bigoplus_{\lambda\in\Lambda} L_\xi^\lambda \quad \text{and} \quad L^\lambda = \bigoplus_{\xi\in\Delta} L_\xi^\lambda.$$

(b) $\operatorname{supp}_{\mathcal{Q}(\Delta)} L = \{\xi \in \mathcal{Q}(\Delta); L_\xi \neq 0\} = \Delta$, so that $L = \bigoplus_{\xi\in\Delta} L_\xi$.

(LT2) (a) If $L_\xi^\lambda \neq 0$ and $\xi \neq 0$, then there exist $e_\xi^\lambda \in L_\xi^\lambda$ and $f_\xi^\lambda \in L_{-\xi}^{-\lambda}$ such that

$$L_\xi^\lambda = ke_\xi^\lambda, \quad L_{-\xi}^{-\lambda} = kf_\xi^\lambda,$$

and for all $\tau \in \Delta$ and $x_\tau \in L_\tau$ we have

$$[[e_\xi^\lambda, f_\xi^\lambda], x_\tau] = \xi^\vee(\tau)x_\tau.$$

(b) $L_\xi^0 \neq 0$ for all $0 \neq \xi \in S$ with $\xi/2 \notin \Delta$.

(LT3) As a Lie algebra, L is generated by $\bigcup_{0\neq\xi\in\Delta} L_\xi$.

(LT4) As abelian group, Λ is generated by $\operatorname{supp}_\Lambda L = \{\lambda \in \Lambda : L^\lambda \neq 0\}$.

We define the *nullity* of a Lie torus L of type (Δ, Λ) as the rank of Λ and the *root-grading type* as the type of Δ. We will say that L is a *Lie torus* (without qualifiers) if L is a Lie torus of type (Δ, Λ) for some pair (Δ, Λ). A Lie torus is called *centreless* if its centre $Z(L) = \{0\}$. If L is a Lie torus, then $L/Z(L)$ is a centreless Lie torus of the same type as L and nullity.

Among the axioms (LT1) – (LT4), the axioms (LT1) and (LT2) are the crucial ones. One can weaken (LT1b) by only assuming $\operatorname{supp}_{\mathcal{Q}(\Delta)} L \subset \Delta$. It then follows that either $\operatorname{supp}_{\mathcal{Q}(\Delta)} L = \Delta$ or $\operatorname{supp}_{\mathcal{Q}(\Delta)} L = \Delta_{\text{ind}} := \{\xi \in \Delta : \xi/2 \notin \Delta\} \cup \{0\}$, in which case L is a Lie torus of type $(\Delta_{\text{ind}}, \Lambda)$. Similarly, if a Lie algebra satisfies (LT1) and (LT2), the subalgebra generated by all $L_\xi, 0 \neq \xi \in \Delta$, satisfies (LT1) – (LT3). The analogous remark applies to (LT4).

An obvious example of a Lie torus of type $(\Delta, \mathbb{Z}^n)$ is a k-Lie algebra $\mathfrak{g} \otimes_k R$ where $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra of type Δ and $R = k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is a Laurent polynomial ring. Another important example, studied in [BGK], is $\mathfrak{sl}_l(k_q)$ for k_q a quantum torus.

Lie tori have been classified, see [Al] for a recent survey of the many papers involved in this classification. Some more background on Lie tori is contained in the papers [ABFP, Ne3].

2.3.2 Some known properties of centreless Lie tori

We review the properties of Lie tori used in the following. This is not a comprehensive survey. The reader can find more information in [ABFP, Ne3, Ne4].

Let L and L' be two Lie tori of type (Δ, Λ) and (Δ', Λ') respectively, thus $L = \bigoplus_{\xi\in\Delta,\lambda\in\Lambda} L_\xi^\lambda$ and $L' = \bigoplus_{\xi'\in\Delta',\lambda'\in\Lambda'} {L'}_{\xi'}^{\lambda'}$. An *isotopy* from L to L' is an isomorphism $f\colon L \to L'$ of Lie algebras for which there exist

1. group isomorphisms $\varphi_r\colon \mathcal{Q}(\Delta) \to \mathcal{Q}(\Delta')$ and $\varphi_e\colon \Lambda \to \Lambda'$, and
2. a group homomorphism $\varphi_s\colon \mathcal{Q}(\Delta) \to \Lambda'$

such that

$$f(L_\xi^\lambda) = (L')_{\varphi_r(\xi)}^{\varphi_e(\lambda)+\varphi_s(\xi)} \tag{2.3.2.1}$$

holds for all $\xi \in \Delta$ and $\lambda \in \Lambda$. One calls L and L' *isotopic* if there exists an isotopy from L to L'. It is immediate that isotopy is an equivalence relation on the class of Lie tori. The maps in (1) and (2) are uniquely determined by (2.3.2.1).

We will need the following result.

2.3.3 Theorem ([Al, Theorem 7.2])**.** *Suppose that L and L' are centreless Lie tori of type (Δ, Λ) and (Δ', Λ') respectively. Let $\mathfrak{h} = L_0^0$ and $\mathfrak{h}' = L_0'^0$. If $\phi : L \to L'$ is an algebra isomorphism, then*

$$\phi \text{ is an isotopy} \iff \phi(\mathfrak{h}) = \mathfrak{h}'.$$

In the remaining part of this section we will assume that L is a centreless Lie torus of type (Δ, Λ) and nullity n.

For e_ξ^λ and f_ξ^λ as in (LT2) we put $h_\xi^\lambda = [e_\xi^\lambda, f_\xi^\lambda] \in L_0^0$ and observe that $(e_\xi^\lambda, h_\xi^\lambda, f_\xi^\lambda)$ is an $\mathfrak{sl}_2$-triple. It follows from (LT3) that weight space L_0^0 of L is equal to

$$L_0^0 = \mathfrak{h} = \operatorname{span}_k\{h_\xi^\lambda\}, \tag{2.3.3.1}$$

and is a toral (= ad-diagonalizable) subalgebra of L whose root spaces are the L_ξ, $\xi \in \Delta$.

Up to scalars, L has a unique nondegenerate symmetric bilinear form $(\cdot|\cdot)$ which is Λ-graded in the sense that $(L^\lambda \mid L^\mu) = 0$ if $\lambda + \mu \neq 0$, [NPPS, Yo3]. Since the subspaces L_ξ are the root spaces of the toral subalgebra $\mathfrak{h}$ we also know $(L_\xi \mid L_\tau) = 0$ if $\xi + \tau \neq 0$.

Recall that the *centroid* of a Lie algebra A defined over k is

$$\mathrm{Ctd}_k(A) = \{\chi \in \mathrm{End}_k(A) |\ [\chi(a_1), a_2] = \chi([a_1, a_2]) \text{ for all } a_1, a_2 \in A\}.$$

The centroid $\mathrm{Ctd}_k(L)$ of a Lie torus L is isomorphic to the group ring $k[\Gamma]$ for a subgroup Γ of Λ, the so-called *central grading group* (see [BN, Prop. 3.13]). Hence $\mathrm{Ctd}_k(L)$ is a Laurent polynomial ring in ν variables, $0 \leq \nu \leq n$, (all possibilities for ν do in fact occur). We will write $\mathrm{Ctd}_k(L) = \bigoplus_{\gamma\in\Gamma} k\chi^\gamma$, where the χ^γ satisfy the multiplication rule $\chi^\gamma\chi^\delta = \chi^{\gamma+\delta}$ and act on L as endomorphisms of Λ-degree γ.

One knows that $\mathrm{Ctd}_k(L)$ acts without torsion on L ([Al, Prop. 4.1]) and as a $\mathrm{Ctd}_k(L)$-module, L is free ([Ne5, Th.7]). If L is fgc (finitely generated module over its centroid), it is a multiloop algebra [ABFP].

If L is not fgc, equivalently $\nu < n$, one knows ([Ne1, Th. 7]) that L has root-grading type A. Lie tori with this root-grading type are classified in [BGK, BGKN, Yo1]. It follows from this classification together with [NY, 4.9] that $L \cong \mathfrak{sl}_l(k_q)$ for k_q a quantum torus in n variables.

Let us recall a construction of a multiloop Lie algebra, since we will work mainly with fgc Lie tori. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over k and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ be an n-tuple of commuting k-automorphisms of $\mathfrak{g}$ satisfying $\sigma_i^{m_i} = 1$. Let

$$R = k[t_1^{\pm1}, \dots, t_n^{\pm1}] \text{ and } S = k[s_1^{\pm1}, \dots, s_n^{\pm1}], \quad s_i = t_i^{\frac{1}{m_i}}, 1 \leq i \leq n.$$

The extension S/R is Galois and we can identify

$$\Gamma := \mathrm{Gal}(S/R) = \mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_n\mathbb{Z}$$

via our choice of compatible roots of unity: if ζ_i is a fixed primitive m_i-th root of unity in k, $1 \leq i \leq n$, the generators of Γ are automorphisms γ_i of S such that $\gamma_i(s^\lambda) = \zeta_i^{\lambda_i} s^\lambda$, $1 \leq i \leq n$.

We have a natural action of Γ on $L_S := \mathfrak{g} \otimes S$ and $\mathrm{Aut}(\mathfrak{g})(S)$ via the second component. Note that the action of Γ on $\mathrm{Aut}(\mathfrak{g})(S)$ can be equivalently described as follows: if $\phi \in \mathrm{Aut}(\mathfrak{g})(S)$ and $\gamma \in \Gamma$ then $\gamma(\phi) : L_S \to L_S$ is given by $l \to \gamma(\phi(\gamma^{-1}(l)))$ where $l \in L_S$.

The family $\boldsymbol{\sigma}$ gives rise to a natural *loop cocycle*

$$\eta = \eta(\boldsymbol{\sigma}) \in Z^1\big(\Gamma, \mathbf{Aut}(\mathfrak{g})(k)\big) \subset Z^1\big(\Gamma, \mathbf{Aut}(\mathfrak{g})(S)\big)$$

defined by $\eta(\boldsymbol{\sigma}) = (a_{\overline{\lambda}})$ where

$$a_{(0,\ldots,1,\ldots,0)} = \sigma_i^{-1} \in \mathrm{Aut}(\mathfrak{g})(S).$$

The cocycle $\eta(\boldsymbol{\sigma})$ in turn gives rise to a new twisted action of Γ on $\mathfrak{g} \otimes S$ and the *multiloop algebra* $L(\mathfrak{g}, \boldsymbol{\sigma})$ *based on* $\mathfrak{g}$ *corresponding to* $\boldsymbol{\sigma}$ is the invariant subalgebra $(\mathfrak{g} \otimes S)^\Gamma$ of $\mathfrak{g} \otimes S$ with respect to the twisted action of Γ. More precisely,

$$L(\mathfrak{g}, \boldsymbol{\sigma}) = \bigoplus_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} \mathfrak{g}_{i_1\ldots i_n} \otimes t_1^{\frac{i_1}{m_1}} \ldots t_n^{\frac{i_n}{m_n}} \subset \mathfrak{g} \otimes_k S, \tag{2.3.3.2}$$

where

$$\mathfrak{g}_{i_1\ldots i_n} = \{x \in \mathfrak{g} : \sigma_j(x) = \zeta_j^{i_j} x \ for\ 1 \leq j \leq n\}.$$

Thus, $L(\mathfrak{g}, \boldsymbol{\sigma})$ is a twisted form of the R–Lie algebra $\mathfrak{g} \otimes_k R$ splitting by S :

$$L(\mathfrak{g}, \boldsymbol{\sigma}) \otimes_R S \simeq \mathfrak{g} \otimes_k S \simeq (\mathfrak{g} \otimes_k R) \otimes_R S.$$

2.3.4 Remark. Let $\widetilde{\mathbf{G}}$ be the simple simply connected algebraic group over k corresponding to $\mathfrak{g}$. Since $\mathbf{Aut}(\mathfrak{g}) \cong \mathbf{Aut}(\widetilde{\mathbf{G}})$ we can also consider by means of the cocycle η the twisted R-group ${}_\eta\widetilde{\mathbf{G}}_R$. It is well known (see for example the proof of [GP1, Prop 4.10]) that the determination of Lie algebras commutes with the twisting process. Thus $L(\mathfrak{g}, \boldsymbol{\sigma})$ is a Lie algebra of ${}_z\widetilde{\mathbf{G}}_R$.

Before giving a nice explicit realization of EALA, we have to introduce certain types of derivations of a Lie torus.

Any $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, k)$ induces a so-called *degree derivation* ∂_θ of L defined by $\partial_\theta(l^\lambda) = \theta(\lambda) l^\lambda$ for $l^\lambda \in L^\lambda$. We put $\mathcal{D} = \{\partial_\theta : \theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, k)\}$ and note that $\theta \mapsto \partial_\theta$ is a vector space isomorphism from $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, k)$ to $\mathcal{D}$, whence $\mathcal{D} \cong k^n$. We define $\mathrm{ev}_\lambda \in \mathcal{D}^*$ by $\mathrm{ev}_\lambda(\partial_\theta) = \theta(\lambda)$. One knows ([Ne1, 8]) that $\mathcal{D}$ induces the Λ-grading of L in the sense that L^λ is the ev_λ-weight space of L.

If $\chi \in \mathrm{Ctd}_k(L)$ then $\chi d \in \mathrm{Der}_k(L)$ for any derivation $d \in \mathrm{Der}_k(L)$. We call

$$\mathrm{CDer}_k(L) := \mathrm{Ctd}_k(L)\mathcal{D} = \textstyle\bigoplus_{\gamma\in\Gamma} \chi^\gamma \mathcal{D}$$

the *centroidal derivations* of L. Since

$$[\chi^\gamma \partial_\theta,\, \chi^\delta \partial_\psi] = \chi^{\gamma+\delta}(\theta(\delta)\partial_\psi - \psi(\gamma)\partial_\theta)$$

it follows that $\mathrm{CDer}(L)$ is a Γ-graded subalgebra of $\mathrm{Der}_k(L)$, a generalized Witt algebra. Note that $\mathcal{D}$ is a toral subalgebra of $\mathrm{CDer}_k(L)$ whose root spaces are the $\chi^\gamma \mathcal{D} = \{d \in \mathrm{CDer}(L) : [t, d] = \mathrm{ev}_\gamma(t) d \text{ for all } t \in \mathcal{D}\}$. One also knows ([Ne1, 9]) that

$$\mathrm{Der}_k(L) = \mathrm{IDer}(L) \rtimes \mathrm{CDer}_k(L). \tag{2.3.4.1}$$

For the construction of EALAs, the Γ-graded subalgebra $\mathrm{SCDer}_k(L)$ of *skew-centroidal derivations* is important:

$$\begin{aligned} \mathrm{SCDer}_k(L) &= \{d \in \mathrm{CDer}_k(L) : (d(l) \mid l) = 0 \text{ for all } l \in L\} \\ &= \textstyle\bigoplus_{\gamma\in\Gamma} \mathrm{SCDer}_k(L)^\gamma, \\ \mathrm{SCDer}_k(L)^\gamma &= \chi^\gamma \{\partial_\theta : \theta(\gamma) = 0\}. \end{aligned}$$

Note $\mathrm{SCDer}_k(L)^0 = \mathcal{D}$ and $[\mathrm{SCDer}_k(L))^\gamma, \mathrm{SCDer}_k(L)^{-\gamma}] = 0$, whence

$$\mathrm{SCDer}_k(L) = \mathcal{D} \ltimes \big(\textstyle\bigoplus_{\gamma\neq 0} \mathrm{SCDer}(L)^\gamma\big).$$

2.4 Construction of EALAs

To construct an EALA we will use data (L, D, τ) described below. Some background material can be found in [Ne3, §6] and [Ne4, §5.5]:

- L is a centreless Lie torus of type (Δ, Λ). We fix a Λ-graded invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$ and let Γ be the central grading group of L.

- $D = \bigoplus_{\gamma\in\Gamma} D^\gamma$ is a graded subalgebra of $\mathrm{SCDer}_k(L)$ such that the evaluation map $\mathrm{ev}_{D^0} : \Lambda \to {D^0}^*$, $\lambda \to \mathrm{ev}_\lambda |_{D^0}$ is injective. We denote by $C = D^{\mathrm{gr}*}$ the graded dual of D. It is well-known that

$$\sigma_D \colon L \times L \to C, \quad \sigma_D(l_1, l_2)(d) = (d \cdot l_1 \mid l_2) \tag{2.4.0.2}$$

  is a central 2-cocycle.

- $\tau : D \times D \to C$ is an *affine cocycle* defined to be a bilinear map satisfying for all $d, d_i \in D$

$$\begin{gathered} \tau(d,d) = 0 \quad \text{and} \quad \textstyle\sum_{\circlearrowleft} d_1 \cdot \tau(d_2, d_3) = \sum_{\circlearrowleft} \tau([d_1, d_2], d_3), \\ \tau(D^0, D) = 0, \quad \text{and} \quad \tau(d_1, d_2)(d_3) = \tau(d_2, d_3)(d_1) \end{gathered}$$

  Here $d \cdot c$ denotes the natural action of D on C. It is important to point out that there do exist non-trivial affine cocycles, see [BGK, Rem. 3.71].

To data (L, D, τ) as above we associate a Lie algebra

$$E = L \oplus C \oplus D$$

with product ($l_i \in L$, $c_i \in C$ and $d_i \in D$)

$$\begin{aligned} [l_1 \oplus c_1 \oplus d_1,\, l_2 \oplus c_2 \oplus d_2] = &\big([l_1, l_2]_L + d_1(l_2) - d_2(l_1)\big) \\ &\oplus \big(\sigma_D(l_1, l_2) + d_1 \cdot c_2 - d_2 \cdot c_1 + \tau(d_1, d_2)\big) \oplus [d_1, d_2]_D. \end{aligned} \tag{2.4.0.3}$$

Here $[.,.]_L$ and $[.,.]_D$ are the Lie algebra products of L and D respectively, and $d_i(l_j)$ is the natural action of D on L sometimes also written as $d_i \cdot l_j$.

It is immediate from the product formula that

(i) $L \oplus D^{\mathrm{gr}*}$ is an ideal of E, and the canonical projection $L \oplus D^{\mathrm{gr}*} \to L$ is a central extension.

(ii) The Lie algebra $D^{\mathrm{gr}*} \oplus D$ is a subalgebra of E.

The Lie algebra E has a subalgebra

$$H = \mathfrak{h} \oplus {D^0}^* \oplus D^0$$

where $\mathfrak{h} = \operatorname{span}_F\{h_\xi^\lambda : \xi \in \Delta^\times, \lambda \in \Lambda\} = \operatorname{span}_F\{h_\xi^0 : 0 \neq \xi \in \Delta_{\mathrm{ind}}\}$. We embed Δ into the dual space $\mathfrak{h}^*$, and extend $\xi \in \Delta \subset \mathfrak{h}^*$ to a linear form of H by $\xi(D^{0\,*} \oplus D^0) = 0$. We embed $\Lambda \subset D^{0\,*}$, using the evaluation map, and then extend $\lambda \in \Lambda \subset D^{0\,*}$ to a linear form of H by putting $\lambda(\mathfrak{h} \oplus C^0) = 0$. Then H is a toral subalgebra of E with root spaces

$$E_{\xi\oplus\lambda} = \begin{cases} L_\xi^\lambda, & \xi \neq 0, \\ L_0^\lambda \oplus (D^{-\lambda})^* \oplus D^\lambda, & \xi = 0. \end{cases}$$

Observe $H = E_0$ since $\mathfrak{h} = L_0^0$. The Lie algebra E has a toral subalgebra

$$H = \mathfrak{h} \oplus C^0 \oplus D^0$$

for $\mathfrak{h}$ as in 2.3.2. The symmetric bilinear form $(\cdot|\cdot)$ on E, defined by

$$\big(l_1 \oplus c_1 \oplus d_1, l_2 \oplus c_2 \oplus d_2\big) = (l_1, l_2)_L + c_1(d_2) + c_2(d_1),$$

is nondegenerate and invariant. Here $(-,-)$ is of course the given bilinear form of the invariant Lie torus L. One can check that thus constructed pair (E, H) satisfies the remaining axioms (EA3)-(EA6) of an EALA, (see [Na, Proposition 5.2.4]). This then shows the part (a) of the following theorem.

2.4.1 Theorem ([Ne2, Theorem 6]). (a) *The pair (E, H) constructed above is an extended affine Lie algebra, denoted* $\mathrm{EA}(L, D, \tau)$. *Its core is $L \oplus D^{\mathrm{gr}\,*}$ and its centreless core is L.*

(b) *Conversely, let (E, H) be an extended affine Lie algebra, and let $L = E_c/Z(E_c)$ be its centreless core. Then there exists a subalgebra $D \subset \mathrm{SCDer}_F(L)$ and an affine cocycle τ such that $E \cong \mathrm{EA}(L, D, \tau)$.*

Chapter 3

Algebraic groups and non-abelian Galois cohomology

This chapter serves as a review of some concepts and results from the theory of algebraic groups and non-abelian Galois cohomology which will be used in the following chapters.

First let us review the definition and basic properties of non-abelian Galois cohomology sets H^0 and H^1. This material is taken from [Ser].

Let Γ denotes a profinite group. A Γ-set E is a discrete topological space on which Γ acts continuously. If $\gamma \in \Gamma$ and $x \in E$, we will denote the image $\gamma(x)$ of x under γ as ${}^{\gamma}x$. If E and E' are two Γ-sets, a *morphism* of E to E' is a map $f : E \to E'$ which commutes with the action of Γ. A Γ-group G is a group in this category; in other words, it is a Γ-set with a group structure invariant under Γ. Notice that when G is commutative, one recovers the notion of a Γ-module. If E is a Γ-set, we put $H^0(\Gamma, E) = E^{\Gamma}$, the set of elements fixed by Γ. If E is a group, $H^0(\Gamma, E)$ is a group.

If G is a Γ-group, one defines a *1-cocycle of* Γ *in* G as a continuous map $a = (a_\gamma) : \Gamma \to G$, $\gamma \mapsto a_\gamma$ such that

$$a_{\delta\gamma} = a_\delta \cdot^{\delta} a_\gamma \quad (\delta, \gamma \in \Gamma).$$

The set of all 1-cocycles will be denoted $Z^1(\Gamma, G)$. Two cocycles a and a' are said to be *cohomologous* if there exists $g \in G$ such that

$$a'_\gamma = g^{-1} a_\gamma^\gamma g.$$

This is an equivalence relation on $Z^1(\Gamma, G)$, and the quotient set is denoted $H^1(\Gamma, G)$. This is the "first cohomology set of Γ in G". It has a distinguished element: the class of the unit cocycle, denoted usually by 0 or 1.

The cohomology sets $H^0(\Gamma, G)$ and $H^1(\Gamma, G)$ are functorial in G and coincide with the usual cohomology groups in dimensions 0 and 1 when G is commutative.

The non-abelian H^1 is a pointed set and therefore the notion of an exact sequence does make sense, i.e. the image of the map is equal to the preimage of a neutral element.

By the functoriality of H^0 and H^1, for a Γ-equivariant exact sequence of groups

$$1 \to G_1 \to G \to G_2 \to 1$$

there are induced sequences

$$H^0(\Gamma, G_1) \to H^0(\Gamma, G) \to H^0(\Gamma, G_2)$$

and

$$H^1(\Gamma, G_1) \to H^1(\Gamma, G) \to H^1(\Gamma, G_2).$$

Also, one may define a *boundary map* $\delta : H^0(\Gamma, G_2) \to H^1(\Gamma, G_1)$ as follows:

for $g_2 \in G_2$ choose $g \in G$ such that g_2 is the image of g. Let $a_\gamma = g^{-1} \cdot^\gamma g$. We define $\delta(g_2)$ as the class of the cocycle $(a_\gamma) \in Z^1(\Gamma, G_1)$.

3.0.2 Proposition. *[Ser, Chapter I, Proposition 38] Let $1 \to G_1 \to G \to G_2 \to 1$ be a Γ-equivariant exact sequence of groups. Then the sequence of pointed sets*

$$1 \to G_1^\Gamma \to G^\Gamma \to G_2^\Gamma \xrightarrow{\delta} H^1(\Gamma, G_1) \to H^1(\Gamma, G) \to H^1(\Gamma, G_2)$$

is exact.

We will now recall the definition of an affine group scheme. Let R be a ring.

An *affine group scheme* $\mathbf{G}$ over R is a representable functor

$$\mathbf{G} : R\!-\!\mathbf{alg} \to \mathbf{grp},$$

where $R\!-\!\mathbf{alg}$ is the category of commutative associative unital R-algebras and $\mathbf{grp}$ is the category of groups. The functor $\mathbf{G}$ is called *representable* if

$$\mathbf{G} = \mathrm{Hom}_{R-\mathbf{rng}}(R[\mathbf{G}], -),$$

for some $R[\mathbf{G}]$ in $R\!-\!\mathbf{rng}$, which is called the ring of regular functions of $\mathbf{G}$. By Yoneda's Lemma, the group structure on $\mathbf{G}$ is translated to the coassociative Hopf algebra structure on $R[\mathbf{G}]$ (cf. [Wat]). For an R-algebra S we call the elements of $\mathbf{G}(S)$ the S-points of $\mathbf{G}$.

3.0.3 Examples. (1) The multiplicative group scheme $\mathbf{G}_m$:

$\mathbf{G}_m(S) = S^\times$ is the group of units in S. $R[\mathbf{G}_m] = R[t^{\pm 1}]$.

(2) The special linear group $\mathbf{SL}_n$ for $n \geq 1$:

$\mathbf{SL}_n(S)$ is the group of $n \times n$-matrices with entries in S and determinant 1. $R[\mathbf{SL}_n] = R[x_{ij}]_{1\leq i,j\leq n}/\langle det(x_{ij}) - 1\rangle$.

If k is an algebraically closed field, an affine group scheme $\mathbf{G}$ over k is *semisimple* if it is smooth connected and if its radical is trivial [Hum]. A semisimple group scheme $\mathbf{G}$ over k is called *simple* if $\mathbf{G}(k)$ does not have any infinite closed normal subgroups. The definition of a simple group scheme over an arbitrary ring R is the following:

3.0.4 Definition. *An affine R-group scheme $\mathbf{G}$ is* simple *if it satisfies the two following requirements:*

(1) $\mathbf{G}$ is smooth.

(2) For each $x \in Spec(R)$, the geometric fiber $\mathbf{G} \times_R \overline{\kappa}(x)$ is simple, where $\overline{\kappa(x)}$ stands for an algebraic closure of the residue field $\kappa(x)$.

Let X be a scheme and $\mathbf{G}$ a group scheme over X. For any scheme Y over X we denote by p_i, for $i = 1, 2$, the corresponding projection $Y \times_X Y \to Y$ on the i-th component and by p_{ij}, for $i, j = 1, 2, 3$, the projection $Y \times_X Y \times_X Y \to Y \times_X Y$ on the ij-th component. These projections naturally induce group homomorphisms

$$\mathbf{G}(Y) \to \mathbf{G}(Y \times_X Y) \text{ and } \mathbf{G}(Y \times_X Y) \to \mathbf{G}(Y \times_X Y \times_X \times Y)$$

which we denote by p_i^* and p_{ij}^* respectively. Assume now that Y/X is an étale cover. For such a covering $Y \to X$, we define the corresponding set of cocycles to be

$$\check{Z}^1(Y/X, \mathbf{G}) := \{g \in \mathbf{G}(Y \times_X Y) \mid p_{23}^*(g)p_{12}^*(g) = p_{13}^*(g)\}$$

and the non-abelian cohomology to be

$$\check{H}^1(Y/X, \mathbf{G}) := \check{Z}^1(Y/X, \mathbf{G})/\mathbf{G}(Y),$$

where $\mathbf{G}(Y)$ acts on $\check{Z}^1(Y/X, \mathbf{G})$ by $g \cdot z = p_2^*(g)zp_1^*(g)^{-1}$. We define

$$\check{H}^1(X, \mathbf{G}) := \varinjlim_{Y} \check{H}^1(Y/X, \mathbf{G}),$$

where the limit is taken over all étale covers $Y \to X$ which are locally of finite type.

3.0.5 Proposition. *[Mil, Proposition 4.5] To any exact sequence of affine group schemes over a scheme X*

$$1 \to \mathbf{G}_1 \to \mathbf{G} \to \mathbf{G}_2 \to 1,$$

there is an associated exact sequence of pointed sets

$$1 \to \mathbf{G}_1(X) \to \mathbf{G}(X) \to \mathbf{G}_2(X) \to \check{H}^1(X, \mathbf{G}_1) \to \check{H}^1(X, \mathbf{G}) \to \check{H}^1(X, \mathbf{G}_2).$$

3.0.6 Proposition. *[Mil, Proposition 4.9]* $H^1(X_{et}, \mathbf{G}_m) \cong \mathrm{Pic(X)}$.

Chapter 4

A cohomological proof of Peterson-Kac's theorem on conjugacy of Cartan subalgebras for affine Kac-Moody Lie algebras

In this chapter[1] we prove the conjugacy theorem for affine Kac-Moody Lie algebras using cohomological techniques.

Let us fix notation that will be used in this chapter and recall the construction of an affine Kac-Moody Lie algebra. R will denote the ring of Laurent polynomials $k[t^{\pm 1}]$ with coefficients in the field k.

Let $\mathfrak{g}$ be a split simple finite dimensional Lie algebra over a field k and let $\mathbf{Aut}(\mathfrak{g})$ be its automorphism group. Let $\widetilde{\mathbf{G}}$ (resp. $\mathbf{G}$) be a simple simply connected (resp. adjoint) algebraic group over k corresponding to $\mathfrak{g}$. If $x, y \in \mathfrak{g}$, we denote their product in $\mathfrak{g}$ by $[x, y]$. We also let $R = k[t^{\pm 1}]$ and $L(\mathfrak{g}) = \mathfrak{g} \otimes_k R$. We again denote the Lie product in $L(\mathfrak{g})$ by $[x, y]$, where $x, y \in L(\mathfrak{g})$.

Split case. The main object under consideration in Chapter 4 is the affine (split or twisted) Kac-Moody Lie algebra $\widehat{L}$ corresponding to $\mathfrak{g}$. Any split

[1] A version of this chapter has been published. V. Chernousov, P. Gille, A. Pianzola and U. Yahorau, "A cohomological proof of Peterson-Kac's theorem on conjugacy of Cartan subalgebras for affine Kac-Moody Lie algebras," Journal of Algebra. 399: 55-78.

affine Kac-Moody Lie algebra is of the form (see [Kac])

$$\widehat{L} = \mathfrak{g} \otimes_k R \oplus k\, c \oplus k\, d.$$

The element c is central and d is a degree derivation for a natural grading of $L(\mathfrak{g})$: if $x \in \mathfrak{g}$ and $p \in \mathbb{Z}$ then

$$[d, x \otimes t^p]_{\widehat{L}} = p\, x \otimes t^p.$$

If $l_1 = x \otimes t^p,\ l_2 = y \otimes t^q \in L(\mathfrak{g})$ are viewed as elements in $\widehat{L}$, their Lie product is given by

$$[x \otimes t^p, y \otimes t^q]_{\widehat{L}} = [x, y] \otimes t^{p+q} + p\, \kappa(x, y)\, \delta_{0,p+q} \cdot c,$$

where κ is the Killing form on $\mathfrak{g}$ and $\delta_{0,p+q}$ is Kronecker's delta.

Twisted case. Let m be a positive integer and let $S = k[t^{\pm\frac{1}{m}}]$ be the ring of Laurent polynomials in the variable $s = t^{\frac{1}{m}}$ with coefficients in k. Let

$$L(\mathfrak{g})_S = L(\mathfrak{g}) \otimes_R S$$

be the Lie algebra obtained from the R-Lie algebra $L(\mathfrak{g})$ by the base change $R \to S$. Similarly, we define Lie algebras

$$\widetilde{L}(\mathfrak{g})_S = L(\mathfrak{g})_S \oplus kc \quad \text{and} \quad \widehat{L}(\mathfrak{g})_S = L(\mathfrak{g})_S \oplus kc \oplus kd.$$ [2]

Fix a primitive root of unity $\zeta \in k$ of degree m. The R-automorphism

$$\zeta^\times : S \to S \quad s \mapsto \zeta s$$

generates the Galois group $\Gamma = \mathrm{Gal}(S/R)$, which we may identify with the abstract group $\mathbb{Z}/m\mathbb{Z}$ by means of $\zeta^\times$. Note that Γ acts naturally on $\mathbf{Aut}(\mathfrak{g})(S) = \mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S)$ and on $L(\mathfrak{g})_S = L(\mathfrak{g}) \otimes_R S$ through the second factor.

Next, let σ be an automorphism of $\mathfrak{g}$ of order m. This gives rise to an S-automorphism of $L(\mathfrak{g})_S$ via $x \otimes s \mapsto \sigma(x) \otimes s$, for $x \in \mathfrak{g}$, $s \in S$. It then easily

[2] Unlike $L(\mathfrak{g})_S$, these object exist over k but not over S.

follows that the assignment

$$\bar{1} \mapsto z_{\bar{1}} = \sigma^{-1} \in \mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S)$$

gives rise to a cocycle $z = (z_{\bar{i}}) \in Z^1(\Gamma, \mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S))$. This cocycle, in turn, gives rise to a twisted action of Γ on $L(\mathfrak{g})_S$. Applying Galois descent formalism, we then obtain the Γ-invariant subalgebra

$$L(\mathfrak{g}, \sigma) := (L(\mathfrak{g})_S)^{\Gamma} = (L(\mathfrak{g}) \otimes_R S)^{\Gamma}.$$

This is a "simple Lie algebra over R" in the sense of [SGA3], which is a twisted form of the "split simple" R-Lie algebra $L(\mathfrak{g}) = \mathfrak{g} \otimes_k R$. Indeed, S/R is an étale extension and from properties of Galois descent we have

$$L(\mathfrak{g}, \sigma) \otimes_R S \simeq L(\mathfrak{g})_S = (\mathfrak{g} \otimes_k R) \otimes_R S.$$

Note that $L(\mathfrak{g}, id) = L(\mathfrak{g})$.

For $\bar{i} \in \mathbb{Z}/m\mathbb{Z}$, consider the eigenspace

$$\mathfrak{g}_{\bar{i}} = \{x \in \mathfrak{g} : \sigma(x) = \zeta^i x\}.$$

Simple computations show that

$$L(\mathfrak{g}, \sigma) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\bar{i}} \otimes k[t^{\pm 1}] s^i.$$

Let

$$\widetilde{L}(\mathfrak{g}, \sigma) := L(\mathfrak{g}, \sigma) \oplus kc \quad \text{and} \quad \widehat{L}(\mathfrak{g}, \sigma) := L(\mathfrak{g}, \sigma) \oplus kc \oplus kd.$$

We give $\widehat{L}(\mathfrak{g}, \sigma)$ a Lie algebra structure such that c is a central element, d is the degree derivation, i.e. if $x \in \mathfrak{g}_{\bar{i}}$ and $p \in \mathbb{Z}$ then

$$[d, x \otimes t^{\frac{p}{m}}] := px \otimes t^{\frac{p}{m}} \tag{4.0.6.1}$$

and if $y \otimes t^{\frac{q}{m}} \in L(\mathfrak{g}, \sigma)$ we get

$$[x \otimes t^{\frac{p}{m}}, y \otimes t^{\frac{q}{m}}]_{\widehat{L}(\mathfrak{g},\sigma)} = [x, y] \otimes t^{\frac{p+q}{m}} + p\,\kappa(\,x, y\,)\,\delta_{0,p+q} \cdot c,$$

where, as before, κ is the Killing form on $\mathfrak{g}$ and $\delta_{0,p+q}$ is Kronecker's delta.

Since $\mathbf{Aut}(\mathfrak{g}) \cong \mathbf{Aut}(\widetilde{\mathbf{G}})$ we can also consider by means of the cocycle z the twisted R-group ${}_z\widetilde{\mathbf{G}}_R$. It is well known (see, for example, the proof of [GP1, Prop 4.10]) that the determination of Lie algebras commutes with the twisting process. Thus $L(\mathfrak{g}, \sigma)$ is a Lie algebra of ${}_z\widetilde{\mathbf{G}}_R$.

The following theorem is the main theorem of this chapter. We keep all the preceeding notations.

4.0.7 Theorem. (Conjugacy theorem for affine Kac-Moody.) *Let $\mathcal{L} = L(\mathfrak{g}, \sigma)$ be an affine Kac-Moody Lie algebra. Let ${}_z\widehat{\mathbf{G}}_R(R)$ be the preimage of $\{Ad(g) : g \in {}_z\widetilde{\mathbf{G}}_R(R)\}$ under the canonical map $\mathrm{Aut}_k(\widehat{\mathcal{L}}) \to \mathrm{Aut}_k(\mathcal{L})$. Then all MADs of $\widehat{\mathcal{L}}$ are conjugate under ${}_z\widehat{\mathbf{G}}_R(R)$.*

* * *

Let $\widehat{L}(\mathfrak{g})$ be an affine Kac-Moody Lie algebra corresponding to a finite dimensional simple Lie algebra $\mathfrak{g}$. Let $\phi \in \mathrm{Aut}_{k-Lie}(\widehat{L}(\mathfrak{g})_S)$. Since $\widetilde{L}(\mathfrak{g})_S$ is the derived subalgebra of $\widehat{L}(\mathfrak{g})_S$ the restriction $\phi|_{\widetilde{L}(\mathfrak{g})_S}$ induces a k-Lie automorphism of $\widetilde{L}(\mathfrak{g})_S$. Furthermore, passing to the quotient $\widetilde{L}(\mathfrak{g})_S/kc \simeq L(\mathfrak{g})_S$ the automorphism $\phi|_{\widetilde{L}(\mathfrak{g})_S}$ induces an automorphism of $L(\mathfrak{g})_S$. This yields a well-defined morphism

$$\mathrm{Aut}_{k-Lie}(\widehat{L}(\mathfrak{g})_S) \to \mathrm{Aut}_{k-Lie}(L(\mathfrak{g})_S).$$

Similar considerations apply to $\mathrm{Aut}_{k-Lie}(\widehat{L}(\mathfrak{g}, \sigma))$. The aim of the next few sections is to show that these two morphisms are surjective.

4.1 S-automorphisms of $L(\mathfrak{g})_S$

In this section we construct a "simple" system of generators of the automorphism group

$$\mathbf{Aut}(\mathfrak{g})(S) = \mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S)$$

which can be easily extended to k-automorphisms of $\widehat{L}(\mathfrak{g})_S$. We produce our list of generators based on a well-known fact that the group in question is generated by S-points of the corresponding split simple adjoint algebraic group and automorphisms of the corresponding Dynkin diagram.

More precisely, let $\widetilde{\mathbf{G}}$ be the split simple simply connected group over k corresponding to $\mathfrak{g}$ and let $\mathbf{G}$ be the corresponding adjoint group. Choose a maximal split k-torus $\widetilde{\mathbf{T}} \subset \widetilde{\mathbf{G}}$ and denote its image in $\mathbf{G}$ by $\mathbf{T}$. The Lie algebra of $\widetilde{\mathbf{T}}$ is a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We fix a Borel subgroup $\widetilde{\mathbf{T}} \subset \widetilde{\mathbf{B}} \subset \widetilde{\mathbf{G}}$.

Let $\Sigma = \Sigma(\widetilde{\mathbf{G}}, \widetilde{\mathbf{T}})$ be the root system of $\widetilde{\mathbf{G}}$ relative to $\widetilde{\mathbf{T}}$. The Borel subgroup $\widetilde{\mathbf{B}}$ determines an ordering of Σ, hence the system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. Fix a Chevalley basis [St67]

$$\{H_{\alpha_1}, \ldots H_{\alpha_n},\ X_\alpha,\ \alpha \in \Sigma\}$$

of $\mathfrak{g}$ corresponding to the pair $(\widetilde{\mathbf{T}}, \widetilde{\mathbf{B}})$. This basis is unique up to signs and automorphisms of $\mathfrak{g}$ which preserve $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{T}}$ (see [St67, §1, Remark 1]).

Since S is a Euclidean ring, by Steinberg [St62] the group $\widetilde{\mathbf{G}}(S)$ is generated by the so-called root subgroups $U_\alpha = \langle x_\alpha(u) \mid u \in S \rangle$, where $\alpha \in \Sigma$ and

$$x_\alpha(u) = \exp(uX_\alpha) = \sum_{n=0}^{\infty} u^n X_\alpha^n\,/n! \qquad (4.1.0.1)$$

We recall also that by [St67, §10, Cor. (b) after Theorem 29], every automorphism σ of the Dynkin diagram $\mathrm{Dyn}(\widetilde{\mathbf{G}})$ of $\widetilde{\mathbf{G}}$ can be extended to an automorphism of $\widetilde{\mathbf{G}}$ (and hence of $\mathbf{G}$) and $\mathfrak{g}$, still denoted by σ, which takes

$$x_\alpha(u) \longrightarrow x_{\sigma(\alpha)}(\varepsilon_\alpha u) \ \text{ and } \ X_\alpha \longrightarrow \varepsilon_\alpha X_{\sigma(\alpha)}.$$

Here $\varepsilon_\alpha = \pm 1$ and if $\alpha \in \Pi$ then $\varepsilon_\alpha = 1$. Thus we have a natural embedding

$$\mathrm{Aut}(\mathrm{Dyn}(\widetilde{\mathbf{G}})) \hookrightarrow \mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S).$$

The group $\mathbf{G}(S)$ acts by S-automorphisms on $L(\mathfrak{g})_S$ through the adjoint representations $\mathrm{Ad} : \mathbf{G} \to \mathbf{GL}(L(\mathfrak{g})_S)$ and hence we also have a canonical embedding

$$\mathbf{G}(S) \hookrightarrow \mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S).$$

As we said before, it is well-known (see [P2] for example) that

$$\mathrm{Aut}_{S-Lie}\,(L(\mathfrak{g})_S) = \mathbf{G}\,(S) \rtimes \mathrm{Aut}(\mathrm{Dyn}(\widetilde{\mathbf{G}})).$$

For later use we need one more fact.

4.1.1 Proposition. *Let* $f : \widetilde{\mathbf{G}} \to \mathbf{G}$ *be the canonical morphism. The group* $\mathbf{G}(S)$ *is generated by the root subgroups* $f(U_\alpha)$, $\alpha \in \Sigma$, *and* $\mathbf{T}(S)$.

Proof. Let $\mathbf{Z} \subset \widetilde{\mathbf{G}}$ be the center of $\widetilde{\mathbf{G}}$. The exact sequence

$$1 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\mathbf{G}} \longrightarrow \mathbf{G} \longrightarrow 1$$

gives rise to an exact sequence in Galois cohomology

$$f(\widetilde{\mathbf{G}}(S)) \hookrightarrow \mathbf{G}(S) \longrightarrow \operatorname{Ker}[H^1(S,\mathbf{Z}) \to H^1(S,\widetilde{\mathbf{G}})] \longrightarrow 1.$$

Since $H^1(S,\mathbf{Z}) \to H^1(S,\widetilde{\mathbf{G}})$ factors through

$$H^1(S,\mathbf{Z}) \longrightarrow H^1(S,\widetilde{\mathbf{T}}) \longrightarrow H^1(S,\widetilde{\mathbf{G}})$$

and since $H^1(S,\widetilde{\mathbf{T}}) = 1$ (because $\operatorname{Pic} S = 1$) we obtain

$$f(\widetilde{\mathbf{G}}(S)) \hookrightarrow \mathbf{G}(S) \longrightarrow H^1(S,\mathbf{Z}) \longrightarrow 1. \tag{4.1.1.1}$$

Similar considerations applied to

$$1 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\mathbf{T}} \longrightarrow \mathbf{T} \longrightarrow 1$$

show that

$$f(\widetilde{\mathbf{T}}(S)) \hookrightarrow \mathbf{T}(S) \longrightarrow H^1(S,\mathbf{Z}) \longrightarrow 1. \tag{4.1.1.2}$$

The result now follows from (4.1.1.1) and (4.1.1.2). □

4.1.2 Corollary. *One has*

$$\operatorname{Aut}_{S-Lie}(L(\mathfrak{g})_S) = \langle \operatorname{Aut}(\operatorname{Dyn}(\widetilde{\mathbf{G}})),\, U_\alpha,\, \alpha \in \Sigma,\, \mathbf{T}(S) \rangle.$$

4.2 k-automorphisms of $L(\mathfrak{g})_S$

We keep the above notation. Recall that for any algebra $\mathfrak{A}$ over a field k the centroid of $\mathfrak{A}$ is

$$\mathrm{Ctd}\,(\mathfrak{A}) = \{\chi \in \mathrm{End}_k(\mathfrak{A}) \; | \chi(a \cdot b) = a \cdot \chi(b) = \chi(a) \cdot b \;\text{ for all }\; a, b \in \mathfrak{A}\,\}.$$

It is easy to check that if $\chi_1, \chi_2 \in \mathrm{Ctd}(\mathfrak{A})$ then both linear operators $\chi_1 \circ \chi_2$ and $\chi_1 + \chi_2$ are contained in $\mathrm{Ctd}\,(\mathfrak{A})$ as well. Thus, $\mathrm{Ctd}\,(\mathfrak{A})$ is a unital associative subalgebra of $\mathrm{End}_k(\mathfrak{A})$. It is also well-known that the centroid is commutative whenever $\mathfrak{A}$ is perfect.

Example. Consider the k-Lie algebra $\mathfrak{A} = L(\mathfrak{g})_S$. For any $s \in S$ the linear k-operator $\chi_s : L(\mathfrak{g})_S \to L(\mathfrak{g})_S$ given by $x \to sx$ satisfies

$$\chi_s([x, y]) = [x, \chi_s(y)] = [\chi_s(x), y],$$

hence $\chi_s \in \mathrm{Ctd}\,(L(\mathfrak{g})_S)$. Conversely, it is known (see [ABP, Lemma 4.2]) that every element in $\mathrm{Ctd}\,(L(\mathfrak{g})_S)$ is of the form χ_s. Thus,

$$\mathrm{Ctd}\,(L(\mathfrak{g})_S) = \{\, \chi_s \mid s \in S\,\} \simeq S.$$

4.2.1 Proposition. ([P2, Proposition 1]) *One has*

$$\begin{aligned} \mathrm{Aut}_{k-Lie}(L(\mathfrak{g})_S) &\simeq \mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S) \rtimes \mathrm{Aut}_k(\mathrm{Ctd}\,(L(\mathfrak{g})_S)) \\ &\simeq \mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S) \rtimes \mathrm{Aut}_k(S). \end{aligned}$$

4.2.2 Corollary. *One has*

$$\mathrm{Aut}_{k-Lie}(L(\mathfrak{g})_S) = \langle\, \mathrm{Aut}_k\,(S), \mathrm{Aut}(\mathrm{Dyn}(\widetilde{\mathbf{G}})),\, U_\alpha,\, \alpha \in \Sigma,\, \mathbf{T}(S)\,\rangle.$$

Proof. This follows from Corollary 4.1.2 and Proposition 4.2.1. ☐

4.3 Automorphisms of $\widetilde{L}(\mathfrak{g})_S$

We remind the reader that the centre of $\widetilde{L}(\mathfrak{g})_S$ is the k-span of c and that $\widetilde{L}(\mathfrak{g})_S = L(\mathfrak{g})_S \oplus kc$. Since any automorphism ϕ of $\widetilde{L}(\mathfrak{g})_S$ takes the centre into itself we have a natural (projection) mapping

$$\mu : \widetilde{L}(\mathfrak{g})_S \to \widetilde{L}(\mathfrak{g})_S/kc \simeq L(\mathfrak{g})_S$$

which induces the mapping

$$\lambda : \mathrm{Aut}_{k-Lie}(\widetilde{L}(\mathfrak{g})_S) \to \mathrm{Aut}_{k-Lie}(L(\mathfrak{g})_S)$$

given by $\phi \to \phi'$ where $\phi'(x) = \mu(\phi(x))$ for all $x \in L(\mathfrak{g})_S$. In the last formula we view x as an element of $\widetilde{L}(\mathfrak{g})_S$ through the embedding $L(\mathfrak{g})_S \hookrightarrow \widetilde{L}(\mathfrak{g})_S$.

4.3.1 Remark. It is straightforward to check that ϕ' is indeed an automorphism of $L(\mathfrak{g})_S$.

4.3.2 Proposition. *The mapping λ is an isomorphism.*

Proof. See [P2, Proposition 4]. □

In what follows if $\phi \in \mathrm{Aut}_{k-Lie}(L(\mathfrak{g})_S)$ we denote its (unique) lifting to $\mathrm{Aut}_{k-Lie}(\widetilde{L}(\mathfrak{g})_S)$ by $\tilde{\phi}$.

4.3.3 Remark. For later use we need an explicit formula for lifts of automorphisms of $L(\mathfrak{g})_S$ induced by some "special" points in $\mathbf{T}(S)$ (those which are not in the image of $\widetilde{\mathbf{T}}(S) \to \mathbf{T}(S)$). More precisely, the fundamental coweights give rise to the decomposition $\mathbf{T} \simeq \mathbf{G}_{m,S} \times \cdots \times \mathbf{G}_{m,S}$. As usual, we have the decomposition $\mathbf{T}(S) \simeq \mathbf{T}(k) \times \mathrm{Hom}\,(\mathbf{G}_m, \mathbf{T})$. The second factor in the last decomposition is the cocharacter lattice of $\mathbf{T}$ and its elements correspond (under the adjoint action) to the subgroup in $\mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S)$ isomorphic to $\mathrm{Hom}(Q, \mathbb{Z})$ where Q is the corresponding root lattice: if $\phi \in \mathrm{Hom}(Q, \mathbb{Z})$ it induces an S-automorphism of $L(\mathfrak{g})_S$ (still denoted by ϕ) given by

$$X_\alpha \to X_\alpha \otimes s^{\phi(\alpha)}, \quad H_{\alpha_i} \to H_{\alpha_i}.$$

It is straightforward to check the mapping $\tilde{\phi} : \widetilde{L}(\mathfrak{g})_S \to \widetilde{L}(\mathfrak{g})_S$ given by

$$H_\alpha \to H_\alpha + \phi(\alpha)\langle X_\alpha, X_{-\alpha}\rangle \cdot c, \quad H_\alpha \otimes s^p \to H_\alpha \otimes s^p$$

if $p \neq 0$ and

$$X_\alpha \otimes s^p \to X_\alpha \otimes s^{p+\phi(\alpha)}$$

is an automorphism of $\widetilde{L}(\mathfrak{g})_S$, hence it is the (unique) lift of ϕ.

4.4 Automorphisms of split affine Kac-Moody Lie algebras

Since $\widetilde{L}(\mathfrak{g})_S = [\widehat{L}(\mathfrak{g})_S, \widehat{L}(\mathfrak{g})_S]$ we have a natural (restriction) mapping

$$\tau : \mathrm{Aut}_{k-Lie}\,(\widehat{L}(\mathfrak{g})_S) \to \mathrm{Aut}_{k-Lie}\,(\widetilde{L}(\mathfrak{g})_S).$$

4.4.1 Proposition. *The mapping τ is surjective.*

Proof. By Proposition 4.3.2 and Corollary 4.2.2 the group $\mathrm{Aut}_{k-Lie}(\widetilde{L}(\mathfrak{g})_S)$ has the distinguished system of generators $\{\,\tilde{\phi}\,\}$ where

$$\phi \in \mathrm{Aut}(\mathrm{Dyn}(\widetilde{\mathbf{G}})),\ \mathbf{T}(S),\ \mathrm{Aut}_k(S),\ U_\alpha.$$

We want to construct a mapping $\hat{\phi} : \widehat{L}(\mathfrak{g})_S \to \widehat{L}(\mathfrak{g})_S$ which preserves the identity

$$[d, x \otimes t^{\frac{p}{m}}]_{\widehat{L}} = p\, x \otimes t^{\frac{p}{m}}$$

for all $x \in \mathfrak{g}$ and whose restriction to $\widetilde{L}(\mathfrak{g})_S$ coincides with $\tilde{\phi}$. These two properties would imply that $\hat{\phi}$ is an automorphism of $\widehat{L}(\mathfrak{g})_S$ lifting $\tilde{\phi}$.

If $\phi \in U_\alpha$ is unipotent we define $\hat{\phi}$, as usual, through the exponential map. If $\phi \in \mathrm{Aut}(\mathrm{Dyn}(\widetilde{\mathbf{G}}))$ we put $\hat{\phi}(d) = d$. If ϕ is as in Remark 6.1.2.2 we extend it by $d \to d - X$ where $X \in \mathfrak{h}$ is the unique element such that $[X, X_\alpha] = \phi(\alpha) X_\alpha$ for all roots $\alpha \in \Sigma$. Note that automorphisms of $L(\mathfrak{g})_S$ given by points in $\mathbf{T}(k)$ are in the image of $\widetilde{\mathbf{T}}(k) \to \mathbf{T}(k)$ and hence they are generated by unipotent elements. Lastly, if $\phi \in \mathrm{Aut}_k\,(S)$ is of the form $s \to as^{-1}$ where $a \in k^\times$ (resp. $s \to as$) we extend $\tilde{\phi}$ by $\hat{\phi}(d) = -d$ (resp. $\hat{\phi}(d) = d$). We leave it to the reader to verify that in all cases $\hat{\phi}$ preserves the above identity and hence $\hat{\phi}$ is an automorphism of $\widehat{L}(\mathfrak{g})_S$. □

4.4.2 Proposition. *One has* $\mathrm{Ker}\,\tau \simeq V$ *where* $V = \mathrm{Hom}_k(kd, kc)$.

Proof. We first embed $V \hookrightarrow \mathrm{Aut}_{k-Lie}(\widehat{L}(\mathfrak{g})_S)$. Let $v \in V$. Recall that any element $x \in \widehat{L}(\mathfrak{g})_S$ can be written uniquely in the form $x = x' + ad$ where $x' \in \widetilde{L}(\mathfrak{g})_S$ and $a \in k$. We define $\hat{v} : \widehat{L}(\mathfrak{g})_S \to \widehat{L}(\mathfrak{g})_S$ by $x \to x + v(ad)$. One checks that $\hat{v}$ is an automorphism of $\widehat{L}(\mathfrak{g})_S$ and thus the required embedding is given by $v \to \hat{v}$.

Since $\hat{v}(x') = x'$ for all $x' \in \widetilde{L}$ we have $\hat{v} \in \operatorname{Ker}\tau$. Conversely, let $\psi \in \operatorname{Ker}\tau$. Then $\psi(x) = x$ for all $x \in \widetilde{L}(\mathfrak{g})_S$. We need to show that $\psi(d) = ac + d$ where $a \in k$. Let $\psi(d) = x' + ac + bd$ where $a, b \in k$ and $x' \in L(\mathfrak{g})_S$. Since $[d, X_\alpha]_{\widehat{L}(\mathfrak{g})_S} = 0$ we get

$$[\psi(d), \psi(X_\alpha)]_{\widehat{L}(\mathfrak{g})_S} = 0.$$

Substituting $\psi(d) = x' + ac + bd$ we obtain

$$[x' + ac + bd, X_\alpha]_{\widehat{L}(\mathfrak{g})_S} = 0$$

or $[x', X_\alpha]_{\widetilde{L}(\mathfrak{g})_S} = 0$. Since this is true for all roots $\alpha \in \Sigma$, the element x' commutes with $\mathfrak{g}$ and this can happen if and only if $x' = 0$.

It remains to show that $b = 1$. To see this we can argue similarly by considering the equality

$$[d, X_\alpha \otimes t^{\frac{1}{m}}]_{\widehat{L}(\mathfrak{g})_S} = X_\alpha \otimes t^{\frac{1}{m}}$$

and applying ψ. $\square$

4.4.3 Corollary. *The sequence of groups*

$$1 \longrightarrow V \longrightarrow \operatorname{Aut}_{k-Lie}(\widehat{L}(\mathfrak{g})_S) \xrightarrow{\lambda\circ\tau} \operatorname{Aut}_{k-Lie}(L(\mathfrak{g})_S) \longrightarrow 1 \qquad (4.4.3.1)$$

is exact.

4.5 Automorphism group of twisted affine Kac-Moody Lie algebras

We keep the notation introduced in the beginning of this Chapter. In particular, we fix an integer m and a primitive root of unity $\zeta = \zeta_m \in k$ of degree m. Consider the k-automorphism $\zeta^\times : S \to S$ such that $s \to \zeta s$ which we view as a k-automorphism of $L(\mathfrak{g})_S$ through the embedding

$$\operatorname{Aut}_k(S) \hookrightarrow \operatorname{Aut}_{k-Lie}(L(\mathfrak{g})_S) \simeq \operatorname{Aut}_{S-Lie}(L(\mathfrak{g})_S) \rtimes \operatorname{Aut}_k(S)$$

(see Proposition 4.2.1). As it is explained in § 4.4 we then get the automorphism $\widehat{\zeta}^{\times}$ (resp. $\widetilde{\zeta}^{\times}$) of $\widehat{L}(\mathfrak{g})_S$ (resp. $\widetilde{L}(\mathfrak{g})_S$) given by

$$x \otimes s^i + ac + bd \longrightarrow x \otimes \zeta^i s^i + ac + bd$$

where $a, b \in k$ and $x \in \mathfrak{g}$.

Consider now the abstract group $\Gamma = \mathbb{Z}/m\mathbb{Z}$ (which can be identified with $\mathrm{Gal}\,(S/R)$ as already explained) and define its action on $\widehat{L}(\mathfrak{g})_S$ (resp. $\widetilde{L}(\mathfrak{g})_S$, $L(\mathfrak{g})_S$) with the use of $\widehat{\zeta}^{\times}$ (resp. $\widetilde{\zeta}^{\times}$, $\zeta^{\times}$). More precisely, for every $l \in \widehat{L}(\mathfrak{g})_S$ we let $\bar{i}(l) := (\widehat{\zeta}^{\times})^i(l)$. Similarly, we define the action of Γ on $\mathrm{Aut}_{k-Lie}\,(\widehat{L}(\mathfrak{g})_S)$ by

$$\bar{i} : \mathrm{Aut}_{k-Lie}\,(\widehat{L}(\mathfrak{g})_S) \longrightarrow \mathrm{Aut}_{k-Lie}\,(\widehat{L}(\mathfrak{g})_S), \quad x \to (\widehat{\zeta}^{\times})^i x (\widehat{\zeta}^{\times})^{-i}.$$

Therefore, $\mathrm{Aut}_{k-Lie}\,(\widehat{L}(\mathfrak{g})_S)$ can be viewed as a Γ-set. Along the same lines one defines the action of Γ on $\mathrm{Aut}_{k-Lie}\,(L(\mathfrak{g})_S)$ and $\mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S)$ with the use of $\zeta^{\times}$. It is easy to see that Γ acts trivially on the subgroup $V \subset \mathrm{Aut}_{k-Lie}\,(\widehat{L}(\mathfrak{g})_S)$ introduced in Proposition 4.4.2. Thus, (6.2.5.1) can be viewed as an exact sequence of Γ-groups.

We next choose an element $\pi \in \mathrm{Aut}(\mathrm{Dyn}(\mathbf{G})) \subset \mathrm{Aut}_k(\mathfrak{g})$ of order m (clearly, m can take value $1, 2$ or 3 only). Like before, we have the corresponding automorphism $\hat{\pi}$ of $\widehat{L}(\mathfrak{g})_S$ given by

$$x \otimes s^i + ac + bd \longrightarrow \pi(x) \otimes s^i + ac + bd$$

where $a, b \in k$ and $x \in \mathfrak{g}$.

Note that $\widehat{\zeta}^{\times}\hat{\pi} = \hat{\pi}\widehat{\zeta}^{\times}$. It then easily follows that the assignment

$$\bar{1} \to z_{\bar{1}} = \hat{\pi}^{-1} \in \mathrm{Aut}_{k-Lie}\,(\widehat{L}(\mathfrak{g})_S)$$

gives rise to a cocycle $z = (z_{\bar{i}}) \in Z^1(\Gamma, \mathrm{Aut}_{k-Lie}\,(\widehat{L}(\mathfrak{g})_S))$.

This cocycle, in turn, gives rise to a (new) twisted action of Γ on $\widehat{L}(\mathfrak{g})_S$ and $\mathrm{Aut}_{k-Lie}\,(\widehat{L}(\mathfrak{g})_S)$. Analogous considerations (with the use of π) are applied to $\mathrm{Aut}_{k-Lie}\,(L(\mathfrak{g})_S)$ and $L(\mathfrak{g})_S$. For future reference note that $\hat{\pi}$ commutes with elements in V, hence the twisted action of Γ on V is still trivial. From now on we view (6.2.5.1) as an exact sequence of Γ-groups, the action of Γ being the twisted action.

4.5.1 Remark. As we noticed before the invariant subalgebra

$$\mathcal{L} = L(\mathfrak{g}, \pi) = (L(\mathfrak{g})_S)^{\Gamma} = ((\mathfrak{g} \otimes_k R) \otimes_R S)^{\Gamma}$$

is a simple Lie algebra over R, a twisted form of a split Lie algebra $\mathfrak{g} \otimes_k R$. The same cohomological formalism also yields that

$$\mathrm{Aut}_{R-Lie}(\mathcal{L}) \simeq (\mathrm{Aut}_{S-Lie}(L(\mathfrak{g})_S))^{\Gamma}. \tag{4.5.1.1}$$

4.5.2 Remark. It is worth mentioning that the canonical embedding

$$\begin{aligned}\iota : (\mathrm{Aut}_{k-Lie}(L(\mathfrak{g})_S))^{\Gamma} \quad \hookrightarrow \quad & \mathrm{Aut}_{k-Lie}((L(\mathfrak{g})_S)^{\Gamma}) = \mathrm{Aut}_{k-Lie}(\mathcal{L}) \simeq \\ & \mathrm{Aut}_{R-Lie}(\mathcal{L}) \rtimes \mathrm{Aut}_k(R),\end{aligned}$$

where the last isomorphism can be established in the same way as in Proposition 4.2.1, is not necessary surjective in general case. Indeed, one checks that if $m = 3$ then the k-automorphism of R given by $t \to t^{-1}$ and viewed as an element of $\mathrm{Aut}_{k-Lie}(\mathcal{L}) \simeq \mathrm{Aut}_{R-Lie}(\mathcal{L}) \rtimes \mathrm{Aut}_k(R)$ is not in $\mathrm{Im}\,\iota$. However (6.4.1.1) implies that the group $\mathrm{Aut}_{R-Lie}(\mathcal{L})$ is in the image of ι.

4.5.3 Remark. The k-Lie algebra $\widehat{\mathcal{L}} = (\widehat{L}(\mathfrak{g})_S)^{\Gamma}$ is a twisted affine Kac–Moody Lie algebra. Conversely, by the Realization Theorem every twisted affine Kac–Moody Lie algebra can be obtained in such a way.

4.5.4 Lemma. *One has $H^1(\Gamma, V) = 1$.*

Proof. Since Γ is cyclic of order m acting trivially on $V \simeq k$ it follows that

$$Z^1(\Gamma, V) = \{\, x \in k \mid mx = 0 \,\} = 0$$

as required. $\square$

The long exact cohomological sequence associated to (6.2.5.1) together with Lemma 4.5.4 imply the following.

4.5.5 Theorem. *The following sequence*

$$1 \longrightarrow V \longrightarrow (\mathrm{Aut}_{k-Lie}(\widehat{L}(\mathfrak{g})_S))^{\Gamma} \xrightarrow{\nu} (\mathrm{Aut}_{k-Lie}(L(\mathfrak{g})_S))^{\Gamma} \longrightarrow 1$$

is exact. In particular, the group $\mathrm{Aut}_{R-Lie}(\mathcal{L})$ is in the image of the canonical *mapping*

$$\mathrm{Aut}_{k-Lie}(\widehat{\mathcal{L}}) \longrightarrow \mathrm{Aut}_{k-Lie}(\mathcal{L}) \simeq \mathrm{Aut}_{R-Lie}(\mathcal{L}) \rtimes \mathrm{Aut}_k(R).$$

Proof. The first assertion is clear. As for the second one, note that as in Remark 4.5.2 we have the canonical embedding

$$(\mathrm{Aut}_{k-Lie}(\widehat{L}(\mathfrak{g})_S))^\Gamma \hookrightarrow \mathrm{Aut}_{k-Lie}((\widehat{L}(\mathfrak{g})_S)^\Gamma) = \mathrm{Aut}_{k-Lie}(\widehat{\mathcal{L}})$$

and the commutative diagram

$$\begin{array}{ccc} (\mathrm{Aut}_{k-Lie}(\widehat{L}(\mathfrak{g})_S))^\Gamma & \xrightarrow{\nu} & (\mathrm{Aut}_{k-Lie}(L(\mathfrak{g})_S))^\Gamma \\ \downarrow & & \downarrow \\ \mathrm{Aut}_{k-Lie}(\widehat{\mathcal{L}}) & \longrightarrow & \mathrm{Aut}_{k-Lie}(\mathcal{L}) \end{array}$$

Then surjectivity of ν and Remark 4.5.2 yield the result.

□

4.6 Some properties of affine Kac-Moody Lie algebras

Henceforth we fix a simple finite dimensional Lie algebra $\mathfrak{g}$ and a (diagram) automorphism σ of finite order m. For brevity, we will write $\widehat{\mathcal{L}}$ and $(\widetilde{\mathcal{L}}, \mathcal{L})$ for $\widehat{L}(\mathfrak{g}, \sigma)$ and $(\widetilde{L}(\mathfrak{g}, \sigma), L(\mathfrak{g}, \sigma))$ respectively.

For all $l_1, l_2 \in \mathcal{L}$ one has

$$[l_1, l_2] - [l_1, l_2]_{\widehat{\mathcal{L}}} = ac \tag{4.6.0.1}$$

for some scalar $a \in k$. Using (4.0.6.1) it is also easy to see that for all $y \in \mathcal{L}$ one has

$$[d, yt^n]_{\widehat{\mathcal{L}}} = mnyt^n + [d, y]_{\widehat{\mathcal{L}}} t^n \tag{4.6.0.2}$$

4.6.1 Remark. Recall that $\mathcal{L}$ has a natural R-module structure: If $y = x \otimes t^{\frac{p}{m}} \in \mathcal{L}$ then

$$yt := x \otimes t^{\frac{p}{m}+1} = x \otimes t^{\frac{p+m}{m}} \in \mathcal{L}.$$

Therefore since $[d,y]_{\widehat{\mathcal{L}}}$ is contained in $\mathcal{L}$ the expression $[d,y]_{\widehat{\mathcal{L}}}t^n$ is meaningful.

Henceforth we will denote by $\mathbf{G}$ the simple simply connected group scheme over R corresponding to $\mathcal{L}$.

The infinite dimensional Lie algebra $\widehat{\mathcal{L}}$ admits a unique (up to non-zero scalar) invariant nondegenerate bilinear form $(\cdot,\cdot)$. Its restriction to $\mathcal{L} \subset \widehat{\mathcal{L}}$ is nondegenerate (see [Kac, 7.5.1 and 8.3.8]) and we have

$$(c,c) = (d,d) = 0,\ \ 0 \neq (c,d) = \beta \in k^{\times}$$

and

$$(c,l) = (d,l) = 0 \ \text{ for all } l \in \mathcal{L}.$$

4.6.2 Remark. It is known that a nondegenerate invariant bilinear form on $\widehat{\mathcal{L}}$ is unique up to nonzero scalar. We may view $\widehat{\mathcal{L}}$ as a subalgebra in the split Kac-Moody Lie algebra $\widehat{L}(\mathfrak{g})_S$. The last one also admits a nondegenerate invariant bilinear form and it is known that its restriction to $\widehat{\mathcal{L}}$ is nondegenerate. Hence this restriction is proportional to the form $(-,-)$.

Let $\mathfrak{h}_{\overline{0}}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}_{\overline{0}}$.
4.6.3 Lemma. *The centralizer of $\mathfrak{h}_{\overline{0}}$ in $\mathfrak{g}$ is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.*

Proof. See [Kac, Lemma 8.1]. □

The algebra $\mathcal{H} = \mathfrak{h}_{\overline{0}} \oplus kc \oplus kd$ plays the role of Cartan subalgebra for $\widehat{\mathcal{L}}$. With respect to $\mathcal{H}$ our algebra $\widehat{\mathcal{L}}$ admits a root space decomposition. The roots are of two types: anisotropic (real) or isotropic (imaginary). This terminology comes from transferring the form to $\mathcal{H}^*$ and computing the "length" of the roots.

The *core* $\widetilde{\mathcal{L}}$ of $\widehat{\mathcal{L}}$ is the subalgebra generated by all the anisotropic roots. In our case we have $\widetilde{\mathcal{L}} = \mathcal{L} \oplus kc$. The correct way to recover $\mathcal{L}$ inside $\widehat{\mathcal{L}}$ is as its core modulo its centre.[3]

If $\mathfrak{m} \subset \widehat{\mathcal{L}}$ is an abelian subalgebra and $\alpha \in \mathfrak{m}^* = \mathrm{Hom}(\mathfrak{m},k)$ we denote the corresponding eigenspace in $\widehat{\mathcal{L}}$ (with respect to the adjoint representation of

[3] In nullity one the core coincides with the derived algebra, but this is not necessarilty true in higher nullities.

$\widehat{\mathcal{L}}$) by $\widehat{\mathcal{L}}_\alpha$. Thus,

$$\widehat{\mathcal{L}}_\alpha = \{\, l \in \widehat{\mathcal{L}} \mid [x,l]_{\widehat{\mathcal{L}}} = \alpha(x)l \text{ for all } x \in \mathfrak{m}\,\}.$$

The subalgebra $\mathfrak{m}$ is called *diagonalizable* in $\widehat{\mathcal{L}}$ if

$$\widehat{\mathcal{L}} = \bigoplus_{\alpha\in\mathfrak{m}^*} \widehat{\mathcal{L}}_\alpha.$$

Every diagonalizable subalgebra of $\mathfrak{m} \subset \widehat{\mathcal{L}}$ is necessarily abelian. We say that $\mathfrak{m}$ is a maximal (abelian) diagonalizable subalgebra (MAD) if it is not properly contained in a larger diagonalizable subalgebra of $\widehat{\mathcal{L}}$.

4.6.4 Remark. Every MAD of $\widehat{\mathcal{L}}$ contains the center kc of $\widehat{\mathcal{L}}$.

4.6.5 Example. The subalgebra $\mathcal{H}$ is a MAD in $\widehat{\mathcal{L}}$ (see [Kac, Theorem 8.5]).

Our aim is to show that an arbitrary maximal diagonalizable subalgebra $\mathfrak{m} \subset \widehat{\mathcal{L}}$ is conjugate to $\mathcal{H}$ under an element of $\mathrm{Aut}_k(\widehat{\mathcal{L}})$. For future reference we record the following facts:

4.6.6 Theorem. (a) *Every diagonalizable subalgebra in $\mathcal{L}$ is contained in a MAD of $\mathcal{L}$ and all MADs of $\mathcal{L}$ are conjugate. More precisely, let $\mathbf{G}$ be the simple simply connected group scheme over R corresponding to $\mathcal{L}$. Then for any MAD $\mathfrak{m}$ of $\mathcal{L}$ there exists $g \in \mathbf{G}(R)$ such that $Ad(g)(\mathfrak{m}) = \mathfrak{h}_{\overline{0}}$.*

(b) *There exists a natural bijection between MADs of $\widetilde{\mathcal{L}}$ and MADs of $\mathcal{L}$. Every diagonalizable subalgebra in $\widetilde{\mathcal{L}}$ is contained in a MAD of $\widetilde{\mathcal{L}}$. All MADs of $\widetilde{\mathcal{L}}$ are conjugate by elements in $Ad(\mathbf{G}(R)) \subset \mathrm{Aut}_k(\mathcal{L}) \simeq \mathrm{Aut}_k(\widetilde{\mathcal{L}})$.*

(c) *The image of the canonical map $\mathrm{Aut}_k(\widehat{\mathcal{L}}) \to \mathrm{Aut}_k(\widetilde{\mathcal{L}}) \simeq \mathrm{Aut}_k(\mathcal{L})$ obtained by restriction to the derived subalgebra $\widetilde{\mathcal{L}}$ contains $\mathrm{Aut}_{R-Lie}(\mathcal{L})$.*

Proof. (a) From the explicit realization of $\mathcal{L}$ one knows that $\mathfrak{h}_{\overline{0}}$ is a MAD of $\mathcal{L}$. Now (a) follows from [CGP].

(b) The correspondence follows from the fact that every MAD of $\widetilde{\mathcal{L}}$ contains kc. A MAD $\widetilde{\mathfrak{m}}$ of $\widetilde{\mathcal{L}}$ is necessarily of the form $\mathfrak{m} \oplus kc$ for some MAD $\mathfrak{m}$ of $\mathcal{L}$ and conversely. The canonical map $\mathrm{Aut}_k(\widetilde{\mathcal{L}}) \to \mathrm{Aut}_k(\mathcal{L})$ is an isomorphism by Proposition 4.3.2.

(c) This was established in Theorem 6.4.1.

$\square$

4.6.7 Lemma. *If $\mathfrak{m} \subset \widehat{\mathcal{L}}$ is a MAD of $\widehat{\mathcal{L}}$ then $\mathfrak{m} \not\subset \widetilde{\mathcal{L}}$.*

Proof. Assume that $\mathfrak{m} \subset \widetilde{\mathcal{L}}$. By Theorem 4.6.6 (b), there exists a MAD $\mathfrak{m}'$ of $\widetilde{\mathcal{L}}$ containing $\mathfrak{m}$. Applying again Theorem 4.6.6 we may assume that up to conjugation by an element of $\mathrm{Aut}_k(\widehat{\mathcal{L}})$, in fact of $\widehat{\mathbf{G}}(R)$, we have $\mathfrak{m} \subset \mathfrak{m}' = \mathfrak{h}_{\overline{0}} \oplus kc$. Then $\mathfrak{m}$ is a proper subalgebra of the MAD $\mathcal{H}$ of $\widehat{\mathcal{L}}$ and this contradicts the maximality of $\mathfrak{m}$. $\square$

In the next three sections we are going to prove some preliminary results related to a subalgebra $\widehat{A}$ of the twisted affine Kac-Moody Lie algebra $\widehat{\mathcal{L}}$ which satisfies the following two conditions:

a) *$\widehat{A}$ is of the form $\widehat{A} = A \oplus kc \oplus kd$, where A is an R-subalgebra of $\mathcal{L}$ such that $A \otimes_R K$ is a semisimple Lie algebra over K where $K = k(t)$ is the fraction field of R.*

b) *The restriction to $\widehat{A}$ of the nondegenerate invariant bilinear form $(-,-)$ of $\widehat{\mathcal{L}}$ is nondegenerate.*

In particular, all these results will be valid for $\widehat{A} = \widehat{\mathcal{L}}$.

4.7 Weights of semisimple operators and their properties

Let $x = x' + d \in \widehat{A}$ where $x' \in A$. It induces a k-linear operator

$$ad(x) : \widehat{A} \to \widehat{A}, \quad y \to ad(x)(y) = [x, y]_{\widehat{A}}.$$

We say that x is a k-diagonalizable element of $\widehat{A}$ if $\widehat{A}$ has a k-basis consisting of eigenvectors of $ad(x)$. Throughout we assume that $x' \neq 0$ and that x is k-diagonalizable.

For any scalar $w \in k$ we let

$$\widehat{A}_w = \{\, y \in \widehat{A} \mid [x, y]_{\widehat{A}} = wy \,\}.$$

We say that w is a *weight* (= eigenvalue) of $ad(x)$ if $\widehat{A}_w \neq 0$. More generally, if O is a diagonalizable linear operator of a vector space V over k (of main interest to us are the vector spaces $\widehat{A}$, $\widetilde{A} = A \oplus kc$, A) and if w is its eigenvalue following standard practice we will denote by $V_w \subset V$ the corresponding eigenspace of O.

4.7.1 Lemma. *(a) If w is a nonzero weight of $ad(x)$ then $\widehat{A}_w \subset \widetilde{A}$.*

(b) $\widehat{A}_0 = \widetilde{A}_0 \oplus \langle\, x \,\rangle$.

Proof. Clearly we have $[\widehat{A}, \widehat{A}] \subset \widetilde{A}$ and this implies $ad(x)(\widetilde{A}) \subset \widetilde{A}$. It then follows that the linear operator $ad(x)|_{\widetilde{A}}$ is k-diagonalizable. Let $\widetilde{A} = \oplus\, \widetilde{A}_{w'}$ where the sum is taken over all weights of $ad(x)|_{\widetilde{A}}$. Since $x \in \widehat{A}_0$ and since $\widehat{A} = \langle\, x \,\rangle \oplus \widetilde{A}$ we conclude that

$$\widehat{A} = \langle\, x, \widetilde{A}_0 \,\rangle \oplus (\oplus_{w' \neq 0} \widetilde{A}_{w'}),$$

so that the result follows. $\square$

The operator $ad(x)|_{\widetilde{A}}$ maps the center $\langle\, c \,\rangle = kc$ of $\widetilde{A}$ into itself, hence it induces a linear operator O_x of $A \simeq \widetilde{A}/kc$ which is also k-diagonalizable. The last isomorphism is induced by a natural (projection) mapping $\lambda : \widetilde{A} \to A$. If $w \neq 0$ the restriction of λ to $\widetilde{A}_w$ is injective (because $\widetilde{A}_w$ does not contain kc). Since $\widetilde{A} = \oplus_w \widetilde{A}_w$ it then follows that

$$\lambda|_{\widetilde{A}_w} : \widetilde{A}_w \longrightarrow A_w$$

is an isomorphism for $w \neq 0$. Thus the three linear operators $ad(x)$, $ad(x)|_{\widetilde{A}}$ and O_x have the same nonzero weights.

4.7.2 Lemma. *Let $w \neq 0$ be a weight of O_x and let $n \in \mathbb{Z}$. Then $w + mn$ is also a weight of O_x and $A_{w+mn} = t^n A_w$.*

Proof. Assume $y \in A_w \subset A$, hence $O_x(y) = wy$. Let us show that $yt^n \in A_{w+mn}$. We have

$$O_x(yt^n) = \lambda(ad(x)(yt^n)) = \lambda([x, yt^n]_{\widehat{A}}). \tag{4.7.2.1}$$

Substituting $x = x' + d$ we get

$$[x, yt^n]_{\widehat{A}} = [x', yt^n]_{\widehat{A}} + [d, yt^n]_{\widehat{A}}$$

Applying (4.6.0.1) and (4.6.0.2) we get that the right hand side is equal to

$$[x', y]\, t^n + ac + [d, y]_{\widehat{A}}\, t^n + mnyt^n$$

where $a \in k$ is some scalar. Substituting this into (4.7.2.1) we get

$$\begin{aligned} O_x(yt^n) &= \lambda([x', y]\, t^n + ac + [d, y]_{\widehat{A}}\, t^n + mnyt^n) \\ &= [x', y]\, t^n + \lambda([d, y]_{\widehat{A}}\, t^n) + mnyt^n \end{aligned}$$

By (4.6.0.1) there exists $b \in k$ such that

$$[x', y]\, t^n = ([x', y]_{\widehat{A}} + bc)\, t^n.$$

Here we view $[x', y]\, t^n$ as an element in $\widehat{A}$. Therefore

$$\begin{aligned} O_x(yt^n) &= mnyt^n + \lambda(([x', y]_{\widehat{A}} + bc)\, t^n + [d, y]_{\widehat{A}}\, t^n) \\ &= mnyt^n + \lambda(([x, y]_{\widehat{A}} + bc)\, t^n). \end{aligned}$$

We now note that by construction $[x, y]_{\widehat{A}} + bc$ is contained in $A \subset \widetilde{A}$. Hence

$$\lambda(([x, y]_{\widehat{A}} + bc)t^n) = \lambda([x, y]_{\widehat{A}} + bc)\, t^n = \lambda([x, y]_{\widehat{A}}))\, t^n.$$

Since $\lambda([x, y]_{\widehat{A}}) = O_x(y) = wy$ we finally get

$$O_x(yt^n) = mnyt^n + wyt^n = (w + mn)yt^n.$$

Thus we have showed that $A_w t^n \subset A_{w+nm}$. By symmetry $A_{w+nm} t^{-n} \subset A_w$ and we are done. $\square$

We now consider the case $w = 0$.

4.7.3 Lemma. *Assume that* $\dim \widetilde{A}_0 > 1$ *and* $n \in \mathbb{Z}$. *Then* mn *is a weight of* $ad(x)$.

Proof. Since $\dim \widetilde{A}_0 > 1$ there exists nonzero $y \in A$ such that $[x, y]_{\widehat{A}} = 0$. Then the same computations as above show that $[x, yt^n]_{\widetilde{A}} = mnyt^n$. $\square$

Our next aim is to show that if w is a weight of $ad(x)$ so is $-w$. We remind the reader that $\widehat{A}$ is equipped with the nondegenerate invariant bilinear form $(-,-)$. Hence for all $y, z \in \widehat{A}$ one has

$$([x, y]_{\widehat{A}}, z) = -(y, [x, z]_{\widehat{A}}). \tag{4.7.3.1}$$

4.7.4 Lemma. *If w is a weight of $ad(x)$ then so is $-w$.*

Proof. If $w = 0$ there is nothing to prove. Assume $w \neq 0$. Consider the root space decomposition

$$\widehat{A} = \bigoplus_{w'} \widehat{A}_{w'}.$$

It suffices to show that for any two weights w_1, w_2 of $ad(x)$ such that $w_1 + w_2 \neq 0$ the subspaces $\widehat{A}_{w_1}$ and $\widehat{A}_{w_2}$ are orthogonal to each other. Indeed, the last implies that if $-w$ were not a weight then every element in $\widehat{A}_w$ would be orthogonal to all elements in $\widehat{A}$, which is impossible.

Let $y \in \widehat{A}_{w_1}$ and $z \in \widehat{A}_{w_2}$. Applying (4.7.3.1) we have

$$w_1(y, z) = ([x, y]_{\widehat{A}}, z) = -(y, [x, z]_{\widehat{A}}) = -w_2(y, z).$$

Since $w_1 \neq -w_2$ we conclude $(y, z) = 0$. $\square$

Now we switch our interest to the operator O_x and its weight subspaces. Since the nonzero weights of $ad(x)$, $ad(x)|_{\widetilde{A}}$ and O_x are the same we obtain, by Lemmas 4.7.2 and 4.7.3, that for every weight w of O_x all elements in the set

$$\{\, w + mn \mid n \in \mathbb{Z} \,\}$$

are also weights of O_x. We call this set of weights by *w-series*. Recall that by Lemma 4.7.2 we have

$$A_{w+mn} = A_w t^n.$$

4.7.5 Lemma. *Let w be a weight of O_x and let $A_w R$ be the R-span of A_w in A. Then the natural map $\nu : A_w \otimes_k R \to A_w R$ given by $l \otimes t^n \mapsto l t^n$ is an isomorphism of k-vector spaces.*

Proof. Clearly, the sum $\sum_n A_{w+mn}$ of vector subspaces A_{w+mn} in A is a direct sum. Hence

$$A_w R = \sum_n A_w t^n = \sum_n A_{w+mn} = \bigoplus_n A_{w+mn} \tag{4.7.5.1}$$

Fix a k-basis $\{e_i\}$ of A_w. Then $\{e_i \otimes t^j\}$ is a k-basis of $A_w \otimes_k R$. Since

$$\nu(e_i \otimes t^n) = e_i t^n \in A_{w+mn}$$

the injectivity of ν easily follows from (4.7.5.1). The surjectivity is also obvious.

□

Notation: We will denote the R-span $A_w R$ by $A_{\{w\}}$.

By our construction $A_{\{w\}}$ is an R-submodule of A and

$$A = \bigoplus_w A_{\{w\}} \tag{4.7.5.2}$$

where the sum is taken over fixed representatives of weight series.

4.7.6 Corollary. $\dim_k A_w < \infty$.

Proof. Indeed, by the above lemma we have

$$\dim_k A_w = \operatorname{rank}_R (A_w \otimes_k R) = \operatorname{rank}_R A_w R = \operatorname{rank}_R A_{\{w\}} \leq \operatorname{rank}_R A < \infty,$$

as required. □

4.7.7 Corollary. *There are finitely many weight series.*

Proof. This follows from the fact that A is a free R-module of finite rank. □

4.7.8 Lemma. *Let w_1, w_2 be weights of O_x. Then $[A_{w_1}, A_{w_2}] \subset A_{w_1+w_2}$.*

Proof. This is straightforward to check. □

4.8 Weight zero subspace

4.8.1 Theorem. $A_0 \neq 0$.

Proof. Assume that $A_0 = 0$. Then, by Lemma 4.7.2, $A_{mn} = 0$ for all $n \in \mathbb{Z}$. It follows that for any weight w, any integer n and all $y \in A_w$, $z \in A_{-w+mn}$ we have $[y, z] = 0$. Indeed

$$[A_w, A_{-w+mn}] \subset A_{w+(-w)+mn} = A_{mn} = 0. \tag{4.8.1.1}$$

For $y \in A$ the operator $ad(y) : A \to A$ may be viewed as a k-operator or as an R-operator. When we deal with the Killing form $\langle -, - \rangle$ on the R-Lie algebra A we will view $ad(y)$ as an R-operator of A.

4.8.2 Lemma. *Let w_1, w_2 be weights of $ad(x)$ such that $\{w_1\} \neq \{-w_2\}$. Then for any integer n and all $y \in A_{w_1}$ and $z \in A_{w_2+mn}$ we have $\langle y, z \rangle = 0$.*

Proof. Let w be a weight of $ad(x)$. By our condition we have $\{w\} \neq \{w + w_1 + w_2\}$. Since $(ad(y) \circ ad(z))(A_{\{w\}}) \subset A_{\{w+w_1+w_2\}}$, in any R-basis of A corresponding to the decomposition (4.7.5.2) the operator $ad(y) \circ ad(z)$ has zeroes on the diagonal, hence $\mathrm{Tr}\,(ad(y) \circ ad(z)) = 0$. □

4.8.3 Lemma. *Let w be a weight of $ad(x)$, n be an integer and let $y \in A_w$. Assume that $ad(y)$ viewed as an R-operator of A is nilpotent. Then for every $z \in A_{-w+mn}$ we have $\langle y, z \rangle = 0$.*

Proof. Indeed, let l be such that $(ad(y))^l = 0$. Since by (4.8.1.1), $ad(y)$ and $ad(z)$ are commuting operators we have

$$(ad(y) \circ ad(z))^l = (ad(y))^l \circ (ad(z))^l = 0.$$

Therefore $ad(y) \circ ad(z)$ is nilpotent and this implies its trace is zero. □

Since the Killing form is nondegenerate, it follows immediately from the above two lemmas that for every nonzero element $y \in A_w$ the operator $ad(y)$ is not nilpotent. Recall that by Lemma 4.7.8 we have $ad(y)(A_{w'}) \subset A_{w+w'}$. Hence taking into consideration Corollary 4.7.7 we conclude that there exits a weight w' and a positive integer l such that

$$ad(y)(A_{\{w'\}}) \neq 0,\ (ad(y) \circ ad(y))(A_{\{w'\}}) \neq 0, \ldots, (ad(y))^l(A_{\{w'\}}) \neq 0$$

and $(ad(y)^l(A_{\{w'\}}) \subset A_{\{w'\}}$. We may assume that l is the smallest positive integer satisfying these conditions. Then all consecutive scalars

$$w',\ w'+w,\ w'+2w,\dots,w'+lw \tag{4.8.3.1}$$

are weights of $ad(x)$, $\{w'+iw\} \neq \{w'+(i+1)w\}$ for $i < l$ and $\{w'\} = \{w'+lw\}$. In particular, we automatically get that lw is an integer (divisible by m) which in turn implies that w is a rational number.

Thus, under our assumption $A_0 = 0$ we have proved that all weights of $ad(x)$ are rational numbers. We now choose (in a unique way) representatives $w_1, \dots, w_s$ of all weight series such that $0 < w_i < m$ and up to renumbering we may assume that

$$0 < w_1 < w_2 < \cdots < w_s < m.$$

4.8.4 Remark. Recall that for any weight w_i, the scalar $-w_i$ is also a weight. Since $0 < -w_i+m < m$ the representative of the weight series $\{-w_i\}$ is $m-w_i$. Then the inequality $m - w_i \geq w_1$ implies $m - w_1 \geq w_i$. Hence out of necessity we have $w_s = m - w_1$.

We now apply the observation (4.8.3.1) to the weight $w = w_1$. Let $w' = w_i$ be as in (4.8.3.1). Choose the integer $j \geq 0$ such that $w_i + jw_1$, $w_i + (j+1)w_1$ are weights and $w_i+jw_1 < m$, but $w_i+(j+1)w_1 \geq m$. We note that since m is not a weight of $ad(x)$ we automatically obtain $w_i + (j+1)w_1 > m$. Furthermore, we have $w_i + jw_1 \leq w_s = m - w_1$ (because $w_i + jw_1$ is a weight of $ad(x)$). This implies

$$m < w_i + (j+1)w_1 \leq w_s + w_1 = m - w_1 + w_1 = m$$

– a contradiction that completes the proof of the theorem. □

4.9 A lower bound of dimensions of MADs in $\widehat{\mathcal{L}}$

4.9.1 Theorem. *Let $\mathfrak{m} \subset \widehat{\mathcal{L}}$ be a MAD. Then $\dim \mathfrak{m} \geq 3$.*

By Lemma 4.6.7, $\mathfrak{m}$ contains an element x of the form $x = x' + d$ where $x' \in \mathcal{L}$ and it also contains c. Since x and c generate a subspace of $\mathfrak{m}$ of dimension 2 the statement of the theorem is equivalent to $\langle\, x, c\,\rangle \neq \mathfrak{m}$.

Assume the contrary: $\langle\, x, c\,\rangle = \mathfrak{m}$. Since $\mathfrak{m}$ is k-diagonalizable we have the weight space decomposition

$$\widehat{\mathcal{L}} = \bigoplus_{\alpha} \widehat{\mathcal{L}}_{\alpha}$$

where the sum is taken over linear mappings $\alpha \in \mathfrak{m}^* = \operatorname{Hom}(\mathfrak{m}, k)$. To find a contradiction we first make some simple observations about the structure of the corresponding eigenspace $\widehat{\mathcal{L}}_0$.

If $\widehat{\mathcal{L}}_{\alpha} \neq 0$, it easily follows that $\alpha(c) = 0$ (because c is in the center of $\widehat{\mathcal{L}}$). Then α is determined uniquely by the value $w = \alpha(x)$ and so instead of $\widehat{\mathcal{L}}_{\alpha}$ we will write $\widehat{\mathcal{L}}_w$.

Recall that by Theorem 4.8.1, $\mathcal{L}_0 \neq 0$. Our aim is first to show that $\mathcal{L}_0$ contains a nonzero element y such that the adjoint operator $ad(y)$ of $\mathcal{L}$ is k-diagonalizable. We will next see that y necessarily commutes with x viewed as an element in $\widehat{\mathcal{L}}$ and that it is k-diagonalizable in $\widehat{\mathcal{L}}$ as well. It then follows that the subspace in $\widehat{\mathcal{L}}$ spanned by c, x and y is a commutative k-diagonalizable subalgebra and this contradicts the fact that $\mathfrak{m}$ is a MAD.

4.9.2 Lemma. *Let $y \in \mathcal{L}$ be nonzero such that $O_x(y) = 0$. Then $[x, y]_{\widehat{\mathcal{L}}} = 0$.*

Proof. Assume that $[x, y]_{\widehat{\mathcal{L}}} = bc \neq 0$. Then

$$(x, [x, y]_{\widehat{\mathcal{L}}}) = (x, bc) = (x' + d, bc) = (d, bc) = \beta b \neq 0.$$

On the other hand, since the form is invariant we get

$$(x, [x, y]_{\widehat{\mathcal{L}}}) = ([x, x]_{\widehat{\mathcal{L}}}, y) = (0, y) = 0$$

– a contradiction which completes the proof. □

4.9.3 Lemma. *Assume that $y \in \mathcal{L}_0$ is nonzero and that the adjoint operator $ad(y)$ of $\mathcal{L}$ is k-diagonalizable. Then $ad(y)$ viewed as an operator of $\widehat{\mathcal{L}}$ is also k-diagonalizable.*

Proof. Choose a k-basis $\{\, e_i \,\}$ of $\mathcal{L}$ consisting of eigenvectors of $ad(y)$. Thus we have $[y, e_i] = u_i e_i$ where $u_i \in k$ and hence

$$[y, e_i]_{\widehat{\mathcal{L}}} = u_i e_i + b_i c$$

where $b_i \in k$.

Case 1: Suppose first that $u_i \neq 0$. Let

$$\tilde{e}_i = e_i + \frac{b_i}{u_i} \cdot c \in \widetilde{\mathcal{L}}.$$

Then we have

$$[y, \tilde{e}_i]_{\widehat{\mathcal{L}}} = [y, e_i]_{\widehat{\mathcal{L}}} = u_i e_i + b_i c = u_i \tilde{e}_i$$

and therefore $\tilde{e}_i$ is an eigenvector of the operator $ad(y) : \widehat{\mathcal{L}} \to \widehat{\mathcal{L}}$.

Case 2: Let now $u_i = 0$. Then $[y, e_i]_{\widehat{\mathcal{L}}} = b_i c$ and we claim that $b_i = 0$. Indeed, we have

$$(x, [y, e_i]_{\widehat{\mathcal{L}}}) = ([x, y]_{\widehat{\mathcal{L}}}, e_i) = (0, e_i) = 0$$

and on the other hand

$$(x, [y, e_i]_{\widehat{A}}) = (x, b_i c) = (x' + d, b_i c) = (d, b_i c) = \beta b_i.$$

It follows that $b_i = 0$ and thus $\tilde{e}_i = e_i$ is an eigenvector of $ad(y)$.

Summarizing, replacing e_i by $\tilde{e}_i$ we see that the set $\{\, \tilde{e}_i \,\} \cup \{\, c, x \,\}$ is a k-basis of $\widehat{\mathcal{L}}$ consisting of eigenvectors of $ad(y)$. □

4.9.4 Proposition. *The subalgebra $\mathcal{L}_0$ contains an element y such that the operator $ad(y) : \mathcal{L} \to \mathcal{L}$ is k-diagonalizable.*

Proof. We split the proof in three steps.

Step 1: Assume first that there exists $y \in \mathcal{L}_0$ which as an element in $\mathcal{L}_K = \mathcal{L} \otimes_R K$ is semisimple. We claim that our operator $ad(y)$ is k-diagonalizable. Indeed, choose representatives $w_1 = 0, w_2, \ldots, w_l$ of the weight series of $ad(x)$. The sets $\mathcal{L}_{w_1}, \ldots, \mathcal{L}_{w_l}$ are vector spaces over k of finite dimension, by Lemma 4.7.6, and they are stable with respect to $ad(y)$ (because $y \in \mathcal{L}_0$). In each k-vector space $\mathcal{L}_{w_i}$ choose a Jordan basis

$$\{e_{ij},\ j = 1, \ldots, l_i\}$$

of the operator $ad(y)|_{\mathcal{L}_{w_i}}$. Then the set

$$\{\, e_{ij},\ i = 1, \cdots, l,\ j = 1, \ldots, l_i \,\} \tag{4.9.4.1}$$

is an R-basis of $\mathcal{L}$, by Lemma 4.7.5 and the decomposition given in (4.7.5.2). It follows that the matrix of the operator $ad(y)$ viewed as a K-operator of $\mathcal{L} \otimes_R K$ is a block diagonal matrix whose blocks corresponds to the matrices of $ad(y)|_{\mathcal{L}_{w_i}}$ in the basis $\{e_{ij}\}$. Hence (4.9.4.1) is a Jordan basis for $ad(y)$ viewed as an operator on $\mathcal{L} \otimes_R K$. Since y is a semisimple element of $\mathcal{L} \otimes_R K$ all matrices of $ad(y)|_{\mathcal{L}_{w_i}}$ are diagonal and this in turn implies that $ad(y)$ is k-diagonalizable operator of $\mathcal{L}$.

Step 2: We next consider the case when all elements in $\mathcal{L}_0$ viewed as elements of the R-algebra $\mathcal{L}$ are nilpotent. Then $\mathcal{L}_0$, being finite dimensional, is a nilpotent Lie algebra over k. In particular its center is nontrivial since $\mathcal{L}_0 \neq 0$. Let $c \in \mathcal{L}_0$ be a nonzero central element of $\mathcal{L}_0$. For any $z \in \mathcal{L}_0$ the operators $ad(c)$ and $ad(z)$ of $\mathcal{L}$ commute. Then $ad(z) \circ ad(c)$ is nilpotent, hence $\langle c, z \rangle = 0$. Furthermore, by Lemma 4.8.2 $\langle c, z \rangle = 0$ for any $z \in \mathcal{L}_{w_i}$, $w_i \neq 0$. Thus $c \neq 0$ is in the radical of the Killing form of $\mathcal{L}$ – a contradiction.

Step 3: Assume now that $\mathcal{L}_0$ contains an element y which as an element of $\mathcal{L}_K$ has nontrivial semisimple part y_s. Let us first show that $y_s \in \mathcal{L}_{\{0\}} \otimes_R K$ and then that $y_s \in \mathcal{L}_0$. By Step 1, the last would complete the proof of the proposition.

By decomposition (4.7.5.2) applied to $A = \mathcal{L}$ we may write y_s as a sum

$$y_s = y_1 + y_2 + \cdots + y_l$$

where $y_i \in \mathcal{L}_{\{w_i\}} \otimes_R K$. In Step 1 we showed that in an appropriate R-basis (4.9.4.1) of $\mathcal{L}$ the matrix of $ad(y)$ is block diagonal whose blocks correspond to the Jordan matrices of $ad(y)|_{\mathcal{L}_{w_i}} : \mathcal{L}_{w_i} \to \mathcal{L}_{w_i}$. It follows that the semisimple part of $ad(y)$ is also a block diagonal matrix whose blocks are semisimple parts of $ad(y)|_{A_{w_i}}$.

Since $\mathcal{L}_K$ is a semisimple Lie algebra over a perfect field we get that $ad(y_s) = ad(y)_s$. Hence for all weights w_i we have

$$[y_s, \mathcal{L}_{w_i}] \subset \mathcal{L}_{w_i}. \tag{4.9.4.2}$$

On the other hand, for any $u \in \mathcal{L}_{w_i}$ we have

$$ad(y_s)(u) = [y_1, u] + [y_2, u] + \cdots + [y_l, u].$$

Since $[y_j, u] \in \mathcal{L}_{\{w_i+w_j\}} \otimes_R K$, it follows that $ad(y_s)(u) \in \mathcal{L}_{\{w_i\}}$ if and only if $[y_2, u] = \cdots = [y_l, u] = 0$. Since this is true for all i and all $u \in \mathcal{L}_{w_i}$ and since the kernel of the adjoint representation of $\mathcal{L}_K$ is trivial we obtain $y_2 = \cdots = y_l = 0$. Therefore $y_s \in \mathcal{L}_{\{0\}} \otimes_R K$.

It remains to show that $y_s \in \mathcal{L}_0$. We may write y_s in the form

$$y_s = \frac{1}{g(t)}(u_0 \otimes 1 + u_1 \otimes t + \cdots + u_m \otimes t^m)$$

where $u_0, \cdots, u_l \in \mathcal{L}_0$ and $g(t) = g_0 + g_1 t + \cdots + g_n t^n$ is a polynomial with coefficients $g_0, \ldots, g_n$ in k with $g_n \neq 0$. The above equality can be rewritten in the form

$$g_0 y_s + g_1 y_s \otimes t + \cdots + g_n y_s \otimes t^n = u_0 \otimes 1 + \cdots + u_m \otimes t^m. \qquad (4.9.4.3)$$

Consider an arbitrary index i and let $u \in \mathcal{L}_{w_i}$. Recall that by (4.9.4.2) we have

$$ad(y_s)(\mathcal{L}_{w_i}) \subset \mathcal{L}_{w_i}.$$

Applying both sides of (4.9.4.3) to u and comparing $\mathcal{L}_{w_i+n}$-components we conclude that $[g_n y_s, u] = [u_n, u]$. Since this is true for all u and all i and since the adjoint representation of $\mathcal{L}_K$ has trivial kernel we obtain $g_n y_s = u_n$. Since $g_n \neq 0$ we get $y_s = u_n / g_n \in \mathcal{L}_0$. □

Now we can easily finish the proof of Theorem 4.9.1. Suppose the contrary. Then $\dim(\mathfrak{m}) < 3$ and hence by Lemma 4.6.7 we have $\mathfrak{m} = \langle c, x' + d\rangle$ with $x' \in \mathcal{L}$. Consider the operator O_x on $\mathcal{L}$. By Theorem 4.8.1 we have $\mathcal{L}_0 \neq 0$. By Propositions 4.9.4 and 4.9.3 there exists a nonzero k-diagonalizable element $y \in \mathcal{L}_0$. Clearly, y is not contained in $\mathfrak{m}$. Furthermore, by Lemma 4.9.2, y viewed as an element of $\widehat{\mathcal{L}}$ commutes with $\mathfrak{m}$ and by Lemma 4.9.3 it is k-diagonalizable in $\widehat{\mathcal{L}}$. It follows that the subspace $\mathfrak{m}_1 = \mathfrak{m} \oplus \langle y\rangle$ is an abelian k-diagonalizable subalgebra of $\widehat{\mathcal{L}}$. But this contradicts maximality of $\mathfrak{m}$.

4.10 All MADs are conjugate

4.10.1 Theorem. *Let $\widehat{\mathbf{G}}(R)$ be the preimage of $\{Ad(g) : g \in \widetilde{\mathbf{G}}(R)\}$ under the canonical map $\mathrm{Aut}_k(\widehat{\mathcal{L}}) \to \mathrm{Aut}_k(\mathcal{L})$. Then all MADs of $\widehat{\mathcal{L}}$ are conjugate* *under $\widehat{\mathbf{G}}(R)$ to the subalgebra $\mathcal{H}$ in 4.6.5.*

Proof. Let $\mathfrak{m}$ be a MAD of $\widehat{\mathcal{L}}$. By Lemma 4.6.7, $\mathfrak{m} \not\subset \widetilde{\mathcal{L}}$. Fix a vector $x = x' + d \in \mathfrak{m}$ where $x' \in \mathcal{L}$ and let $\mathfrak{m}' = \mathfrak{m} \cap \mathcal{L}$. Thus we have $\mathfrak{m} = \langle\, x, c, \mathfrak{m}' \,\rangle$. Note that $\mathfrak{m}' \neq 0$, by Theorem 4.9.1. Furthermore, since $\mathfrak{m}'$ is k-diagonalizable in $\mathcal{L}$, without loss of generality we may assume that $\mathfrak{m}' \subset \mathfrak{h}_{\overline{0}}$ given that by Theorem 4.6.6(b) there exists $g \in \mathbf{G}(R)$ such that $Ad(g)(\mathfrak{m}') \subset \mathfrak{h}_{\overline{0}}$ and that by Theorem 6.4.1 g has lifting to $\mathrm{Aut}_{k-Lie}(\widehat{\mathcal{L}})$.

Consider the weight space decomposition

$$\mathcal{L} = \bigoplus_i L_{\alpha_i} \tag{4.10.1.1}$$

with respect to the k-diagonalizable subalgebra $\mathfrak{m}'$ of $\mathcal{L}$ where $\alpha_i \in (\mathfrak{m}')^*$ and as usual

$$L_{\alpha_i} = \{\, z \in \mathcal{L} \mid [t, z] = \alpha_i(t)z \text{ for all } t \in \mathfrak{m}'\}.$$

4.10.2 Lemma. *L_{α_i} is invariant with respect to the operator O_x.*

Proof. The k-linear operator O_x commutes with $ad(t)$ for all $t \in \mathfrak{m}'$ (because x and $\mathfrak{m}'$ commute in $\widehat{\mathcal{L}}$), so the result follows. $\square$

4.10.3 Lemma. *We have $x^{'} \in L_0$.*

Proof. By our construction $\mathfrak{m}'$ is contained in $\mathfrak{h}_0$, hence d commutes with the elements of $\mathfrak{m}'$. But x also commutes with the elements of $\mathfrak{m}'$ and so does $x' = x - d$. $\square$

$L_0 = C_{\mathcal{L}}(\mathfrak{m}')$, being the Lie algebra of the reductive group scheme $C_{\mathbf{G}}(\mathfrak{m}')$ (see [CGP]), is of the form $L_0 = \mathbf{z} \oplus A$ where $\mathbf{z}$ and A are the Lie algebras of the central torus of $C_{\mathbf{G}}(\mathfrak{m}')$ and its semisimple part respectively. Our next goal is to show that $A = 0$.

Suppose this is not true. To get a contradiction we will show that the subset $\widehat{A} = A \oplus kc \oplus kd \subset \widehat{\mathcal{L}}$ is a subalgebra satisfying conditions a) and b) stated at the end of § 4.6 and that it is stable with respect to $ad(x)$. This, in turn, will allow us to construct an element $y \in A$ which viewed as an element of $\widehat{\mathcal{L}}$ commutes with x and $\mathfrak{m}'$ and is k-diagonalizable. The last, of course, contradicts the maximality of $\mathfrak{m}$.

Let $\mathbf{H}$ denote the simple simply connected Chevalley-Demazure algebraic k-group corresponding to $\mathfrak{g}$. Since $\mathbf{G}$ is split over S we have

$$\mathbf{H}_S = \mathbf{H} \times_k S \simeq \mathbf{G}_S = \widetilde{\mathbf{G}} \times_R S.$$

Let $C_{\mathfrak{g}}(\mathfrak{m}') = \mathbf{t} \oplus \mathbf{r}$ where $\mathbf{t}$ is the Lie algebra of the central torus of the reductive k-group $C_{\mathbf{H}}(\mathfrak{m}')$ and $\mathbf{r}$ is the Lie algebra of its semisimple part. Since centralizers commute with base change, we obtain that

$$\mathbf{t}_S = \mathbf{t} \otimes_k S = \mathbf{z} \otimes_R S = \mathbf{z}_S, \quad \mathbf{r}_S = \mathbf{r} \times_k S = A \otimes_R S = A_S.$$

4.10.4 Lemma. *We have $ad(d)(A) \subset A$ and in particular $\widehat{A}$ is a subalgebra of $\widehat{\mathcal{L}}$.*

Proof. Since $\mathbf{r}$ consists of "constant" elements we have $[d, \mathbf{r}]_{\widehat{L}(\mathfrak{g})_S} = 0$, and this implies that $[d, \mathbf{r}_S]_{\widehat{L}(\mathfrak{g})_S} \subset \mathbf{r}_S$. Also, viewing $\mathcal{L}$ as a subalgebra of $\widehat{L}(\mathfrak{g})_S$ we have $[d, \mathcal{L}]_{\widehat{\mathcal{L}}} \subset \mathcal{L}$. Furthermore, S/R is faithfully flat, hence $A = A_S \cap \mathcal{L} = \mathbf{r}_S \cap \mathcal{L}$. Since both subalgebras $\mathbf{r}_S$ and $\mathcal{L}$ are stable with respect to $ad(d)$, so is their intersection. ☐

4.10.5 Lemma. *The restriction of the nondegenerate invariant bilinear form $(-,-)$ on $\widehat{\mathcal{L}}$ to L_0 is nondegenerate.*

Proof. We mentioned before that the restriction of $(-,-)$ to $\mathcal{L}$ is nondegenerate. Hence in view of decomposition (4.10.1.1) it suffices to show that for all $a \in L_0$ and $b \in L_{\alpha_i}$ with $\alpha_i \neq 0$ we have $(a, b) = 0$.

Let $l \in \mathfrak{m}'$ be such that $\alpha_i(l) \neq 0$. Using the invariance of $(-,-)$ we get

$$\alpha_i(l)(a, b) = (a, \alpha_i(l)b) = (a, [l, b]) = ([a, l], b) = 0.$$

Hence $(a, b) = 0$ as required. ☐

4.10.6 Lemma. *The restriction of $(-,-)$ to A is nondegenerate.*

Proof. By lemma(4.10.5) it is enough to show that $\mathbf{z}$ and A are orthogonal in $\widehat{\mathcal{L}}$. Moreover, viewing $\mathbf{z}$ and A as subalgebras of the split affine Kac-Moody Lie algebra $\widehat{L}(\mathfrak{g})_S$ and using Remark 4.6.2 we conclude that it suffices to verify that $\mathbf{z}_S = \mathbf{t}_S$ and $A_S = \mathbf{r}_S$ are orthogonal in $\widehat{L}(\mathfrak{g})_S$.

Let $a \in \mathbf{t}$ and $b \in \mathbf{r}$. We know that

$$(at^{\frac{i}{m}}, bt^{\frac{j}{m}}) = \kappa(a,b)\delta_{i+j,0}$$

where κ is a Killing form of $\mathfrak{g}$. Since $\mathbf{r}$ is a semisimple algebra we have $\mathbf{r} = [\mathbf{r}, \mathbf{r}]$. It follows that we can write b in the form $b = \sum[a_i, b_i]$ for some $a_i, b_i \in \mathbf{r}$. Using the facts that $\mathbf{t}$ and $\mathbf{r}$ commute and that the Killing form is invariant we have

$$\kappa(a,b) = \kappa(a, \sum[a_i, b_i]) = \sum \kappa([a, a_i], b_i) = \sum \kappa(0, b_i) = 0.$$

Thus $(at^{\frac{i}{m}}, bt^{\frac{j}{m}}) = 0$. □

4.10.7 Remark. It follows immediately from Lemma 4.10.6 that the restriction to $\widehat{A}$ of the nondegenerate invariant bilinear form $(-,-)$ is nondegenerate. Indeed, we have $\widehat{A} = A \oplus \langle\, c, d\,\rangle$. We know that the restriction of our form to $\langle\, c, d\,\rangle$ is non-degenerate. By the lemma its restriction to A is also non-degenerate. Since A and $\langle\, c, d\,\rangle$ are orthogonal to each other our assertion follows.

4.10.8 Lemma. *The k-subspace $A \subset \mathcal{L}$ is invariant with respect to O_x.*

Proof. Let $a \in A$. We need to verify that

$$[x, a]_{\widehat{\mathcal{L}}} \in A \oplus kc \subset \widehat{\mathcal{L}}.$$

But $[d, A]_{\widehat{\mathcal{L}}} \subset A + kc$ by Lemma 4.10.4. We also have

$$[x', A]_{\widehat{\mathcal{L}}} \subset A \oplus kc$$

(because $x' \in L_0$, by Lemma 4.10.3, and A viewed as a subalgebra in L_0 is an ideal). Since $x = x' + d$ the result follows. □

According to Lemma 4.10.3 we can write $x' = x'_0 + x'_1$ where $x'_0 \in \mathbf{z}$ and $x'_1 \in A$.

4.10.9 Lemma. *We have $O_x|_A = O_{x'_1+d}|_A$. In particular, the operator $O_{x'_1+d}|_A$ of A is k-diagonalizable.*

Proof. By Lemma 4.10.8, we have $O_x(A) \subset A$. Since O_x is k-diagonalizable (as an operator of $\mathcal{L}$), so is the operator $O_x|_A$ of A. Therefore the last assertion of the lemma follows from the first one.

Let now $a \in A$. Using the fact that x'_0 and a commute in $\mathcal{L}$ we have

$$[x', a]_{\widehat{\mathcal{L}}} = [x'_0, a]_{\widehat{\mathcal{L}}} + [x'_1, a]_{\widehat{\mathcal{L}}} = [x'_1, a]_{\widehat{\mathcal{L}}} + bc$$

for some $b \in k$. Thus $O_x(a) = O_{x'_1+d}(a)$. □

4.10.10 Lemma. *The operator $ad(x'_1 + d) : \widehat{A} \to \widehat{A}$ is k-diagonalizable.*

Proof. Since by Lemma 4.10.9 $O_{x'_1+d}|_A : A \to A$ is k-diagonalizable we can apply the same arguments as in Lemma 4.9.3. □

Now we can produce the required element y. It follows from Lemma 4.10.6 that the Lie algebra $\widehat{A}$ satisfies all the conditions stated at the end of Section 4.6. By Lemma 4.10.10, $ad(x'_1+d)$ is k-diagonalizable operator of $\widehat{A}$. Hence arguing as in Theorem 4.9.1 we see that there exists a nonzero $y \in A$ such that $[y, x'_1 + d]_{\widehat{\mathcal{L}}} = 0$ and $ad(y)$ is a k-diagonalizable operator on $\widehat{A}$. Then by Lemma 4.10.9 we have $O_x(y) = O_{x'_1+d}(y) = 0$ and hence, by Lemma 4.9.2, x and y commute in $\widehat{\mathcal{L}}$.

According to our plan it remains to show that y is k-diagonalizable in $\widehat{\mathcal{L}}$. To see this we need

4.10.11 Lemma. *Let $z \in \mathfrak{m}'$. Then $[z, y]_{\widehat{\mathcal{L}}} = 0$.*

Proof. Since $y \in A \subset C_{\mathcal{L}}(\mathfrak{m}')$ we have $[z, y]_{\mathcal{L}} = 0$. Then $[z, y]_{\widehat{\mathcal{L}}} = bc$ for some $b \in k$. It follows

$$0 = (0, y) = ([x, z]_{\widehat{\mathcal{L}}}, y) = (x, [z, y]_{\widehat{\mathcal{L}}}) = (x^{'} + d, bc) = (d, bc) = \beta b.$$

This yields $b = 0$ as desired. □

4.10.12 Proposition. *The operator $ad(y) : \widehat{\mathcal{L}} \to \widehat{\mathcal{L}}$ is k-diagonalizable.*

Proof. According to Lemma 4.9.3, it suffices to prove that $ad(y) : \mathcal{L} \to \mathcal{L}$ is k-diagonalizable. Since y viewed as an element of A is semisimple it is still semisimple viewed as an element of $\mathcal{L}$. In particular, the R-operator $ad(y) : \mathcal{L} \to \mathcal{L}$ is also semisimple.

Recall that we have the decomposition of $\mathcal{L}$ into the direct sum of the weight spaces with respect to O_x :

$$\mathcal{L} = \bigoplus_w \mathcal{L}_w = \bigoplus_i \bigoplus_n \mathcal{L}_{w_i+mn} = \bigoplus_i \mathcal{L}_{\{w_i\}}.$$

Since y and x commute in $\widehat{\mathcal{L}}$, for all weights w we have $ad(y)(\mathcal{L}_w) \subset \mathcal{L}_w$. If we choose any k-basis of $\mathcal{L}_w$ it is still an R-basis of $\mathcal{L}_{\{w\}} = \mathcal{L}_w \otimes_k R$ and in this basis the R-operator $ad(y)|_{\mathcal{L}_{\{w\}}}$ and the k-operator $ad(y)|_{\mathcal{L}_w}$ have the same matrices. Since the R-operator $ad(y)|_{\mathcal{L}_{\{w\}}}$ is semisimple, so is $ad(y)|_{\mathcal{L}_w}$, i.e. $ad(y)|_{\mathcal{L}_w}$ is a k-diagonalizable operator. Thus $ad(y) : \mathcal{L} \to \mathcal{L}$ is k-diagonalizable. $\square$

Summarizing, assuming $A \neq 0$ we have constructed the k-diagonalizable element

$$y \notin \mathfrak{m} = \langle\, \mathfrak{m}', x, c\, \rangle$$

in $\widehat{\mathcal{L}}$ which commutes with $\mathfrak{m}'$ and x in $\widehat{\mathcal{L}}$. Then the subalgebra $\langle\, \mathfrak{m}, y\, \rangle$ in $\widehat{\mathcal{L}}$ is commutative and k-diagonalizable which is impossible since $\mathfrak{m}$ is a MAD. Thus A is necessarily trivial and this implies $C_{\mathcal{L}}(\mathfrak{m}')$ is the Lie algebra of the R-torus $C_{\mathbf{G}}(\mathfrak{m}')$, in particular $C_{\mathcal{L}}(\mathfrak{m}')$ is abelian.

Note that $x' \in C_{\mathcal{L}}(\mathfrak{m}')$, by Lemma 4.10.3, and that $\mathfrak{h}_{\bar{0}} \subset C_{\mathcal{L}}(\mathfrak{m}')$ (because $\mathfrak{m}' \subset \mathfrak{h}_{\bar{0}}$, by construction). Since $C_{\mathcal{L}}(\mathfrak{m}')$ is abelian and since $x = x' + d$ it follows that $ad(x)(\mathfrak{h}_{\bar{0}}) = 0$. Hence $\langle\, \mathfrak{h}_{\bar{0}}, x, c\, \rangle$ is a commutative k-diagonalizable subalgebra in $\widehat{\mathcal{L}}$. But it contains our MAD $\mathfrak{m}$. Therefore $\mathfrak{m} = \langle\, \mathfrak{h}_{\bar{0}}, x, c\, \rangle$. To finish the proof of Theorem 4.10.1 it now suffices to show that $x' \in \mathfrak{h}_{\bar{0}}$. For that, in turn, we may view x' as an element of $L(\mathfrak{g})_S$ and it suffices to show that $x' \in \mathfrak{h}$ because $\mathfrak{h} \cap \mathcal{L} = \mathfrak{h}_{\bar{0}}$.

4.10.13 Lemma. $x' \in \mathfrak{h}$.

Proof. Consider the root space decomposition of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$:

$$\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha).$$

Every k-subspace $\mathfrak{g}_\alpha$ has dimension 1. Choose a nonzero elements $X_\alpha \in \mathfrak{g}_\alpha$. It follows from $\mathfrak{m}' = \mathfrak{h}_{\bar{0}}$ that $C_{L(\mathfrak{g})_S}(\mathfrak{m}') = \mathfrak{h}_S$. Thus $x' \in \mathfrak{h}_S$. Then $\mathfrak{g}_\alpha \otimes_k S$ is stable with respect to $ad(x')$ and clearly it is stable with respect to $ad(d)$. Hence it is also stable with respect to O_x.

Arguing as in Lemma 4.7.2 one can easily see that the operator O_x, viewed as an operator of $L(\mathfrak{g})_S$, is k-diagonalizable. Since $\mathfrak{g}_\alpha \otimes_k S$ is stable with respect to O_x, it is the direct sum of its weight subspaces. Hence

$$\mathfrak{g}_\alpha \otimes_k S = \bigoplus_w (L(\mathfrak{g})_S)_{\{w\}}$$

where $\{w\} = \{w + j/m \mid j \in \mathbb{Z}\}$ is the weight series corresponding to w. But $\mathfrak{g}_\alpha \otimes_k S$ has rank 1 as an S-module. This implies that in the above decomposition we have only one weight series $\{w\}$ for some weight w of O_x.

We next note that automatically we have $\dim_k (L(\mathfrak{g})_S)_w = 1$. Any its nonzero vector which is a generator of the S-module $\mathfrak{g}_\alpha \otimes_k S$ is of the form $X_\alpha t^{\frac{j}{m}}$. It follows from Lemma 4.7.2 that $\mathfrak{g}_\alpha = \langle X_\alpha \rangle$ is also a weight subspace of O_x. Thus for every root α we have

$$[x, X_\alpha]_{\widehat{L}(\mathfrak{g})_S} = [x' + d, X_\alpha]_{\widehat{L}(\mathfrak{g})_S} = [x', X_\alpha] = b_\alpha X_\alpha$$

for some scalar $b_\alpha \in k$. Since $x' \in \mathfrak{h}_S$ this can happen if and only if $x' \in \mathfrak{h}$. $\square$

By the previous lemma we have $x' \in \mathfrak{h}_{\bar{0}}$, hence

$$\mathfrak{m} = \langle\, \mathfrak{h}_{\bar{0}}, c, d \,\rangle = \mathcal{H}.$$

The proof of Theorem 4.10.1 is complete. $\square$

Chapter 5

Two properties of a core of EALA

Now we go on towards the conjugacy result for EALA. But before we do this, we shall establish some properties of the core of EALA in this chapter.

5.1 Core is the same for two structures

Let (E, H) and (E, H') be two extended affine Lie algebra structures on E and let

$$\pi : E_c \longrightarrow E_{cc}$$

be the natural (quotient) map. Here E_c (resp. E_{cc}) is the core (resp. centreless core) of E with respect to the first structure.

In this section we want to show the following.

5.1.1 Theorem. $E_c = E'_c$.

We split the proof of the proposition in a few steps.

Since E_c and E'_c are ideals of E, $E_c \cap E'_c$ is an ideal of E_c. We realize E as in Theorem 2.4.1:

$$E \simeq L \oplus D^{gr^*} \oplus D,$$

where L is a Lie torus of type (Δ, Λ) (recall that Λ is a free abelian group of finite rank), $D \subset \mathrm{SCDer}_k(L)$ is a graded subalgebra. Using this notation we have

$$E_c \simeq L \oplus D^{gr^*}, \quad E_{cc} \simeq L$$

and π is identified with a canonical projection

$$pr_L : L \oplus D^{gr^*} \longrightarrow L.$$

5.1.2 Lemma. *Let $I = pr_L(E_c \cap E'_c)$. Then $I \neq 0$.*

Proof. Let $e' \in E'_c$ be an arbitrary element. Write it in the form $e' = l+c+d \in E'_c$ where $l \in L$, $c \in D^{gr*}$, $d \in D$. Pick an arbitrary $a \in L$. Then

$$[e', a]_E = [l + c + d, a]_E = [l, a]_E + [c, a]_E + [d, a]_E = ad_L(l)(a) + d(a) + c_1$$

for some $c_1 \in D^{gr*}$.(Recall that $d \in \mathrm{Der}_k(L)$, hence $d(a)$ is well-defined.) It follows that

$$pr_L([e', a]_E) = (ad_L(l) + d)(a).$$

By (2.3.4.1)

$$\mathrm{Der}_k(L) = \mathrm{IDer}_k(L) \rtimes \mathrm{CDer}_k(L).$$

Clearly we have $ad_L(l) \in \mathrm{IDer}_k(L)$ and

$$d \in \mathrm{SCDer}_k(L) \subset \mathrm{CDer}_k(L).$$

Therefore, if $0 \neq l + d \in E$, then there exists $a \in L$ s.t. $pr_L([e', a]_E) \neq 0$.

Notice that E'_c is not a subset of D^{gr*} (since E'_c is not commutative). Hence E'_c contains an element $e' = l + c + d$ with $l + d \neq 0$. As we saw above this in turn implies that $pr_L(E'_c \cap E_c) \neq 0$. □

5.1.3 Lemma. *$d(x) \in I$ for all $d \in D$ and $x \in I$.*

Proof. By the definition of I there exists $c \in D^{gr*}$ such that

$$\widetilde{x} := x + c \in E_c \cap E'_c.$$

Since $E_c \cap E'_c$ is an ideal of E we have $[d, \widetilde{x}]_E \in E_c \cap E'_c$. Hence $pr_L([d, \widetilde{x}]_E) \in I$. On the other side, since $[d, c]_E \in D^{gr*}$ we obtain that

$$pr_L([d, \widetilde{x}]_E) = pr_L([d, x]_E) = pr_L(d(x)) = d(x).$$

The assertion follows. $\square$

5.1.4 Lemma. *I is a graded ideal with respect to the degree grading on L.*

Proof. We argue by induction. Let $l \geq 2$ be a positive integer. Assume that for any element $x' \in I$ containing at most $l-1$ weight components (with respect to the degree grading on L) all these components are contained in I. Let now

$$x = \sum_{j=1}^{l} x^{\lambda_j} \in I$$

where $\lambda_j \in \Lambda$ and $0 \neq x^{\lambda_j} \in L^{\lambda_j}$.

Notice that there exists $d_0 \in D^0$ such that $\mathrm{ev}_{\lambda_1}(d_0) \neq \mathrm{ev}_{\lambda_2}(d_0)$. Indeed, this is obvious since by construction the map

$$\mathrm{ev} : \Lambda \to D^{0*}, \quad \lambda \mapsto ev_\lambda$$

is injective.

According to Lemma 5.1.3 for any $d \in D^0$ we have

$$d(x) = \sum_{j=1}^{l} ev_{\lambda_j}(d) x^{\lambda_j} \in I.$$

Therefore

$$d_0(x) - ev_{\lambda_2}(d_0)x \in I$$

and this element has at most $l-1$ nonzero components with a non-zero component $(ev_{\lambda_1}(d_0) - ev_{\lambda_2}(d_0))x^{\lambda_1}$. By induction hypothesis this implies that $x^{\lambda_1} \in I$. Then $x - x^{\lambda_1} \in I$ and the induction completes the proof. $\square$

5.1.5 Lemma. *$I = L$.*

Proof. Recall that L, being a centreless Lie torus of type (Δ, Λ), is a centreless division (Δ, Λ)-graded Lie algebra. Therefore, by [Yo2, Lemma 4.4], L is a Λ-graded simple. Now the statement of the lemma follows from 5.1.2 and 5.1.4. $\square$

Proof of Proposition 5.1.1. Let us first show that $E_c \subset E_c'$. Since $[L, L]_E = E_c$ by perfectness of E_c, it is enough to see that $L \subset E_c'$.

Let $c \in D^{gr*}$ be an arbitrary element. Since $[L, L]_E = E_c$ we can write c in the form

$$c = \sum_{i=1}^{s} [x_i, y_i]_E$$

where $x_i, y_i \in L$. By Lemma 5.1.5 there exist elements $e_i, f_i \in E_c \cap E'_c$ such that

$$pr_L(e_i) = x_i, \quad pr_L(f_i) = y_i.$$

Let $e_i = x_i + z_i,\ f_i = y_i + t_i$ where $z_i, t_i \in D^{gr*}$. Then

$$[e_i, f_i]_E = [x_i + z_i, y_i + t_i]_E = [x_i, y_i]_E,$$

since $z_i, t_i \in Z(E_c) = C$. Therefore

$$c = \sum_{i=1}^{s} [e_i, f_i]_E \in E_c \cap E'_c.$$

Since $c \in D^{gr*}$ is arbitrary we conclude that $D^{gr*} \subset E'_c$.

Take now arbitrary $l \in L$. Again, by Lemma 5.1.5, there exists $e' \in E_c \cap E'_c$ such that $e' = l + c$ with $c \in D^{gr*}$. But $c \in E'_c$; this implies $l \in E'_c$ and therefore $L \subset E'_c$.

To sum up, we have showed that $E_c = L \oplus D^{gr*} \subset E'_c$. Similarly $E'_c \subset E_c$ and we are done.

5.2 Core is automorphism stable

In this section we draw one important corollary of the Proposition 5.1.1 (which allows us to talk about the core E_c of an EALA E without specifying a choice of a toral subalgebra H).

5.2.1 Corollary. *The core E_c of an EALA E is stable under automorphisms of E, i.e. $\phi(E_c) = E_c$ for any $\phi \in \mathrm{Aut}_{k-Lie}(E)$.*

Proof. Let $\phi \in \mathrm{Aut}_{k-Lie}(E)$. Denote $H' = \phi(H)$. Let $(-|-)'$ be a bilinear form on E given by

$$(x \,|\, y)' = (\phi^{-1}(x) \,|\, \phi^{-1}(y)).$$

Clearly, $(E, H', (-|-)')$ is another structure of an EALA on E. Therefore, by Proposition 5.1.1, we have that the core E'_c of (E, H') is equal to E_c. It remains to show that $E'_c = \phi(E_c)$.

Let $\alpha \in R$ be a root with respect to H. There exists a unique element t_α in H such that $(t_\alpha \,|\, h) = \alpha(h)$ for all $h \in H$. Recall that α is called anisotropic if $(t_\alpha \,|\, t_\alpha) \neq 0$ and that E_c is generated (as an ideal) by $\cup_{\alpha \in R^{an}} E_\alpha$.

Let R' be the set of roots of (E, H'). A mapping ${}^t\phi^{-1}_{|H} : H^* \to H'^*$ satisfies ${}^t\phi^{-1}_{|H}(R) = R'$. Notice that $\phi(t_\alpha) = t_{({}^t\phi)^{-1}(\alpha)}$. Indeed, this follows from

$$(\phi(t_\alpha) \,|\, h')' = (t_\alpha \,|\, \phi^{-1}(h')) = \alpha(\phi^{-1}(h')) = (\alpha \circ \phi^{-1})(h') = {}^t\phi^{-1}(\alpha)(h').$$

We next have

$$(t_{({}^t\phi)^{-1}(\alpha)} \,|\, t_{({}^t\phi)^{-1}(\alpha)})' = (\phi(t_|\alpha) \,|\, \phi(t_\alpha))' = (t_\alpha \,|\, t_\alpha).$$

Therefore,

$${}^t\phi^{-1}(R^{an}) = (R')^{an}, \quad \phi(E_\alpha) = E'_{{}^t\phi^{-1}}(\alpha),$$

and this implies $\phi(E_c) = E'_c = E_c$. $\square$

Chapter 6

Lifting of automorphisms

The corollary 5.2.1 allows us to introduce a well-defined natural map

$$res_{cc} : \mathrm{Aut}_{k-Lie}(E) \to \mathrm{Aut}_{k-Lie}(E_{cc}),$$

which is a composite of a natural restriction map

$$res_c : \mathrm{Aut}_{k-Lie}(E) \longrightarrow \mathrm{Aut}_{k-Lie}(E_c)$$

and a natural map $\pi : \mathrm{Aut}_{k-Lie}(E_c) \to \mathrm{Aut}_{k-Lie}(E_c/Z(E_c))$.

In this chapter we will study the image of this map.

6.1 Lifting automorphisms in the split case

Let E be an extended affine Lie algebra whose centreless core E_{cc} is split. Thus $E = L \oplus C \oplus D$ where $L = \mathfrak{g} \otimes R$, $C = D^{gr*}$, $\mathfrak{g}$ is a simple Lie algebra over an algebraically closed field k of characteristic 0 and $R = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. For the puprose of this chapter we may assume that an invariant form $(-,-)_L$ on L is given by

$$(x \otimes p, y \otimes q)_L = \kappa(x, y)\mathrm{CT}(pq), \quad x, y \in \mathfrak{g}, \ p, q \in R,$$

where CT is a constant term function.

The following theorem is the main result of this section, and it says that all

R-linear automorphisms of $\mathfrak{g} \otimes R$ can be lifted to automorphisms of the entire algebra E, if $\mathfrak{g}$ is not isomorphic to $sl_2(k)$.

6.1.1 Theorem. $\mathrm{Aut}_{R-Lie}(\mathfrak{g} \otimes R)$ *is in the image of*

$$res_{cc} : \mathrm{Aut}_{k-Lie}(E) \longrightarrow \mathrm{Aut}_{k-Lie}(\mathfrak{g} \otimes R),$$

if $\mathfrak{g}$ *is not isomorphic to* $sl_2(k)$.

Proof. Notice that

$$\mathrm{Aut}_{R-Lie}(\mathfrak{g} \otimes R) = \mathbf{G}(R) \rtimes \mathrm{Aut}(\mathrm{Dyn}(\mathfrak{g})),$$

where $\mathbf{G}$ is an adjoint simple group corresponding to $\mathfrak{g}$.

We will proceed in 3 steps.

Let $\widetilde{\mathbf{G}}$ denote a simple simply connected group corresponding to $\mathfrak{g}$.

Step 1. Lifting of automorphisms from the image of a natural map $\widetilde{\mathbf{G}}(R) \to \mathrm{Aut}_{R-Lie}(\mathfrak{g} \otimes R)$.

Choose a maximal split k-torus $\widetilde{\mathbf{T}} \subset \widetilde{\mathbf{G}}$. The Lie algebra of $\widetilde{\mathbf{T}}$ is a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We fix a Borel subgroup $\widetilde{\mathbf{T}} \subset \widetilde{\mathbf{B}} \subset \widetilde{\mathbf{G}}$.

Let $\Sigma = \Sigma(\widetilde{\mathbf{G}}, \widetilde{\mathbf{T}})$ be the root system of $\widetilde{\mathbf{G}}$ relative to $\widetilde{\mathbf{T}}$. The Borel subgroup $\widetilde{\mathbf{B}}$ determines an ordering of Σ, hence the system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. Fix a Chevalley basis [St67]

$$\{H_{\alpha_1}, \ldots H_{\alpha_n},\ X_\alpha,\ \alpha \in \Sigma\}$$

of $\mathfrak{g}$ corresponding to the pair $(\widetilde{\mathbf{T}}, \widetilde{\mathbf{B}})$.

In [Sta], A. Stavrova showed that the group $\widetilde{\mathbf{G}}(R)$ is generated by the so-called root subgroups $U_\alpha = \langle x_\alpha(p) \mid p \in R \rangle$, where $\alpha \in \Sigma$ and

$$x_\alpha(p) = \exp(pX_\alpha) = \sum_{n=0}^{\infty} p^n X_\alpha^n / n!\,. \qquad (6.1.1.1)$$

6.1.2 Lemma. $\mathrm{ad}(X_\alpha \otimes p)$ *is a nilpotent endomorphism of a Lie algebra* E.

Proof. It follows easily from Serre's relations that $\mathrm{ad}(X_\alpha \otimes p)$ is a nilpotent endomorphism of a Lie algebra $\mathfrak{g} \otimes R$. Let $a \in \mathbb{Z}$ be such that

$$(\mathrm{ad}_{\mathfrak{g}\otimes R}(X_\alpha \otimes p))^a = 0.$$

It follows from the multiplication rule on E that $\mathrm{ad}(X_\alpha \otimes p)(e) \in \mathfrak{g} \otimes R \oplus C$ for any $e \in E$. Therefore it suffices to show, that $\mathrm{ad}(X_\alpha \otimes p)|\mathfrak{g} \otimes R \oplus C$ is nilpotent. Since C is the center of $\mathfrak{g} \otimes R \oplus C$ we have $\mathrm{ad}(X_\alpha \otimes p)(c) = 0$ for any $c \in C$. Finally, for any $l \in \mathfrak{g} \otimes R$, we have

$$\begin{aligned}
&(\mathrm{ad}_E(X_\alpha \otimes p))^{a+1}(l) = \\
&= \mathrm{ad}_E(X_\alpha \otimes p)((\mathrm{ad}_E(X_\alpha \otimes p))^a(l)) \\
&= \mathrm{ad}_E(X_\alpha \otimes p)((\mathrm{ad}_{\mathfrak{g}\otimes R}(X_\alpha \otimes p))^a(l) + c) \\
&= \mathrm{ad}_E(X_\alpha \otimes p)(c) = 0,
\end{aligned}$$

where $c \in C$.

This proves the lemma. $\square$

Now we are ready to prove the lifting result.

Since $\widetilde{\mathbf{G}}(R)$ is generated by root subgroups U_α's we can write

$$g = \prod_\alpha \exp(X_\alpha \otimes p_\alpha). \tag{6.1.2.1}$$

By Lemma 6.1.2, $\mathrm{ad}_E(X_\alpha \otimes p_\alpha)$ is a nilpotent operator on E. Therefore the sum

$$\exp(\mathrm{ad}_E(X_\alpha \otimes p_\alpha)) = \sum_{i \geq 0} (\mathrm{ad}(X_\alpha \otimes p_\alpha))^i / i!$$

is finite and, hence $\exp(\mathrm{ad}_E(X_\alpha \otimes p_\alpha)) \in \mathrm{Aut}_{k-Lie}(E)$.

Notice, that for any $i \geq 0$ and $l \in \mathfrak{g} \otimes R$ we have

$$pr_{\mathfrak{g}\otimes R} \circ (\mathrm{ad}_E(X_\alpha \otimes p_\alpha))^i(l) = \mathrm{ad}_{\mathfrak{g}\otimes R}(X_\alpha \otimes p_\alpha))^i(l).$$

Therefore

$$(pr_{\mathfrak{g}\otimes R} \circ \exp(\mathrm{ad}_E(X_\alpha \otimes p_\alpha)))|\mathfrak{g} \otimes R = \exp(\mathrm{ad}_{\mathfrak{g}\otimes R}(X_\alpha \otimes p_\alpha)) = \mathrm{Ad}(\exp(X_\alpha \otimes p_\alpha)).$$

For an element $g \in \widetilde{\mathbf{G}}(R)$ fix its presentation as in 6.1.2.1. Define

$$\mathrm{Ad}_E(g) = \prod_\alpha \exp(\mathrm{ad}_E(X_\alpha \otimes p_\alpha)). \tag{6.1.2.2}$$

Then

$$\begin{aligned}
&(pr_{\mathfrak{g}\otimes R} \circ \mathrm{Ad}_E(g))|\mathfrak{g} \otimes R \\
&= (pr_{\mathfrak{g}\otimes R} \circ \textstyle\prod_\alpha \exp(\mathrm{ad}_E(X_\alpha \otimes p_\alpha)))|\mathfrak{g} \otimes R \\
&= \textstyle\prod_\alpha (pr_{\mathfrak{g}\otimes R} \circ \exp(\mathrm{ad}_E(X_\alpha \otimes p_\alpha)))|\mathfrak{g} \otimes R \\
&= \textstyle\prod_\alpha \mathrm{Ad}(\exp(X_\alpha \otimes p_\alpha)) \\
&= \mathrm{Ad}(\textstyle\prod_\alpha \exp(X_\alpha \otimes p_\alpha)) \\
&= \mathrm{Ad}(g),
\end{aligned}$$

therefore we can lift automorpisms of the form $\mathrm{Ad}(g)$, $g \in \widetilde{\mathbf{G}}(R)$, to automorphisms $\mathrm{Ad}_E(g)$ of E.

Step 2. Lifting of automorphisms from the image of a natural map $\mathbf{G}(R) \to \mathrm{Aut}_{R-Lie}(\mathfrak{g} \otimes R)$.

6.1.3 Lemma. *Let $\pi : \widetilde{\mathbf{G}} \to \mathbf{G}$ be the canonical covering. There exists an étale extension R'/R such that $\mathbf{G}(R) \subset \pi(\widetilde{\mathbf{G}}(R')) \subset \mathbf{G}(R')$.*

Proof. Let $\widetilde{\mathbf{T}} \subset \widetilde{\mathbf{G}}$ be a split maximal torus and $\mathbf{T} = \pi(\widetilde{\mathbf{T}})$ be its image under π. Consider an exact sequence

$$1 \longrightarrow Z \longrightarrow \widetilde{\mathbf{T}} \longrightarrow \mathbf{T} \longrightarrow 1$$

where Z is the kernel of π. For an arbitrary étale extension R'/R it gives rise to an exact cohomological sequence

$$\widetilde{\mathbf{T}}(R') \longrightarrow \mathbf{T}(R') \xrightarrow{\alpha_{R'}} H^1(R', Z) \longrightarrow H^1(R', \widetilde{\mathbf{T}}) = 1.$$

(The last equality is due to the fact that $\widetilde{\mathbf{T}}$ is split and $\mathrm{Pic}(R') = 1$). Since Z has finite exponent, say m, so is $H^1(R', Z)$. Take $R' = k[s_1^{\pm 1}, \ldots, s_n^{\pm 1}]$ with $s_i = t_i^{\frac{1}{m}}$, $i = 1, \ldots, n$.

Consider now a similar exact cohomological sequence

$$\widetilde{\mathbf{G}}(R) \longrightarrow \mathbf{G}(R) \xrightarrow{\beta} H^1(R, Z) \xrightarrow{\gamma} H^1(R, \widetilde{\mathbf{G}}).$$

The image of γ is trivial since γ factors through

$$H^1(R, Z) \longrightarrow H^1(R, \widetilde{\mathbf{T}}) \longrightarrow H^1(R, \widetilde{\mathbf{G}})$$

and as we explained above $H^1(R, \widetilde{\mathbf{T}}) = 1$. Thus α_R and β are surjective and this implies

$$\mathbf{G}(R) = \pi(\widetilde{\mathbf{G}}(R)) \cdot \mathbf{T}(R).$$

Fix a decomposition $\mathbf{T} \simeq \mathbf{G}_m \times \cdots \times \mathbf{G}_m$. It allows us to identify

$$\mathbf{T}(R) \simeq R^\times \times \cdots \times R^\times.$$

It follows that $\mathbf{T}(R)$ is generated by $\mathbf{T}(k)$ and elements of the form

$$(1, \ldots, 1, \prod t_i^{n_i}, 1, \ldots, 1)$$

where n_i are integers. Clearly, all these generators are in $\mathbf{T}(R')^m$, hence $\mathbf{T}(R) \subset \mathbf{T}(R')^m$. Since $\alpha_{R'}(\mathbf{T}(R')^m) = 1$ we get

$$\mathbf{T}(R) \subset \mathbf{T}(R')^m \subset \pi(\widetilde{\mathbf{T}}(R'))$$

and the assertion follows. □

Let R' be as in Lemma 6.1.3. Consider a vector space

$$E_{R'} := \mathfrak{g} \otimes R' \oplus D^{gr*} \oplus D.$$

We want to equip it with a structure of an extended affine Lie algebra such that E becomes a Lie subalgebra of $E_{R'}$. First we extend a non-degenerate symmetric invariant bilinear form on $\mathfrak{g} \otimes R$ to such a form on $\mathfrak{g} \otimes R'$, putting

$$(x \otimes s^m | y \otimes s^n)_{\mathfrak{g}\otimes R'} := \kappa(x, y)\delta_{m,-n}.$$

One checks immediately that $(x \,|\, y)_{\mathfrak{g}\otimes R'}$ is a non-degenerate symmetric invariant bilinear form and its restriction to $\mathfrak{g} \otimes R$ coincides with the original form $(-|-)_{\mathfrak{g}\otimes R}$.

Now let us define an action of D on $\mathfrak{g} \otimes R'$ by skew-centroidal derivations. By the Realization Theorem, $D = \oplus_\gamma D^\gamma$ is a graded subalgebra of $\mathrm{SCDer}_k(\mathfrak{g} \otimes R)$.

Any $d^\gamma \in D^\gamma$ is of the form $\chi^\gamma d_\theta$, where $\chi_\gamma \in \mathrm{Ctd}(\mathfrak{g}\otimes R)^\gamma = R^\gamma$, $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, k)$ and

$$d^\gamma(l^\lambda) = \theta(\lambda)\chi^\gamma(l^\lambda)$$

for any $l^\lambda \in (\mathfrak{g}\otimes R)^\lambda$. There is an obvious extension $\theta_{R'} \in \mathrm{Hom}_{\mathbb{Z}}((\frac{1}{m}\mathbb{Z})^n, k)$ of θ. Also, we have an inclusion $R = \mathrm{Ctd}_k(\mathfrak{g}\otimes R) \hookrightarrow R' = \mathrm{Ctd}_k(\mathfrak{g}\otimes R')$. We now define an action of d^γ on $\mathfrak{g}\otimes R'$ as follows: for $l \in (\mathfrak{g}\otimes R')^\lambda$, $\lambda \in (\frac{1}{m}\mathbb{Z})^n$ we let

$$d^\gamma_{R'}(l) = \theta_{R'}(\lambda)\chi^\gamma(l).$$

It follows from the construction that $d^\gamma_{R'}|_{\mathfrak{g}\otimes R} = d^\gamma$. Also, $d^\gamma_{R'}$ is a centroidal derivation, which is still skew-centroidal. Indeed, the criterion for $\chi^\gamma d_{\theta_{R'}}$ to be skew-centroidal derivation is $\theta_{R'}(\gamma) = 0$. This holds, since $\theta_{R'}(\gamma) = \theta(\gamma) = 0$, the latter because $\chi^\gamma d_\theta$ is a skew-centroidal derivation of $\mathfrak{g}\otimes R$.

It follows from the Realization Theorem for EALAs that $E_{R'}$ equipped it with the Lie bracket structure

$$\begin{aligned}&[l_1 + c_1 + d_1, l_2 + c_2 + d_2]_{E_{R'}} = ([l_1, l_2]_{\mathfrak{g}\otimes R'} + d_1(l_2) - d_2(l_1)) + \\ &+ (\sigma_{R'}(l_1, l_2) + d_1 \cdot c_2 - d_2 \cdot c_1 + \tau(d_1, d_2)) + ([d_1, d_2]_D),\end{aligned}$$

where $\sigma_{R'}(l_1, l_2)(d) = (d_S(l_1) \,|\, l_2)_{\mathfrak{g}\otimes R'}$ is an EALA.

6.1.4 Lemma. *Let $E \hookrightarrow E_{R'}$ be a natural embedding. Then E is a Lie subalgebra of $E_{R'}$.*

Proof. We need only to show that $\sigma_{R'}|_{\mathfrak{g}\otimes R} = \sigma$. If $d \in D$ and $l_1, l_2 \in \mathfrak{g}\otimes R$ then we have

$$\sigma_{R'}(l_1, l_2)(d) = (d_{R'}(l_1) \,|\, l_2)_{\mathfrak{g}\otimes R'} = (d(l_1) \,|\, l_2)_{\mathfrak{g}\otimes R} = \sigma(l_1, l_2),$$

as required. $\square$

Let $g \in \mathbf{G}(R)$ and $\widetilde{g} \in \widetilde{\mathbf{G}}(R')$, where R' is as in Lemma 6.1.3, be such that $\pi(\widetilde{g}) = g$. Clearly,

$$\mathrm{Ad}_{\mathfrak{g}\otimes R'}(g) = \mathrm{Ad}_{\mathfrak{g}\otimes R'}(\widetilde{g}). \tag{6.1.4.1}$$

We know from *Step 1* that there is a lifting

$$\mathrm{Ad}_{E_{R'}}(\widetilde{g}) \in \mathrm{Aut}_{k-Lie}(E_{R'})$$

of $\mathrm{Ad}_{\mathfrak{g}\otimes R'}(\widetilde{g})$.

6.1.5 Lemma. *One has* $\mathrm{Ad}_{E_{R'}}(\widetilde{g})(E) = E$.

Proof. It follows from the construction in *Step 1* and from (6.1.4.1) that

$$\mathrm{Ad}_{E_{R'}}(\widetilde{g})(\mathfrak{g}\otimes R \oplus D^{gr*}) = \mathfrak{g}\otimes R \oplus D^{gr*}.$$

It remains to check only that

$$\mathrm{Ad}_{E_{R'}}(\widetilde{g})(d) \in \mathfrak{g}\otimes R \oplus D^{gr*} \oplus D.$$

We can write $\mathrm{Ad}_{E_{R'}}(\widetilde{g})(d) = l + c + d$, for some $l \in \mathfrak{g}\otimes R'$, $c \in D^{gr*}$. Therefore, it is enough to prove that $l \in \mathfrak{g}\otimes R$.

Let $l_1 \in \mathfrak{g}\otimes R$. Then $\mathrm{Ad}_{E_{R'}}(\widetilde{g})(l_1) = \mathrm{Ad}(\widetilde{g})(l_1) + c_1$, for some $c_1 \in C$. Since $\mathrm{Ad}_{E_{R'}}(\widetilde{g})$ is a Lie algebra automorphism we have

$$\mathrm{Ad}_{E_{R'}}(\widetilde{g})([d, l_1]_{E_{R'}}) = [\mathrm{Ad}_{E_{R'}}(\widetilde{g})(d), \mathrm{Ad}_{E_{R'}}(\widetilde{g})(l_1)]_{E_{R'}}.$$

That is

$$\mathrm{Ad}_{E_{R'}}(\widetilde{g})(d(l_1)) = [l + c + d, Ad(\widetilde{g})(l_1) + c_1]_{E_{R'}}.$$

Comparing $\mathfrak{g}\otimes R'$-components we get

$$\mathrm{Ad}(\widetilde{g})(d(l_1)) = [l, \mathrm{Ad}(\widetilde{g})(l_1)]_{\mathfrak{g}\otimes R'} + d(\mathrm{Ad}(\widetilde{g})(l_1)).$$

Since, of course,

$$\mathrm{Ad}(\widetilde{g})(d(l_1)) = \mathrm{Ad}(g)(d(l_1)) \in \mathfrak{g}\otimes R$$

and

$$d(\mathrm{Ad}(\widetilde{g})(l_1)) = d(\mathrm{Ad}(g)(l_1)) \in \mathfrak{g}\otimes R,$$

we get that $[l, \mathrm{Ad}(\widetilde{g})(l_1)]_{\mathfrak{g}\otimes R'} \in \mathfrak{g}\otimes R$ for any $l_1 \in \mathfrak{g}\otimes R$, i.e. $[l, l_1]_{\mathfrak{g}\otimes R'} \in \mathfrak{g}\otimes R$ for any $l_1 \in \mathfrak{g}\otimes R$.

Let us show that $l \in \mathfrak{g}\otimes R$. As before, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\Sigma = \{\alpha\}$ be a root system of $(\mathfrak{g}, \mathfrak{h})$, $\{\alpha_i\}$ be simple roots. Let $\omega_j \in \mathfrak{h}$ be such that $\alpha_i(\omega_j) = \delta_{i,j}$.

Then $\{X_\alpha, \omega_i\}$ form a basis of $\mathfrak{g}$. Write $l = \sum_{\alpha\in\Sigma} X_\alpha \otimes p_\alpha + \sum_i \omega_i \otimes r_i$, where $p_\alpha, r_i \in R'$.

We have $[l, X_{\alpha_i}]_{\mathfrak{g}\otimes R'} \in \mathfrak{g}\otimes R$ and has X_{α_i}-coordinate equal to $\sum_j \omega_j(\alpha_i) r_j = r_i$. Hence $r_i \in R$.

For $\alpha \in \Sigma$ choose $h \in \mathfrak{h}$ such that $\alpha(h) \neq 0$. We have $[l, h]_{\mathfrak{g}\otimes R'} \in \mathfrak{g}\otimes R$ and its X_α-component is $-\alpha(h)p_\alpha$. Hence $p_\alpha \in R$.

Therefore, $l \in \mathfrak{g}\otimes R$ and hence $\mathrm{Ad}_{E_{R'}}(\widetilde{g})(d) \in E$. This proves the lemma. □

Summarizing, $\mathrm{Ad}_{\mathfrak{g}\otimes R}(g)$ admits a lifting $\mathrm{Ad}_E(g) := \mathrm{Ad}_{E_{R'}}(\widetilde{g})|_E$.

Step 3. Lifting automorphisms from $\mathrm{Aut}(\mathrm{Dyn}(\mathfrak{g}))$.

We view automorphisms of $\mathfrak{g}$ also as automorphisms of $\mathfrak{g}\otimes R$ by identifying an automorphism f of $\mathfrak{g}$ with the automorphism $f\otimes \mathrm{Id}_R$ of $\mathfrak{g}\otimes R$. It is shown in [NPPS, Cor. 7.4] that

6.1.6 Lemma. *The form* $(-|-)_{\mathfrak{g}\otimes R}$ *is invariant under any automorphism of* $\mathfrak{g}$.

Proof. It is well known that $(-\,|\,-)_{\mathfrak{g}\otimes R} = \nu\circ\kappa$ for some $\nu \in \mathrm{Hom}_k(R,k)$, where κ is the Killing form on $\mathfrak{g}\otimes R$. Since κ is invariant with respect to any automorphism of $\mathfrak{g}$ so is $(-|-)$. □

Let $\phi \in \mathrm{Aut}(\mathrm{Dyn}(\mathfrak{g}))$. Abusing notation we denote a (graph) automorphism of $\mathfrak{g}\otimes R$ corresponding to ϕ by the same letter. Note that since ϕ is "constant" one has $(\phi\circ d)(l) = (d\circ\phi)(l)$ for all $l \in L = \mathfrak{g}\otimes R$.

6.1.7 Lemma. *Let* $\tilde{\phi} : E \to E$ *be a map given by*

$$\tilde{\phi}(l + c + d) = \phi(l) + c + d$$

where $l \in \mathfrak{g}\otimes R$*,* $c \in C$*,* $d \in D$*. Then* $\tilde{\phi}$ *is an automorphism of* E *extending* ϕ*.*

Proof. Since $\widetilde{\phi}$ acts trivially on $C\oplus D$ it suffices to show

$$\widetilde{\phi}([l_1, l_2]_E) = [\widetilde{\phi}(l_1), \widetilde{\phi}(l_2)]_E$$

for all $l_1, l_2 \in \mathfrak{g} \otimes R$. But

$$\widetilde{\phi}([l_1, l_2]_E) - [\widetilde{\phi}(l_1), \widetilde{\phi}(l_2)]_E = \sigma(l_1, l_2) - \sigma(\phi(l_1), \phi(l_2)).$$

For $d \in D$ we have

$$\sigma(l_1, l_2)(d) = (d(l_1) \,|\, l_2)_{\mathfrak{g}\otimes R}$$

and

$$\sigma(\phi(l_1), \phi(l_2))(d) = (d(\phi(l_1)) \,|\, \phi(l_2))_{\mathfrak{g}\otimes R} = (\phi(d(l_1)) \,|\, \phi(l_2))_{\mathfrak{g}\otimes R} = (d(l_1) \,|\, l_2)_{\mathfrak{g}\otimes R}.$$

The last equality is due to Lemma 6.1.6. □

This completes the proof of Theorem 6.1.1 as well. □

6.2 Automorphisms of EALA with split centreless core

Let $E = \mathfrak{g} \otimes R \oplus D^{gr*} \oplus D$ be an EALA with centreless core split multiloop algebra. The following statement is well known.

6.2.1 Proposition. *One has*

$$\mathrm{Aut}_{k-Lie}(E_{cc}) \cong \mathrm{Aut}_{R-Lie}(E_{cc}) \rtimes \mathrm{Aut}_k(R).$$

Recall that we denote by res_c a natural restriction map

$$res_c : \mathrm{Aut}_{k-Lie}(E) \to \mathrm{Aut}_{k-Lie}(E_c)$$

and by π a natural map

$$\pi : \mathrm{Aut}_{k-Lie}(E_c) \to \mathrm{Aut}_{k-Lie}(E_{cc})$$

so that $res_{cc} = \pi \circ res_c$.

6.2.2 Lemma. *π is injective.*

Proof. Let $\theta_1,\ \theta_2 \in \mathrm{Aut}_{k-Lie}(E_c)$ be such that $\pi(\theta_1) = \pi(\theta_2)$. Then for all $x, y \in E_c$ there are $x_c, y_c \in Z(E_c) = D^{gr*}$ such that $\theta_1(x) = \theta_2(x) + x_c$ and

$\theta_1(y) = \theta_2(y) + y_c$. Therefore

$$\begin{aligned}
\theta_1([x,y]_{E_c}) &= [\theta_1(x), \theta_1(y)]_{E_c} \\
&= [\theta_2(x) + x_c, \theta_2(y) + y_c]_{E_c} \\
&= [\theta_2(x), \theta_2(y)]_{E_c} \\
&= \theta_2([x,y]_{E_c}).
\end{aligned}$$

Since $E_c = [E_c, E_c]$ and $x, y \in E_c$ were arbitrary we get that $\theta_1 = \theta_2$. ☐

6.2.3 Corollary. $\mathrm{Ker}(f) = \mathrm{Ker}(res_c)$

6.2.4 Proposition. $\mathrm{Ker}(res_c)$ *is a torsion-free abelian group.*

Proof. Let ϕ be an automorphism of E such that $\phi|_{E_c} = \mathrm{Id}_{E_c}$. Let $x \in D$. Write $\phi(x) = l + c + d$ where $l \in L$, $c \in D^{gr*}$, $d \in D$. For every $y \in \mathfrak{g}$ we have

$$[\phi(x), \phi(y)]_E = [\phi(x), y]_E = \phi([x,y]_E).$$

Since y is constant, $[x,y]_E = x(y) = 0$. Therefore

$$[l, y]_E = [l + c + d, y]_E = \phi(0) = 0.$$

It follows $[l,y]_L = 0$. Since $Z(\mathfrak{g} \otimes R) = 0$ this implies $l = 0$. Thus

$$\phi(x) = c + d \in D^{gr*} \oplus D.$$

We next show that $d = x$. For any $0 \neq y \in \mathfrak{g}$ we have

$$[\phi(x), \phi(y \otimes t_i)]_E = \phi([x, y \otimes t_i]_E),$$

$1 \leq i \leq n$, or equivalently

$$[c + d, y \otimes t_i]_E = [x, y \otimes t_i]_E$$

or

$$[d - x, y \otimes t_i]_E = 0 \; for \; 1 \leq i \leq n. \tag{6.2.4.1}$$

Write $d - x = \sum_\lambda t^\lambda d_{\theta_\lambda}$. Substituting this expression in (6.2.4.1) we get that $\theta_\lambda = 0$. This, in turn, implies $d_{\theta_\lambda} = 0$ and therefore $x - d = 0$.

Now we define a homomorphism

$$\mu : \mathrm{Ker}(res_c) \to \mathrm{Hom}_k(D, D^{gr*}), \quad \phi \to \mu(\phi)$$

where $\mu(\phi) : z \mapsto pr_{D^{gr*}}(\phi(z))$ for $z \in D$. From what we have already seen it follows that μ is injective. Since $\mathrm{Hom}_k(D, D^{gr*})$ is a torsion-free abelian group so is $\mathrm{Ker}(res_c)$. $\square$

6.2.5 Corollary. *The sequence of groups*

$$1 \longrightarrow \mathrm{Ker}(res_c) \longrightarrow \mathrm{Aut}_{k-Lie}(E) \xrightarrow{res_{cc}} \mathrm{Im}(res_{cc}) \longrightarrow 1 \qquad (6.2.5.1)$$

is exact.

6.2.6 Remark. We proved already that the group $\mathrm{Aut}_{R-Lie}(E_{cc})$ is contained in $\mathrm{Im}(res_{cc})$, if E_{cc} is not isomorphic to $sl_2(R)$. This will be an important fact in the next sections.

6.3 EALA as a subalgebra of a "split" EALA

Let E be an extended affine Lie algebra. We know from the Realization Theorem (see [Ne5], Th. 5.15) that

$$E = L \oplus D^{gr*} \oplus D,$$

where $L \cong E_{cc}$ is a centreless Lie torus, $D \subset \mathrm{SCDer}_k(L)$ is a graded subalgebra of the Lie algebra of skew-centroidal derivations of L. Recall that L, being fgc, is a multiloop Lie algebra (twisted or split), hence of the form $L = L(\mathfrak{g}, \boldsymbol{\sigma})$ as in 2.3.3.2.

We now consider the vector space $E_S = L_S \oplus D^{gr*} \oplus D$ and we would like first to equip it with a structure of an extended affine Lie algebra. Recall that $L = L(\mathfrak{g}, \boldsymbol{\sigma})$ (resp. D) has a natural Λ-grading where $\Lambda = \mathbb{Z}^n$ (resp. $m\mathbb{Z}^n$-grading). Here $m\mathbb{Z}^n := m_1\mathbb{Z} \oplus \cdots \oplus m_n\mathbb{Z}$.

We define the action of D on L_S by the skew-centroidal derivations (such that the induced action on L coincides with the original one) as follows. Any homogeneous element $d \in D^\lambda$ is of the form $d = \chi^\lambda d_\theta$ where $\lambda \in m\mathbb{Z}^n$,

$\theta \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, k)$. For any homogeneous element $x \otimes s^{\mu} \in L_S = \mathfrak{g} \otimes S$ we let

$$d(x \otimes s^{\mu}) := \theta(\mu) x \otimes \chi^{\lambda} s^{\mu}.$$

It is straightforward to check that this formula indeed defines an action of D on L_S by the skew-centroidal derivations.

Furthermore, it is known that there exists a unique (up to scalar) graded non-degenerated invariant bilinear symmetric form on $\mathfrak{g} \otimes S$ and its restriction to L has the same properties. It then follows that a canonical invariant symmetric form $(-|-)$ on L has a unique extension to $\mathfrak{g} \otimes S$.

The action of D on L_S and the invariant symmetric form on L_S defined above allow us to define a structure of extended affine Lie algebra on E_S. (Notice that the evaluation map $ev : \mathbb{Z}^n \to D^{0*}$ is injective since it is the same for L_S as for L.)

We next remark that the standard action of Γ on $L_S = \mathfrak{g} \otimes S$ (through the second factor) can be extended to its standard action on E_S: if $\gamma \in \Gamma$ and $l + c + d \in E_S$ we let

$$\gamma(l + c + d) := \gamma(x) + c + d.$$

This in turn induces the standard action of Γ on $\mathrm{Aut}(E_S)$: if $\gamma \in \Gamma$ and $\phi \in \mathrm{Aut}(E_S)$ then

$$\gamma(\phi)(l + c + d) = \gamma(\phi(\gamma^{-1}(l + c + d))).$$

Furthermore, the cocycle

$$\eta(\boldsymbol{\sigma}) = (a_{\overline{\lambda}}) \in Z^1(\Gamma, \mathrm{Aut}(\mathfrak{g})(S))$$

can be lifted to a cocycle

$$\widehat{\eta(\boldsymbol{\sigma})} = (\widehat{a}_{\overline{\lambda}}) \in Z^1(\Gamma, \mathrm{Aut}_{k-Lie}(E_S))$$

where $\widehat{a}_{\overline{\lambda}}$ is an automorphism of E_S given by

$$\widehat{a}_{\overline{\lambda}}(x + c + d) = a_{\overline{\lambda}}(x) + c + d.$$

(It is straightforward to verify that $\widehat{a}_{\overline{\lambda}}$ is an automorphism of E_S and that $\widehat{\eta(\boldsymbol{\sigma})}$ is a cocycle.) As a consequence of all our constructions we obtain a twisted action of Γ (through the cocycle $\widehat{\eta(\boldsymbol{\sigma})}$) on E_S which satisfies $(E_S)^\Gamma = E$.

6.4 The automorphism group of EALA with twisted centreless core

Throughout the section we assume that the centreless core E_{cc} of E is not isomorphic to $sl_2(R)$.

We keep the notation from the last section. Recall that res_c is the restriction map

$$res_c : \mathrm{Aut}_{k-Lie}(E_S) \to \mathrm{Aut}_{k-Lie}((E_S)_c).$$

It is easy to see that the standard action of Γ on $\mathrm{Ker}(res_c)$ is trivial and hence so is the twisted one; in particular $\mathrm{Ker}(res_c)$ is a Γ-group. Also $\mathrm{Im}(res_{cc})$ contains $\mathrm{Aut}_{R-Lie}(L_S)$ and hence stable with respect to the twisted action of Γ. Thus (6.2.5.1) can be viewed as an exact sequence of Γ-groups (The action is twisted!). The long exact cohomological sequence associated to (6.2.5.1) implies the following.

6.4.1 Theorem. *(a) The following sequence*

$$1 \longrightarrow \mathrm{Ker}(res_c) \longrightarrow (\mathrm{Aut}_{k-Lie}(E_S))^\Gamma \xrightarrow{res'_{cc}} \mathrm{Im}(res_{cc})^\Gamma \longrightarrow 1$$

is exact. Here res'_{cc} is the restriction of res_{cc}.

(b) The group $\mathrm{Aut}_{R-Lie}(E_{cc})$ is in the image of a canonical mapping

$$\mathrm{Aut}_{k-Lie}(E) \longrightarrow \mathrm{Aut}_{k-Lie}(E_{cc}).$$

Proof. Since Γ acts trivially on $\mathrm{Ker}(res_c)$ and since $\mathrm{Ker}(res_c)$ is a torsion free abelian group we have $H^1(\Gamma, \mathrm{Ker}(res_c)) = 1$ and so the first assertion is clear.

As for the second one, note that we have a canonical embedding

$$(\mathrm{Aut}_{k-Lie}(E_S))^\Gamma \hookrightarrow \mathrm{Aut}_{k-Lie}(E_S^\Gamma) = \mathrm{Aut}_{k-Lie}(E)$$

and similarly

$$(\mathrm{Aut}_{k-Lie}\,(L_S))^\Gamma \hookrightarrow \mathrm{Aut}_{k-Lie}\,((L_S)^\Gamma) = \mathrm{Aut}_{k-Lie}\,(E);$$

hence a commutative diagram

$$\begin{array}{ccc} (\mathrm{Aut}_{k-Lie}\,(E_S))^\Gamma & \xrightarrow{res'_{cc}} & \mathrm{Im}(res_{cc})^\Gamma \subset (\mathrm{Aut}_{k-Lie}\,(L_S))^\Gamma \\ \downarrow & & \downarrow \\ \mathrm{Aut}_{k-Lie}\,(E) & \longrightarrow & \mathrm{Aut}_{k-Lie}\,(E_{cc}) \end{array}$$

Recall that due to Galois descent for group schemes we have

$$\mathrm{Aut}_{R-Lie}\,(E_{cc}) \simeq (\mathrm{Aut}_{S-Lie}\,(L_S))^\Gamma. \tag{6.4.1.1}$$

This, together with Remark 6.2.6 and surjectivity of res'_{cc} yield the result. □

6.5 Lifting of k-automorphisms

We will see that in contrast to the case of R-automorphisms, it is not always possible to lift all the k-automorphisms of $\mathfrak{g} \otimes R$ to the entire algebra E. In this section we discuss a lifting problem for k-automorphisms of $\mathfrak{g}\otimes R$, induced by k-automorphisms of R.

As before, let $L = \mathfrak{g} \otimes R$ be a centreless core of an EALA E, which we write as $E = L \oplus C \oplus D$. Let $\mathcal{D} = \mathrm{SCDer}(L)$ and $\mathcal{C} = \mathcal{D}^{gr*}$. By [Ne1], $L \oplus \mathcal{C}$ is a universal central extension of L. More precisely, for $x \otimes r \in L$, $y \otimes s \in L$ we define

$$[x \otimes r, y \otimes s]_{L\oplus\mathcal{C}} = [x, y] \otimes rs + \sigma(x \otimes r, y \otimes s),$$

where $\sigma(x\otimes r, y\otimes s)(d) = (d(x\otimes r), y\otimes r)$. Then $pr_L : L\oplus\mathcal{C} \to L$ is a universal central extension of L. By [VKL, Proposition 1.3], any $\phi \in \mathrm{Aut}_{k-Lie}(L)$ has a unique lifting $\widetilde{\phi} \in Aut_{k-Lie}(L \oplus \mathcal{C})$, i.e. such that the diagram

$$\begin{array}{ccc} L \oplus \mathcal{C} & \xrightarrow{pr_L} & L \\ {\scriptstyle\widetilde{\phi}}\downarrow & & \downarrow{\scriptstyle\phi} \\ L \oplus \mathcal{C} & \xrightarrow[pr_L]{} & L \end{array}$$

commutes.

Let $\phi \in \mathrm{Aut}_k(R)$. Then ϕ induces an automorphism of L given by

$$x \otimes r \mapsto x \otimes \phi(r)$$

which we still will denote by ϕ.

6.5.1 Remark. Notice, that

$$\mathrm{Aut}_k(R) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, k^\times) \ltimes \mathrm{GL}_n(\mathbb{Z}).$$

For $\chi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, k^\times)$, $A \in \mathrm{GL}_n(\mathbb{Z})$ and $t^\lambda \in R$ one has $(\chi,\ A)(t^\lambda) = \chi(\lambda)t^{A\lambda}$.

6.5.2 Lemma. *The bilinear form $(-,-) = \kappa \otimes (-,-)_R$ on L is invariant under $\phi \in \mathrm{Aut}_k(R)$.*

Proof. Let $\phi = (\chi, A) \in \mathrm{Aut}_k(R)$, $x \otimes t^\lambda, y \otimes t^\mu \in L$. Then

$$\begin{aligned}
(\phi(x \otimes t^\lambda), \phi(y \otimes t^\mu)) &= (x \otimes \phi(t^\lambda), y \otimes \phi(t^\mu)) \\
&= \kappa(x,y)(\phi(t^\lambda), \phi(t^\mu))_R \\
&= \kappa(x,y)(\chi(\lambda)t^{A\lambda}, \chi(\mu)t^{A\mu})_R \\
&= \kappa(x,y)\chi(\lambda+\mu)\delta_{A(\lambda+\mu),0} \\
&= \kappa(x,y)\delta_{\lambda+\mu,0} \\
&= \kappa(x,y)(t^\lambda, t^\mu)_R \\
&= (x \otimes t^\lambda, y \otimes t^\mu),
\end{aligned}$$

thus the claim. □

6.5.3 Corollary. *Let $d \in \mathcal{D}$. Let $\phi \in \mathrm{Aut}_{k-Lie}(R)$. Then $\phi \circ d \circ \phi^{-1} \in \mathcal{D}$.*

Proof. Using Lemma 6.5.2 , for any $l_1, l_2 \in L$ we get

$$\begin{aligned}
&(\phi \circ d \circ \phi^{-1}(l_1), l_2) + (\phi \circ d \circ \phi^{-1}(l_2), l_1) = \\
&(d(\phi^{-1}(l_1)), \phi^{-1}(l_2)) + (d(\phi^{-1}(l_2)), \phi^{-1}(l_1)) = 0,
\end{aligned}$$

thus the claim. □

Let $\phi_{\mathcal{D}} \in \mathrm{Aut}_{k-Lie}(\mathcal{D})$ be a map $d \mapsto \phi \circ d \circ \phi^{-1}$ and $\phi_{\mathcal{C}} \in Aut_k(\mathcal{C})$ be a map $\alpha \mapsto \alpha \circ \phi_{\mathcal{D}}^{-1}$.

6.5.4 Lemma.

$$(\chi, A)_{\mathcal{D}}(t^\lambda d_\theta) = t^{A\lambda} d_{\chi(\lambda)\theta\circ A^{-1}}.$$

Proof. Straightforward computations. □

6.5.5 Lemma. *Let $\phi \in Aut_{k-Lie}(R)$. Then the lifting $\widetilde{\phi}$ is defined by $\widetilde{\phi}(l+c) = \phi(l) + \phi_{\mathcal{C}}(c)$, for any $l \in L$, $c \in \mathcal{C}$.*

Proof. We only have to check that a mapping $\widetilde{\phi}$ is a Lie algebra homomorphism. Since $\mathcal{C}$ is a centre of $L \oplus \mathcal{C}$, it is enough to show

$$\widetilde{\phi}([l_1, l_2]_{\widetilde{L}}) = [\widetilde{\phi}(l_1), \widetilde{\phi}(l_2)]_{\widetilde{L}} \tag{6.5.5.1}$$

for any $l_1, l_2 \in L$. The left hand side of (6.5.5.1) is

$$\widetilde{\phi}([l_1, l_2]_L + \sigma(l_1, l_2)) = [l_1, l_2]_L + \phi_{\mathcal{C}}(\sigma(l_1, l_2))$$

and the right hand side is

$$\begin{aligned} [\phi(l_1), \phi(l_2)]_{\widetilde{L}} &= [\phi(l_1), \phi(l_2)]_L + \sigma(\phi(l_1), \phi(l_2)) \\ &= \phi([l_1, l_2]_L) + \sigma(\phi(l_1), \phi(l_2)). \end{aligned}$$

Hence we have to check

$$\phi_{\mathcal{C}}(\sigma(l_1, l_2)) = \sigma(\phi(l_1), \phi(l_2)).$$

For any $d \in \mathcal{D}$ we have

$$\begin{aligned} \phi_{\mathcal{C}}(\sigma(l_1, l_2))(d) &= \sigma(l_1, l_2)(\phi_{\mathcal{D}}^{-1}(d)) \\ &= (\phi_{\mathcal{D}}^{-1}(d)(l_1), l_2) \\ &= (\phi^{-1}(d(\phi(l_1))), l_2) \\ &= (d(\phi(l_1)), \phi(l_2)) \\ &= \sigma(\phi(l_1), \phi(l_2)), \end{aligned}$$

thus the claim. □

6.5.6 Lemma. *Let D be a graded subalgebra of $\mathcal{D}$. Let $C = D^{gr*}$. Let $i : D \to \mathcal{D}$ be an inclusion map. And let $i^* : \mathcal{D}^{gr*} \to D^{gr*}$ be a corresponding map between the graded duals. The map i^* is surjective, i.e. $D^{gr*} \cong \mathcal{D}^{gr*} / \operatorname{Ker}(i^*)$.*

Proof. Obvious. □

By the universal property of a universal central extension $\widetilde{L} \to L$ there is a unique Lie algebra map $\pi : \widetilde{L} \to L \oplus C$ over L.
6.5.7 Lemma. $\pi = id_L \oplus i^*$.

Proof. Obvious. $\square$

6.5.8 Lemma. *π is a universal central extension of $L \oplus C$.*

Proof. We have to check that for any central extension

$$\theta : K = (L \oplus C) \oplus C' \longrightarrow L \oplus C$$

there is a unique map $\widetilde{\pi}$ such that the diagram

$$\begin{tikzcd} & \widetilde{L} \arrow[dl, "\widetilde{\pi}"'] \arrow[d, "\pi"] \\ (L \oplus C) \oplus C' \arrow[r, "\theta"] & L \oplus C \end{tikzcd}$$

commutes.

Let ψ be a 2-cocycle corresponding to an extension $\theta : K \to L \oplus C$.

First notice that $K \xrightarrow{pr_L \circ \theta} L$ is a central extension. For this we have to show that $C \oplus C' \subset Z(K)$. Of course, $C' \subset Z(K)$. Let $c \in C$, $x \in L \oplus C$. Since $L \oplus C$ is a core of an extended affine Lie algebra E, it is perfect. Therefore $c = \sum [x_i, y_i]_{L \oplus C}$, $x_i, y_i \in L \oplus C$.

We get

$$\begin{aligned} [c, x]_K &= \sum [[x_i, y_i]_{L \oplus C}, x]_K \\ &= \sum \psi([x_i, y_i]_{L \oplus C}, x) \\ &= -\sum (\psi([y_i, x]_{L \oplus C}, x_i) + \psi([x, x_i]_{L \oplus C}, y_i)) \\ &= 0, \end{aligned}$$

thus $K \to L$ is indeed central.

Therefore, by the universal property of $pr_L : \widetilde{L} \to L$, there is a unique map λ such that the outer triangle of the diagram

$$\begin{tikzcd} & \widetilde{L} \arrow[dl, "\lambda"'] \arrow[d, "\pi"] \arrow[dr, "\mathrm{pr}_L"] & \\ (L \oplus C) \oplus C' \arrow[r, "\theta"] & L \oplus C \arrow[r, "pr_L"] & L \end{tikzcd}$$

commutes. But then again by the universal property of $pr_L : \widetilde{L} \to L$ we get that the left triangle commutes and λ is a unique map with this property. This proves the lemma. $\square$

From now on we assume that $\tau = 0$.

6.5.9 Lemma. *Let $\phi \in \mathrm{Aut}_{k-Lie}(R)$. Then ϕ has a lifting to the core $L \oplus C$ of E iff $\phi_{\mathcal{D}}(D) = D$. In this case ϕ has a lifting to the entire EALA E.*

Proof. It follows from [VKL, Proposition 1.3] that ϕ has a lifting to $\mathrm{Aut}_{k-Lie}(L \oplus C)$ iff $\widetilde{\phi}(\mathrm{Ker}(i^*)) = \mathrm{Ker}(i^*)$. By Lemma 6.5.5 $\widetilde{\phi}(\mathrm{Ker}(i^*)) = \phi_{\mathcal{C}}(\mathrm{Ker}(i^*))$.

Let us show that $\phi_{\mathcal{C}}(\mathrm{Ker}(i^*)) = \mathrm{Ker}(i^*)$ iff $\phi_{\mathcal{D}}(D) = D$.

Assume $\phi_{\mathcal{C}}(\mathrm{Ker}(i^*)) = \mathrm{Ker}(i^*)$. Let $d \in D$. We want to show that $\phi_{\mathcal{D}}(d) \in D$ and $\phi_{\mathcal{D}}^{-1}(d) \in D$. For any $\alpha \in \mathrm{Ker}(i^*)$ we have

$$0 = \phi_{\mathcal{C}}(\alpha)(d) = \alpha(\phi_{\mathcal{D}}^{-1}(d)).$$

This shows that $\phi_{\mathcal{D}}^{-1}(d)$ is in a kernel of any functional $\alpha \in Ker(i^*)$. It follows that $\phi_{\mathcal{D}}^{-1}(d) \in D$. Similarly, for any $\alpha \in \mathrm{Ker}(i^*)$ we have

$$0 = \phi_{\mathcal{C}}^{-1}(\alpha)(d) = \alpha(\phi_{\mathcal{D}}(d)).$$

Hence $\phi_{\mathcal{D}}(d) \in D$.

Conversely, assume that $\phi_{\mathcal{D}}(D) = D$. Let $\alpha \in \mathrm{Ker}(i^*)$. We want to show that $\phi_{\mathcal{C}}(\alpha) \in \mathrm{Ker}(i^*)$ and $\phi_{\mathcal{C}}^{-1}(\alpha) \in \mathrm{Ker}(i^*)$. For any $d \in D$ we have $\phi_{\mathcal{D}}(d) \in D$ and $\phi_{\mathcal{D}}^{-1}(d) \in D$. Therefore

$$\phi_{\mathcal{C}}(\alpha)(d) = \alpha(\phi_{\mathcal{D}}^{-1}(d)) = 0$$

and

$$\phi_{\mathcal{C}}^{-1}(\alpha)(d) = \alpha(\phi_{\mathcal{D}}(d)) = 0$$

Therefore $\phi_{\mathcal{C}}(\alpha) \in \mathrm{Ker}(i^*)$. Thus we proved the first part of the lemma.

Now assume that $\phi_{\mathcal{D}}(D) = D$. Let us show that ϕ can be lifted to an automorphism ϕ_E of E. From the above it follows that the map $\phi_{\mathcal{C}}$ induces an isomorphism $\phi_C : C \to C$. The lifting ϕ' of ϕ to $Aut_{k-Lie}(L \oplus C)$ is given by

$$\phi'(l + c) = \phi(l) + \phi_C(c),$$

for any $l \in L$, $c \in C$.

We define ϕ_E by

$$\phi_E(l + c + d) = \phi'(l + c) + \phi_{\mathcal{D}}(d),$$

for any $l \in L$, $c \in C$, $d \in D$.

Of course, ϕ_E is an automorphism of E as a k-vector space. We want to check that it preserves the Lie bracket. For this it is enough to check

$$\phi_E([d, l]_E) = [\phi_E(d), \phi_E(l)]_E \tag{6.5.9.1}$$

and

$$\phi_E([d, c]_E) = [\phi_E(d), \phi_E(c)]_E \tag{6.5.9.2}$$

for any $d \in D$, $l \in L$, $c \in C$.

The left hand side of 6.5.9.1 is

$$\phi_E([d, l]) = \phi(d(l)) = (\phi \circ d)(l).$$

The right hand side of 6.5.9.1 is

$$[\phi_{\mathcal{D}}(d), \phi(l)]_E = \phi_{\mathcal{D}}(d)(\phi(l)) = (\phi \circ d \circ \phi^{-1})(\phi(l)) = (\phi \circ d)(l).$$

Hence 6.5.9.1 holds.

The left hand side of 6.5.9.2 evaluated at $d_1 \in D$ is

$$\phi_E([d, c]_E)(d_1) = \phi_C(d \cdot c)(d_1) = (d \cdot c)(\phi_{\mathcal{D}}^{-1}(d_1)) = c([\phi_{\mathcal{D}}^{-1}(d_1), d]_D).$$

The right hand side of 6.5.9.2 evaluated at $d_2 \in D$ is

$$\begin{aligned}
[\phi_E(d), \phi_E(c)]_E(d_1) &= (\phi_{\mathcal{D}}(d) \cdot \phi_C(c))(d_1) \\
&= \phi_C(c)([d_1, \phi_{\mathcal{D}}(d)]_D) \\
&= c(\phi_{\mathcal{D}}^{-1}([d_1, \phi_{\mathcal{D}}(d)]_D)) \\
&= c([\phi_{\mathcal{D}}^{-1}(d_1), d]_D).
\end{aligned}$$

Hence 6.5.9.2 holds. This ends the proof.

$\square$

6.5.10 Corollary.
- *The map* $\mathrm{Aut}_{k-Lie}(E) \to \mathrm{Aut}_{k-Lie}(\mathfrak{g} \otimes R)$ *is surjective if* $D = \mathcal{D}$ *or* $D = \mathcal{D}^0$.

- *If $\phi = (\chi, 1)$ then ϕ always has a lifting to E.*

Proof. Follows immediately from 6.5.4 and 6.5.9. $\square$

6.5.11 Example. Let $R = k[t_1^{\pm 1}, t_2^{\pm 1}]$. Let $D = kd_\theta \oplus kt_1\partial_2$, where $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, k)$ is given by $\theta(e_1) = 1$, $\theta(e_2) = \sqrt{2}$. Then $(1, A)$ can be lifted if and only if $A = I_2$.

Chapter 7

A conjugacy theorem of Cartan subalgebras in extended affine Lie algebras

In this chapter we prove the main result of this dissertation.

Throughout we assume that all extended affine Lie algebras are of *fgc type*, i.e. their centreless cores are finitely generated modules over the corresponding centroids.

7.0.12 Theorem. (Conjugacy theorem for EALAs.) *Let (E, H) be an extended affine Lie algebra of fgc type, which is not isomorphic to $sl_2(R)$, where R is a ring of Laurent polynomials in more then 1 variable. Assume E admits a second structure (E, H') of an extended affine Lie algebra. Then H and H' are conjugate, i.e., there exists an automorphism $\phi \in \mathrm{Aut}_{k-Lie}(E)$ such that $\phi(H) = H'$.*

7.1 Some auxiliary properties

From now on we assume (E, H) and (E, H') are two structures of EALA on E. We set $H_c = H \cap E_c$ and $H'_c = H' \cap E'_c = H' \cap E_c$. Let $\pi : E_c \to E_{cc}$ be a canonical projection. We denote $H_{cc} = \pi(H_c)$ and $H'_{cc} = \pi(H'_c)$. On $L = E_{cc}$ we have two Lie torus structures coming from (E, H) and (E', H'), say L and L'. Then $L_0^0 = H_{cc}$ and $(L')_0^0 = H'_{cc}$.

Both subalgebras H_{cc} and H'_{cc} of L are MADs for which the conjugacy theorem in [CGP] can be applied. Hence there exists $g \in \mathrm{Aut}_{R-Lie}(L)$ such that $g(H'_{cc}) = H_{cc}$; moreover, g can be choosen in the image of a natural map ${}_\eta\widetilde{\mathbf{G}}_R(R) \to \mathrm{Aut}_{R-Lie}(L)$ where ${}_\eta\widetilde{\mathbf{G}}_R$ is the simple simply connected group scheme over R corresponding to L as defined in Remark 2.3.4.

According to Theorem 6.4.1, part (b), $\mathrm{Ad}_L(g) \in \mathrm{Aut}_{R-Lie}(L) \subset \mathrm{Aut}_{k-Lie}(L)$ can be lifted to an automorphism, say ϕ, of E. So replacing the second structure $(E, H', (-|-)')$ by $(E, \phi(H'), (-|-)' \circ (\phi^{-1} \times \phi^{-1})$ we may assume without loss of generality that $H_{cc} = H'_{cc}$.

7.1.1 Lemma. $H_c = H'_c$.

Proof. Take any $x \in H_c$. Since $H_{cc} = H'_{cc}$ there exists $y \in H'_c$ such that $\pi(x) = \pi(y)$. Then $c = x - y \in Ker(\pi) = C$. Since $C = Z(E_c)$ the elements x and y commute and by construction both of them are k-ad-diagonalizable in E. It follows that c is also k-ad-diagonalizable in E.

We now note that it follows from the inclusion $[C, D]_E \subset C$ and from $[C, E_c]_E = 0$ that any eigenvector of $ad(c)$ with a nonzero D-component necessarily commutes with c. Therefore $c \in Z(E) \subset H'_c$ implying $x = y + c \in H'_c$. Thus we have showed that $H_c \subset H'_c$ and similarly $H'_c \subset H_c$. □

The above lemma says that $H_c = H'_c = H_{cc} \oplus C^0$ and we have decompositions $H = H_{cc} \oplus C^0 \oplus D^0$ and $H' = H_{cc} \oplus C^0 \oplus D'^0$.

7.1.2 Lemma. *One has $D'^0 \subset L_0 \oplus C \oplus D$ where $L_0 = C_L(H_{cc})$.*

Proof. Let $x \in D'^0$. By Lemma 7.1.1 we have $[x, y]_E = 0$ for all $y \in H_{cc}$. Let $x = l + c + d$ where $l \in L$, $c \in C$, $d \in D$. Then

$$0 = [x, y]_E = [l, y]_E + [c, y]_E + [d, y]_E = [l, y]_E.$$

Therefore, $l \in L_0$, as required. □

7.1.3 Lemma. *Let $d = d_\theta \in D^0$ and assume that $y \in L$ satisfies $[d, y]_E = \omega y$ for some $\omega \in k$. Then*

$$[d, \chi^\gamma y]_E = (\theta(\gamma) + \omega)\chi^\gamma y$$

for any $\chi^\gamma \in R^\gamma = \mathrm{Ctrd}(L)^\gamma$.

Proof. Indeed, we have

$$[d, \chi^\gamma y]_E = d(\chi^\gamma y) = \theta(\gamma)\chi^\gamma y + \omega\chi^\gamma y = (\theta(\gamma) + \omega)\chi^\gamma y.$$

□

7.1.4 Lemma. $D'^0 \subset L_0 \oplus C \oplus D^0$.

Proof. Let $d'^0 \in D'^0$. Then by Lemma 7.1.2 we can write $d'^0 = l_0 + c + d$ with $l_0 \in L_0$, $c \in C$ and $d \in D$. Let $\chi^\gamma \in R^\gamma$. By Lemma 7.1.3 we know that $[d'^0, \chi^\gamma h]_E \in k\chi^\gamma h$ for any $h \in L_0^0$. On the other hand

$$[d'^0, \chi^\gamma h]_E = [l_0, \chi^\gamma h]_E + [c, \chi^\gamma]_E + [d, \chi^\gamma h]_E$$

and

$$\begin{aligned} [c, \chi^\gamma h]_E &= [c, \chi^\gamma h]_{E_c} = 0, \\ [l_0, \chi^\gamma h]_E &= [l_0, \chi^\gamma h]_L + \sigma(l_0, \chi^\gamma h), \\ [d, \chi^\gamma h]_E &= d(\chi^\gamma h). \end{aligned}$$

Since $[d', \chi^\gamma h]_E \in k\chi^\gamma h$ we get $\sigma(l_0, \chi^\gamma h) = 0$; also $[l_0, \chi^\gamma h]_L = \chi^\gamma [l_0, h]_L = 0$. Hence $d(\chi^\gamma h) \in k\chi^\gamma h$.

Write $d = \sum_\mu d^\mu$. Comparing degrees we have that $d^\mu(\chi^\gamma h) = 0$ for all $\mu \neq 0$. Let $d^\mu = \chi^{\gamma'} d_\theta$ where $\mu \neq 0$ and $\chi^{\gamma'} \in R^{\gamma'}$. Then we have $\chi^{\gamma+\gamma'}\theta(\gamma)h = 0$, whence $\theta(\gamma) = 0$ for all γ in the grading group Γ of R. Since $[\Lambda/\Gamma] < \infty$, this implies $\theta = 0$ and so $d^\mu = 0$ for all $\mu \neq 0$. Thus $d = d^0$ and we are done. □

7.1.5 Lemma. $D'^0 \subset H_{cc} \oplus C \oplus D^0$.

Proof. Let $d'^0 \in D'^0$. By Lemma 7.1.4 we can write $d'^0 = l_0 + c + d^0$ with $l_0 \in L_0$, $c \in C$, $d^0 \in D^0$. Since E_c is an ideal of E we have $[d'^0, E_c]_E \subset E_c$. Since the operator $ad_E(d'^0)$ is k-diagonalizable it then follows that E_c has a basis consisting of eigenvectors. This implies that the operator $ad_L(l_0) + d^0$ on L is also k-diagonalizable.

We already mentioned that we have two Lie tori structures on L, the second one is denoted by L'; the L'-structure has a Λ'-grading $L' = \oplus_{\lambda' \in \Lambda'} L^{\lambda'}$, induced by D'^0. Similarly, $L = \oplus_{\lambda \in \Lambda} L^\lambda$ is induced by D^0. Applying 2.3.3 with $\phi = \mathrm{Id}$ we get that Lie tori L and L' are isotopic, i.e.

$$L_\alpha^\lambda = (L')_{\phi_r(\alpha)}^{\phi_\Lambda(\lambda) + \phi_s(\alpha)}.$$

Then $(ad_L l_0 + d^0)(L_\alpha^\lambda) \subset L_\alpha^\lambda$ (because this holds for $(L')_{\phi_r(\alpha)}^{\phi_\Lambda(\lambda)+\phi_s(\alpha)}$), and it acts by a scalar multiplication on each space L_α^λ. But d^0 also acts by a scalar multiplication on L_α^λ, hence the same is true for $ad_L l_0$. In particular $ad_L l_0$ is a k-ad-diagonalizable operator, commuting with H_{cc}, whence $l_0 \in H_{cc}$ since H_{cc} is a MAD by [Al]. $\square$

7.2 Final conjugacy

We keep the above notation. Let $C^{\neq 0} = \oplus_{\mu \neq 0} C^\mu$.

7.2.1 Lemma. *There exists a subspace $V \subset H'$ such that*

(i) $H' = H_c \oplus V$, $V \subset C^{\neq 0} \oplus D^0$, *and*

(ii) V *is a graph of some linear mapping* $\psi \in \mathrm{Hom}(D^0, C^{\neq 0})$.

Proof. By Lemma 7.1.1, $H' = H_c' \oplus D'^0 = H_c \oplus D'^0$ and by Lemma 7.1.5, $D'^0 \subset H_{cc} \oplus C \oplus D^0$. We decompose

$$H_{cc} \oplus C \oplus D^0 = (H_{cc} \oplus C^0) \oplus (C^{\neq 0} \oplus D^0).$$

Let $p : H_{cc} \oplus C \oplus D^0 \to C^{\neq 0} \oplus D^0$ be a natural projection and put $V = p(D'^0)$.

Since $D'^0 \cap (H_{cc} \oplus C^0) \subset D'^0 \cap E_c = 0$, we get $V \cong D'^0$ as vector spaces. Moreover the inclusions

$$V \subset C^{\neq 0} \oplus D^0 \subset H_{cc} \oplus C \oplus D^0$$

imply $V \cap (H_{cc} \oplus C^0) = 0$. Note also that $V \subset H'$. Indeed, every $v \in V$ is of the form $v = p(d'^0)$ for some $d'^0 \in D'^0$, whence $d'^0 = h + c^0 + v$ for unique $c^0 \in C^0, h \in H_{cc}$. Since $h, c_0 \in H'$ it follows that $v = d'^0 - c^0 - h \in H'$.

By dimension argument we now get $H' = H_c \oplus V$. That $V \subset C^{\neq 0} \oplus D^0$ is clear from the construction. Finally, to see (ii) we observe that

$$E = [E, E]_E \oplus V = [E, E]_E \oplus D^0$$

and this implies that a natural projection $C^{\neq 0} \oplus D^0 \to D^0$ restricted to V is an isomorphism. $\square$

7.2.2 Lemma. *Let* $\xi \in \mathrm{Hom}_k(D^0, C^{\neq 0})$ *be such that*

$$V = \{\xi(d^0) + d^0 \mid d^0 \in D^0\}$$

is a toral subalgebra of $C \oplus D$*. Then*

(a) The weights of V *in* $C \oplus D$ *are of the form* $ev'_\lambda \in V^*$ *where*

$$ev'_\lambda(\xi(d^0) + d^0) = ev_\lambda(d^0).$$

(b) Every C^μ *is contained in the* ev'_μ*-weight space.*

(c) There exists a unique linear map $\psi : D \to C$ *such that for every* $\mu \in \Lambda$ *and* $d^\mu \in D^\mu$*, the element* $\psi(d^\mu) + d^\mu$ *is an eigenvector for* V *with weight* ev'_μ *and*

$$\psi(d^\mu) \in C^{\neq \mu} = \oplus_{\lambda \neq \mu} C^\lambda.$$

(d) The map $\phi : C \oplus D \to C \oplus D$ *given by*

$$\phi(c + d) = (c + \psi(d)) + d$$

is an automorphism of $C \oplus D$ *mapping* $C^0 \oplus D^0$ *onto* $C^0 \oplus V$*.*

Proof. (b) Let $c \in C^\mu$ and $d_0 \in D^0$. Then

$$[\xi(d_0) + d_0, c]_E = d_0 \cdot c = ev_\mu(d_0)c,$$

as required.

(a) Let $c + d$ be a nonzero eigenvector for $ad(V)$. Since $\tau(D^0, D) = 0$ we get

$$[\xi(d^0) + d^0, c + d]_E = (d^0 \cdot c - d \cdot \xi(d^0)) + [d^0, d]_D.$$

If $d = 0$, then substituting the decomposition $c = \sum c^\mu$ we get

$$d^0 \cdot c = \sum_\mu ev_\mu(d^0)c^\mu$$

for all $d^0 \in D^0$. Since $d^0 \cdot c$ is proportional to c it follows immediately $c = c^\mu$ for some μ.

If $d \neq 0$, it follows from $[d_0, d]_D \in kd$ that $d = d^\mu$ for some μ whence the weight of $c + d$ is as defined in (a).

As a by-product we note that if $c + d^\mu$ and $c' + d^\mu$ are two eigenvectors for V then λ-components of c and c' are the same for each $\lambda \neq \mu$. Indeed, as we have already seen $c + d^\mu$ and $c' + d^\mu$ are contained in the same weight subspace $(C \oplus D)_{ev'_\mu}$, hence so is $(c + d^\mu) - (c' + d^\mu) = c - c'$. But then we have automatically that $c - c' \in C^\mu$, as required.

(c) It follows from the above that

$$\begin{aligned} ev'_\mu(\xi(d^0) + d^0)(c + d^\mu) &= ev_\mu(d^0)(c + d^\mu) = [\xi(d^0) + d^0, c + d^\mu]_E \\ &= (d^0 \cdot c - d^\mu \cdot \xi(d^0)) + ev_\mu(d^0)d^\mu, \end{aligned}$$

whence $ev_\mu(d^0)c = d^0 \cdot c - d^\mu \cdot \xi(d^0)$. Substituting decomposion $c = \sum_{\lambda \in \Lambda} c^\lambda$ we have

$$ev_\mu(d^0)c^\lambda = ev_\lambda(d^0)c^\lambda - (d^\mu \cdot \xi(d^0))^\lambda$$

for every $\lambda \in \Lambda$, hence

$$(d^\mu \cdot \xi(d^0))^\lambda = ev_{\lambda-\mu}(d^0)c^\lambda. \tag{7.2.2.1}$$

If $\lambda \neq \mu$ there exists $d^0 \in D^0$ such that $ev_{\lambda-\mu}(d^0) \neq 0$. Thus c^λ is defined uniquely by (7.2.2.1) (uniqueness follows from the argument at the end of part (a)). As for $\lambda = \mu$ we note that the μ-component of $d^\mu \cdot \xi(d^0)$ is zero, because $\xi(d^0) \in C^{\neq 0}$, and so no conditions on c^μ.

We now define $D^\mu \to C^{\neq \mu}$ by (7.2.2.1) and then extend it linearly to all of $\psi : D \to C$. It follows from the above discussions that ψ has all the required properties.

(d) ϕ is clearly bijective and takes $C^0 \oplus D^0$ onto $C^0 \oplus V$. Let us check that ϕ is a homomorphism. Let $c_1, c_2 \in C$. Since $[C, C]_E = 0$ we obviously have $\phi([c_1, c_2]) = [\phi(c_1), \phi(c_2)]$. Let now $c \in C$ and $d \in D$. Since $\phi(c) = c$ and $[C, D]_E \subset C$ the equality

$$\phi([c, d]_E) = [\phi(c), \phi(d)]_E$$

holds if and only if

$$-d \cdot c = [c, \psi(d) + d]_E = -d \cdot c.$$

To finish the proof it remains to check that

$$\phi([d^\lambda, d^\mu]_E) = [\phi(d^\lambda), \phi(d^\mu)]_E$$

for all λ, μ and all nonzero d^μ, d^λ.

Case (1): $[d^\lambda, d^\mu]_D \neq 0$. Notice that by construction $\phi(d^\mu) = \psi(d^\mu) + d^\mu$ is the unique ev'_μ-eigenvector for V whose projection on D is d^μ and $\psi(d^\mu) \in C^{\neq\mu}$. Then we have

$$[\phi(d^\lambda), \phi(d^\mu)]_E = (d^\lambda \cdot \psi(d^\mu) - d^\mu \cdot \psi(d^\lambda) + \tau(d^\lambda, d^\mu)) + [d^\lambda, d^\mu]_D$$

is an eigenvector of weight $ev'_{\lambda+\mu}$ since this holds for the left side. Its $C^{\lambda+\mu}$-component is $\tau(d^\lambda, d^\mu)$ since

$$\psi(d^\lambda) \in C^{\neq\lambda}, \quad \psi(d^\mu) \in C^{\neq\mu}, \quad \tau(d^\lambda, d^\mu) \in C^{\lambda+\mu} \text{ and } D^\alpha \cdot C^\beta \subset C^{\alpha+\beta}.$$

It follows that

$$\begin{aligned} [\phi(d^\lambda), \phi(d^\mu)]_E - \tau(d^\lambda, d^\mu) &= (d^\lambda \cdot \psi(d^\mu) - d^\mu \cdot \psi(d^\lambda)) + [d^\lambda, d^\mu]_D \\ &= \psi([d^\lambda, d^\mu]_D) + [d^\lambda, d^\mu]_D \\ &= \phi([d^\lambda, d^\mu]_E) - \tau(d^\lambda, d^\mu). \end{aligned}$$

Case (2): $[d^\lambda, d^\mu]_D = 0$. As before,

$$[\phi(d^\lambda), \phi(d^\mu)]_E = (d^\lambda \cdot \psi(d^\mu) - d^\mu \cdot \psi(d^\lambda) + \tau(d^\lambda, d^\mu)) + 0$$

is an eigenvector for V with weight $ev'_{\lambda+\mu}$. Also,

$$\phi([d^\lambda, d^\mu]_E) = \phi(\tau(d^\lambda, d^\mu)) = \tau(d^\lambda, d^\mu).$$

Hence ϕ is an automorphism if and only if

$$d^\lambda \cdot \psi(d^\mu) = d^\mu \cdot \psi(d^\lambda) \tag{7.2.2.2}$$

holds for all $d^\lambda \in D^\lambda, d^\mu \in D^\mu$ with $[d^\lambda, d^\mu]_D = 0$.

Both sides of (7.2.2.2) are contained in C. To check this condition we compare their homogeneous components with respect to the decomposition $C = \bigoplus C^\rho$.

Both sides do not contain components in $C^{\lambda+\mu}$. Let now $\rho \neq \lambda + \mu$. Recall that by construction of ψ if $\tau \in \Lambda$ and $d^0 \in D^0$ then

$$\begin{aligned} ev_{\tau-\mu}(d^0)\psi(d^\mu)^\tau &= d^\mu \cdot (\xi(d^0)^{\tau-\mu}) \;\; \text{if} \;\; \tau \neq \mu, \\ ev_{\tau-\lambda}(d^0)\psi(d^\lambda)^\tau &= d^\lambda \cdot (\xi(d^0)^{\tau-\lambda}) \;\; if \;\; \tau \neq \lambda. \end{aligned}$$

Choose $d^0 \in D^0$ such that the scalar

$$e^{-1} := ev_{\rho-\lambda-\mu}(d^0) \neq 0.$$

Then with the use of $[d^\lambda, d^\mu]_D = 0$ we get

$$\begin{aligned} d^\lambda \cdot \psi(d^\mu)^{\rho-\lambda} &= d^\lambda \cdot (e\, d^\mu \cdot \xi(d^0)^{\rho-\lambda-\mu}) \\ &= e\, d^\mu \cdot (d^\lambda \cdot \xi(d^0)^{\rho-\lambda-\mu}) \\ &= e\, d^\mu \cdot (ev_{\rho-\lambda-\mu}(d^0)\psi(d^\lambda)^{\rho-\mu}) \\ &= d^\mu \cdot \psi(d^\lambda)^{\rho-\lambda}, \end{aligned}$$

thus proving 7.2.2.2. □

7.2.3 Theorem. *The map $\nu : E \to E$ given by*

$$\nu(l + c + d) = l + \phi(c + d) = l + (c + \psi(d)) + d$$

for all $l \in L,\ c \in C,\ d \in D$ is an automorphism of E such that $\nu(H) = H'$.

Proof. It is obvious that

$$\nu([l, c]_E) = [\nu(l), \nu(c)]_E, \;\; \nu([l_1, l_2]_E) = [\nu(l_1), \nu(l_2)]_E, \; \nu([l, d]_E) = [\nu(l), \nu(d)]_E$$

for all $l, l_1, l_2 \in L$, $c \in C$ and $d \in D$. Now the rest follows from Lemma 7.2.2, part (d). □

Chapter 8

A counterexample to conjugacy

The aim of this section is to give a counterexample to the conjugacy of maximal k-ad-diagonalizable subalgebras in extended affine Lie algebras.

But first we will give another example of an EALA; we will use it for a construction of a counterexample.

8.1 One more example of an Extended Affine Lie Algebra

Let $Q = (t_1, t_2)$ be a quaternion Azumaya algebra over $R = k[t_1^{\pm 2}, t_2^{\pm 2}]$. Thus, it has generators $\imath, \jmath$ and relations $\imath^2 = t_1, \jmath^2 = t_2$ and $\imath\jmath = -\jmath\imath$.

Recall, that a unital associative Λ-graded k-algebra $A = \oplus_{\lambda\in\Lambda} A^\lambda$ is called an *associative Lie torus of type* Λ if it satisfies the following axioms:

(AT1): Every non-zero A^λ contains an invertible element.

(AT2): $\dim A^\lambda \leq 1$ for all $\lambda \in \Lambda$.

(AT3): $span_{\mathbb{Z}}(supp_\Lambda A) = \Lambda$.

One easily checks that Q has a structure of an associative torus ([Ne3], Definition 4.20) of type $\mathbb{Z}^2$. Indeed,

$$Q = \bigoplus_{a,b\in\mathbb{Z}} Q^{(a,b)}$$

is a $\mathbb{Z}^2$-graded Lie algebra over k with one-dimensional graded components $Q^{(a,b)} = k\imath^a\jmath^b$, $a, b \in \mathbb{Z}$.

In general, if A is an associative torus of type M then $sl_n(A)$ is a Lie torus of type (A_{n-1}, M) (see, for example, [Ne5], Example 4.21). Thus, $L = sl_2(Q)$ is a Lie torus of type $(A_1, \mathbb{Z}^2)$. In particular, it has a double grading such that

$$L_0^{(a,b)} = k \begin{bmatrix} \imath^a\jmath^b & 0 \\ 0 & -\imath^a\jmath^b \end{bmatrix}$$

if a, b are even and

$$L_0^{(a,b)} = k \begin{bmatrix} \imath^a\jmath^b & 0 \\ 0 & -\imath^a\jmath^b \end{bmatrix} \oplus k \begin{bmatrix} \imath^a\jmath^b & 0 \\ 0 & \imath^a\jmath^b \end{bmatrix}$$

otherwise; also we have

$$L_{-2}^{(a,b)} = k \begin{bmatrix} 0 & 0 \\ \imath^a\jmath^b & 0 \end{bmatrix},$$

$$L_{2}^{(a,b)} = k \begin{bmatrix} 0 & \imath^a\jmath^b \\ 0 & 0 \end{bmatrix},$$

where $a, b \in \mathbb{Z}$.

An invariant bilinear form $(-,-)$ on L is given by

$$\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \right) = (x_{11}y_{11} + x_{12}y_{21} + x_{21}y_{12} + x_{22}y_{22})_0,$$

where a_0 for $a \in Q$ denotes the Q^0-component of a. This form is symmetric, nondegenerate, invariant, graded, i.e. $(L_\xi^\lambda, L_\tau^\mu) = 0$ if $\lambda + \mu \neq 0$ or $\xi + \tau \neq 0$. (Recall, that a form $(-,-)$ is invariant if $([a,b],c) = (a,[b,c])$ for all a, b, c.)

Choose $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, k)$ such that $\theta(1,0) = 1$ and $\theta(0,1)$ is not in $\mathbb{Q}$. Consider $d = d_\theta \in \mathrm{End}_k(L)$ where $d(l) = \theta(\lambda)l$ for $l \in L^\lambda$, $\lambda \in \mathbb{Z}^2$. Let $D = kd$ and $C = D^* = \mathrm{Hom}_k(D, k)$. Finally we set $E = L \oplus C \oplus D$ and equip it with the multiplication given by

$$[l_1 \oplus c_1 \oplus d_1, l_2 \oplus c_2 \oplus d_2]_E = ([l_1, l_2]_L + d_1(l_2) - d_2(l_1)) \oplus \sigma(l_1, l_2).$$

It follows easily from 2.4, that E is an extended affine Lie algebra.

The symmetric bilinear form on E given by

$$(l_1 \oplus c_1 \oplus d_1, l_2 \oplus c_2 \oplus d_2) = (l_1, l_2) + c_1(d_2) + c_2(d_1)$$

is nondegenerate and invariant.

8.2 Construction of the Counterexample

Let $Q = (t_1, t_2)$ be as above and let $A = M_2(Q)$. We may view A as the Q-endomorphism algebra of a free right Q-module $V = Q \oplus Q$ of rank 2.

Let $m : V = Q \oplus Q \to Q$ be a Q-linear map given given by

$$(u, v) \mapsto (1 + \imath)u - (1 + \jmath)v.$$

Denote its kernel by W. It is shown in [GP2] that m is split and that W is a projective Q-module of rank 1 which is not free. Since m is split there is a decomposition $V = W \oplus U$ where U is a free Q-module of rank 1.

Let $s \in A$ be the matrix (in the standard basis) of the Q-linear endomorphism $\mathcal{S}$ of V given by $\mathcal{S}(w) = -w$ and $\mathcal{S}(u) = u$, $w \in W$, $u \in U$.

8.2.1 Lemma. $\mathrm{ad}(s) : A \to A$ *is a k-diagonalizable operator.*

Proof. Notice that there is a canonical isomorphism

$$\mathrm{End}_Q(W \oplus U) \cong \mathrm{End}_Q(W) \oplus \mathrm{Hom}_Q(W, U) \oplus \mathrm{Hom}_Q(U, W) \oplus \mathrm{End}_Q(U).$$

Let $\phi \in \mathrm{End}_Q(W)$. Then

$$[\phi, S] = \phi \circ S - S \circ \phi = -\phi - (-\phi) = 0.$$

Similarly, if $\phi \in \mathrm{End}_Q(W)$ then $[\phi, S] = 0$. It follows

$$\mathrm{End}_Q(W) \oplus \mathrm{End}_Q(U) \subset \mathrm{End}_Q(V)_0.$$

Let now $\phi \in \mathrm{Hom}_Q(W, U)$. Then

$$[\phi, S] = \phi \circ S - S \circ \phi = -\phi - \phi = -2\phi$$

implying

$$\mathrm{Hom}_Q(W,U) \subset \mathrm{End}_Q(V)_{-2}.$$

Similarly,

$$\mathrm{Hom}_Q(U,W) \subset \mathrm{End}_Q(V)_2$$

and the assertion follows. $\square$

Since $s \in sl_2(Q) \subset A$ we get immediately

8.2.2 Corollary. *s is a k-ad-diagonalizable element of $sl_2(Q)$ whose eigenvalues are $0, \pm 2$.*

Let E be the EALA from Example 8.1 so that $L = sl_2(Q)$. Our next goal is to show that s is k-ad-diagonalizable considered as an element of E. The following lemma shows that s is k-ad-diagonalizable as an element of E_c.

8.2.3 Lemma. *Let $s \in L$ be a k-ad-diagonalizable element in L, i.e.*

$$L = \bigoplus_\alpha L_\alpha,$$

where as usual

$$L_\alpha = \{x \in L \mid [s,x]_L = \alpha x\}.$$

Then it is also k-ad-diagonalizable viewed as an element of E_c.

Proof. We denote by L_α the eigenspace of $\mathrm{ad}\, s$ with eigenvalue $\alpha \in k$. For $l_\alpha \in L_\alpha$ and $l_\beta \in L_\beta$ we have $\sigma(s, [l_\alpha, l_\beta]_L) = (\alpha+\beta)\sigma(l_\alpha, l_\beta)$, since

$$\begin{aligned} 0 &= \sigma(s,[l_\alpha,l_\beta]) + \sigma(l_\alpha,[l_\beta,s]) + \sigma(l_\beta,[s,l_\alpha]) \\ &= \sigma(s,[l_\alpha,l_\beta]) - \beta\sigma(l_\alpha,l_\beta) + \alpha\sigma(l_\beta,l_\alpha). \end{aligned}$$

Hence

$$\begin{aligned} \big[s,[l_\alpha,l_\beta]_{E_c}\big]_{E_c} &= \big[s,[l_\alpha,l_\beta]_L + \sigma(l_\alpha,l_\beta)\big]_{E_c} = \big[s,[l_\alpha,l_\beta]_L\big]_{E_c} \\ &= \big[s,[l_\alpha,l_\beta]_L\big]_L + \sigma(s,[l_\alpha,l_\beta]_L) = (\alpha+\beta)[l_\alpha,l_\beta]_{E_c} \end{aligned}$$

proving $[L_\alpha, L_\beta]_{E_c} \subset (E_c)_{\alpha+\beta}$. It then follows from $E_c = [E_c,E_c]_{E_c} = [L,L]_{E_c} = \sum_{\alpha,\beta\in k}[L_\alpha,L_\beta]_{E_c} \subset \sum_{\gamma\in k}(E_c)_\gamma$ that E_c is spanned by eigenvectors of $\mathrm{ad}\, s$, whence the result. $\square$

Let $d \in D$ and let $[s,d]_E = y = y_0 + y_2 + y_{-2}$ where $y_\lambda \in L_\lambda$.

8.2.4 Lemma. *One has $y_0 = 0$.*

Proof. Assume $y_0 \neq 0$. Since $(-,-)|_{L_0}$ is nondegenerate there is $u \in L_0$ such that $(u, y_0) \neq 0$. Then taking into consideration the fact that $(L_0, L_2) = (L_0, L_{-2}) = 0$ we get

$$0 \neq (y_0, u) = (y_0 + y_2 + y_{-2}, u) = ([s,d]_E, u) = -(d, [s,u]_E).$$

But it follows from Lemma 8.2.3, that $[s,u]_E = 0$, – a contradiction. □

Let $d' = d - \frac{1}{2}y_2 + \frac{1}{2}y_{-2}$.

8.2.5 Lemma. *One has $[s,d']_E = 0$.*

Proof. We first observe that

$$\begin{aligned} [s,d']_E &= [s, d - \tfrac{1}{2}y_2 + \tfrac{1}{2}y_{-2}]_E \\ &= y - \tfrac{1}{2}[s,y_2]_E + \tfrac{1}{2}[s,y_{-2}]_E \\ &= y - y_2 - \tfrac{1}{2}\psi(s,y_2) - y_{-2} + \tfrac{1}{2}\psi(s,y_{-2}) \\ &= \tfrac{1}{2}\psi(s, y_{-2} - y_2) \in C. \end{aligned}$$

Also, using the invariance of the form $(-,-)$ we get

$$([s,d']_E, d') = (s, [d',d']_E) = (s,0) = 0.$$

But $(c,d) = c(d)$ is 0 if and only if $c = 0$. Then since $[s,d']_E \in C$ it follows that $[s,d']_E = 0$. □

8.2.6 Corollary. *s is a k-ad-diagonalizable element of E.*

Proof. This follows from Lemma 8.2.3 and Lemma 8.2.5. □

Let $p = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ where $a \in k^\times$ is an arbitrary scalar and $H = k \cdot p \oplus C \oplus D$ be the "standard" MAD in E. Let H' be any MAD of E which contains s (such a MAD does exist). We are going to show that H and H' are not conjugate in E. We will need some auxiliary lemmas.

8.2.7 Lemma. *Two elements s and p in $sl_2(Q)$ are not conjugate by an R-linear automorphism of $sl_2(Q)$.*

Proof. Any R-linear automorphism of $sl_2(Q)$ is a conjugation with some matrix in $\mathrm{GL}_2(Q)$ or a map

$$\pi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{bmatrix}^{-1}$$

(which is a nontrivial outer R-automorphism of $sl_2(Q)$) followed by a conjugation. Assume that $\phi(p) = s$ for some $\phi \in \mathrm{Aut}_{R-Lie}(sl_2(Q))$.

Case 1: ϕ is a conjugation. Then the eigenspaces in $V = Q \oplus Q$ of the Q-linear transformation $\phi(p)$ are free Q-modules of rank 1 (because they are images of those of p). Since W is an eigenspace of s which is not free Q-module we get a contradiction.

Case 2: ϕ is π followed by a conjugation. But $\pi(p) = p$, hence we are reduced to the previous case. □

We may identify $R = \mathrm{Ctd}(sl_2(Q))$.

8.2.8 Lemma. *s and p are not conjugate by a k-automorphism of $sl_2(Q)$.*

Proof. Assume the contrary. Let $\phi \in \mathrm{Aut}_{k-Lie}(sl_2(Q))$ be such that $\phi(p) = s$. It induces an automorphism $C(\phi) : R \to R$ of the centroid of $sl_2(Q)$. Consider a new Lie algebra $L' = sl_2(Q) \otimes_\phi R$ over R. As a set it coincides with $sl_2(Q)$. Also, the Lie bracket in L' is the same as in $sl_2(Q)$, but the action of R on L' is given by the composition of $C(\phi)$ and the standard action of R on $sl_2(Q)$. Thus we have a natural k-linear Lie algebra isomorphism

$$\psi : L' = sl_2(Q) \otimes_\phi R \to sl_2(Q)$$

which takes p into p. It follows from the construction that $\phi \circ \psi : L' \to sl_2(Q)$ is an R-linear isomorphism.

Note that since the action of R on $sl_2(Q)$ is componentwise we have a natural identification

$$L' = sl_2(Q) \otimes_\phi R \cong sl_2(Q \otimes_\phi R)$$

and it easily follows from the construction that $Q \otimes_\phi R$ is a quaternion algebra $(\phi(t_1), \phi(t_2))$ over R. Thus $sl_2(Q)$ and $sl_2(Q \otimes_\phi R)$ are R-isomorphic and they are R-forms of $sl_4(R)$. Moreover they are inner forms, hence correspond to an element $[\xi] \in H^1(R, \mathrm{PGL}_4)$.

The boundary map $H^1(R, \mathrm{PGL}_4) \to H^2(R, \mathbf{G}_m)$ maps $[\xi]$ to the Brauer equivalence class of both Q and $Q \otimes_\phi R$. Since $[Q] = [Q \otimes_\phi R]$ it follows that there is an R-algebra isomorphism $\overline{\theta} : Q \otimes_\phi R \to Q$ which in turn induces a canonical R-Lie algebra isomorphism

$$\theta : sl_2(Q \otimes_\phi R) \to sl_2(Q)$$

by componentwise application of $\overline{\theta}$. Clearly, $\theta(p) = p$.

Finally, consider an R-linear automorphism

$$\phi' = \phi \circ \psi \circ \theta^{-1} \in \mathrm{Aut}_{R-Lie}(sl_2(Q)).$$

We have

$$\phi'(p) = \phi(\psi(\theta^{-1}(p))) = \phi(\psi(p)) = \phi(p) = s$$

which contradicts Lemma 8.2.7. $\square$

8.2.9 Theorem. *There is no $\phi \in \mathrm{Aut}_{k-Lie}(E)$ such that $\phi(H') = H$.*

Proof. Assume the contrary. Let $\phi \in \mathrm{Aut}_k(E)$ be such that $\phi(H') = H$. We proved before that the core E_c is ϕ-stable. Hence

$$\phi(H' \cap E_c) = H \cap E_c = k \cdot p \oplus C. \tag{8.2.9.1}$$

Of course, $k \cdot s \oplus C \subset H' \cap E_c$, because C is the center of E. Therefore by dimension reasons we have $k \cdot s \oplus C = H' \cap E_c$.

The automorphism ϕ induces a k-automorphism $\phi_{cc} : E_{cc} \to E_{cc}$. In our example $E_{cc} = sl_2(Q)$. By (8.2.9.1), $\phi_{cc}(k \cdot s) = k \cdot p$. Hence there exists a scalar $\alpha \in k^\times$ such that $\phi_{cc}(s) = \alpha \cdot p$. But this contradicts Lemma 8.2.8. $\square$

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