Some geometric properties on Banach spaces associated to hypergroups

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

In

Mathematics

Department of Mathematical and Statistical Sciences

University of Alberta

Abstract

This thesis is dedicated to the study of some geometric properties on Banach spaces associated to hypergroups. This thesis contains three major parts.

The purpose of the first part is to initiate a systematic approach to the study of the class of invariant complemented subspaces of $L_\infty(K)$ and $C_0(K)$, the class of left translation invariant W^*-subalgebras of $L_\infty(K)$ and finally the class of non-zero left translation invariant C^*-subalgebras of $C_0(K)$ in the hypergroup context with the goal of finding some relations between these function spaces.

The second part consists of two themes; fixed point properties for non-expansive and affine maps. The first theme provides a condition when a non-expansive self map on a weak (weak*) compact convex subset of several function spaces over K has a fixed point while the second theme present some applications of common fixed point properties for affine actions of K.

The main concentration of the third part is on initiating the study of inner amenable hypergroups extending amenable hypergroups and inner amenable locally compact groups.

Acknowledgements

First and foremost, I would like to express my most sincere gratitude to my supervisor, Professor Anthony To-Ming Lau, for his constant encouragement and for his excellent guidance.

My special thank is to my yoga instructor, Mrs. Maya Machawe. Without her support, I would certainly have lost the track of my life.

I would also like to express my appreciation to Professor Kenneth A. Ross whose joy of life and immeasurable enthusiasm for mathematics have truly inspired me.

Thanks are due to the friendly staff of the Department of Mathematical and Statistical Sciences at the University of Alberta, and the financial support they provided me is gratefully acknowledged. Furthermore, I would like to acknowledge my supervisor, through his NSERC grant, for the summer support he provided me.

Last but not least, my immense thanks and gratitude for my parents, my sister and my brother, for their love and understanding. This thesis is naturally dedicated to them.

Contents

Chapter 1

Introduction and Backgrounds

1.1 Introduction

A hypergroup is a locally compact Hausdorff space equipped with a convolution product which maps any two points to a probability measure with a compact support. Hypergroups generalize locally compact groups in which the above convolution reduces to a point mass measure. It was in the 1970's that Dunkl [15], Jewett [30] and Spector [66] began the study of hypergroups (in [30] it is called a convo). The theory of hypergroups then developed in various directions, namely in the area of commutative hypergroups ([9], [16], [59], [61]), specifically orthogonal polynomials ([38] and [77]), function spaces [75] and weighted hypergroups ([21], [22]). It is worthwhile to mention that there are some axiomatic differences in the definition of hypergroups given by these three authors, however, the core idea remains the same. Since almost all of the analysis on hypergroups has been based on the definition of Jewett, we shall base our work on his definition. For a complete history, we refer the interested reader to ([60] and [61]). Throughout, K will denote a hypergroup with a left Haar measure λ.

The idea behind amenable hypergroups came from a well-known object called F-algebras, also known as Lau algebras. The class of F-algebras was created and analysed in 1983 [41]. The construction was made to generalize the notion of several algebras. This rich structure contains the Fourier algebra $A(G)$, the Fourier-Stieltjes algebra $B(G)$, the group algebra $L_1(G)$ and the measure algebra $M(G)$ of a locally compact group G. It also contains the hypergroup algebra $L_1(K)$ and the measure algebra $M(K)$. Lau introduced left amenable F-algebras ([41], § 4) and provided various characterizations of this object (see also [20] and [51]), from which one can obtain in particular that $L_1(K)$ is left amenable if and only if $L_1(K)^*$ has a topological left invariant mean ([41], Theorem 4.1). With this groundwork, the class of amenable hypergroups came to existence [64]. By ([64], Theorem 3.2) K is amenable if and only if $L_1(K)$ is left amenable. On the other hand, $L_1(K)$ demonstrates different behaviours in terms of amenability and weak amenability ([64] and [37]) in comparison with its group counterpart. See ([31], Theorem 2.5) for amenability and [32] for weak amenability of the group algebra $L_1(G)$. Amenability of hypergroups has attracted the attention of many authors ([2] and [74]).

Let $L_\infty(K)$ be the W^*-algebra of all essentially bounded measurable complex-valued functions on K with essential supremum norm and point-wise multiplication and let Y be a closed, translation invariant subspace of $L_\infty(K)$. A closed, left translation invariant subspace X of Y is said to be invariantly complemented in Y if X is the range of a continuous projection on Y, which commutes with all left translation operators on Y or equivalently if Y has a closed left translation complement in X. This concept was introduced by Lau [42] for locally compact groups and was studied in ([19] and [18]). However, this area was open in the theory of hypergroups on which we started an investigation. Motivated by the harmonic analysis considered by Lau [42] and Lau-Losert [46], we initiate the study of the class of invariant complemented subspaces of $L_\infty(K)$ and $C_0(K)$, the class of non-zero left translation invariant C^*-subalgebras of $C_0(K)$ and finally the class of left translation invariant W^*-subalgebras of $L_\infty(K)$ in the hypergroup context with the goal of finding some relations between these function spaces.

Let X be a left translation invariant W^*-subalgebra of $L_\infty(K)$. Takesaki and Tatsuuma in 1971 showed that $X = \{f \in L_\infty(K) \mid R_n f = f,\ \forall n \in N\}$, for a unique closed subgroup N of a locally compact group K ([70], Theorem 2, see also [42], Lemma 3.2 for a different proof). A decade later, Lau ([42], Theorem 3.3) proved that K is amenable if and only if X is invariantly complemented in $L_\infty(K)$, where K is a locally compact group.

In section 2.2 we shall initiate a formal study of the class of left translation invariant W^*-subalgebras of $L_\infty(K)$ in the hypergroup setting. In the process of building a bridge between this class and closed subhypergroups, by the nature of our framework, we encounter a feature which is dormant in the group context. We note that the constructed W^*-subalgebra X has a certain property that we assume for obtaining a reasonable correspondence. This new notion, "local translation property TB", extends the notion of translation property TB, which was considered by Voit [73]. We say that X has the local translation property TB if for each element $k_0 \in K$

$$\begin{aligned} & k_0 * \{g \in K \mid f|_{k*g} \equiv f(k), \quad \forall f \in X, k \in K\} \\ & = \{g \in K \mid R_g f = R_{k_0} f, \quad \forall f \in X\}. \end{aligned}$$

After providing this definition we demonstrate our main theorem of this section and we prove that X is a left translation invariant W^*-subalgebra of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB if and only if there exists a unique closed Weil subhypergroup N such that $X = \{f \in L_\infty(K) \mid R_g f = R_k f, \quad \forall g \in k * N, \quad k \in K\}$. Furthermore, the normality of N is characterized by X being translation invariant and inversion invariant (Theorem 2.2.5). For compact hypergroups we even have $X = \{f \in L_\infty(K) \mid R_n f = f, \ \forall n \in N\}$, for a unique compact subhypergroup. As a consequence, then we prove that every left translation invariant W^*-subalgebra of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB is invariantly complemented in $L_\infty(K)$, where K is a compact hypergroup (Corollary 2.2.6).

Let X be a non-zero left translation invariant C^*-subalgebra of $C_0(K)$. DeLeeuw ([12], Theorem 5.1) proved that X is the algebra $C_0(K/N)$, for some subgroup N of K, if K is a commutative locally compact group. Lau-Losert

([46], Lemma 12) extended his result to any locally compact group K. In section 2.3, we commence an investigation of the class of non-zero left translation invariant C^*-subalgebras of $C_0(K)$, when K is furnished with a hypergroup structure. We first set up a basis by providing a characterization of hypergroups admitting an invariant mean on the space $WAP(K)$, the space of all continuous weakly almost periodic functions on K (Lemma 2.3.2). As another fundamental result, we endow X with the local translation property TB and we prove that X is a non-zero left translation invariant C^*-subalgebra of $C_0(K)$ with the local translation property TB if and only if there exists a unique compact subhypergroup N of K such that $X = \{f \in C_0(K) \mid R_n f = f, \ \ \forall n \in N\}$ (Lemma 2.3.5). Then in one of our major results we show in particular that X is invariantly complemented in $C_0(K)$ provided that X has the local translation property TB (Theorem 2.3.6).

Let C be a non-empty closed bounded convex subset of a Banach space E. Then C is said to have the fixed point property if every nonexpansive mapping from C into C has a fixed point. It is shown by Browder ([8], Theorem 1) that a nonempty closed bounded convex subset of a uniformly convex Banach space has the fixed point property. However, not every non-empty closed bounded convex subset of l_1 has the fixed point property [34]. In addition, there is a weakly compact convex subset D of $L_1[0,1]$ and an isometry T from D into D without a fixed point ([1], see also [5], Theorem 4.2 and [4]). In 1965 Kirk proved that if C is weakly compact convex and has normal structure, then C has the fixed point property [33]. Furthermore, if E is a dual Banach space, C is weak*-compact convex and has normal structure, then C has the fixed point property [54]. Therefore, it is useful to determine whether a (dual) Banach space E has weak (weak*)-normal structure (every weakly (weak*) compact convex subset of E has normal structure). In 1988, Lau and Mah studied conditions under which various dual of function spaces over a locally compact group G have weak*-normal structure or certain geometric properties related to weak*-normal structure [48] (for other related geometric properties of Banach spaces see [71]). Inspired by their work, in section 3.2 we shall initiate a formal study of interconnection between the structure of a hypergroup and weak*-normal structure and the weak (weak*) fixed point property (every weakly (weak*) compact convex subset has the fixed point property) for (dual of) function spaces or algebras associated with the hypergroup. The central result of this section states that if weak* convergence and weak convergence for sequences agree on the unit sphere of X^*, then K is discrete, here X is a closed subspace of $CB(K)$ containing $C_0(K)$ (Theorem 3.2.1). With the help of Theorem 3.2.1, among other results we provide a necessary and sufficient condition for $M(K)$ to have weak*-normal structure or the weak* fixed point property, for $L_1(K)$ ($M(K)$, where K is a separable hypergroup) to have the weak (weak*) fixed point property for left reversible semigroups (Corollaries 3.2.3 and 3.2.4).

Let (E, τ) be a separated locally convex space. In 1973 Lau [43] gave a characterization of topological semigroups S for which $AP(S)$ has a left invariant mean; if S acts on a compact convex subset C of a locally convex topological vector space (E, τ) and the action is separately continuous equicontinuous and affine, then C contains a common fixed point for S ([43], Theorem 3.2, see also [28]). Moreover, in 2008 Lau and Zhang [52] characterized separable semitopological semigroups S having a left invariant mean on the space $WAP(S)$; whenever S acts on a weakly compact convex subset Y of (E, τ) and the action is weakly separately continuous, weakly quasi-equicontinuous and τ-nonexpansive, Y has a common fixed point for S ([52], Theorem 3.4). One of the purposes of section 3.3 is to commence an investigation on common fixed point properties of affine actions of a hypergroup K possessing an invariant mean on the space $WAP(K)$ or $AP(K)$. We prove in Lemma 3.3.1 that $AP(K)$ has an invariant mean if and only if for any separately (jointly) continuous, equicontinuous and affine action of K on a compact convex subset Y of (E, τ), Y has a common fixed point for K. In addition, we show in Theorem 3.3.4 that $WAP(K)$ has an invariant mean if and only if for any separately (jointly) weakly continuous, quasi weakly equicontinuous, weakly almost periodic linear representation $\{T_g \mid g \in K\}$ of K on (E, τ) and for any weakly compact convex T_K-invariant subset Y of E, there is in Y a common fixed point for K.

Let $LUC(K)$ be the space of all bounded left uniformly continuous complex-valued functions on K. In 1982, Lau proved that a locally compact group G is amenable if and only if every weak*-closed left translation invariant subspace of $L_\infty(G)$ which is contained and complemented in $LUC(G)$ is invariantly complemented in $LUC(G)$ ([42], Corollary 4.4). Four years later, Lau showed in ([47]) in particular that a locally compact group G is amenable if and only if every weak*-closed left translation invariant complemented subspace of $L_\infty(G)$ is invariantly complemented in $L_\infty(G)$ ([47], Corollary 2). With these backgrounds, we also launch in section 3.3 an investigation to find similar outcomes when the ground work is a hypergroup. As a result, we provide three important applications of common fixed point properties for affine actions on an amenable hypergroup (Lemma 3.3.5 and Theorems 3.3.7 and 3.3.11). Being equipped with these fertile seeds, we conclude that K is amenable if and only if for every weak*-closed left translation invariant subspace X of $L_\infty(K)$ which is contained and complemented in $LUC(K)$ with norm $\leq \gamma$, there is a bounded linear operator P from $LUC(K)$ into X with $||P|| \leq \gamma$ such that $Pf \in \overline{co(L_K f)}^{W^*}$, for $f \in X$ and that $PL_g = P$, for all $g \in K$ (Corollary 3.3.6), K is amenable if and only if for every weak*-closed left translation invariant complemented subspace X of $L_\infty(K)$, there is a bounded linear operator P from $L_\infty(K)$ into X such that $Pf \in \overline{co(L_K f)}^{W^*}$, for $f \in X$ and that $PL_g = P$, for all $g \in K$ (Corollary 3.3.12). As another application of Theorem 3.3.7 we state in Corollary 3.3.10 that if A and X are closed left translation invariant subspaces of $L_p(K)$ with $1 < p < \infty$ such that X is contained and complemented in A, then there exists a continuous projection P from A onto $\{f \in X \mid L_n f = f, \ \ \forall n \in N\}$ such that $PL_g = P$, for all $g \in K$, where N is a closed Weil subhypergroup of an amenable hypergroup K.

Inner amenable locally compact groups G are ones possessing a mean m on $L_\infty(G)$ such that $m(R_g L_{g^{-1}} f) = m(f)$, for all $f \in L_\infty(G)$ and $g \in G$. This concept was introduced by Effros in 1975 for discrete groups and was studied by several authors ([11], [10], [17], [49], [50], [56] and [57]). It has been shown by Losert and Rindler [56] that the existence of an inner invariant mean on $L_\infty(G)$ is equivalent to the existence of an asymptotically central net in $L_1(G)$ which is in the case of groups equivalent to the existence of a quasi central net in $L_1(G)$. In section 4.2 we define the notion of inner amenable hypergroups extending amenable hypergroups and inner amenable locally compact groups. We say that a hypergroup K is inner amenable and m is an inner invariant mean if m is a mean on $L_\infty(K)$ and $m(L_g f) = m(R_g f)$ for all $f \in L_\infty(K)$ and all $g \in K$. An inner invariant mean m on a discrete hypergroup K is nontrivial if $m(f) \neq f(e)$ for $f \in l_\infty(K)$. In the process of constructing a discrete hypergroup with no nontrivial inner invariant mean we also define the concept of strong ergodicity of an action of a locally compact group on a hypergroup. Then we prove a relation between nontrivial inner invariant means on bounded functions of the semidirect product $K \rtimes_\tau G$ of a discrete hypergroup K and a discrete group G and strong ergodicity of the action τ; If K is commutative and τ is not strongly ergodic, then $l_\infty(K \rtimes_{\tau|_S} S)$ possesses a nontrivial inner invariant mean for each subgroup S of G, however, if τ is strongly ergodic and $l_\infty(G)$ has no nontrivial inner invariant mean, then $l_\infty(K \rtimes_\tau G)$ has no nontrivial inner invariant mean (Theorem 4.2.4). Then we prove that inner amenability is an asymptotic property; there is a positive norm one net $\{\phi_\alpha\}$ in $L_1(K)$ such that $||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0$, for all $g \in K$ if and only if K is inner amenable (Lemma 4.2.1), while the existence of a positive norm one net $\{\phi_\alpha\}$ in $L_2(K)$ such that $||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2 \to 0$, for all $g \in K$ only implies the inner amenability of K (Lemma 4.2.5) and implies the existence of a state m on $B(L_2(K))$ such that $m(L_g) = m(\Delta^{\frac{1}{2}}(g)R_g)$, for all $g \in K$ (Theorem 4.2.6). Furthermore, in Corollary 4.2.8 we characterize inner amenability of a hypergroup K in terms of compact operators; K is inner amenable if and only if there is a non-zero compact operator T in $B(L_\infty(K))$ such that $TL_g = TR_g$, for all $g \in K$ and $T(f) \geq 0$, for $f \geq 0$.

Classical Hahn-Banach extension theorem and monotone extension property are well known and are widely used in several areas of mathematics. As one deals with (positive normalized) anti-actions of a semigroup on a real (partially ordered) topological vector space (with a topological vector unit), it is also interesting to know the condition under which the extension of an invariant (monotonic) linear functional is also invariant (and monotonic). In 1974 Lau characterized left amenable semigroups with these properties ([40], Theorems 1 and 2). In section 4.3 we shall be concerned about hypergroup version of Hahn-Banach extension and monotone extension properties and we prove in Theorem 4.3.1 that $RUC(K)$ has a right invariant mean if and only if whenever $\{T_g \in B(E) \mid g \in K\}$ is a separately continuous representation of K on a Banach space E and F is a closed T_K-invariant subspace of E. If p is a continuous seminorm on E such that $p(T_g x) \leq p(x)$ for all $x \in E$ and $g \in K$ and Φ is a continuous T_K-invariant linear functional on F such that $|\Phi(x)| \leq p(x)$, then there is a continuous T_K-invariant linear functional $\tilde{\Phi}$ on E extending Φ such that $|\tilde{\Phi}(x)| \leq p(x)$, for all $x \in E$, if and only if for any positive normalized separately continuous linear representation $\mathscr{T}$ of K on a partially ordered real Banach space E with a topological order unit 1, if F is a closed $\mathscr{T}$-invariant subspace of E containing 1, and Φ is a $\mathscr{T}$-invariant monotonic linear functional on F, then there exists a $\mathscr{T}$-invariant monotonic linear functional $\tilde{\Phi}$ on E extending Φ. The three statements above are also equivalent to an algebraic property: for any positive normalized separately continuous linear representation $\mathscr{T}$ of K on a partially ordered real Banach space E with a topological order unit 1, E contains a maximal proper $\mathscr{T}$-invariant ideal. As an application of these important geometric properties we provide a new proof of the known result; if K is a commutative hypergroup, then $UC(K)$ has an invariant mean (Corollary 4.3.2).

Let X be a weak*-closed left translation invariant subspace of $L_\infty(K)$. The concentration of section 4.4 is mainly on weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations. It turns out that similar to the locally compact groups ([42], Lemma 5.2), if X is an invariant complemented subspace of $L_\infty(K)$, then there is a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations if and only if $X \cap C_0(K)$ is weak*-dense in X (Theorem 4.4.1). This theorem has a major consequence; if K is compact, then X is invariantly complemented in $L_\infty(K)$ if and only if there is a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations (Corollary 4.4.2). Furthermore, we also characterize compact hypergroups; K is compact if and only if K is amenable and for every weak*-closed left translation invariant, invariant complemented subspace X of $L_\infty(K)$, there exists a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations (Corollary 4.4.4).

Finally, in chapter 5 we provide some related remarks and open problems.

1.2 Notations

Throughout, K will denote a hypergroup with a left Haar measure λ. For basic notations we refer to Jewett [30] and the book of Bloom and Heyer [7]. The involution on K is denoted by $x \mapsto \check{x}$.

Let $L_\infty(K)$ be the W^*-algebra of all essentially bounded measurable complex-valued functions on K with essential supremum norm and point-wise multiplication and let $CB(K)$ denote the Banach space of all bounded continuous complex-valued functions on K and $C_c(K)$ denote the space of all continuous bounded functions on K with compact support. Let $LUC(K)$ ($RUC(K)$) be the space of all bounded left (right) uniformly continuous functions on K, i.e. all $f \in CB(K)$ such that the map $g \mapsto L_g f$ ($g \mapsto R_g f$) from K into $CB(K)$ is continuous when $CB(K)$ has the norm topology and let $AP(K)$ ($WAP(K)$) denote the space of continuous (weakly) almost periodic functions on K, that is the collection of all $f \in CB(K)$ for which the set $\{L_g f \mid g \in K\}$ is relatively compact in the norm (weak) topology of $CB(K)$. Each of the spaces $AP(K)$, $WAP(K)$ and $LUC(K)$ is a norm closed, conjugate closed, translation invariant subspace of $CB(K)$ containing constant functions and we have that $AP(K), C_0(K) \subseteq WAP(K) \subseteq LUC(K)$ [65].

We denote by $W^*.o.t$, the weak*-operator topology on $B(X^*)$ determined by the family of seminorms:

$$\{p_{z,\phi} \mid z \in X^*,\ \phi \in X\}, \quad \text{where } p_{z,\phi}(T) := |< Tz, \phi >|, \quad T \in B(X^*),$$

where $B(X^*)$ is the space of bounded linear operators on X^*.

1.3 Basic definitions

Definition 1.3.1. *[30] A pair $(K, *)$ is called a hypergroup if the following are satisfied:*

1. *K is a non-empty locally compact Hausdorff space.*

2. *The symbol $*$ denotes a binary operation $(x, y) \mapsto \delta_x * \delta_y$ from $K \times K$ to $M(K)$ which extends to an operator $* : (\mu, \nu) \mapsto \mu * \nu$ from $M(K) \times M(K)$ to $M(K)$ such that for $f \in C_0(K)$,*

   $$\mu * \nu(f) = \int_K f d\mu * \nu = \int_K \int_K \int_K f d(\delta_x * \delta_y) d\mu(x) d\nu(y).$$

   This identity will then hold for all bounded Borel measurable functions f on K.

3. *With this operation, $M(K)$ is a complex (associative) algebra.*

4. *The map $(g, k) \mapsto \delta_g * \delta_k$ from $K \times K$ to $M(K)$ is continuous when $M(K)$ is given the cone topology, i.e, the weak topology $\sigma(M(K), C_c(K) \cup \{1\})$, which is the weak*-topology if and only if K is compact*

5. *$\delta_x * \delta_y$ is a probability measure for $x, y \in K$ and supp$(\delta_x * \delta_y)$ is compact.*

6. *The map $(x, y) \mapsto$ supp$(\delta_x * \delta_y)$ from $K \times K$ to $\mathscr{C}(K)$, the space of all non-empty compact subsets of K, where $\mathscr{C}(K)$ is given the Michael topology, is continuous.*

7. *There exists a unique element $e \in K$ such that $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$, for each $x \in K$.*

8. *There exists a homeomorphism* $x \mapsto \check{x}$ *of* K *such that for* $x \in K$, $\check{\check{x}} = x$,

9. *For* $x, y \in K$, $e \in supp(\delta_x * \delta_y)$ *iff* $y = \check{x}$.

Definition 1.3.2. *Let* (X, Γ) *be a Hausdorff topological space and* $\mathscr{C}(X)$ *be the space of all non-empty compact subsets of* X. *The Michael topology on* $\mathscr{C}(X)$ *is the topology generated by collections of the form* $\{A \in \mathscr{C}(X) \mid A \cap U_i \neq \emptyset$ *and* $A \subseteq \bigcup_{i=1}^{n} U_i, \quad i = 1, 2, .., n\}$, *where* $U_1, U_2, ..., U_n$ *are open subsets of* X *[30].*

Definition 1.3.3. *[30]*

1. *A hypergroup* K *is said to be commutative if* $\delta_g * \delta_k = \delta_k * \delta_g$ *for all* $g, k \in K$.

2. *If* A, B *are subsets of* K *then the set* $A * B$ *is defined by* $A * B := \cup_{\substack{x \in A \\ y \in B}} supp(\delta_x * \delta_y)$.

3. *A closed non-empty subset* H *of* K *is called a subhypergroup if* $H^{\check{}} = H$ *and* $H * H \subseteq H$.

4. *A subhypergroup* N *of* K *is said to be normal in* K *if* $g * N = N * g$ *for all* $g \in K$.

5. *A subhypergroup* H *is called a subgroup, if* $\delta_h * \delta_{\check{h}} = \delta_e = \delta_{\check{h}} * \delta_h$, *for all* $h \in H$.

6. *The maximal subgroup* $G(K)$ *is defined by* $\{g \in K \mid \delta_g * \delta_{\check{g}} = \delta_{\check{g}} * \delta_g = \delta_e\}$

Example 1.3.1.

1. *All locally compact groups are hypergroups. A hypergroup is a locally compact group precisely when the convolution product of every two point measures is again a point measure [30].*

2. *Let* G *be a locally compact group. Let* H *be a compact, non-normal subgroup of* G. *The space of double cosets,* $G//H := \{HxH \mid x \in G\}$ *with the quotient topology and the convolution given by* $\delta_{HxH} * \delta_{HyH} = \int_H \delta_{HxtyH} dt$, *for* $x, y \in K$ *is a hypergroup [30].*

3. *The cosine-hypergroup on $\mathbb{R}_+$ is given by the convolution*

$$\delta_x * \delta_y := 1/2(\delta_{x+y} + \delta_{|x-y|}), \qquad x, y \in \mathbb{R}_+.$$

4. *The cosh-hypergroup on $\mathbb{R}_+$ is given by the convolution*

$$\delta_k * \delta_m = \frac{cosh(k-m)}{2coshk.coshm}\delta_{|k-m|} + \frac{cosh(k+m)}{2coshk.coshm}\delta_{k+m}, \qquad k, m \in \mathbb{R}_+.$$

Definition 1.3.4. *Suppose H is a compact hypergroup and J is a discrete hypergroup with $H \cap J = \{e\}$ where e is the identity of both hypergroups. Let $K := H \cup J$ have the unique topology for which H and J are closed subspaces of K. Let σ be the normalized Haar measure on H and define the operation . on K as follows:*

1. *If $s, t \in H$ then $\delta_s.\delta_t := \delta_s * \delta_t$.*

2. *If $a, b \in J$ and $a \neq \check{b}$ then $\delta_a.\delta_b := \delta_a * \delta_b$.*

3. *If $s \in H$ and $a \in J$ $(a \neq e$ $)$ then $\delta_a.\delta_s = \delta_s.\delta_a = \delta_a$.*

4. *If $a \in J$ and $a \neq e$ and $\delta_{\check{a}} * \delta_a = \sum_{b\in J} \delta_{\check{a}} * \delta_a(b)\delta_b$, then*

$$\delta_{\check{a}}.\delta_a = \delta_{\check{a}} * \delta_a(e)\sigma + \sum_{t\in K\setminus\{e\}} \delta_{\check{a}} * \delta_a(t)\delta_t.$$

The hypergroup K is called the join of H and J and is written by $K = H \vee J$.

Definition 1.3.5. *A (left) Haar measure λ on a hypergroup K is a non-zero, non-negative, possibly unbounded, regular Borel measure which is left translation-invariant, i.e, $\delta_x * \lambda = \lambda$ [30].*

Remark 1.3.1. *It remains an open question whether every hypergroup admits a left Haar measure. However if K admits a left Haar measure, it is unique up to a scalar multiple by Jewett ([30], 5.2). It has been shown that every compact, commutative, or discrete hypergroup admits a left Haar measure. (Jewett 1975, Spector 1978, Jewett 1975 resp.)*

Remark 1.3.2. *In contrast to the group case, $AP(K)$ and $WAP(K)$ need not be algebras even for discrete commutative hypergroups ([75], Remark 2.8). By ([64], Remark 2.3 and Proposition 2.4) if the maximal subgroup $G(K)$ of K is open or if K is the hypergroup join of a compact hypergroup H and a discrete hypergroup J with $H \cap J = \{e\}$, where e is the identity of H and J, then $LUC(K)$ is an algebra.*

Definition 1.3.6. *For any $f \in L_\infty(K)$ we define the left (right) translation operator L_x and R_y by $R_y f(x) = L_x f(y) = \int_K f(u) d\delta_x * \delta_y(u)$, for any $x, y \in K$ if this integral exists though it may not be finite. In addition we define an inversion operator $\check{f}$ by $\check{f}(x) = f(\check{x})$ [30].*

Note that in contrast to the group case, the operators L_x and R_x are not isometry if we deal with hypergroups.

Definition 1.3.7. *Let $\phi * \mu(g) = \int R_{\check{k}} \phi(g) d\mu(k)$ and $\phi \circledast \mu(g) = \int \Delta(\check{k}) R_{\check{k}} \phi(g) d\mu(k)$, for $\mu \in M(K)$ and $\phi \in L_1(K)$. Then $(\phi \circledast \mu)\lambda = \phi\lambda * \mu$.*

We note that $\phi \circledast \mu$ is denoted by $\phi * \mu$ in the group setting.

Definition 1.3.8. *Let N be a closed subhypergroup of K. Then $K/N = \{g * N \mid g \in K\}$ is a locally compact space when it is equipped with the quotient topology. N is called a Weil subhypergroup if the mapping $f \mapsto T_N f$, where $(T_N f)(g * N) = \int R_n f(g) d\lambda_N(n)$ and λ_N is a left Haar measure on N is a well defined map from $C_c(K)$ onto $C_c(K/N)$ [26]. This map can be extended to a well-defined surjective map from $C_0(K)$ onto $C_0(K/N)$ by the density of $C_c(K)$ in $C_0(K)$.*

Remark 1.3.3. *It is well known that the class of Weil subhypergroups include the class of subgroups and the class of compact subhypergroups ([26], p 250).*

Definition 1.3.9. *If N is a closed normal subhypergroup, then K/N is said to be a hypergroup if the convolution $\delta_{g*N} * \delta_{k*N}(f) = \int f(u * N) d\delta_g * \delta_k(u)$ ($f \in C_c(K/N)$) is independent of the representatives $g * N$ and $k * N$ [72].*

Remark 1.3.4. *It has been proved that K/N is a hypergroup if and only if N is a closed normal Weil subhypergroup of K ([72], Theorems 2.3 and 2.6).*

Definition 1.3.10. *Let $(K, *)$ and $(J, .)$ be hypergroups. Then a continuous mapping $p : K \to J$ is said to be a hypergroup homomorphism if $\delta_{p(g)}.\delta_{p(k)^{\check{}}} = p(\delta_g * \delta_{\check{k}})$, for all $g, k \in K$ ([72], p 291).*

Definition 1.3.11. *Let G be a locally compact group and let τ be a continuous group homomorphism from G into the topological group $Aut(K)$ of all hypergroup homomorphisms on K (with the topology of pointwise convergence). The semidirect product $K \rtimes G$ of K and G is the locally compact space $K \times G$ equipped with the product topology, the convolution $\delta_{(k_1,g_1)} * \delta_{(k_2,g_2)} = \delta_{k_1} * \delta_{\tau_{g_1}(k_2)} \times \delta_{g_1 g_2}$ [74].*

Definition 1.3.12. *Let X be a closed translation invariant subspace of $L_\infty(K)$ containing constants. A left invariant mean on X is a positive norm one linear functional, which is invariant under left translations by elements of K. A hypergroup K is said to be amenable if there is a left invariant mean on $L_\infty(K)$ [64].*

Remark 1.3.5. *Amenable hypergroups include all compact hypergroups and all commutative hypergroups [64].*

Remark 1.3.6. *It is still an open question as to whether $WAP(K)$ ($AP(K)$) always has an invariant mean. It is clear by an application of the Ryll-Nardzewski fixed point theorem [62] that $WAP(G)$ ($AP(G)$) has a unique invariant mean when G is a locally compact group.*

Definition 1.3.13. *Let Y be a closed, translation invariant subspace of $L_\infty(K)$. A closed, left translation invariant subspace X of Y is said to be invariantly complemented in Y if X is the range of a continuous projection on Y, which commutes with all left translation operators on Y or equivalently if Y has a closed left translation invariant complement in X.*

Definition 1.3.14. *The action $\mathscr{T} = \{T_g \mid g \in K\}$ is a separately continuous representation of K on a Banach space X if $T_g : X \to X$, $T_e = I$, $||T_g|| \leq 1$, for each $g \in K$, the mapping $(g, x) \mapsto T_g x$ from $K \times X$ to X is separately* *continuous, and* $T_{g_1}T_{g_2}x = \int T_u x d\delta_{g_1} * \delta_{g_2}(u)$*, for* $x \in X$ *and* $g_1, g_2 \in K$*. If* $\mathscr{T}$ *is a separately continuous representation of* K *on* X*, then for* $g \in K$*,* $\mu \in M(K)$*,* $f \in X^*$ *and* $\phi \in X$ *define* $f \,.\, g = M_g f$ *by* $< f \,.\, g, \phi > = < f, T_g\phi >$ *and* $f \,.\, \mu = M_\mu f$ *by* $< f \,.\, \mu, \phi > = \int < f, T_g\phi > d\mu(g)$*. Then* $f \,.\, \mu \in X^*$*,* $f \,.\, \delta_g = f \,.\, g$ *and* $(f \,.\, \mu) \,.\, \nu = f \,.\, (\mu * \nu)$*, for* $\mu, \nu \in M(K)$*. We say that a subspace* Y *of* X^* *is* K*-invariant if* $Y \,.\, g \subseteq Y$*, for all* $g \in K$*. Moreover, let* $< N_g m, f > = < m, M_g f >$*,* $< N_\mu m, f > = < m, f \,.\, \mu >$ *and* $N_\phi = N_{\phi\lambda}$*, for* $\mu \in M(K)$*,* $\phi \in L_1(K)$*,* $m \in X^{**}$*,* $f \in X^*$ *and* $g \in K$*. Then* $N_\mu N_\nu = N_{\mu*\nu}$ *and* $N_\phi N_\mu = N_{\phi \circledast \mu}$*, for each* $\mu, \nu \in M(K)$*. In addition,* $||M_g|| \leq 1$*,* $||N_g|| \leq 1$*,* $||M_\mu|| \leq ||\mu||$ *and* $||N_\mu|| \leq ||\mu||$*, for all* $\mu \in M(K)$ *and* $g \in K$*.*

Remark 1.3.7. *A hypergroup* K *is amenable if and only if whenever there is a jointly (separately) continuous representation* $\mathscr{T} = \{T_g \mid g \in K\}$ *of* K *on a non-empty compact convex subset* C *of a locally convex topological vector space such that* $T_e x = x$*, for* $x \in C$*, the mapping* $g \mapsto T_g x$ *is affine for each* $x \in C$ *and* $T_g T_k x = \delta_g * \delta_k \,.\, x$*, for all* $g, k \in K$ *and* $x \in C$*, then there is a point* $x_0 \in C$ *such that* $T_g x_0 = x_0$*, for all* $g \in K$ *[65].*

Chapter 2

Hypergroups and invariant complemented subspaces.[1]

2.1 Introduction

The purpose of the present chapter is to initiate a systematic approach to the study of the class of invariant complemented subspaces of $L_\infty(K)$ and $C_0(K)$, the class of left translation invariant W^*-subalgebras of $L_\infty(K)$ and finally the class of non-zero left translation invariant C^*-subalgebras of $C_0(K)$ in the hypergroup context with the goal of finding some relations between these function spaces.

Among other results, we construct two correspondences: one, between closed Weil subhypergroups and certain left translation invariant W^*-subalgebras of $L_\infty(K)$, and another between compact subhypergroups and a specific subclass of the class of left translation invariant C^*-subalgebras of $C_0(K)$. By the help of these two characterizations, we extract some results about invariant complemented subspaces of $L_\infty(K)$ and $C_0(K)$.

[1] A version of this chapter has been published. N. Tahmasebi, Hypergroups and invariant complemented subspaces, J. Math. Anal. Appl. 414 (2014) 641-655.[68]

2.2 Invariant complemented subspaces of $L_\infty(K)$

The strict topology on $CB(K)$ is a locally convex topology determined by the seminorms:

$$\{p_\phi \mid \phi \in C_0(K)\}, \text{ where } p_\phi(f) = ||f\phi||, \quad f \in CB(K),$$

and the relative weak*-topology on $CB(K)$ is defined by the seminorms:

$$\{p_\phi \mid \phi \in L_1(K)\}, \text{ where } p_\phi(f) = |\int f(g)\phi(g)d\lambda(g)|, \quad f \in CB(K).$$

Remark 2.2.1. *Let X be a locally compact Hausdorff space and A be a closed self-adjoint subalgebra of $CB(X)$ (with the strict topology) which separates X in the sense that for any pair x_1, x_2 of distinct points of X there is an f in A with $f(x_1) = 1$, $f(x_2) = 0$. Then $A = CB(X)$ ([23], Corollary).*

Definition 2.2.1. *The representation $\{T_g \mid g \in K\}$ of K on a Banach space X is (weakly) almost periodic if the orbit $\{T_g x \in X \mid g \in K\}$ is a relatively (weakly) compact subset of X, for each $x \in X$.*

The goal of this section is to originate a systematic approach to the class of left translation invariant W^*-subalgebras of $L_\infty(K)$.

Lemma 2.2.1. *Let K be an amenable hypergroup and let the representation $\{T_g \in B(X) \mid g \in K\}$ be a separately continuous representation of K on a Banach space X. Then there exists a continuous projection P from X^* onto the weak*-closed subspace $F = \{f \in X^* \mid M_g f = f, \ \ \forall g \in K\}$ of X^* and P commutes with any weak*-weak*-continuous linear operator from X^* into X^* which commutes with operators $\{M_g \mid g \in K\}$.*

Proof. Let K be an amenable hypergroup and fix a non-zero function $f \in X^*$. Then $\overline{coM_K f}^{W^*}$ is nonempty , weak*-compact convex subset of X^*. Define an affine action $(g, \phi) \mapsto M_g\phi$ from $K \times \overline{coM_K f}^{W^*}$ into $\overline{coM_K f}^{W^*}$. This action is separately continuous when $\overline{coM_K f}^{W^*}$ has the $weak^*$-topology of X^*.

Thus, there is some $\psi_0 \in \overline{coM_K f}^{W^*} \cap F$, since K is amenable ([65], Theorem 3.3.1). We will show that $\overline{coM_K}^{W^*.o.t}$ is a semigroup of operators. If $\{P_\alpha\}$ and $\{P_\beta\}$ are nets in coM_K such that $P_\alpha \to P_1$ and $P_\beta \to P_2$ in $W^*.o.t$, where $P_\alpha = \sum_{i=1}^{n} \lambda_{i,\alpha} M_{g_{i,\alpha}}$ and $P_\beta = \sum_{i=1}^{m} \lambda_{i,\beta} M_{g_{i,\beta}}$, then for $\phi \in X$ and $f \in X^*$

$$< P_\alpha(P_\beta f), \phi >= \sum_{i=1}^{n} \lambda_{i,\alpha} < M_{g_{i,\alpha}} P_\beta f, \phi > \to \sum_{i=1}^{n} \lambda_{i,\alpha} < M_{g_{i,\alpha}} P_2 f, \phi >=< P_\alpha P_2 f, \phi >,$$

since M_g is weak*-weak*-continuous. In addition, $P_\alpha(P_\beta f) \in \overline{coM_K f}^{W^*}$. Consequently, $P_1 \circ P_2 f \in \overline{coM_K f}^{W^*}$, since $< P_\alpha P_2 f, \phi > \to < P_1 P_2 f, \phi >$, for each $\phi \in X$. Therefore, there exists a continuous projection P from X^* onto F such that P commutes with any weak*-weak*-continuous linear operator from X^* into X^* which commutes with $\{M_g \in B(X^*) \mid g \in K\}$ ([45], Theorem 2.1). In ([45], Theorem 2.1) it is the part of assumption that M_K is a semigroup, however, by a close look at the proof, one will see that the only requirement is that the representation is norm-decreasing and $\overline{coM_K}^{W^*.o.t}$ is a semigroup. □

Corollary 2.2.2. *Let N be a closed amenable subhypergroup of K. Then for each $1 < p \leq \infty$ there exists a continuous projection P from $L_p(K)$ onto $\{f \in L_p(K) \mid R_n f = f, \quad \forall n \in N\}$ and P commutes with any weak*-weak*-continuous linear operator from $L_p(K)$ into $L_p(K)$ which commutes with right translations $\{R_g \mid g \in N\}$. In particular, P commute with left translations.*

Proof. Consider the continuous representation of N on $L_q(K)$ $(1 \leq q < \infty)$ given by $\{R_n \in B(L_q(K)) \mid n \in N\}$. Then apply Lemma 2.2.1 to get the required projection. □

Next definition is originally due to Voit [73].

Definition 2.2.2. *Let N be a closed subhypergroup of K. Then N has the translation property TB if*

$$\begin{aligned} X: \ &= \{f \in CB(K) \mid R_n f = f, \ \forall n \in N\} \\ &= \{f \in CB(K) \mid R_g f = R_k f, \quad \forall g \in k * N, k \in K\} \end{aligned}$$

We note that in this case $X = \{f \in CB(K) \mid f|_{g*N} \equiv f(g), \quad \forall g \in K\}$ and X is a left translation invariant subalgebra of $CB(K)$.

Definition 2.2.3. *Let X be a closed subspace of $CB(K)$. We say that X has the local translation property TB if for each element $k_0 \in K$*

$$\begin{aligned} &k_0 * \{g \in K \mid f|_{k*g} \equiv f(k), \quad \forall f \in X, k \in K\} \\ &= \{g \in K \mid R_g f = R_{k_0} f, \quad \forall f \in X\}. \end{aligned}$$

It turns out that there is a connection between the local translation property TB and the translation property TB. The relation appears as we change our perspective from subhypergroups to subspaces of $CB(K)$; the translation property TB of a subhypergroup N is an equivalent condition of the local translation property TB of $\{f \in CB(K) \mid R_g f = f, \quad \forall g \in N\}$. Therefore, this property can be considered as an extension of the translation property TB.

Remark 2.2.2. *Let N be a closed subhypergroup of K. Then N has the translation property TB if and only if $X = \{f \in CB(K) \mid R_g f = f, \ \forall g \in N\}$ has the local translation property TB.*

Proof. Let $k_0 \in K$, $k_0 * N_1 = k_0 * \{g \in K \mid f|_{k*g} \equiv f(k), \quad \forall f \in X, k \in K\}$ and $N_2^{k_0} = \{g \in K \mid R_g f = R_{k_0} f, \quad \forall f \in X\}$. Then $k_0 * N_1 \subseteq N_2^{k_0}$ since X is left translation invariant. In addition, $N_2^{k_0} \subseteq k_0 * N$. In fact if $g_0 \in N_2^{k_0} \setminus k_0 * N$, let $f \in CB(K/N)$ such that $f(g_0 * N) \neq f(k_0 * N)$ and let $\widetilde{f}(k) = f(k * N)$, then $\widetilde{f} \in X$ and $R_{g_0} \widetilde{f}(e) \neq R_{k_0} \widetilde{f}(e)$. This statement contradicts the definition of $N_2^{k_0}$. Now it is easy to check the equivalence. □

Example 2.2.1. *Let $K = H \vee J$, where H is a compact hypergroup and J is a discrete hypergroup with $H \cap J = \{e\}$. Then by Remark 2.2.2, ([64], Example 3.3) and Corollary 2.2.2, $X = \{f \in L_\infty(K) \mid R_g f = f, \quad \forall g \in H\}$ is a left translation invariant W^*-subalgebra of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB and X is invariantly complemented in $L_\infty(K)$.*

In 1993 Hermann [26] defined the notion of a Weil subhypergroups on which the Weil formula may be applied. The definition of a Weil subhypergroup involves its left Haar measure, however, a more concrete equivalent condition can be obtained from [73]; see Remark 2.2.3. As a consequence, in Lemma 2.2.3 we are able to provide a strong connection between a Weil subhypergroup N and the algebra $\{f \in CB(K) \mid f|_{g*N} \equiv f(g), \ \ \forall g \in K\}$. This relation plays a prominent role in the main Theorem of this section.

Remark 2.2.3. *:*

1. *Let N be a closed subhypergroup of K, $p_1 : g \mapsto g*N$ and $p_2 : g \mapsto N*g$ be canonical mappings from K to the corresponding quotient spaces. Then by taking the hypergroup involution it is easy to see that*

   $$\{L_k(f \circ p_1) \mid f \in C_c(K/N), k \in K\} \subseteq \{f \in CB(K) \mid f|_{g*N} \equiv f(g), \ \ \forall g \in K\}$$

   if and only if

   $$\{R_k(f \circ p_2) \mid f \in C_c(K/N), k \in K\} \subseteq \{f \in CB(K) \mid f|_{N*g} \equiv f(g), \ \forall g \in K\}.$$

2. *Using part 1 of this Remark, we can remove the normality condition in Theorems 2.3 and 2.6 of [73] to obtain that N is a Weil subhypergroup if and only if*

   $$\{L_k(f \circ p_1) \mid f \in C_c(K/N), k \in K\} \subseteq \{f \in CB(K) \mid f|_{g*N} \equiv f(g), \ \ \forall g \in K\}.$$

Lemma 2.2.3. *Let N be a closed subhypergroup of K. If we put X to be $X = \{f \in CB(K) \mid f|_{g*N} \equiv f(g), \ \ \forall g \in K\}$, then the following are equivalent:*

1. *N is a Weil subhypergroup.*

2. *X is left translation invariant.*

3. *X has the local translation property TB.*

4. $\{f \in CB(K) \mid f|_{N*g} \equiv f(g), \quad \forall g \in K\}$ *is right translation invariant.*

Proof. (2) $\to$ (1) follows from Remark 2.2.3.

For (1) $\to$ (2), let $f \in X$ and view f as a function $\bar{f}$ on the locally compact space K/N, where $\bar{f}(g*N) = f(g)$. Then $\bar{f} \in CB(K/N)$. Let $\{f_\alpha\}$ be a net in $C_c(K/N)$ converging to $\bar{f}$ in the strict topology of $CB(K/N)$ and put $\tilde{f}_\alpha(g) := f_\alpha(g*N)$. Then for $g, k \in K$

$$\begin{aligned} |L_g\tilde{f}_\alpha(k) - L_g f(k)| &= |\textstyle\int \tilde{f}_\alpha(u) - f(u) d\delta_g * \delta_k(u)| \\ &\leq \textstyle\int |\tilde{f}_\alpha(u) - f(u)| d\delta_g * \delta_k(u) \to 0, \end{aligned}$$

since strict convergent implies pointwise convergent. Let $g, k \in K$, $k_0 \in k*N$ and let $\epsilon > 0$ be given. Pick α_0 such that $|L_g\tilde{f}_\alpha(k_0) - L_g f(k_0)| < \frac{\epsilon}{2}$ and $|L_g\tilde{f}_\alpha(k) - L_g f(k)| < \frac{\epsilon}{2}$, for $\alpha \geq \alpha_0$. Then

$$\begin{aligned} |L_g f(k_0) - L_g f(k)| &\leq |L_g\tilde{f}_\alpha(k_0) - L_g f(k_0)| + |L_g\tilde{f}_\alpha(k_0) - L_g f_\alpha(k)| \\ &+ |L_g\tilde{f}_\alpha(k) - L_g f(k)| < \epsilon, \end{aligned}$$

by Remark 2.2.3 (2). Hence, $L_g f(k_0) = L_g f(k)$ since $\epsilon > 0$ is arbitrary.

For (2) $\leftrightarrow$ (3), let $k_0 \in K$,

$$k_0 * N_1 = k_0 * \{g \in K \mid f|_{k*g} \equiv f(k), \quad \forall f \in X, k \in K\}$$

and $k_0 * N_2 = \{g \in K \mid R_g f = R_{k_0} f, \quad \forall f \in X\}$. Then by (2),

$$k_0 * N_1 = k_0 * \{g \in K \mid R_v f = R_k f, \ \forall f \in X, k \in K, v \in k*g\}$$

Thus, $k_0 * N \subseteq k_0 * N_1 \subseteq k_0 * N_2$. If $g_0 \in k_0 * N_2 \setminus k_0 * N$, let $f \in CB(K/N)$ such that $f(g_0 * N) \neq f(k_0 * N)$ and let $\widetilde{f}(k) = f(k*N)$. Then $\widetilde{f} \in X$ and $R_{g_0}\widetilde{f}(e) \neq R_{k_0}\widetilde{f}(e)$. Hence, $k_0 * N = k_0 * N_1 = k_0 * N_2$, i.e. X has the local translation property TB. Conversely, if X has the local translation property TB, $u \in k_0 * n$, $n \in N$ and $f \in X$, then $u \in \{g \in K \mid R_g f = R_{k_0} f, \quad \forall f \in X\}$. Thus, $L_k f|_{k_0 * N} \equiv L_k f(k_0)$, for $k \in K$.

The implication (2) $\leftrightarrow$ (4) is by taking the hypergroup involution. $\square$

Consider the set $\mathscr{D}$ of all left translation invariant W^*-subalgebras X of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB. Then it reveals to the author that there is a one-to-one correspondence between elements of $\mathscr{D}$ and closed Weil subhypergroups of K. This characterization is given in Theorem 2.2.5.

Lemma 2.2.4. *Let X be a weak*-closed left translation invariant subspace of $L_\infty(K)$. Then $X \cap LUC(K)$ is weak*-dense in X.*

Proof. Let $\{e_\alpha\}$ be a bounded approximate identity for $L_1(K)$ and let $f \in X$. Then $\{e_\alpha * f\} \subseteq X \cap LUC(K)$, since X is left translation invariant and the net $e_\alpha * f$ converge to f in the weak*-topology of $L_\infty(K)$ ([65], Lemma 2.2.5, ii). $\square$

Remark 2.2.4. *Let X be a translation invariant inversion invariant subspace of $CB(K)$. Then X has the local translation property TB if and only if for each element $k_0 \in K$*

$$\{g \in K \mid f|_{g*k} \equiv f(k), \ \ \forall f \in X, k \in K\} * \check{k}_0 = \{g \in K \mid L_g f = L_{\check{k}_0} f, \quad \forall f \in X\}.$$

We are now ready to prove the main Theorem of this section

Theorem 2.2.5. *X is a left translation invariant W^*-subalgebra of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB if and only if there exists a unique closed Weil subhypergroup N of K such that*

$$X = \{f \in L_\infty(K) \mid R_g f = R_k f, \quad \forall g \in k * N, \quad k \in K\}.$$

Furthermore, N is normal if and only if X is inversion invariant and translation invariant.

Proof. Let N be a closed Weil subhypergroup of K and let

$$X := \overline{\{f \in CB(K) \mid f|_{g*N} \equiv f(g), \quad \forall g \in K\}}^{W^*}.$$

Then $X \cap CB(K)$ has the local translation property TB and by Lemma 2.2.3, $X = \{f \in L_\infty(K) \mid R_g f = R_k f, \quad \forall g \in k * N, \quad k \in K\}$. Thus, X is a $weak^*$-closed, conjugate-closed, left translation invariant subspace of $L_\infty(K)$. In addition, X is also an algebra, since $X \cap CB(K)$ is $weak^*$-dense in X.

Conversely, let X be a left translation invariant W^*-subalgebra of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB. Then $X \cap CB(K)$ is strictly closed in $CB(K)$ since strict convergence implies pointwise convergence. Set $N := \{g \in K \mid f|_{k*g} \equiv f(k), \quad \forall f \in X \cap CB(K), \quad k \in K\}$. Then $N = \{n \in K \mid R_n f = f, \quad \forall f \in X \cap CB(K)\}$ since $X \cap CB(K)$ has local translation property. Hence, N is closed since $X \cap CB(K)$ is closed in the relative weak*-topology of $CB(K)$. Next we prove that N is a subhypergroup. To this end let $f \in X \cap CB(K)$, $g \in K$ and $a \in n_1 * n_2$, for some $n_1, n_2 \in N$. Then $f(g) = f(b) = f(c)$, for $b \in g * n_1$ and $c \in b * n_2$. Hence, $f(u) = f(g)$, for $u \in g * a \subseteq g * n_1 * n_2 = \bigcup_{b \in g * n_1} b * n_2$, i.e, $N * N \subseteq N$. Let $k \in K$, $n \in N$ and $u \in k * \check{n}$ that is to say that $k \in u * n$. Then $f(u) = f(k)$, for $f \in X \cap CB(K)$ by the definition of N.

Let $A = \{f \in L_\infty(K) \mid R_g f = R_k f, \quad \forall g \in k * N, \quad k \in K\}$ and let $Y = \overline{\{f \in CB(K) \mid f|_{g*N} \equiv f(g), \quad \forall g \in K\}}^{W^*}$. Then $X \cap CB(K) \subseteq A \cap CB(K) \subseteq Y \cap CB(K)$ since $X \cap CB(K)$ is left translation invariant . We will next show that $Y \cap CB(K) \subseteq X \cap CB(K)$. First observe that each function f in $Y \cap CB(K)$ or $X \cap CB(K)$ can be regarded as a continuous function $\bar{f}$ on the locally compact Hausdorff space K/N, where $\bar{f}$ is given by $\bar{f}(g * N) = f(g)$. Define a function $p : f \mapsto \tilde{f}$ from $CB(K/N)$ to $CB(K)$, where, $\tilde{f}(g) = f(g * N)$. Then p is continuous when $CB(K/N)$ and $CB(K)$ equipped with their strict topologies, hence, p is a strict-isomorphism from $CB(K/N)$ onto $Y \cap CB(K)$ and p preserve the conjugation. In addition, $X \cap CB(K)$ is closed when $CB(K)$ equipped with the relative weak*-topology, and so it is strictly closed. Thus, $p^{-1}(X \cap CB(K)) = \{\bar{f} \mid f \in X \cap CB(K)\}$ is a strictly closed subalgebra of $CB(K/N)$ which separate points of K/N. In fact suppose $f(x) = f(y)$, for all $f \in X \cap CB(K)$. Then $R_x f(k) = L_k f(x) = L_k f(y) = R_y f(k)$, for $k \in K$ and $f \in X \cap CB(K)$, since $X \cap CB(K)$ is left translation invariant. Thus,

$x \in y*N$ because $X \cap CB(K)$ has the local translation property TB. Therefore, $p^{-1}(X \cap CB(K)) = CB(K/N)$ by the strict Stone-Weierstrass Theorem ([23], Corollary). Consequently, $X \cap CB(K) = Y \cap CB(K)$ and an application of Lemma 2.2.4 shows that $X = A = Y$. Moreover, N is a Weil subhypergroup by Lemma 2.2.3 since $A \cap CB(K) = Y \cap CB(K)$.

Suppose to the contrary that N_0 is another closed Weil subhypergroup of K such that $X = \overline{\{f \in CB(K) \mid f|_{g*N_0} \equiv f(g), \ \ \forall g \in K\}}^{W^*}$. Then $N_0 \subseteq N$. If $g \notin N \setminus N_0$, let $f \in CB(K/N_0)$ such that $f(g * N_0) \neq f(N_0)$ and let $\widetilde{f}(k) := f(k * N_0)$, then $\widetilde{f} \in X \cap CB(K)$ and $\widetilde{f}(g) \neq \widetilde{f}(e)$ which contradicts the definition of N. Therefore, $N_0 = N$.

Finally, if N is normal, then

$$\begin{aligned} X &= \overline{\{f \in CB(K) \mid f|_{g*N} \equiv f(g), \ \ \forall g \in K\}}^{W^*} \\ &= \overline{\{f \in CB(K) \mid f|_{N*g} \equiv f(g), \ \ \forall g \in K\}}^{W^*}. \end{aligned}$$

Thus X is right translation invariant by Lemma 2.2.3. In addition, X is also inversion invariant, let $u \in g * N$, for some $g \in K$ and $f \in X \cap CB(K)$, then $\check{u} \in N * \check{g} = \check{g} * N$, hence $\check{f}(u) = \check{f}(g)$. Thus, $\check{f} \in X \cap CB(K)$. Conversely, if X is inversion invariant and translation invariant, then by the same process we find a unique closed subhypergroup N_1 of K such that $X = \overline{\{f \in CB(K) \mid f|_{g*N_1} \equiv f(g), \ \ \forall g \in K\}}^{W^*}$, where

$$N_1 = \{g \in K \mid f|_{g*k} \equiv f(k), \ \ \forall f \in X \cap CB(K), \ \ k \in K\}.$$

Then $N_1 = N$. In fact if $g_0 \in N \setminus N_1$, then by Remark 2.2.4, there is some $h \in X \cap CB(K)$ such that $L_g h \neq h$ since $X \cap CB(K)$ has the local translation property TB. Thus, $R_{\check{g}}\check{h} \neq \check{h}$ and $\check{h} \in X \cap CB(K)$ since X and $CB(K)$ are inversion invariant which is a contradiction by the definition of N. Thus, $N_1 \subseteq N$. The converse inclusion follows similarly. To see that N is normal, let $u \in N * g$, then there is some $g_u \in K$ such that $u \in g_u * N_1 = g_u * N$. For any $f \in X \cap CB(K)$ we have that $f(g) = f(u) = f(g_u)$ by the definition of N and N_1. Thus, $g_u * N = g * N$ since $\{\bar{f} \in CB(K/N) \mid f \in X \cap CB(K)\}$

separates points of K/N. Hence, $N * g \subseteq g * N$. Using the same argument we obtain the equality. $\square$

Remark 2.2.5. *Let X be a left translation invariant W^*-subalgebra of $L_\infty(K)$, $N = \{g \in K \mid f|_{k*g} \equiv f(k), \quad \forall f \in X \cap CB(K), k \in K\}$ and also let $A = \{f \in L_\infty(K) \mid R_g f = R_k f, \quad \forall g \in k * N, k \in K\}$. Then*

1. *$X \cap CB(K)$ has the local translation property TB if and only if $\{\bar{f} \mid f \in X \cap CB(K)\}$ separate points of K/N.*

2. *$X \cap CB(K)$ has the local translation property TB if and only if $X = A$.*

If $X \cap CB(K)$ does not have the local translation property TB and $x_0 \in x * N_2 \setminus x * N$ for some $x \in K$, where

$$x * N_2 := \{g \in K \mid R_g f = R_x f, \quad \forall f \in X \cap CB(K)\},$$

then $f(x_0) = R_{x_0} f(e) = R_x f(e) = f(x)$, for all $f \in X \cap CB(K)$. Hence $\{\bar{f} \mid f \in X \cap CB(K)\}$ does not separate points of K/N. In addition, let $f \in CB(K/N)$ with $f(x_0 * N) \neq f(x * N)$ and put $\tilde{f}(g) = f(g * N)$. Then $\tilde{f} \in A$ and $R_{x_0}\tilde{f}(e) \neq R_x \tilde{f}(e)$. Thus, $\tilde{f} \notin X$ since $x_0 \in x * N_2$. For other parts see the proof of Theorem 2.2.5

Corollary 2.2.6. *Let K be a compact hypergroup. Then every left translation invariant W^*-subalgebra X of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB is invariantly complemented in $L_\infty(K)$.*

Proof. By Theorem 2.2.5, $X = \{f \in L_\infty(K) \mid R_n f = f, \quad \forall n \in N\}$, for a compact subhypergroup N of K since compact subhypergroups have the translation property TB ([73], Lemma 1.5). Now the result follows from Lemma 2.2.2. $\square$

Example 2.2.2. *Consider the example of Jewett ([30], 9.1.D), where K is the conjugacy class of A_4, the subgroup of even permutation of S_4. Let X be a proper translation invariant W^*-subalgebra of $L_\infty(K)$ such that the character $\xi \in X$. Then $X \cap CB(K)$ does not have the local translation property TB.*

Proof. Suppose that X has the local translation property TB. Then by ([73], Lemma 1.5) and Theorem 2.2.5, there is a compact subhypergroup N such that $X = \{f \in L_\infty(K) \mid R_n f = f, \;\; \forall n \in N\}$. In particular, $R_n\xi = \xi$, for all $n \in N$. Thus, $N = \{e\}$ since $0 = R_b\xi \neq \xi$, $0 = R_c\xi \neq \xi$ and $R_a\xi(a) \neq \xi(a)$. Therefore, $X = L_\infty(K)$. $\square$

Example 2.2.3. *:*

1. *Let K be the cosh-hypergroup on $\mathbb{R}_+$ or $\mathbb{N}_0$. Then in view of Theorem 2.2.5 and ([73], 4.3), $\{f \in L_\infty(K) \mid R_g f = f, \;\; \forall g \in H_2\}$ is the only non-trivial translation invariant W^*-subalgebra of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB, where $H_2 = \{0, 2, 4, ...\}$.*

2. *Let K be the cosine-hypergroup on $\mathbb{R}_+$ or $\mathbb{N}_0$. Then by Theorem 2.2.5, ([73], 4.2) and ([77], Remark 5.3) all translation invariant W^*-subalgebra of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB are of the form $X_n = \{f \in L_\infty(K) \mid R_g f = f, \;\; \forall g \in H_n\}$, where $H_n = \{0, n, 2n, 3n, ...\}$ and $n \in \mathbb{N}_0$.*

3. *Let K be a hypergroup on $\mathbb{R}_+$ with supp$(\delta_x * \delta_y) = [|x - y|, x + y]$. Then an application of Theorem 2.2.5 together with ([77], Remark 5.3) show that for any non-trivial translation invariant W^*-subalgebra of $L_\infty(K)$, $X \cap CB(K)$ does not have the local translation property TB.*

Remark 2.2.6. *Skantharajah ([64], Proposition 3.5) proved that every closed subgroup N of an amenable hypergroup is amenable and if in addition N is normal, then K/N is also amenable ([64], Proposition 3.6). However, from the proofs of these results it follows that these Propositions not only hold for closed subgroups, but in fact hold for any closed Weil subhypergroups.*

2.3 Invariant complemented subspaces of $C_0(K)$

In this section we initiate the study of the class of non-zero left translation invariant C^*-subalgebras of $C_0(K)$. Our investigation is based on two important characterizations: one for hypergroups admitting an invariant mean on the space $WAP(K)$ (Lemma 2.3.2) and one for the class of non-zero left translation invariant C^*-subalgebras of $C_0(K)$ with the local translation property TB (Lemma 2.3.5).

Let X be a linear subspace of $L_\infty(K)$. X is said to be left introverted if for any $n \in X^*$ and $h \in X$ the function $n_l h : g \mapsto n(L_g h)$ belongs to X. If X is left introverted, then it is easy to see that X^*, with the multiplication given by $< m \odot n, h >=< m, n_l h >$, for $m, n \in X^*$ and $h \in X$ becomes a Banach algebra and the set $Mean(X)$ of all means on X renders a semigroup structure. Moreover, we can embed K into X^* via point evaluations, δ_g, for $g \in K$. It is known that spaces $WAP(K)$ and $AP(K)$ are left introverted ([65], p 101).

Lemma 2.3.1. *Let X be a Banach space and let $\{T_g \in B(X) \mid g \in K\}$ be a (weakly) almost periodic separately continuous representation. Then $\phi_{T_K} f$ is an element of $AP(K)$ ($WAP(K)$), for each $f \in X$ and $\phi \in X^*$, where $(\phi_{T_K} f)(g) =< \phi, T_g f >$.*

Furthermore, for each $m \in Mean(WAP(K))$ and each $f \in X$, there is a unique $P_m f$ in the closed convex hull of $T_K f := \{T_g f \mid g \in K\}$, where $P_m f$ is given via $< \phi, P_m f >=< m, \phi_{T_K} f >$, for $\phi \in X^*$.

Proof. We first note that for each $f \in X$, the function $\phi_{T_K} f \in CB(K)$, since $g \mapsto T_g f$ is continuous from K to X. Fix $\phi \in X^*$ and define a bounded linear operator T_ϕ from X into $CB(K)$ via $T_\phi(f) = \phi_{T_K} f$. Then T_ϕ is also weakly continuous and $R_k(\phi_{T_K} f)(g) = \int < \phi, T_u f > d\delta_g * \delta_k(u) =< \phi, T_g T_k f >= \phi_{T_K}(T_k f)(g)$, for each $f \in X$. Therefore, $R_K(\phi_{T_K} f) = T_\phi(T_K f)$ is relatively (weakly) compact, since the action is (weakly) almost periodic.

The rest follows by the point of view taken in ([65], Lemma 4.3.1). □

Lemma 2.3.2. *$WAP(K)$ admits an invariant mean if and only if for any Banach space X and any weakly almost periodic separately continuous representation $\{T_g \in B(X) \mid g \in K\}$ of K on X, there exists a representation $\{P_m \in B(X) \mid m \in Mean(WAP(K))\}$ such that:*

1. *For each* $m \in Mean(WAP(K))$, $< \phi, P_m f > = < m, \phi_{T_K} f >$, *where* $(\phi_{T_K} f)(g) = < \phi, T_g f >$ *and* $\phi \in X^*$.

2. $||P_m|| \leq 1$, *for each* $m \in Mean(WAP(K))$.

3. $P_{\delta_g} = T_g$, *for each* $g \in K$.

4. $P_{m \odot n} = P_m \circ P_n$, *for* $m, n \in Mean(WAP(K))$.

5. *There exists an element* $m_0 \in Mean(WAP(K))$ *such that* P_{m_0} *is a continuous projection from* X *onto* $F = \{f \in X \mid T_g f = f, \quad \forall g \in K\}$.

6. P_{m_0} *commutes with any continuous linear operator* T *from* X *to* X *which commutes with* $\{T_g \in B(X) \mid g \in K\}$.

A similar statement is true for $AP(K)$ *if* $\{T_g \in B(X) \mid g \in K\}$ *is a separately continuous almost periodic representation.*

Proof. Let X be a Banach space, $\{T_g \in B(X) \mid g \in K\}$ be a weakly almost periodic separately continuous representation. For each $f \in X$, $\phi \in X^*$ and $m \in Mean(WAP(K))$, we let $< \phi, P_m f > = < m, \phi_{T_K} f >$, where $(\phi_{T_K} f)(g) = < \phi, T_g f >$ (Lemma 2.3.1). Then P_m is a bounded linear operator on X (Lemma 2.3.1), $||P_m|| \leq 1$ and $P_{\delta_g} = T_g$. We will show that $P_{m \odot n} = P_m \circ P_n$. For $f \in X$, $\phi \in X^*$ and $g, x \in K$,

$$\begin{aligned} L_g \phi_{T_K} f(x) &= \int \phi_{T_K} f(u) d\delta_g * \delta_x(u) \\ &= \int < \phi, T_u f > d\delta_g * \delta_x(u) \\ &= < \phi, T_g T_x f > = < T_g^* \phi, T_x f > \\ &= (T_g^* \phi)_{T_K} f(x). \end{aligned}$$

Hence, for $m, n \in Mean(WAP(K))$

$$\begin{aligned} n_l \phi_{T_K} f(g) &= < n, L_g \phi_{T_K} f > = < n, (T_g^* \phi)_{T_K} f > \\ &= < T_g^* \phi, P_n f > = < \phi, T_g P_n f > = \phi_{T_K} P_n f(g). \end{aligned}$$

Therefore,

$$\begin{aligned} < \phi, P_{m \odot n} f > &=< m \odot n, \phi_{T_K} f >=< m, n_l \phi_{T_K} f > \\ &=< m, \phi_{T_K} P_n f >=< \phi, P_m(P_n f) > . \end{aligned}$$

since $WAP(K)$ is left introverted. i.e, $P_{m \odot n} = P_m \circ P_n$, because X^* separates points of X. If m_K is an invariant mean on $WAP(K)$, then by an idea similar to that of ([46], Lemma 10) P_{m_K} is a continuous projection from X onto F, commuting with any continuous linear operator T from X to X which commutes with $\{T_g \in B(X) \mid g \in K\}$.

Conversely, let $X = WAP(K)$ and consider weakly almost periodic separately continuous representation $\{R_k \in B(WAP(K)) \mid k \in K\}$ ([65], Lemma 4.2.1 and Proposition 4.2.4). If $m_0 \in Mean(WAP(K))$ such that P_{m_0} is a continuous projection from $WAP(K)$ onto $\mathbb{C}.1$ with $||P_{m_0}|| \leq 1$ commuting in particular with left translation operators on $WAP(K)$, then $P(L_g f) = L_g(Pf) = Pf$, for $f \in WAP(K)$ and $g \in K$. In addition, for $\phi \in WAP(K)^*$

$$< \phi, P_{m_0} 1 >=< m_0, \phi_{R_{\check{k}}} 1 >=< m_0, \phi(1) >=< \phi, 1 > .$$

Hence, $P_{m_0} 1 = 1$, since $WAP(K)^*$ separate points of $WAP(K)$. Thus, $||P_{m_0}|| = 1$. Therefore, P_{m_0} is a left invariant mean on $WAP(K)$. $\square$

Remark 2.3.1. *Let X be a Banach space, $\{T_g \in B(X) \mid g \in K\}$ be a (weakly) almost periodic separately continuous representation of K on X and let m_K be an invariant mean on $AP(K)$ ($WAP(K)$). Then by an argument similar to that of ([65], Proposition 4.3.2, see [24], Theorem 3.8.4 for semigroups) we have*

1. $X = \{f \in X \mid T_g f = f, \ \ \forall g \in K\} \oplus \overline{< \{T_g f - f \mid g \in K, \ f \in X\} >}.$
2. $P_{m_K} f = \overline{co(T_K f)} \cap \{f \in X \mid T_g f = f, \ \ \forall g \in K\}.$

Where $\overline{< \{T_g f - f \mid g \in K, \ f \in X\} >}$ denote the closed subspace of X spanned by $\{T_g f - f \mid g \in K, \ f \in X\}$.

It follows from ([72], Theorem 1.6 and [65], Proposition 4.3.4, i) that if N is a closed normal Weil subhypergroup of a hypergroup K possessing an invariant mean on the space $WAP(K)$ $(AP(K))$, then there is an invariant mean on $WAP(K/N)$ $(AP(K/N))$ since the natural mapping $K \mapsto K/N$ is a hypergroup homomorphism and $WAP(K)$ $(AP(K))$ has an invariant mean. Next we will prove the converse, in case that $WAP(N)$ $(AP(N))$ has also an invariant mean and N has the translation property TB. This result is one of the applications of Lemma 2.3.2 which is of independent interest.

Corollary 2.3.3. *Let N be a closed normal subhypergroup of K with the translation property TB. If $WAP(N)$ has an invariant mean, then $WAP(K/N)$ when viewed as a subspace of $WAP(K)$ is invariantly complemented in $WAP(K)$. If, in addition, $WAP(K/N)$ has an invariant mean, then there is an invariant mean on $WAP(K)$.*

This result can also be obtained for $AP(K)$.

Proof. Let N be a closed normal subhypergroup of K with the translation property TB. Then the canonical mapping π from K onto K/N is an open hypergroup homomorphism ([72], Theorem 1.6 and [73], Lemma 1.7). Hence by ([65], Proposition 4.2.11) we have

$$W\widetilde{AP(K}/N) \cong A := \{f \in WAP(K) \mid R_g f = f, \quad \forall g \in N\},$$

where for $f \in WAP(K/N)$ we let $\tilde{f}(g) := f(g * N)$. Now the first assertion follows from Lemma 2.3.2 by taking $X = WAP(K)$ and the representation $\{R_n \in B(X) \mid n \in N\}$ which is weakly almost periodic and separately continuous. Next let m_N and $m_{K/N}$ be invariant means on $WAP(N)$ and $WAP(K/N)$, respectively. If P_{m_N} is a continuous projection from $WAP(K)$ onto A commuting with left translations, then set $\overline{P_{m_N}}(f) := \overline{P_{m_N} f}$, where $\overline{P_{m_N} f}(g * N) = (P_{m_N} f)(g)$ and $f \in WAP(K)$. Now it is easy to see that $m := m_{K/N} \circ \overline{P_{m_N}}$ is an invariant mean on $WAP(K)$, since $m(1_K) = ||m|| = 1$ and $\overline{L_g P_{m_N} f} = L_{g*N} \overline{P_{m_N} f}$, for $g \in K$. $\square$

Lemma 2.3.4. *Let X be a closed translation invariant subspace of $WAP(K)$ and let N be a closed subhypergroup of K. If $WAP(N)$ has an invariant mean, then there exists a continuous projection P from X onto the closed subspace $\{f \in X \mid R_g f = f, \quad \forall g \in N\}$ of X with $||P|| \leq 1$ and P commutes with any continuous linear operator from X into X which commutes with right translations $\{R_g \mid g \in N\}$. In particular, P commutes with any left translation.*

Proof. This is a direct consequence of Lemma 2.3.2 by considering the representation $\{R_n \in B(X) \mid n \in N\}$. □

Lemma 2.3.5. *X is a non-zero left translation invariant C^*-subalgebra of $C_0(K)$ with the local translation property TB if and only if there exists a unique compact subhypergroup N of K such that*

$$X = \{f \in C_0(K) \mid R_n f = f, \quad \forall n \in N\}.$$

Furthermore, N is normal if and only if X is inversion invariant and translation invariant

Proof. Let N be a compact subhypergroup of K and define X as $X := \{f \in C_0(K) \mid R_n f = f, \quad \forall n \in N\}$. Then X can also be written as $X = \{f \in C_0(K) \mid f|_{k*N} \equiv f(k), \quad \forall k \in K\}$ ([73], Lemma 1.5). Thus, X is a left translation invariant C^*-subalgebra of $C_0(K)$. In addition, since $X \cong C_0(K/N)$, in which we identify $f \in X$ by $f \circ \pi \in X$, where $\pi : g \mapsto g * N$ is the canonical mapping from K onto K/N, X is non-zero.

To see that X has the local translation property TB, for $k_0 \in K$ we let

$$k_0 * N_1 = k_0 * \{g \in K \mid f|_{k*g} \equiv f(k), \quad \forall f \in X, k \in K\}$$

and let $k_0 * N_2 = \{g \in K \mid R_g f = R_{k_0} f, \quad \forall f \in X\}$. Then, $k_0 * N \subseteq k_0 * N_1 \subseteq k_0 * N_2$ since X is left translation invariant. If $g_0 \in k_0 * N_2 \setminus k_0 * N$, let $f \in C_c(K/N)$ such that $f(g_0 * N) \neq f(k_0 * N)$. Then find some $h \in C_c(K)$ such that $\int R_n h(x) d\lambda_N(n) = f(x * N)$, where λ_N is a normalized Haar measure on N ([26]) and let $\widetilde{f}(x) := f(x * N)$. Then $\widetilde{f} \in X$ and $R_{g_0}\widetilde{f}(e) \neq R_{k_0}\widetilde{f}(e)$. This shows that X has the local translation property TB.

Conversely, let X be a non-zero left translation invariant C^*-subalgebra of $C_0(K)$ with the local translation property TB and let

$$N := \{g \in K \mid f|_{k*g} \equiv f(k), \quad \forall f \in X, k \in K\}.$$

Then by the same argument as in the proof of Theorem 2.2.5 we can show that N is a closed subhypergroup of K. Set $Y := \{f \in C_0(K) \mid R_g f = f, \quad \forall g \in N\}$. Then $X \subseteq Y$. For the converse inclusion we first note that each f in Y or X can be regarded as a function $\bar{f}$ in $C_0(K/N)$ ([73], Lemma 1.5), where $\bar{f}(g * N) := f(g)$. Let $\mathscr{A} := \{\bar{f} \in C_0(K/N) \mid f \in X\}$ and $\mathscr{B} := \{\bar{f} \in C_0(K/N) \mid f \in Y\}$. Then $\mathscr{A} \subseteq \mathscr{B}$. By the Stone-Weierstrass Theorem $\mathscr{A} = \mathscr{B}$. We note that $\mathscr{A}$ separates points of K/N, because X has the local translation property TB (see the proof of Theorem 2.2.5 for details). Consequently, $X = Y$.

Suppose N_0 is another compact subhypergroup of K such that X is also given by $X = \{f \in C_0(K) \mid R_g f = f, \quad \forall g \in N_0\}$. Then $N_0 \subseteq N$. In addition, if $g \notin N \setminus N_0$, then we will have a contradiction. In fact let $f \in C_c(K/N_0)$ such that $f(g * N_0) \neq f(N_0)$. Choose some $h \in C_c(K)$ such that $\int R_n h(x) d\lambda_{N_0}(n) = f(x * N_0)$, where λ_{N_0} is a normalized Haar measure on N_0 ([26]) and let $\widetilde{f}(x) := f(x * N_0)$. By this setup, one can see that $\widetilde{f} \in X$, while $R_g\widetilde{f}(e) \neq \widetilde{f}(e)$. Therefore, $N_0 = N$.

The last assertion also can be proved similar to that of Theorem 2.2.5 with the observation that $\mathscr{A}$ separates points of K/N. $\square$

Remark 2.3.2. *Let X be a left translation invariant C^*-subalgebra of $C_0(K)$. Let $N = \{g \in K \mid f|_{k*g} \equiv f(k), \quad \forall f \in X, k \in K\}$ and consider A as $A = \{f \in C_0(K) \mid R_n f = f, \quad \forall n \in N\}$. Then*

1. *X has the local translation property TB if and only if the subspace $\{\bar{f} \in C_0(K/N) \mid f \in X\}$ separates point of K/N.*

2. X has the local translation property TB if and only if $X = A$.

Proof is similar to that of Remark 2.3.2.

Theorem 2.3.6. *Let X be a non-zero left translation invariant C^*-subalgebra of $C_0(K)$ with the local translation property TB. Then there exists a continuous projection P from $C_0(K)$ onto X and P commutes with any continuous linear operator from $C_0(K)$ into $C_0(K)$ which commutes with right translations.*

In particular, P commutes with any left translation operator on $C_0(K)$.

Proof. This follows directly from Lemmas 2.3.4 and 2.3.5 with the observation that compact hypergroups are amenable ([64], Example 3.3). □

Remark 2.3.3. *Let X be a non-zero left translation invariant C^*-subalgebra of $C_0(K)$ with the local translation property TB and N be a compact subhypergroup of K given by $N = \{g \in K \mid R_g f = f, \quad \forall f \in X\}$. Then we have a correspondence between the projection P in Theorem 2.3.6 and the mapping $f \mapsto T_N f$, where $(T_N f)(g) = \int R_n f(g) d\lambda_N(n)$ and λ_N is a normalized Haar measure on N which maps $C_0(K)$ onto $C_0(K/N)$ when $C_0(K/N)$ viewed as a closed subspace of $C_0(K)$.*

Chapter 3

Fixed point properties, invariant means and invariant projections related to hypergroups.[1]

3.1 Introduction

A closed bounded subset C of a Banach space E is said to have normal structure if for every nontrivial convex subset B of C, there is some $x_0 \in B$ such that $\sup_{y \in B} ||x_0 - y|| < diam(B)$, where $diam(B) = \sup_{x,y \in B} ||x - y||$. In addition, a (dual) Banach space E has weak (weak*)-normal structure, if every weakly (weak*) compact convex subset of E has normal structure. It is known that a C^*-algebra A has weak-normal structure if and only if it is finite dimensional ([53], Theorem 4.5). The concept of weak*-normal structure was introduced by Lim who showed that l_1 has weak*-normal structure ([54], p 189). A self mapping T on a subset F of a (dual) Banach space is said to be nonexpansive if $||Tx - Ty|| \leq ||x - y||$, for $x, y \in F$. Furthermore, a (dual) Banach space E has the weak (weak*) fixed point property weak-FPP (FPP*) if for every non-empty weakly (weak*) compact convex subset C of E and for

[1] A version of this chapter is under review. N. Tahmasebi, Fixed point properties, invariant means and invariant projections related to hypergroups, [67].

every nonexpansive map T from C into C, T has a fixed point in C. It follows from ([53], Theorem 4.5) that $C_0(K)$ has weak-normal structure if and only if K is finite.

Definition 3.1.1. *$\mathfrak{T} = \{T_g \mid g \in K\}$ is a continuous representation of K on a compact convex subset X of a separated locally convex topological vector space (E, τ) if $T_g : X \to X$, $T_e = I$ and $< f, T_{g_1}T_{g_2}x >= \int < f, T_u x > d\delta_{g_1} * \delta_{g_2}(u)$, for $x \in X$, $f \in E^*$ and $g_1, g_2 \in K$. Moreover, $\mathfrak{T}$ is called equicontinuous if for each $y \in X$ and $\epsilon > 0$ there is some $\delta > 0$ such that whenever $p_\alpha(x - y) < \delta$ for all $p_\alpha \in Q$ where Q is the set of seminorms that generate the topology τ, one has $p_\alpha(T_g x - T_g y) < \epsilon$ for all $g \in K$ and $p_\alpha \in Q$.*

3.2 Fixed point properties for nonexpansive mappings

In this section we initiate the study of various geometric property including weak (weak*)-normal structure and the weak (weak*) fixed point property weak-FPP (FPP*) on several function spaces over K.

Theorem 3.2.1. *Let X be a closed subspace of $CB(K)$ containing $C_0(K)$. If weak* convergence and weak convergence for sequences agree on the unit sphere of X^*, then K is discrete.*

Proof. If K is not discrete then $\lambda(\{e\}) = 0$ ([30], Theorem 7.1.B). Let U_n be a sequence of open neighbourhoods of e such that $\bar{U}_n$ is compact and $\lambda(U_n) \to 0$. Define a function $h_1 \in C_0(K)$ via

$$h_1(x) = \begin{cases} 1 & \text{if } x = e, \\ 0 & \text{if } x \in U_1^c \end{cases} \tag{3.1}$$

and $0 \leq h_1(x) \leq 1$, for $x \in K$ and let $V_1 := \{g \in K \mid h_1(g) \neq 0\}$. Then

$V_1 \subseteq U_1$. Define $h_2 \in C_0(K)$ via

$$h_2(x) = \begin{cases} 1 & \text{if } x = e, \\ 0 & \text{if } x \in U_2^c \cup V_1^c \end{cases} \tag{3.2}$$

and $0 \leq h_2(x) \leq 1$, for $x \in K$. Let $V_2 := \{g \in K \mid h_2(g) \neq 0\}$. Then $V_2 \subseteq U_1 \cap V_1$. Define $h_{n+1} \in C_0(K)$ recursively by

$$h_{n+1}(x) = \begin{cases} 1 & \text{if } x = e, \\ 0 & \text{if } x \in U_{n+1}^c \cup V_n^c \end{cases} \tag{3.3}$$

and $0 \leq h_{n+1}(x) \leq 1$, for $x \in K$, where $V_n = \{g \in K \mid h_n(g) \neq 0\}$. Now for each n define a function d_n on $K \times K$ by

$$d_n(a,e) := \sup_{k,g \in K} |h_n|_{a*k}(g) - h_n(k)|$$

and

$$d_n(x,y) := \inf_{a \in x * \check{y}} d_n(a,e).$$

We will show that d_n is a pseudo-metric. Clearly $d_n(x,x) = 0$, since $e \in x * \check{x}$. In addition,

$$\begin{aligned} d_n(a,e) &= \sup_{k,g \in K} |h_n|_{a*k}(g) - h_n(k)| \\ &= \sup_{k,g \in K} |h_n(g) - h_n|_{\check{a}*g}(k)| \\ &= d_n(\check{a},e), \end{aligned}$$

since $g \in a * k$ and $k \in \check{a} * g$ are equivalent. Thus,

$$d_n(x,y) = \inf_{a \in x * \check{y}} d_n(a,e) = \inf_{\check{a} \in y * \check{x}} d_n(a,e) = \inf_{\check{a} \in y * \check{x}} d_n(\check{a},e) = d_n(y,x).$$

To see the triangle inequality, let $\epsilon > 0$ be given, $a_1 \in x * \check{y}$ and $a_2 \in y * \check{z}$ are chosen such that $d_n(a_1,e) \leq d_n(x,y) + \epsilon$ and $d_n(a_2,e) \leq d_n(y,z) + \epsilon$. Hence, $a_1 * a_2 \cap x * \check{z} \neq \emptyset$ since $a_1 \in x * \check{z} * \check{a}_2$. Take $a_3 \in a_1 * a_2 \cap x * \check{z}$ and let $k_1 \in K$ and $g_1 \in a_3 * k_1 \subseteq a_1 * a_2 * k_1$ be such that $d_n(a_3,e) - \epsilon \leq |h_n(g_1) - h_n(k_1)|$.

Choose $a_4 \in \check{a}_1 * g_1 \cap a_2 * k_1$. Then

$$\begin{aligned} d_n(a_3, e) - \epsilon &\leq |h_n(g_1) - h_n(k_1)| \\ &\leq |h_n(a_4) - h_n(g_1)| + |h_n(a_4) - h_n(k_1)| \\ &\leq d_n(\check{a}_1, e) + d_n(a_2, e) \\ &= d_n(a_1, e) + d_n(a_2, e) \\ &\leq d_n(x, y) + d_n(y, z) + 2\epsilon. \end{aligned}$$

Thus, $d_n(a_3, e) \leq d_n(x, y) + d_n(y, z)$. Therefore, $d_n(x, z) \leq d_n(x, y) + d_n(y, z)$, since $a_3 \in x * \check{z}$. For each n let

$$\begin{aligned} C_n : &= \{x \in K \mid d_n(x, e) = 0\} \\ &= \{x \in K \mid \sup_{k,g \in K} |h_n|_{x*k}(g) - h_n(k)| = 0\} \\ &= \{x \in K \mid h_n|_{x*k}(g) \equiv h_n(k), \quad \forall k, g \in K\} \end{aligned}$$

Then C_n is a compact subhypergroup of K (see the proof of Lemma 2.3.5). Let $C := \bigcap_{n=1}^{\infty} C_n$. Then $d_n(C * x, C * y) := d_n(x, y)$ defines a metric on K/C. In fact if $C * x \neq C * y$, then $C * x * \check{y} \cap C = \emptyset$. Thus, for each $a \in x * \check{y}$, $C * a \neq C$. Hence, $d_n(a, e) > 0$, for all n. Therefore, $d_n(C * x, C * y) = \inf_{a \in x * \check{y}} d_n(a, e) > 0$. Observe that $\{V_n\}$ is a decreasing sequence, $e \in V_n$, $V_n \subseteq U_n$, $C \subseteq V_n$, for each n and finally $\lambda(V_n) \to 0$ as $n \to \infty$. Since K/C has a countable base at the identity C, let W_n be a decreasing sequence of relatively compact neighbourhoods of C such that $C = \bigcap_{n=1}^{\infty} W_n$. Let $\phi_n = \frac{\chi_{W_n}}{\lambda(W_n)}$, where χ_{W_n} is the characteristic function on W_n. Then $\phi_n \in L_1(K)$ and for every $f \in CB(K)$ we have $\phi_n(f) = \int f \phi_n d\lambda \to \int f d\lambda_C$, where λ_C is the normalized Haar measure on C. In particular, ϕ_n converges to λ_C in the weak* topology of X^*, hence, it also converges in the weak topology of X^*. Since the weak topology on X^* is the relative weak topology for a subset of $M(K)$, ϕ_n converges to λ_C in the weak topology in $M(K)$. Then $\lambda_C \in L_1(K)$, since $\phi_n \in L_1(K)$ and $L_1(K)$ is weakly sequentially complete. Thus, $\lambda_C(C) = 1$ which is a contradiction since $\lambda(C) = 0$. □

Corollary 3.2.2. *The following are equivalent:*

1. *K is discrete.*

2. *$M(K)$ is isometrically isomorphic to $l_1(K)$.*

3. *Weak* convergence and weak convergence of sequences agree on the unit sphere of $M(K)$.*

4. *$M(K)$ has weak*-normal structure.*

5. *$M(K)$ has the weak* fixed point property FPP*.*

Proof. $3 \Rightarrow 1$ follows from Theorem 3.2.1 and other parts are obvious (for details see [48], Theorem 1). $\square$

Let S be a semitopological semigroup, i.e, a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto as$ and $s \mapsto sa$ are continuous from S into S. Then S is called left reversible if $\overline{aS} \cap \overline{bS} \neq \emptyset$, for any $a, b \in S$. It is known that all commutative semigroups and all semitopological semigroups which are algebraically groups are left reversible. In 1973, Lau proved that if $\mathscr{T} = \{T_s \mid s \in S\}$ is a separately continuous nonexpansive representation of a left reversible semitopological semigroup S (i.e, $T_{s_1 s_2} = T_{s_1} T_{s_2}$, for $s_1, s_2 \in S$) on a nonempty compact convex subset C of a Banach space E, then C has a common fixed point for S([43], Theorem 4.1). A (dual) Banach space E is said to have the weak (weak*) fixed point property for left reversible semigroups if whenever $\mathscr{T}$ is a separately continuous nonexpansive representation of a left reversible semitopological semigroup S on a nonempty weakly (weak*) compact convex subset C of E, C contains a common fixed point for S. Lim showed that E has the weak fixed point property for left reversible semigroups if it has weak-normal structure ([55], Theorem 3), l_1 and any uniformly convex Banach space have the weak* fixed point property for left reversible semigroups ([54], p 189 and [55], Theorem 3).

Remark 3.2.1. *Let C be a nonempty, weakly compact convex subset of E and assume that C has normal structure. Let S be a topological semigroup and $\{T_g \mid g \in S\}$ be a left reversible semigroup of nonexpansive, separately* *continuous actions on C. Then C contains a common fixed point for S, i.e. there exists $x \in C$ such that $T_g x = x$ for every $s \in S$ ([55], Theorem 3).*

Corollary 3.2.3. *The following are equivalent:*

1. *K is discrete.*
2. *$L_1(K)$ has the weak fixed point property for left reversible semigroups.*

Proof. $1 \Rightarrow 2$ follows from Corollary 3.2.2, ([55], Theorem 3) and the fact that weak*-normal structure implies weak-normal structure. Conversely, if 2 holds while K is not discrete, then $L_1(K)$ has weak fixed point property and contains an isometric copy of $L_1[0,1]$ ([35], Corollary, p 136) which is not possible ([1], The example). $\square$

Corollary 3.2.4. *Let K be a separable hypergroup. The following are equivalent:*

1. *K is discrete.*
2. *$M(K)$ has the weak* fixed point property for left reversible semigroups.*

Proof. $2 \Rightarrow 1$ is a consequence of Corollary 3.2.2 and $1 \Rightarrow 2$ can be derived from ([54], p 189) since if K is discrete and hence countable, then $M(K) = l_1$. $\square$

The following is a consequence of ([48], Lemma 3).

Lemma 3.2.5. *Let X be a C*-subalgebra of $CB(K)$ containing $C_0(K)$*

1. *If X^* has weak-normal structure then K is discrete.*
2. *If X contains the constants and X^* has weak*-normal structure then K is finite.*

Remark 3.2.2.
1. *Weak* convergence and weak convergence for sequences agree on the unit sphere of $LUC(K)^*$ if and only if K is discrete.*
2. *Suppose that $LUC(K)$ is an algebra. Then*

(a) *$LUC(K)^*$ has weak*-normal structure if and only if K is finite.*

(b) *$LUC(K)$ has weak-normal structure if and only if K is finite.*

Proof. The first part follows from Theorem 3.2.1 and ([13], Theorem 15, page 103). For the second part apply Lemma 3.2.5, and ([53], Theorem 4.5) respectively and note that $C_0(K) \subseteq LUC(K)$. □

Example 3.2.1. *1. Let $K = H \vee J$ be the hypergroup join of a compact hypergroup H and a discrete hypergroup J with $H \cap J = \{e\}$, where e is the identity of H and J. Then*

(a) *Weak* convergence and weak convergence for sequences agree on the unit sphere of $LUC(K)^*$ if and only if H is finite.*

(b) *$LUC(K)^*$ has weak*-normal structure if and only if H and J are finite.*

(c) *$LUC(K)$ has weak-normal structure if and only if H and J are finite.*

(d) *$L_1(K)$ has the weak fixed point property for left reversible semigroups if and only if H is finite.*

(e) *$M(K)$ has weak* fixed point property FPP* if and only if H is finite.*

2. Let K be any hypergroup structure on $\mathbb{N}_0$. Then

(a) *$M(K)$ has weak*-normal structure.*

(b) *$CB(K)^*$ does not have weak*-normal structure.*

(c) *$CB(K)$ does not have weak-normal structure.*

(d) *$L_1(K)$ has the weak fixed point property for left reversible semigroups.*

Proof. This follows from Corollaries 3.2.2 and 3.2.3 and Remark 3.2.2 since in both cases $LUC(K) = CB(K)$ is an algebra ([64], Proposition 2.4). □

3.3 Common fixed point properties for affine maps

It is the goal of this section to present several applications of common fixed point properties for affine actions of a hypergroup K.

Note that if E is any locally convex space, and A is any equicontinuous subset of the dual E^*, then the weak* topology and the topology of uniform convergence on totally bounded subsets of E coincide when restricted to A.

Lemma 3.3.1. *$AP(K)$ has an invariant mean if and only if for any separately continuous equicontinuous and affine representation $\{T_g \mid g \in K\}$ of K on a compact convex subset Y of a separated locally convex topological vector space (E, τ) one of the following holds:*

1. *Y has a common fixed point for K.*

2. *There exists a uniformly continuous retraction P from Y onto $\{\phi \in Y \mid \phi = T_g\phi, \quad \forall g \in K\}$ and P commutes with any affine continuous operator from Y into Y which commutes with $\{T_g \mid g \in K\}$.*

Proof. Let m be an invariant mean on $AP(K)$ and let $f \in E^*$ be fixed. For each $y \in Y$ define a continuous bounded function $h_{y,f}$ on K via $h_{y,f}(g) =< f, T_g y >$ and consider an operator $T : Y \to (CB(K), ||.||)$, where $T(y) = h_{y,f}$. Then T is continuous, since the action is equicontinuous. Thus, $h_{y,f} \in AP(K)$ because $R_K(h_{y,f}) = T(T_K y)$. If m is an invariant mean on $AP(K)$ and for each $\phi \in Y$, let $P\phi$ be an accumulation point of the net $\{\sum_{i=1}^{n_\alpha} \lambda_{i,\alpha} T_{g_{i,\alpha}}\phi\}$, where $\{\sum_{i=1}^{n_\alpha} \lambda_{i,\alpha}\delta_{g_{i,\alpha}}\}_\alpha$ is a net of convex combination of point evaluations on $AP(K)^*$ converging to m in weak*-topology of $AP(K)^*$. Then $P\phi \in Y$ since Y is compact. In addition, $< f, P\phi >= m(h_{\phi,f}) = m(L_g h_{\phi,f}) = m(h_{\phi,f\ .\ g}) =< f\ .\ g, P\phi >=< f, T_g P\phi >$ since the action is affine. Thus, $T_g P\phi = P\phi$ because E^* separates the points of E. Let $Q = \{q_\beta \mid \beta \in I\}$ be the family of seminorms generating the topology τ of E and let $\epsilon > 0$ be given. Then there is some $\delta > 0$ such that $q_\beta(T_g\phi_1 - T_g\phi_2) < \epsilon$ for all $g \in K$, whenever $\phi_1, \phi_2 \in Y$ and

$q_\beta(\phi_1 - \phi_2) < \delta$. Hence,

$$q_\beta(P\phi_1 - P\phi_2) \leq \lim_\alpha \sum_{i=1}^{n_\alpha} \lambda_{i,\alpha} q_\beta(T_{g_{i,\alpha}}\phi_1 - T_{g_{i,\alpha}}\phi_2) < \epsilon,$$

which means that P is uniformly continuous. Moreover, it is easy to see that P is a retraction and commutes with any affine continuous function from Y into Y which commutes with $\{T_g \mid g \in K\}$. Conversely, let $E = AP(K)^*$ and $Y = Mean(AP(K))$, where Y is equipped with the weak*-topology of E. Consider an affine representation $\mathscr{T} = \{L_g^* \mid g \in K\}$ of K on Y. Then $\mathscr{T}$ is jointly continuous since $AP(K) \subseteq LUC(K)$. For each $f \in AP(K)$, define a seminorm p_f on $AP(K)^*$ by $p_f(\Psi) := \sup_{g\in K} \mid < \Psi, L_g f > \mid$, $\Psi \in AP(K)^*$ and let $Q = \{p_f \mid f \in AP(K)\}$ be the family of such seminorms and τ be the locally convex topology determined by Q. Then $\mathscr{T}$ is τ-equicontinuous. In fact let $\epsilon > 0$, $f \in WAP(K)$ be given and choose $\delta := \epsilon$. If $m_1, m_2 \in WAP(K)^*$ such that $\sup_{x\in K} \mid < m_1 - m_2, L_x f > \mid < \delta$, then for each $g_0 \in K$,

$$\begin{aligned}
&\sup_{x\in K} \mid < L_{g_0}^* m_1 - L_{g_0}^* m_2, L_x f > \mid \\
&\leq \sup_{x\in K} \int \mid < m_1 - m_2, L_u f > \mid d\delta_{g_0} * \delta_x(u) \\
&< \delta = \epsilon.
\end{aligned}$$

In addition, τ agrees with the weak* topology on Y. Hence, the action of K on Y is equicontinuous. Therefore, the common fixed point of the action is an invariant mean on $AP(K)$. □

Theorem 3.3.2. *The following are equivalent:*

1. *$AP(K)$ has an invariant mean.*

2. *K has the following fixed point property:*

   Let $\{T_g \in B(E) \mid g \in K\}$ be a separately continuous representation of K as contractions on a Banach space E such that $T_K f$ is relatively compact in E for each $f \in E$. If X is a closed T_K-invariant subspace of E and if there exists a bounded linear operator Q on E^* with $||Q|| \leq \gamma$ such that

$Q\Phi_0|_{\overline{coT_K f}} \equiv f$, for some $\Phi_0 \in E^*$ and for all $f \in X$, then there exists a linear operator P on E^* with $||P|| \leq \gamma$ such that $P\Phi_0|_{\overline{coT_K f}} \equiv f$, for all $f \in X$ and that $M_g P = P$.

Proof. Let

$$\mathscr{P} = \{P \in B(E^*) \mid \ ||P|| \leq \gamma \text{ and } P\Phi_0|_{\overline{coT_K f}} \equiv f, \text{ for all } f \in X\}$$

and let τ_1 and τ_2 denote locally convex topologies defined on E^* and $B(E^*)$ respectively by the family of seminorms:

$$\{p_f \mid f \in E\}, \quad \text{where } p_f(\Phi) = \sup_{g \in K} | < \Phi, T_g f > |, \quad \Phi \in E^*$$

and

$$\{p_{\Phi,f} \mid \Phi \in E^*, \ f \in E\}, \quad \text{where } p_{\Phi,f}(T) = p_f(T\Phi), \quad T \in B(E^*).$$

Then $(\mathscr{P}, W^*.o.t)$ is a non-empty closed convex subset of $\{P \in B(E^*) \mid ||P|| \leq \gamma\}$, and hence $(\mathscr{P}, W^*.o.t)$ is also compact. Furthermore, since weak* topology on $\mathscr{P}\Phi$ agrees with the topology of uniform convergence on totally bounded subsets of E and $T_K f$ is totally bounded because it is relatively compact for each $f \in E$, τ_1 agrees with the weak* topology on $\mathscr{P}\Phi$, for each $\Phi \in E^*$ (note that $\mathscr{P}\Phi$ is an equicontinuous subset of E^*). Thus, τ_2 agrees with the weak* operator topology on $\mathscr{P}$. Hence, the action $(g, P) \mapsto M_g P$ from $K \times (\mathscr{P}, W^*.o.t)$ into $(\mathscr{P}, W^*.o.t)$ is separately continuous, affine and equicontinuous. For let $\epsilon > 0$, $\Phi \in E^*$ and $f \in E$ be given and choose $\delta := \epsilon$. If $T_1, T_2 \in B(E^*)$ such that $p_{\Phi,f}(T_1 - T_2) < \delta$, then for each $g \in K$,

$$\begin{aligned} p_{\Phi,f}(M_g T_1 - M_g T_2) &= \sup\nolimits_{k \in K} | < M_g(T_1 - T_2)\Phi, T_k f > | \\ &\leq \sup\nolimits_{k \in K} \int | < (T_1 - T_2)\Phi, T_u f > | d\delta_g * \delta_k(u) \\ &< \delta = \epsilon. \end{aligned}$$

In addition, for $\Phi \in E^*$ and $f \in E$,

$$\begin{aligned} < M_g M_k P\Phi, f > &=< P\Phi, T_k T_g f > \\ &= \int < P\Phi, T_u f > d\delta_k * \delta_g(u) \\ &= \int < M_u P\Phi, f > d\delta_k * \delta_g(u) \end{aligned}$$

and $(B(E^*), W^*.o.t)^* = \{\delta_{\Phi,f} \mid \Phi \in E^*,\ f \in E\}$, where $\delta_{\Phi,f}(P) =< P\Phi, f >$, for $P \in B(E^*)$. Therefore, by Lemma 3.3.1 there exists $P_0 \in \mathscr{P}$ such that $M_g P_0 = P_0$, for all $g \in K$.

Conversely, let $E = AP(K)$ and $X = \mathbb{C}.1$. Consider the representation $\{L_{\check{g}} \in B(AP(K)) \mid g \in K\}$ of K on $AP(K)$. Then $M_g\Phi = L_g^*\Phi$ for $\Phi \in AP(K)^*$ and $g \in K$. In addition, the map $(g, f) \to L_g f$ is jointly continuous from $K \times (E, ||.||)$ to $(E, ||.||)$ and $L_K f$ is relatively compact, for $f \in AP(K)$. Let Q be an identity operator on E^* and $m \in Mean(AP(K))$. Then $< Qm, h >=< m, f >= f$ for each $f \in X$ and $h \in \overline{coL_K f} = f$. By assumption, there exists a continuous linear operator P on $AP(K)^*$ with $||P|| \leq 1$ such that $Pm(f) = f$, for all $f \in X$ and that $L_g^* P = P$. Then $Pm(1) = 1 = ||Pm||$ and hence, Pm in an invariant mean on $AP(K)$. □

Definition 3.3.1. *A collection of separately weakly continuous mappings $\mathscr{T} = \{T_g \mid g \in K\}$ as linear maps from a separated locally convex topological vector space (E, τ) into (E, τ) is said to be weakly almost periodic if $coT_K x$ is relatively weakly compact in E, for each $x \in E$. In this case the weak vector valued integral $\int T_u x d\delta_{g_1} * \delta_{g_2}(u)$ exists and is a unique element in E and hence $\mathscr{T}$ defines a representation on E. i.e, $< f, T_{g_1} T_{g_2} x >= \int < f, T_u x > d\delta_{g_1} * \delta_{g_2}(u)$, for $x \in E$, $f \in E^*$ and $g_1, g_2 \in K$. $\mathscr{T}$ is called quasi weakly equicontinuous if the closure of T_K in the product space E^E, consists of only weakly continuous mappings.*

The following Lemma is a consequence of ([6], Theorem 4.2.3):

Lemma 3.3.3. *For $f \in CB(K)$ the following are equivalent:*

1. *$f \in WAP(K)$.*

2. *$L_K f$ is $\sigma(CB(K), \Delta(CB(K)))$-relatively compact.*

3. *$R_K f$ is $\sigma(CB(K), \Delta(CB(K)))$-relatively compact.*

Where, $\Delta(CB(K))$ is the spectrum of $CB(K)$.

Theorem 3.3.4. *The following are equivalent:*

1. *$WAP(K)$ has an invariant mean.*

2. *Whenever $\{T_g \mid g \in K\}$ is a separately weakly continuous, quasi weakly equicontinuous, weakly almost periodic linear representation of K on a separated locally convex topological vector space (E, τ) and Y is a weakly compact convex T_K-invariant subset of E, there is in Y a common fixed point for K.*

Proof. Let m_0 be an invariant mean on $WAP(K)$ and $\{T_g \mid g \in K\}$ be separately weakly continuous. For each $f \in E^*$ define $h_{y,f}(g) =< f, T_g y >$, for $g \in K$ and $y \in Y$ and let T be an operator from (Y, w) to $CB(K)$ defined by $T(y) = h_{y,f}$, where $CB(K)$ has $\sigma(CB(K), \Delta(CB(K)))$-topology. Let $m \in \Delta(CB(K))$ and let $\{\delta_{g_\alpha}\}$ be a net in $CB(K)^*$ converging to m in the weak*-topology. Without loss of generality assume that T_{g_α} converges to some continuous function $\phi \in (Y, w)^Y$. If $\{y_\beta\}$ is a net in Y converging to some $y \in Y$, then $< m, T(y_\beta) >=< f, \phi(y_\beta) > \to < f, \phi(y) >=< Ty, m >$. Therefore, T is continuous. In addition, for each $y_1 \in Y$ we have that $h_{T_g y_1, f}(k) =< f, T_k T_g y_1 >= \int < f, T_u y_1 > d\delta_k * \delta_g(u) = R_g h_{y_1,f}(k)$, for $f \in E^*$ and $k \in K$. Thus, $R_K h_{y_1,f} = T(T_K y_1)$. Consequently, $h_{y_1,f} \in WAP(K)$ since $T_K y_1$ is relatively weakly compact and T is continuous (Lemma 3.3.3). Let $\phi_0 \in Y$ be fixed and let ψ be a weak cluster point of the net $\{\sum_{i=1}^{n_\alpha} \lambda_{i,\alpha} T_{g_{i,\alpha}} \phi_0\}_\alpha$ in Y, where $\{\sum_{i=1}^{n_\alpha} \lambda_{i,\alpha} \delta_{g_{i,\alpha}}\}_\alpha$ is a net of convex combination of point evaluations on $WAP(K)$ converging to m_0 in weak*-topology of $WAP(K)^*$. Then

$$\begin{aligned}< f_1, \psi >= m_0(h_{\phi_0,f_1}) &= m_0(L_k h_{\phi_0,f_1}) \\ &= \lim \textstyle\sum_{i=1}^{n_\alpha} \lambda_{i,\alpha} \int < f_1, T_u\phi_0 > d\delta_k * \delta_{g_{i,\alpha}}(u) \\ &= \lim \textstyle\sum_{i=1}^{n_\alpha} \lambda_{i,\alpha} < f_1, T_k T_{g_{i,\alpha}}\phi_0 > \\ &= \lim \textstyle\sum_{i=1}^{n_\alpha} \lambda_{i,\alpha} < f_1 \ . \ k, T_{g_{i,\alpha}}\phi_0 > \\ &=< f_1 \ . \ k, \psi >=< T_k\psi, f_1 >,\end{aligned}$$

for each $f_1 \in E^*$, since $f_1 \ . \ k \in E^*$ by the linearity of the action. Conversely, let $E = WAP(K)^*$, $Y = Mean(WAP(K))$ and define a family of seminorms Q on E by:

$$\{p_f \mid f \in WAP(K)\}, \text{ where } p_f(\psi) = \sup_{g\in K} | < \psi, L_g f > |, \quad \psi \in WAP(K)^*.$$

Let τ be the topology determined by Q. Then, the weak topology of (E, τ) and the weak*-topology on $WAP(K)^*$ coincide, by Mackey-Arens Theorem on E. Hence, Y is a weakly compact convex subset of (E, τ). Consider the separately weakly continuous weakly almost periodic representation $\mathscr{T} = \{L_g^* \mid g \in K\}$ of K on $(WAP(K)^*, \tau)$. In addition, the action is τ-equicontinuous (see the proof of Lemma 3.3.1). Thus, $\mathscr{T}$ is also quasi weakly equicontinuous since it is affine and weakly continuous ([52], p 2541). Consequently, the fixed point in $Mean(WAP(K))$ under this action is an invariant mean on $WAP(K)$. $\square$

By a similar argument as in Theorem 3.3.4 we have the following common fixed property:

Remark 3.3.1. *The following are equivalent:*

1. *$WAP(K)$ has an invariant mean.*
2. *Whenever $\{T_g \mid g \in K\}$ is a separately weakly continuous, equicontinuous, weakly almost periodic linear representation of K on a separated locally convex topological vector space (E, τ) and let Y be a weakly compact convex T_K-invariant subset of E, there is some $\phi \in Y$ such that $T_g\phi = \phi$, for all $g \in K$.*

Lemma 3.3.5. *The following are equivalent:*

1. *K is amenable.*

2. *K has the following property:*

   Let $\{T_g \in B(E^*) \mid g \in K\}$ be a separately weak-weak*-continuous representation of K as contractions on the dual of a Banach space E. Let A be a closed T_K-invariant subspace of E^* such that the mapping $(g, x) \mapsto T_g x$ is separately continuous from $K \times (A, ||.||)$ to $(A, ||.||)$. If X is a weak*-closed T_K-invariant subspace of E^* contained in A and if there exists a continuous projection Q from A onto X with $||Q|| \leq \gamma$, then there exists a linear operator P from A into X with $||P|| \leq \gamma$ such that $Pf \in \overline{co(T_K f)}^{W^*}$, for $f \in X$ and that $PT_g = P$, for all $g \in K$.*

Proof. Let

$$\mathscr{P} = \{P \in B(A, X) \mid \ ||P|| \leq \gamma \text{ and } \ Pf \in \overline{co(T_K f)}^{W^*}, \text{ for } \quad f \in X\}$$

and let τ be the relative weak*-operator topology on $B(A, X)$ defined by the family of seminorms:

$$\{p_{f,\phi} \mid f \in A, \ \phi \in E\}, \quad \text{where } p_{f,\phi}(T) = |< Tf, \phi >|, \quad T \in B(A, X).$$

Then $(\mathscr{P}, \tau)$ is a non-empty subset of $B(A, X)$. In addition, $\overline{co(T_K f)}^{W^*}$ is a weak*-compact convex subset of X for each $f \in X$ because the representation is contractive. Hence, $(\mathscr{P}, \tau)$ is compact and convex. Consider the action $(g, P) \mapsto PT_g$ from $K \times (\mathscr{P}, \tau)$ to $(\mathscr{P}, \tau)$. Then for $g \in K$ the map $P \mapsto PT_g$ is clearly affine and continuous from $(\mathscr{P}, \tau)$ to $(\mathscr{P}, \tau)$. Also, for $P \in \mathscr{P}$ the map $g \mapsto PT_g$ is continuous from K to $(\mathscr{P}, \tau)$. Moreover, since the map $g \mapsto T_g f$ is continuous from K to $(A, ||.||)$ and A is a Banach space, the weak vector valued integral $< T_k T_g f, \Phi >= \int < T_u f, \Phi > d\delta_k * \delta_g(u)$ exists, for each

$\Phi \in A^*$, $g, k \in K$ and $f \in A$. Thus, for $f \in A$, $\phi \in E$ and $g, k \in K$

$$\begin{aligned} < PT_kT_gf, \phi > &=< T_kT_gf, P^*\phi > \\ &= \int < T_uf, P^*\phi > d\delta_k * \delta_g(u) \\ &= \int < PT_uf, \phi > d\delta_k * \delta_g(u), \end{aligned}$$

since $P^*\phi \in A^*$. By Skantharajah ([65], Theorem 3.3.1) there exists $P_2 \in \mathscr{P}$ such that $P_2T_g = P_2$, for all $g \in K$.

Conversely, let $A = LUC(K)$, $X = \mathbb{C}.1$ and consider the representation $\{L_{\check{g}} \in B(L_\infty(K)) \mid g \in K\}$ of K on $L_\infty(K)$. Then A is a closed left translation invariant subspace of $L_\infty(K)$ containing X ([64], Lemma 2.2). In addition, the map $(g, f) \to L_{\check{g}}f$ is jointly continuous from $K \times (A, ||.||)$ to $(A, ||.||)$. Define a continuous norm one projection Q from A onto X by $Q(f) = f(e)1$, for $f \in A$. Then by assumption, there exists a continuous linear operator P from A into X with $||P|| \leq 1$ such that $Pf \in \overline{co(L_Kf)}^{W^*}$, for $f \in X$ and that $PL_g = P$, for all $g \in K$ which in this case is also a projection. Let $m(f) = (Pf)(e)$, for each $f \in A$. Then $m(1) = 1 = ||m||$ and $m(f) = m(L_gf)$, for each $g \in K$, $f \in A$. Thus, K is amenable ([64], Theorem 3.2). □

As an application of Lemma 3.3.5 one has:

Corollary 3.3.6. *K is amenable if and only if for every weak*-closed left translation invariant subspace X of $L_\infty(K)$ which is contained and complemented in $LUC(K)$ with norm $\leq \gamma$, there is a bounded linear operator P from $LUC(K)$ into X with $||P|| \leq \gamma$ such that $Pf \in \overline{co(L_Kf)}^{W^*}$, for $f \in X$ and that $PL_g = P$, for all $g \in K$.*

Theorem 3.3.7. *Let N be a closed Weil subhypergroup of K. Then the following are equivalent:*

1. *K is amenable.*

2. *K has the following fixed point property:*

   Let $\{T_g \in B(E^*) \mid g \in K\}$ be a separately weak-weak*-continuous representation of K as contractions on the dual of a Banach space E. Let A be* *a closed T_K-invariant subspace of E^* such that the mapping $(g, x) \mapsto T_g x$ is jointly continuous from $K \times (A, ||.||)$ to $(A, ||.||)$. If X is a weak*-closed T_K-invariant subspace of E^* contained in A and if there exists a continuous projection Q from A onto X with $||Q|| \leq \gamma$, then there exists a continuous projection P from A onto $\{f \in X \mid T_n f = f, \quad \forall n \in N\}$ with $||P|| \leq \gamma$ such that $T_n P T_g = P$, for all $g \in K$ and $n \in N$.*

Proof. Let $\mathscr{P}$ be as in Lemma 3.3.5 and let τ be the relative weak*-operator topology on $B(A, X)$. Then $(\mathscr{P}, \tau)$ is a non-empty compact convex subset of $B(A, X)$ (Lemma 3.3.5). Consider the action $((n, k), P) \mapsto T_n P T_{\check{k}}$ from $(N \times K) \times (\mathscr{P}, \tau)$ to $(\mathscr{P}, \tau)$. Then for $k \in K$ and $n \in N$ the map $P \mapsto T_n P T_{\check{k}}$ is affine and continuous from $(\mathscr{P}, \tau)$ to $(\mathscr{P}, \tau)$. Also, for $P \in \mathscr{P}$ the map $(n, k) \mapsto T_n P T_{\check{k}}$ is continuous from $N \times K$ to $(\mathscr{P}, \tau)$ by the joint continuity of the action on A. Moreover, since the map $g \mapsto T_g f$ is continuous for $f \in A$ and is weak*-continuous for $f \in E$, the weak vector valued integral $< T_k T_g f, \Phi >= \int < T_u f, \Phi > d\delta_k * \delta_g(u)$ and $< T_k T_g h, \phi >= \int < T_u h, \phi > d\delta_k * \delta_g(u)$ exists, for each $\Phi \in A^*$, $g, k \in K$, $f \in A$, $\phi \in E$ and $h \in E^*$. Thus, for $f \in A$, $\phi \in E$, $g_1, k_1 \in N$ and $g_2, k_2 \in K$

$$
\begin{aligned}
&\int < T_u P T_{\check{v}} f, \phi > d\delta_{(k_1,k_2)} * \delta_{(g_1,g_2)}(u, v) \\
&= \int \int < T_u P T_{\check{v}} f, \phi > d\delta_{k_1} * \delta_{g_1}(u) d\delta_{k_2} * \delta_{g_2}(v) \\
&= \int < T_{k_1} T_{g_1} P T_{\check{v}} f, \phi > d\delta_{k_2} * \delta_{g_2}(v) \\
&= \int < T_{\check{v}} f, P^* M_{g_1} M_{k_1} \phi > d\delta_{k_2} * \delta_{g_2}(v) \\
&=< T_{\check{g}_2} T_{\check{k}_2} f, P^* M_{g_1} M_{k_1} \phi > \\
&=< T_{k_1} T_{g_1} P T_{\check{g}_2} T_{\check{k}_2} f, \phi >
\end{aligned}
$$

since $P^* M_{g_1} M_{k_1} \phi \in A^*$. By Skantharajah ([65], Theorem 3.3.1) there exists $P_2 \in \mathscr{P}$ such that $T_n P_2 T_{\check{k}} = P_2$, for $k \in K$ and $n \in N$ since $N \times K$ is amenable ([64], Proposition 3.8 and Remark 2.2.6). In this case, P_2 is an operator from A into $F = \{f \in X \mid T_n f = f, \quad \forall n \in N\}$ and if $h \in F$, then $P_2 h \in \overline{co(T_N h)}^{W^*} = h$. Thus, P_2 is a continuous projection from A onto F. For the converse see the proof of Lemma 3.3.5. $\square$

Corollary 3.3.8. *Let N be a closed Weil subhypergroup of an amenable hypergroup K and let X be a weak*-closed translation invariant subspace of $L_\infty(K)$ which is contained in $WAP(K)$. Then there is a continuous projection P from X onto $\{f \in X \mid L_n f = f = R_n f, \ \ \forall n \in N\}$ such that $PL_g = P$, for all $g \in K$.*

Proof. By Remark 2.2.6 N is an amenable subhypergroup. Thus, $WAP(N)$ has an invariant mean. In addition, $F := \{f \in X \mid R_n f = f, \ \ \forall n \in N\}$ is a weak*-closed left translation invariant subspace of $L_\infty(K)$ and F is contained and complemented in X (Lemma 2.3.5). Consider the representation $\{L_{\check{g}} \in B(L_\infty(K)) \mid g \in K\}$. Then the mapping $(g, f) \mapsto L_{\check{g}} f$ is jointly continuous from $K \times (WAP(K), ||.||)$ to $(WAP(K), ||.||)$ since $WAP(K) \subseteq LUC(K)$. Therefore, there is a continuous projection P from X onto $\{f \in F \mid L_n f = f, \ \ \forall n \in N\}$ such that $PL_g = P$, for all $g \in K$ (Theorem 3.3.7). □

Corollary 3.3.9. *Let N be a closed Weil subhypergroup of an amenable hypergroup K and let $1 < p < \infty$. Then there is a continuous projection P from $L_p(K)$ onto $\{f \in L_p(K) \mid L_n f = f = R_n f, \ \ \forall n \in N\}$ such that $PL_g = P$, for all $g \in K$.*

Proof. This follows from Theorem 3.3.7 and Corollary 2.2.2. □

Using Lemma 3.3.5, Theorem 3.3.7 and Remark 2.2.6 we have the following fixed point properties:

Corollary 3.3.10. *Let K be an amenable hypergroup and let A and X be closed left translation invariant subspaces of $L_p(K)$ ($1 < p < \infty$) such that X is contained and complemented in A. Then there exists a bounded linear operator P_1 from A into X such that $P_1 f \in \overline{co(L_K f)}$, for $f \in X$ and that $P_1 L_g = P_1$, for all $g \in K$.*

In addition, for any closed Weil subhypergroup N of K, there exists a continuous projection P_2 from A onto $\{f \in X \mid L_n f = f, \ \ \forall n \in N\}$ such that $P_2 L_g = P_2$, for all $g \in K$.

Theorem 3.3.11. *The following are equivalent:*

1. *K is amenable.*

2. *Let $\{T_g \in B(X) \mid g \in K\}$ be a separately continuous representation of K on a Banach space X as contractions and Y be a weak*-closed K-invariant subspace of X^*. If there is a continuous projection Q from X^* onto Y with $||Q|| \leq \gamma$, then there exists a bounded linear operator P from X^* into Y with $||P|| \leq \gamma$ such that $Pf \in \overline{co\{f\ .\ g \mid g \in K\}}^{W^*}$, for $f \in Y$ and $Pf = P(f\ .\ g)$, for $f \in X^*$ and $g \in K$.*

Proof. Let m be a left invariant mean on $L_\infty(K)$ and $\{\phi_\alpha\}$ be a bounded approximate identity for $L_1(K)$. For each $\phi \in X$ and $f \in X^*$ define $h_{\alpha,\phi,f}(g) =< Q(f\ .\ L_{\check{g}}\phi_\alpha), \phi >$. Then $h_{\alpha,\phi,f}$ is bounded by $||Q||\ ||\phi||$ and is continuous since the mapping $g \mapsto L_{\check{g}}\phi_\alpha$ is continuous. Define a bounded linear operator P on X^* by $< Pf, \phi >= \lim_\alpha < P_\alpha f, \phi >$, where $< P_\alpha f, \phi >= m(h_{\alpha,\phi,f})$. Then $||P|| \leq ||Q||$ and $Pf \in Y$, for each $f \in X^*$ since $< P_\alpha f, \phi >= 0$, for $\phi \in Y_\perp$ and $f \in X^*$. We will show that $Pf = P(f\ .\ g)$, for $f \in X^*$ and $g \in K$. To this end, define a continuous function $c_{f,\alpha}$ from K into X^* by $c_{f,\alpha}(g) = f\ .\ L_{\check{g}}\phi_\alpha$. Then the Bochner integral $\int (f\ .\ L_{\check{u}}\phi_\alpha) d\delta_x * \delta_g(u)$ exists since

$$\begin{aligned}\int ||(f\ .\ L_{\check{u}}\phi_\alpha)|| d\delta_x * \delta_g(u) &\leq \int ||L_{\check{u}}\phi_\alpha||\ ||f||\ d\delta_x * \delta_g(u)\\ &\leq \int ||\phi_\alpha||\ ||f||\ d\delta_x * \delta_g(u)\\ &= ||\phi_\alpha||\ ||f||\ < \infty.\end{aligned}$$

In addition, for each $m \in X^{**}$

$$< \int (f\ .\ L_{\check{u}}\phi_\alpha) d\delta_x * \delta_g(u), m >= \int < f\ .\ L_{\check{u}}\phi_\alpha, m > d\delta_x * \delta_g(u).$$

Thus, for $\phi \in X$

$$\begin{aligned}
<\int (f\ .\ L_{\check{u}}\phi_\alpha) d\delta_x * \delta_g(u), \phi> &= \int < f\ .\ L_{\check{u}}\phi_\alpha, \phi > d\delta_x * \delta_g(u) \\
&= \int\int < f, L_k\phi > L_{\check{u}}\phi_\alpha(k) d\lambda(k) d\delta_x * \delta_g(u) \\
&= \int\int < f, L_k\phi > L_{\check{u}}\phi_\alpha(k) d\delta_x * \delta_g(u) d\lambda(k) \\
&= \int < f, L_k\phi > L_{\check{x}} L_{\check{g}}\phi_\alpha(k) d\lambda(k) \\
&=< f\ .\ L_{\check{x}} L_{\check{g}}\phi_\alpha, \phi >
\end{aligned}$$

Therefore, $f\ .\ L_{\check{x}} L_{\check{g}}\phi_\alpha = \int (f\ .\ L_{\check{u}}\phi_\alpha) d\delta_x * \delta_g(u)$. For $x \in K$, $f \in X^*$ and $\phi \in X$

$$\begin{aligned}
L_x h_{\alpha,\phi,f}(g) &= \int < Q(f\ .\ L_{\check{u}}\phi_\alpha), \phi > d\delta_x * \delta_g(u) \\
&= \int < f\ .\ L_{\check{u}}\phi_\alpha, Q^*\phi > d\delta_x * \delta_g(u) \\
&=< \int (f\ .\ L_{\check{u}}\phi_\alpha) d\delta_x * \delta_g(u), Q^*\phi > \\
&=< f\ .\ L_{\check{x}} L_{\check{g}}\phi_\alpha, Q^*\phi > \\
&=< f\ .\ (\delta_x * L_{\check{g}}\phi_\alpha), Q^*\phi > \\
&=< Q((f\ .\ x)\ .\ L_{\check{g}}\phi_\alpha), \phi > \\
&= h_{\alpha,\phi,f\ .\ x}(g).
\end{aligned}$$

Hence,

$$< P_\alpha f, \phi >= m(h_{\alpha,\phi,f}) = m(L_x h_{\alpha,\phi,f}) = m(h_{\alpha,\phi,f\ .\ x}) =< P_\alpha(f\ .\ x), \phi > .$$

Let $\{\sum_{i=1}^n \lambda_{i,\beta}\delta_{g_{i,\beta}}\}_\beta$ be a net of convex combination of point evaluations on $L_\infty(K)$ converging to m in weak*-topology of $L_\infty(K)^*$. Then for each $\phi \in X$ and $f \in Y$

$$\begin{aligned}
< Pf, \phi > &= \lim_\alpha \lim_\beta \textstyle\sum_{i=1}^n \lambda_{i,\beta} < Q(f\ .\ L_{\check{g}_{i,\beta}}\phi_\alpha), \phi > \\
&= \lim_\alpha \lim_\beta \textstyle\sum_{i=1}^n \lambda_{i,\beta} < f\ .\ L_{\check{g}_{i,\beta}}\phi_\alpha, \phi > \\
&= \lim_\alpha \lim_\beta \textstyle\sum_{i=1}^n \lambda_{i,\beta} < f\ .\ (\delta_{g_{i,\beta}} * \phi_\alpha), \phi > \\
&= \lim_\alpha \lim_\beta \textstyle\sum_{i=1}^n \lambda_{i,\beta} < (f\ .\ g_{i,\beta})\ .\ \phi_\alpha, \phi > \\
&= \lim_\beta \lim_\alpha \textstyle\sum_{i=1}^n \lambda_{i,\beta} \int < f\ .\ g_{_{i,\beta}}, L_k\phi > \phi_\alpha(k) d\lambda(k) \\
&= \lim_\beta \textstyle\sum_{i=1}^n \lambda_{i,\beta} < f\ .\ g_{_{i,\beta}}, \phi >,
\end{aligned}$$

since $f \,.\, L_g\phi_\alpha \in Y$, for $g \in K$ ([47], Lemma 2). That is $Pf \in \overline{co\{f \,.\, g \mid g \in K\}}^{W^*}$. Conversely, let $X = L_1(K)$, $Y = \mathbb{C}.1$ and let $\phi_0 \in L_1(K)$ with $||\phi_0|| = 1$ be fixed. Define the representation $\{L_{\check{g}} \in B(L_1(K)) \mid g \in K\}$ of K on $L_1(K)$. Then $f \,.\, g = L_g f$, for $g \in K$ and $f \in L_\infty(K)$ ([30], Theorem 5.1.D). Consider a weak*-continuous norm one projection Q from $L_\infty(K)$ onto $\mathbb{C}.1$ defined by $Qf = f(\phi_0).1$. By assumption, there exists a continuous linear operator P from $L_\infty(K)$ into Y with $||P|| \leq 1$ such that $Pf \in \overline{co(L_K f)}^{W^*}$, for $f \in Y$ and that $PL_g = P$, for all $g \in K$. We note that P is also a projection since $\overline{co(L_K f)}^{W^*} = f$, for $f \in Y$. Define a bounded linear functional on $L_\infty(K)$ by $m(f) = c$ if $Pf = c.1$. Then $m(1) = 1 = ||m||$ and $m(f) = m(L_g f)$, for $f \in L_\infty(K)$ and $g \in K$. Thus, K is amenable. □

Corollary 3.3.12. *The following are equivalent:*

1. *K is amenable.*

2. *Let Y be a weak*-closed left translation invariant subspace of $L_\infty(K)$. If there is a continuous projection Q from $L_\infty(K)$ onto Y with $||Q|| \leq \gamma$, then there exists a bounded linear operator P from $L_\infty(K)$ into Y with $||P|| \leq \gamma$ such that $Pf \in \overline{coL_K f}^{W^*}$, for $f \in Y$ and $P = PL_g$, for $g \in K$.*

Proof. This follows directly from Theorem 3.3.11 by considering the representation $\{L_{\check{g}} \mid g \in K\}$ of K on $L_1(K)$. □

Corollary 3.3.13. *Let K be an amenable hypergroup. Let X be a closed left translation invariant subspace of $LUC(K)$ and Y be a weak*-closed left translation invariant subspace of X^*. If there is a continuous projection Q from X^* onto Y with $||Q|| \leq \gamma$, then there exists a bounded linear operator P from X^* into Y with $||P|| \leq \gamma$ such that $Pf \in \overline{coL_K^* f}^{W^*}$, for $f \in Y$ and $P = PL_g^*$, for $g \in K$.*

Proof. Apply Theorem 3.3.11 to the representation $\{L_{\check{g}} \mid g \in K\}$ of K on X. □

Chapter 4

Inner amenable hypergroups, Invariant projections and Hahn-Banach extension theorem related to hypergroups.[1]

4.1 Introduction

Let G be a locally compact group. A mean m on $L_\infty(G)$ is called inner invariant and G is called inner amenable if $m(L_g R_{g^{-1}} f) = m(f)$, for all $g \in G$ and $f \in L_\infty(G)$ (see [17] for discrete case) which is equivalent to saying that $L_g^* m = R_g^* m$, for all $g \in G$. However, this equivalence relation breaks down when one deals with hypergroups. We say that a hypergroup K is inner amenable if there exists a mean m on $L_\infty(K)$ such that $m(R_g f) = m(L_g f)$ for all $g \in K$ and $f \in L_\infty(K)$. Of course amenable hypergroups are inner amenable. An inner invariant mean m on a non-trivial discrete hypergroup is called non-trivial if $m \neq \delta_e$, the point evaluation function on $l_\infty(K)$. If this is the case, then $m_1 = \frac{m - m(\{e\})\delta_e}{1 - m(\{e\})}$ is an inner invariant mean on $l_\infty(K)$ and

[1] A version of this chapter is under review. N. Tahmasebi, Inner amenable hypergroups, Invariant projections and Hahn-Banach extension theorem related to hypergroups, [69].

$m_1(\{e\}) = 0$. Any invariant mean on $l_\infty(K)$ is a non-trivial inner invariant mean and hence any non-trivial discrete amenable hypergroup possesses a non-trivial inner invariant mean.

4.2 Inner amenable hypergroups

Example 4.2.1. *Let H be a discrete amenable hypergroup and J be a discrete non-amenable hypergroup. Then $K = H \times J$ is a non-amenable hypergroup and $l_\infty(K)$ has a non-trivial inner invariant mean.*

Proof. Let H be a discrete amenable hypergroup and J be a discrete non-amenable hypergroup. Let $K = H \times J$ with the identity (e_1, e_2). If m is an invariant mean on $l_\infty(H)$ and $f \in l_\infty(K)$, then for each $k \in J$ define a function $f_k \in l_\infty(H)$ via $f_k(g) = f(g, k)$. Furthermore, define a mean m_1 on $l_\infty(K)$ by $m_1(f) = m(f_{e_2})$. Then $m_1(f) = m(f_{e_2}) \neq f_{e_2}(e_1) = f(e_1, e_2)$. In addition, for $(g_1, g_2) \in K$ and $k \in H$ we have

$$\begin{aligned}
(L_{(g_1,g_2)}f)_{e_2}(k) &= L_{(g_1,g_2)}f(k, e_2) \\
&= \textstyle\sum_{(u,v)\in K} f(u, v)\delta_{(g_1,g_2)} * \delta_{(k,e_2)}(u, v) \\
&= \textstyle\sum_{u\in H} \sum_{v\in J} f(u, v)\delta_{g_1} * \delta_k(u)\delta_{g_2} * \delta_{e_2}(v) \\
&= \textstyle\sum_{u\in H} f_{g_2}(u)\delta_{g_1} * \delta_k(u) \\
&= L_{g_1} f_{g_2}(k).
\end{aligned}$$

Hence, $(L_{(g_1,g_2)}f)_{e_2} = L_{g_1} f_{g_2}$. Similarly, $(R_{(g_1,g_2)}f)_{e_2} = R_{g_1} f_{g_2}$. Thus,

$$\begin{aligned}
m_1(L_{(g_1,g_2)}f) = m((L_{(g_1,g_2)}f)_{e_2}) &= m(L_{g_1} f_{g_2}) = m(R_{g_1} f_{g_2}) \\
&= m((R_{(g_1,g_2)}f)_{e_2}) = m_1(R_{(g_1,g_2)}f).
\end{aligned}$$

□

The modular function Δ is defined by $\lambda * \delta_{\check{g}} = \Delta(g)\lambda$, where λ is a left Haar measure on K and $g \in K$.

The following result shows that similar to the locally compact groups ([57], Proposition 1), inner amenability of a hypergroup is also an asymptotic property.

Lemma 4.2.1. *The following are equivalent:*

1. *K is inner amenable.*
2. *There is a net $\{\phi_\alpha\}$ in $L_1(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_1 = 1$ such that $||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0$, for all $g \in K$.*
3. *There is a net $\{\psi_\beta\}$ in $L_1(K)$ with $\psi_\beta \geq 0$ such that $\frac{1}{||\psi_\beta||}||L_g\psi_\beta - \Delta(g)R_g\psi_\beta||_1 \to 0$, for all $g \in K$.*

Proof. For $3 \Rightarrow 2$ put $\phi_\alpha = \frac{\psi_\alpha}{||\psi_\alpha||}$. We will prove the equivalence of 1 and 2. Let m be a mean on $L_\infty(K)$ such that $m(L_gf) = m(R_gf)$, for $f \in L_\infty(K)$ and $g \in K$. Then there is a net of positive norm one elements $\{q_\gamma\}$ in $L_1(K)$ such that $< L_gq_\gamma - \Delta(g)R_gq_\gamma, f >\to 0$, for each $f \in L_\infty(K)$. Let T be a map from $L_1(K)$ into $L_1(K)^K$ defined by $T\phi(g) = \Delta(g)R_g\phi - L_g\phi$, for $f \in L_\infty(K)$, $\phi \in L_1(K)$ and $g \in K$. Thus, $0 \in \overline{T(P_1(K))}$ by Mazur's theorem, where $P_1(K) = \{\phi \in L_1(K) \mid \phi \geq 0, \quad ||\phi|| = 1\}$. Therefore, there is a net of positive norm one elements $\{\phi_\alpha\}$ in $L_1(K)$ such that $||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha|| \to 0$. Conversely, let m be any weak*-cluster point of $\{\phi_\alpha\}$ in $L_\infty(K)^*$. Then m is a mean on $L_\infty(K)$ such that $m(R_gf) = m(L_gf)$ for all $g \in K$ and $f \in L_\infty(K)$. □

Corollary 4.2.2. *Let K be a discrete hypergroup. Then the following are equivalent:*

1. *There is an inner invariant mean m on $l_\infty(K)$ such that $m(\{e\}) = 0$.*
2. *There is a net $\{\phi_\alpha\}$ in $l_1(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_1 = 1$ such that $\phi_\alpha(e) = 0$ and that $||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0$, for all $g \in K$.*

Let G be a locally compact group and let τ be a continuous group homomorphism from G into the topological group $Aut(K)$ of all hypergroup homomorphisms on K with the topology of pointwise convergence. The semidirect product $K \rtimes_\tau G$ of K and G is the locally compact space $K \times G$ equipped with the product topology, the convolution $\delta_{(k_1,g_1)} * \delta_{(k_2,g_2)} = \delta_{k_1} * \delta_{\tau_{g_1}(k_2)} \times \delta_{g_1 g_2}$ [74]. In this case, there is a natural action τ of G on $L_p(K)$ ($1 \leq p \leq \infty$) defined by $\tau_g f(k) = f(\tau_g k)$ for $f \in L_p(K)$, $g \in G$ and $k \in K$. If G and K are discrete, then we say that τ is strongly ergodic if the condition $||\tau_g \phi_\alpha - \phi_\alpha||_2 \to 0$, for some positive norm one net $\{\phi_\alpha\}$ in $l_2(K)$ and all $g \in G$ implies that $\phi_\alpha(e_1) \to 1$, where e_1 is the identity of K. In addition, a mean m on $l_\infty(K)$ is τ-invariant if $m(\tau_g f) = m(f)$, for all $g \in G$ and $f \in l_\infty(K)$. The trivial τ-invariant mean on $l_\infty(K)$ is given by $\delta_{e_1}(f) = f(e_1)$, for $f \in l_\infty(K)$ (for the corresponding definitions in the countable group setting see [10]).

The following three results are inspired by [10].

Lemma 4.2.3. *Let G be a discrete group and let τ be a continuous group homomorphism from G into the topological group $Aut(K)$ of all hypergroup homomorphisms on a discrete hypergroup K. Then there is a non-trivial τ-invariant mean m on $l_\infty(K)$ if and only if τ is not strongly ergodic.*

Proof. Let m be a non-trivial τ-invariant mean on $l_\infty(K)$. Without loss of generality assume $m(\delta_e) = 0$, where e is the identity of K. By a standard argument (see the proof of Lemma 4.2.1 for example) find a positive norm one net $\{\psi_\alpha\}$ in $l_1(K)$ such that $||\tau_g \psi_\alpha - \psi_\alpha|| \to 0$ for all $g \in G$ and $\lim_\alpha \psi_\alpha(e) = 0$. Then $\{\phi_\alpha = \psi_\alpha^{\frac{1}{2}}\}$ is a positive norm one net in $l_2(K)$, $\lim_\alpha \phi_\alpha(e) = 0$ and for $g \in G$

$$||\tau_g \phi_\alpha - \phi_\alpha||_2^2 = ||\tau_g(\psi_\alpha^{\frac{1}{2}}) - \psi_\alpha^{\frac{1}{2}}||_2^2 = ||(\tau_g \psi_\alpha)^{\frac{1}{2}} - \psi_\alpha^{\frac{1}{2}}||_2^2 \leq ||\tau_g \psi_\alpha - \psi_\alpha||_1 \to 0.$$

Therefore, τ is not strongly ergodic. Conversely, let $\{\phi_\alpha\}_{\alpha \in I}$ be a positive norm one net in $l_2(K)$ such that $||\tau_g \phi_\alpha - \phi_\alpha||_2^2 \to 0$ and that $\lim_\alpha \phi_\alpha(e) \neq 1$. Choose $\alpha_0 \in I$ such that $\phi_\alpha(e) \neq 1$ for all $\alpha \geq \alpha_0$ and put $I_1 = \{\alpha \in I \mid \alpha \geq \alpha_0\}$. Then $\{\psi_\alpha = \frac{\phi_\alpha - \phi_\alpha(e)\delta_e}{1-\phi_\alpha(e)}\}_{\alpha \in I_1}$ is a positive norm one net in $l_2(K)$ such that $||\tau_g \psi_\alpha - \psi_\alpha||_2^2 \to 0$ and $\psi_\alpha(e) = 0$ for all $\alpha \in I_1$. Let m be a weak*-cluster point of $\{\psi_\alpha^2\}_{\alpha \in I_1}$ in $l_\infty(K)^*$ and by passing possibly to a subnet assume $m(f) = \lim < \psi_\alpha^2, f >$. Then m is a nontrivial τ-invariant mean on $l_\infty(K)$. $\square$

Theorem 4.2.4. *Let $K \rtimes_\tau G$ be the semidirect product hypergroup of a discrete hypergroup K and a discrete group G.*

1. *If K is commutative and τ is not strongly ergodic, then for each subgroup S of G, $l_\infty(K \rtimes_{\tau|_S} S)$ possesses a non-trivial inner invariant mean.*

2. *If τ is strongly ergodic and $l_\infty(G)$ has no non-trivial inner invariant mean, then $l_\infty(K \rtimes_\tau G)$ has no non-trivial inner invariant mean.*

Proof. Assume that there exists a subgroup S of G such that $l_\infty(K \rtimes_{\tau|_S} S)$ has no non-trivial inner invariant mean. Let m be a mean on $l_\infty(K)$ such that $m(\tau_g f) = m(f)$, for all $g \in S$ and $f \in l_\infty(K)$. We will show that m is trivial. For $f \in l_\infty(K \rtimes_{\tau|_S} S)$ and $g \in S$ define a function $f_g \in l_\infty(K)$ by $f_g(k) = f(k, g)$, $(k \in K)$. Let $M(f) = m(f_{e_2})$, for $f \in l_\infty(K \rtimes_{\tau|_S} S)$. Then M is a mean on $l_\infty(K \rtimes_{\tau|_S} S)$. For $f \in l_\infty(K \rtimes_{\tau|_S} S)$, $(k_1, g_1) \in K \rtimes_{\tau|_S} S$ and $k \in K$

$$\begin{aligned}
(L_{(k_1,g_1)}f)_{e_2}(k) &= L_{(k_1,g_1)}f(k, e_2) \\
&= \textstyle\sum_{(u,v)} f(u,v)\delta_{(k_1,g_1)} * \delta_{(k,e_2)}(u,v) \\
&= \textstyle\sum_u \sum_v f(u,v)\delta_{k_1} * \delta_{\tau_{g_1}k}(u)\delta_{g_1 e_2}(v) \\
&= \textstyle\sum_u f(u, g_1)\delta_{k_1} * \delta_{\tau_{g_1}k}(u) \\
&= \textstyle\sum_u f_{g_1}(u)\delta_{k_1} * \delta_{\tau_{g_1}k}(u) \\
&= L_{k_1} f_{g_1}(\tau_{g_1}k) \\
&= \tau_{g_1}(L_{k_1} f_{g_1})(k).
\end{aligned}$$

Moreover,

$$\begin{aligned}
(R_{(k_1,g_1)}f)_{e_2}(k) &= R_{(k_1,g_1)}f(k, e_2) \\
&= \textstyle\sum_{(u,v)} f(u,v)\delta_{(k,e_2)} * \delta_{(k_1,g_1)}(u,v) \\
&= \textstyle\sum_u \sum_v f(u,v)\delta_k * \delta_{\tau_{e_2}k_1}(u)\delta_{e_2 g_1}(v) \\
&= \textstyle\sum f_{g_1}(u)\delta_k * \delta_{k_1}(u) \\
&= L_{k_1} f_{g_1}(k),
\end{aligned}$$

since K is commutative. Hence,

$$\begin{aligned} M(L_{(k_1,g_1)}f) = m((L_{(k_1,g_1)}f)_{e_2}) &= m(\tau_{g_1}(L_{k_1}f_{g_1})) = m(L_{k_1}f_{g_1}) \\ &= m((R_{(k_1,g_1)}f)_{e_2}) = M(R_{(k_1,g_1)}f). \end{aligned}$$

Therefore, M is inner invariant. Then M is trivial, i.e, $M(f) = f(e_1, e_2)$. For $f \in l_\infty(K)$ let $f_1(k,g) = f(k)$ if $g = e_2$ and zero otherwise, $((k,g) \in K \rtimes_{\tau|_S} S)$. Then $(f_1)_{e_2}(k) = f_1(k, e_2) = f(k)$. Thus, $f(e_1) = f_1(e_1, e_2) = M(f_1) = m((f_1)_{e_2}) = m(f)$ which means that m is trivial. Consequently, τ is strongly ergodic by Lemma 4.2.3.

Suppose m is a non-trivial inner invariant mean on $l_\infty(K \rtimes_\tau G)$ and assume without loss of generality that $m(\delta_{(e_1,e_2)}) = 0$, where (e_1, e_2) is the identity of $K \rtimes_\tau G$. Then $m(R_{(e_1,g^{-1})}L_{(e_1,g)}h) = m(h)$, for all $h \in l_\infty(K \rtimes_\tau G)$ and $(e_1, g) \in K \rtimes_\tau G$. For $f \in l_\infty(K)$ let $f_1(k, g) = f(k)$ if $g = e_2$ and zero otherwise, $((k, g) \in K \rtimes_\tau G)$. Then $f_1 \in l_\infty(K \rtimes_\tau G)$. We will show that $m(\chi_{K \rtimes_\tau e_2}) = 0$. If not, then m_1 with $m_1(f) = \frac{m(f_1)}{m(\chi_{K\rtimes_\tau e_2})}$, $(f \in l_\infty(K))$ is a mean on $l_\infty(K)$ and $m_1(\delta_{e_1}) = 0$. For $(k_1, g_1), (e_1, g) \in K \rtimes_\tau G$ and $f \in l_\infty(K)$

$$\begin{aligned} R_{(e_1,g)}(\tau_g f)_1(k_1, g_1) &= \textstyle\sum_{(u,v)} (\tau_g f)_1(u, v)\delta_{(k_1,g_1)} * \delta_{(e_1,g)}(u, v) \\ &= \textstyle\sum_u \sum_v (\tau_g f)_1(u, v)\delta_{k_1} * \delta_{e_1}(u)\delta_{g_1 g}(v) \\ &= (\tau_g f)_1(k_1, g_1 g) \end{aligned}$$

Hence,

$$R_{(e_1,g)}(\tau_g f)_1(k_1, g_1) = \begin{cases} \tau_g f(k_1) = f(\tau_g(k_1)) & \text{if} \quad g_1 g = e_2, \\ 0 & \text{if} \quad g_1 g \neq e_2. \end{cases} \tag{4.1}$$

In addition,

$$\begin{aligned} L_{(e_1,g)}f_1(k_1, g_1) &= \textstyle\sum_{(u,v)} f_1(u, v)\delta_{(e_1,g)} * \delta_{(k_1,g_1)}(u, v) \\ &= \textstyle\sum_u \sum_v f_1(u, v)\delta_{e_1} * \delta_{\tau_g(k_1)}(u)d\delta_{gg_1}(v) \\ &= f_1(\tau_g(k_1), gg_1) \end{aligned}$$

Thus,

$$L_{(e_1,g)}(f)_1(k_1,g_1) = \begin{cases} f(\tau_g(k_1)) & \text{if} \quad gg_1 = e_2, \\ 0 & \text{if} \quad gg_1 \neq e_2. \end{cases} \tag{4.2}$$

Therefore, $R_{(e_1,g)}(\tau_g f)_1 = L_{(e_1,g)} f_1$. In other words $(\tau_g f)_1 = R_{(e_1,g^{-1})} L_{(e_1,g)} f_1$. Now observe that

$$m_1(\tau_g f) = \frac{m((\tau_g f)_1)}{m(\chi_{K \rtimes_\tau e_2})} = \frac{m(R_{(e_1,g^{-1})} L_{(e_1,g)} f_1)}{m(\chi_{K \rtimes_\tau e_2})} = \frac{m(f_1)}{m(\chi_{K \rtimes_\tau e_2})} = m(f).$$

A contradiction with the strong ergodicity of τ (Lemma 4.2.3). Consequently, $m(\chi_{K \rtimes_\tau e_2}) = 0$. For a subset C of G let $m_2(\chi_C) = m(\chi_{K \rtimes_\tau C})$ and let m_3 be an extension of m_2 to a mean on $l_\infty(G)$. Then m_3 is a mean on $l_\infty(G)$ and $m_3(\delta_{e_2}) = m(\chi_{K \rtimes_\tau e_2}) = 0$. Furthermore, m_3 is also inner invariant since m_3 is an extension of m_2 and $(K \times gCg^{-1}) = (e_1, g)(K \times C)(e_1, g^{-1})$ for each $g \in G$ and each subset C of G. □

Lemma 4.2.5. *The following conditions hold:*

1. *If there is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_2 = 1$ such that $||L_g \phi_\alpha - \Delta^{\frac{1}{2}}(g) R_g \phi_\alpha||_2 \to 0$, for all $g \in K$, then K is inner amenable.*

2. *If K is unimodular and there is a net $\{V_\alpha\}$ of Borel subsets of K with $0 < \lambda(V_\alpha) < \infty$ such that $||\frac{L_g \chi_{V_\alpha}}{\lambda(V_\alpha)} - \frac{R_g \chi_{V_\alpha}}{\lambda(V_\alpha)}||_1 \to 0$ for all $g \in K$, then there is a net $\{\psi_\alpha\}$ in $L_2(K)$ with $\psi_\alpha \geq 0$ and $||\psi_\alpha||_2 = 1$ such that $||L_g \psi_\alpha - R_g \psi_\alpha||_2 \to 0$, for all $g \in K$.*

Proof. (1): For each α put $\psi_\alpha = \phi_\alpha^2$. Then for $g, k \in K$

$$\begin{aligned}
&\int\int (\phi_\alpha(u) - \Delta^{\frac{1}{2}}(g)\phi_\alpha(v))^2 d\delta_g * \delta_k(u) d\delta_k * \delta_g(v) \\
&= L_g \phi_\alpha^2(k) + \Delta(g) R_g \phi_\alpha^2(k) - 2\Delta^{\frac{1}{2}}(g) L_g \phi_\alpha(k) R_g \phi_\alpha(k) \\
&= (L_g \phi_\alpha(k) - \Delta^{\frac{1}{2}}(g) R_g \phi_\alpha(k))^2 + L_g \phi_\alpha^2(k) + \Delta(g) R_g \phi_\alpha^2(k) - (L_g \phi_\alpha)^2(k) - \Delta(g)(R_g \phi_\alpha)^2(k)
\end{aligned}$$

Hence,

$$\begin{aligned}
&-[\int\int\int(\phi_\alpha(u)-\Delta^{\frac{1}{2}}(g)\phi_\alpha(v))^2 d\delta_g*\delta_k(u)d\delta_k*\delta_g(v)d\lambda(k)]\\
&=-[\int(L_g\phi_\alpha(k)-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k))^2 d\lambda(k)+\int L_g\phi_\alpha^2(k)d\lambda(k)+\int\Delta(g)R_g\phi_\alpha^2(k)d\lambda(k)\\
&-\int(L_g\phi_\alpha)^2(k)d\lambda(k)-\int\Delta(g)(R_g\phi_\alpha)^2(k)d\lambda(k)]\\
&\leq -||L_g\phi_\alpha(k)-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k)||_2^2-||\phi_\alpha||_2^2-||\phi_\alpha||_2^2+||\phi_\alpha||_2^2+||\phi_\alpha||_2^2\to 0,
\end{aligned}$$

because $\int\Delta(g)(R_g\phi_\alpha)^2(k)d\lambda(k) =< \Delta(g)R_g\phi_\alpha, R_g\phi_\alpha >=< \phi_\alpha, R_{\check{g}}R_g\phi_\alpha >\leq ||\phi_\alpha||_2^2$ and each ϕ_α is positive. In addition,

$$\begin{aligned}
\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha(k)R_g\phi_\alpha(k)-\Delta(g)R_g\phi_\alpha^2(k) &\leq \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha(k)R_g\phi_\alpha(k)-\Delta(g)(R_g\phi_\alpha)^2(k)\\
&=[L_g\phi_\alpha(k)-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k)]\ \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k),
\end{aligned}$$

by Holder's inequality. Thus,

$$\begin{aligned}
&\int|\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha(k)R_g\phi_\alpha(k)-\Delta(g)R_g\phi_\alpha^2(k)|d\lambda(k)\\
&\leq \Delta^{\frac{1}{2}}(g)||R_g\phi_\alpha||_2\ ||L_g\phi_\alpha(k)-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k)||_2\to 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
||L_g\psi_\alpha-\Delta(g)R_g\psi_\alpha||_1 &= \int|L_g\phi_\alpha^2(k)-\Delta(g)R_g\phi_\alpha^2(k)|d\lambda(k)\\
&\leq \int|\int\int(\phi_\alpha(u)-\Delta^{\frac{1}{2}}(g)\phi_\alpha(v))^2 d\delta_g*\delta_k(u)d\delta_k*\delta_g(v)|d\lambda(k)\\
&+\int|2\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha(k)R_g\phi_\alpha(k)-2\Delta(g)R_g\phi_\alpha^2(k)|d\lambda(k)\to 0,
\end{aligned}$$

since,

$$\begin{aligned}
&\int\int(\phi_\alpha(u)-\Delta^{\frac{1}{2}}(g)\phi_\alpha(v))^2 d\delta_g*\delta_k(u)d\delta_k*\delta_g(v)\\
&=\int\int[\phi_\alpha^2(u)+\Delta(g)\phi_\alpha^2(v)-2\Delta^{\frac{1}{2}}(g)\phi_\alpha(u)\phi_\alpha(v)]d\delta_g*\delta_k(u)d\delta_k*\delta_g(v)\\
&=L_g\phi_\alpha^2(k)-\Delta(g)R_g\phi_\alpha^2(k)+2\Delta(g)R_g\phi_\alpha^2(k)-2\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha^2(k)L_g\phi_\alpha^2(k).
\end{aligned}$$

By Lemma 4.2.1 then K is inner amenable. The rest follows by a similar argument as in ([64], Theorem 4.3) if K is unimodular. □

Remark 4.2.1. *Let K be a discrete hypergroup. If there is a positive norm* *one net $\{\phi_\alpha\}$ in $l_2(K)$ with $\phi_\alpha(e) \to 0$ such that $||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2 \to 0$, for all $g \in K$, $l_\infty(K)$ has a non-trivial inner invariant mean.*

Theorem 4.2.6. *The following are equivalent:*

1. *There is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_2 = 1$ such that $||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2 \to 0$, for all $g \in K$.*

2. *There is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_2 = 1$ such that for each $g \in K$*

$$|\ ||L_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)| \to 0 \ \text{ and } \ |\ ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)| \to 0.$$

In this case K is inner amenable and there is a state m on $B(L_2(K))$ such that $m(L_g) = m(\Delta^{\frac{1}{2}}(g)R_g)$, for all $g \in K$, where L_g (R_g) is the left (right) translation operator on $L_2(K)$.

Proof. If (1) holds, then for $g \in K$

$$\begin{aligned} |\ ||L_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)\ | &= |\ < L_g\phi_\alpha, L_g\phi_\alpha > - < L_g\phi_\alpha, \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha >\ | \\ &= |\ < L_g\phi_\alpha, L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha >\ | \\ &\leq ||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha|| \to 0. \end{aligned}$$

Similarly, $|\ ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)\ | \to 0$, for $g \in K$. Conversely, for each $g \in K$ we have

$$\begin{aligned} ||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 &=< L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha, L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha > \\ &= ||L_g\phi_\alpha||_2^2 + ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - 2 < L_g\phi_\alpha, \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha > \\ &= ||L_g\phi_\alpha||_2^2 + ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - 2\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e) \\ &\leq |\ ||L_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)\ | \\ &+|\ ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)\ | \to 0. \end{aligned}$$

For each $T \in B(L_2(K))$ let $m_\alpha T =< T\phi_\alpha, \phi_\alpha >$ and let m be a weak*-cluster point of the net $\{m_\alpha\}$ in $B(L_2(K))^*$. Without loss of generality assume that

$mT = \lim_\alpha m_\alpha(T)$. Then m is a state on $B(L_2(K))$ and for $g \in K$

$$\begin{aligned}
|m(L_g) - m(\Delta^{\frac{1}{2}}(g)R_g)| &= |\lim_\alpha < L_g\phi_\alpha, \phi_\alpha > - \lim_\alpha < \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha, \phi_\alpha > | \\
&= |\lim_\alpha < L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha, \phi_\alpha > | \\
&\leq \lim_\alpha ||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha|| = 0.
\end{aligned}$$

In addition, K is inner amenable by Lemma 4.2.5. □

Let G be a locally compact group. Then G is an $[IN]$-group if and only if G possesses a compact neighborhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in G$. However, one may not expect this equivalence relation to hold in the hypergroup setting. A hypergroup K is called $[IN]$-hypergroup if there is a compact neighbourhood V of e such that $g*V = V*g$, for all $g \in K$. It is easy to see that each of compact or commutative hypergroups are $[IN]$-hypergroups and possess a compact neighborhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in K$. For a discrete hypergroup K the situation is quite different: although K is an $[IN]$-hypergroup, we have that $L_g\delta_e = R_g\delta_e$, for all $g \in K$ if and only if $\delta_g * \delta_{\check{g}}(e) = \delta_{\check{g}} * \delta_g(e)$, for all $g \in K$.

Corollary 4.2.7. *Let K be a hypergroup possessing a compact neighborhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in K$. Let Q_V be the operator on $L_2(K)$ given by $Q_Vf =< f, \chi_V > .\chi_V$ for $f \in L_2(K)$. Then the following are equivalent:*

1. *There is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha \geq 0$, $< \phi_\alpha, \chi_V >= 0$ and $||\phi_\alpha||_2 = 1$ such that $||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2 \to 0$, for all $g \in K$.*

2. *There is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha \geq 0$, $< \phi_\alpha, \chi_V >= 0$ and $||\phi_\alpha||_2 = 1$ such that for $g \in K$*

$$| \, ||L_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)| \to 0, \ \ \textit{and} \ | \, ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)| \to 0.$$

In this case

a. *There is an inner invariant mean m on $L_\infty(K)$ with $m(\chi_V) = 0$.*

b. *There is a state m on $B(L_2(K))$ such that $m(Q_V) = 0$ and $m(L_g) = m(\Delta^{\frac{1}{2}}(g)R_g)$, for all $g \in K$.*

c. *The operators $id - Q_V$ and $id + Q_V$ are not in the C^*-algebra generated by $\{L_g - \Delta^{\frac{1}{2}}(g)R_g \mid g \in K\}$.*

Proof. We will show b $\Rightarrow$ c, for all other parts we refer to the proof of Theorem 4.2.6. Let $T = \sum_{i=1}^{n} \lambda_i (L_{g_i} - \Delta^{\frac{1}{2}}(g_i)R_{g_i})$. Then $m(T) = 0$ and hence

$$||T - (id - Q_V)|| \geq |m(T) - m(id - Q_V)| = 1.$$

Similarly, $||T - (id + Q_V)|| \geq 1$. Thus, $id - Q_V$ and $id + Q_V$ are not in the C^*-algebra generated by $\{L_g - \Delta^{\frac{1}{2}}(g)R_g \mid g \in K\}$. $\square$

Remark 4.2.2. *Let K be a unimodular hypergroup possessing a compact neighbourhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in K$ and let $1 \leq p < \infty$. Then there is a compact operator T in $B(L_p(K))$ such that $L_gT = R_gT$, $L_{\check{k}}TL_g = R_{\check{k}}TR_g$ and $TL_g = TR_g$, for all $g, k \in K$.*

Proof. Let $Tf := < \chi_V, f > \chi_V$. Then for $f \in L_p(K)$ and $g, k \in K$,

$$\begin{aligned} L_{\check{k}}TL_gf &= < \chi_V, L_gf > L_{\check{k}}\chi_V \\ &= < L_{\check{g}}\chi_V, f > L_{\check{k}}\chi_V \\ &= < R_{\check{g}}\chi_V, f > R_{\check{k}}\chi_V \\ &= < \chi_V, R_gf > R_{\check{k}}\chi_V \\ &= R_{\check{k}}TR_gf. \end{aligned}$$

Hence, $L_{\check{k}}TL_g = R_{\check{k}}TR_g$, for all $g, k \in K$. Similarly we can prove other parts. $\square$

Example 4.2.2. *1. Let $K = H \vee J$ be the hypergroup join of a compact group H and a discrete commutative hypergroup J with $H \cap J = \{e\}$. Then there is a compact neighbourhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in K$.*

2. *Let $K = H \vee J$ be the hypergroup join of a finite commutative hypergroup H and a discrete group J with $H \cap J = \{e\}$. Then $\delta_g * \delta_{\check{g}}(e) = \delta_{\check{g}} * \delta_g(e)$, for all $g \in K$ and hence $L_g\delta_e = R_g\delta_e$, for all $g \in K$.*

Lau and Paterson in ([50], Theorem 2) proved that a locally compact group G is inner amenable if and only if there exists a non-zero compact operator in $\mathscr{A}'_\infty$, where $\mathscr{A}'_\infty = \{T \in B(L_\infty(G)) \mid L_{g^{-1}}R_gT = TL_{g^{-1}}R_g, \ \ \forall g \in G\}$. We note that $\mathscr{A}'_\infty = \{T \in B(L_\infty(G)) \mid R_gTR_{g^{-1}} = L_gTL_{g^{-1}}, \ \ \forall g \in G\}$ which is not the case as we step beyond the groundwork of locally compact groups. The following is an extension of ([50], Theorem 2):

Remark 4.2.3. *The following conditions hold:*

1. *If K is inner amenable, then there is a compact operator T in $B(L_\infty(K))$ such that $T(h) = 1$, for some $h \in L_\infty(K)$, $L_{\check{n}}TL_g = R_{\check{m}}TR_g$, $TL_g = TR_g$, for all $g, n, m \in K$ and $T(f) \geq 0$, for $f \geq 0$.*

2. *If there is a non-zero operator T in $B(L_\infty(K))$ such that $TL_g = TR_g$, for all $g \in K$ and $T(f) \geq 0$, for $f \geq 0$, then K is inner amenable and $T(f) \geq 0$, for $f \geq 0$.*

Proof. 1. If m is an inner invariant mean on $L_\infty(K)$, then the operator T in $B(L_\infty(K))$ defined by $T(f) = m(f)1$, for $f \in L_\infty(K)$ is the desired operator.

2. Let m be a mean on $L_\infty(K)$. Then $m \circ T$ is an inner invariant positive linear functional on $L_\infty(K)$. Let $f_0 \in L_\infty(K)$ such that $T(f_0) > 0$. Then f_0 can be decomposed into positive elements and if $f \geq 0$, then $T(f) \leq ||f||T(1)$. Hence, $m \circ T(1) \neq 0$ and $\frac{m \circ T}{m \circ T(1)}$ is an inner invariant mean on $L_\infty(K)$.

□

Corollary 4.2.8. *K is inner amenable if and only if there is a non-zero compact operator T in $B(L_\infty(K))$ such that $TL_g = TR_g$, for all $g \in K$ and $T(f) \geq 0$, for $f \geq 0$.*

Corollary 4.2.9. *Let G be a locally compact group. Then G is inner amenable if and only if there is a non-zero operator T in $\mathscr{A}_\infty'$ such that $TL_g = TR_g$, for all $g \in G$ and $T(f) \geq 0$, for $f \geq 0$.*

We say that K satisfies central Reiter's condition P_1, if there is a net $\{\phi_\alpha\}$ in $L_1(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_1 = 1$ such that $||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0$ uniformly on compact subsets of K. By Lemma 4.2.1 if K satisfies central Reiter's condition P_1, then K is inner amenable. Sinclair ([63], page 47) in particular called a net $\{\phi_\alpha\}$ in $L_1(G)$ quasi central if $||\mu * \phi_\alpha - \phi_\alpha * \mu|| \to 0$, for all $\mu \in M(G)$, where G is a locally compact group. We say that the net $\{\phi_\alpha\}$ in $L_1(K)$ is quasi central if $||\mu * \phi_\alpha - \phi_\alpha \circledast \mu|| \to 0$, for all $\mu \in M(K)$.

Remark 4.2.4. *If the net $\{\phi_\alpha\}$ in $L_1(K)$ satisfies central Reiter's condition P_1, then*

1. *For given $\{\psi_i\}_{i=1}^n \subseteq L_1(K)$ and $\epsilon > 0$, there is an element $\phi \in L_1(K)$ such that $||\psi_i * \phi - \phi * \psi_i|| < \epsilon$, for $i = 1, 2, ..., n$.*

2. *The net $\{\phi_\alpha\}$ is a quasi central net in $L_1(K)$.*

Proof. (1): Let $\epsilon > 0$ be given and let C_i be compact subsets of K such that $\int_{K\setminus C_i} |\psi_i|(g)d\lambda(g) < \epsilon$. Let $C = \bigcup_{i=1}^n C_i$ and let $\alpha \in I$ be such that $||L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha(k)|| < \epsilon$, for all $g \in C$. Then

$$
\begin{aligned}
&||\psi_i * \phi_\alpha - \phi_\alpha * \psi_i||_1 \\
&= \int | \int \psi_i(g)L_{\check{g}}\phi_\alpha(k)d\lambda(g) - \int \psi_i(g)\Delta(\check{g})R_{\check{g}}\phi_\alpha(k)d\lambda(g)|d\lambda(k) \\
&\leq \int |\psi_i(g)| \int |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha(k)|d\lambda(k)\ d\lambda(g) \\
&= \int_{K\setminus C} |\psi_i(g)| \int |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha(k)|d\lambda(k)\ d\lambda(g) \\
&+ \int_C |\psi_i(g)| \int |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha(k)|d\lambda(k)\ d\lambda(g) < \epsilon^2 + \epsilon\ Max_{i=1,\ldots,n}||\psi_i||_1
\end{aligned}
$$

(2): Without loss of generality assume that $\mu \in M(K)$ has a compact support C. Let $\epsilon > 0$ be given and let $\alpha \in I$ be such that $||L_{\check{g}}\phi_\alpha - \Delta(\check{g})R_{\check{g}}\phi_\alpha|| <$

ϵ, for all $g \in C$. Then

$$
\begin{aligned}
||\mu * \phi_\alpha - \phi_\alpha \circledast \mu|| &= \int | \int (L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g}) R_{\check{g}}\phi_\alpha) d\mu(g) | d\lambda(k) \\
&\leq \int \int_C |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g}) R_{\check{g}}\phi_\alpha| d\mu(g) d\lambda(k) \\
&+ \int \int_{K\setminus C} |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g}) R_{\check{g}}\phi_\alpha| d\mu(g) d\lambda(k) \\
&\leq \epsilon ||\mu||.
\end{aligned}
$$

□

Losert and Rindler called a net $\{\phi_\alpha\}$ in $L_1(G)$, G is a locally compact group, asymptotically central if $\frac{1}{||\phi_\alpha||}(\Delta(g) R_g L_{g^{-1}}\phi_\alpha - \phi_\alpha) \to 0$ weakly for all $g \in G$ ([56]). We say that the net $\{\phi_\alpha\}$ in $L_1(K)$ is (hypergroup) asymptotically central if $\frac{1}{||\phi_\alpha||}(\Delta(g) R_g L_{\check{g}}\phi_\alpha - \phi_\alpha) \to 0$ ($\frac{1}{||\phi_\alpha||}(\Delta(g) R_g \phi_\alpha - L_g\phi_\alpha) \to 0$) weakly for all $g \in K$. The reason for our definition is that $Z(L_1(K)) = \{\phi \in L_1(K) \mid \Delta(g) R_g\phi = L_g\phi, \ \ \forall g \in K\}$, where $Z(L_1(K))$ is the algebraic center of the hypergroup algebra $L_1(K)$. Then it is easy to see that if K is discrete and unimodular or commutative, then any approximate identity in $L_1(K)$ is hypergroup asymptotically central.

Remark 4.2.5. *If $L_1(K)$ has an asymptotically central bounded approximate identity, then K is an inner amenable locally compact group.*

Proof. Let $\{\phi_\alpha\}$ be an asymptotically central bounded approximate identity for $L_1(K)$ and m be a weak*-cluster point of $\{\phi_\alpha\}$ in $L_\infty(K)^*$. Without loss of generality assume that ϕ_α's are real-valued and $\lim < \phi_\alpha, f >=< m, f >$ for each $f \in L_\infty(K)$. Then $m(L_g R_{\check{g}} f) = m(f)$, for each $f \in L_\infty(K)$ and $g \in K$. In addition, $m(\phi * f) = \lim < \phi_\alpha, \phi * f >= \lim < \check{\Delta}\check{\phi} * \phi_\alpha, f >=< \check{\Delta}\check{\phi}, f >= \phi * f(e)$, for $\phi \in L_1(K)$ and $f \in L_\infty(K)$. Thus, $m(f) = f(e)$, for each $f \in C_0(K)$ ([64], Lemma 2.2). Therefore,

$$\delta_g * \delta_{\check{g}}(f) = R_{\check{g}} f(g) = L_g R_{\check{g}} f(e) = m(L_g R_{\check{g}} f) = m(f) = \delta_e(f),$$

for $f \in C_0(K)$. i.e. $\delta_g * \delta_{\check{g}} = \delta_e$, for all $g \in K$ and hence $G(K) = K$. It follows then by the proof of ([56], Theorem 2) that the locally compact group K is also inner amenable. $\square$

In 1991, Lau and Paterson characterized inner amenable locally compact groups G in terms of a fixed point property of an action of G on a Banach space ([49], Theorem 5.1). This characterization can be extended naturally to hypergroups and we have:

Remark 4.2.6. *The following are equivalent:*

1. *K is inner amenable.*

2. *Whenever $\{T_g \in B(E) \mid g \in K\}$ is a separately continuous representation of K on a Banach space E as contractions, there is some $T \in \overline{\{N_\phi \mid \phi \in L_1(K), \ \ ||\phi|| = 1, \ \ \phi \geq 0\}}^{W^*.o.t}$ such that $N_g T = T N_g$, for all $g \in K$.*

Remark 4.2.7. *Let N be a closed normal Weil subhypergroup of K. If K is inner amenable, then K/N is also inner amenable.*

Proof. Define a linear isometry ϕ from $L_\infty(K/N)$ to the subspace $\{f \in L_\infty(K) \mid R_g f = R_k f, \ \ g \in k * N, \ \ k \in K\}$ of $L_\infty(K)$ by $\phi(f) = f \circ \pi$, where π is the quotient map from K onto K/N. Then

$$\begin{aligned} \int |L_g(\phi f)(k) - \phi(L_{g*N} f)(k)| d\lambda(k) &= \int | \int f(u * N) d\delta_g * \delta_k(u) - (L_{g*N} f) \circ \pi(k)| d\lambda(k) \\ &= \int | \int f(u * N) d\delta_{g*N} * \delta_{k*N}(u * N) - L_{g*N} f(k * N)| d\lambda(k) \\ &= 0, \end{aligned}$$

since N is a Weil subhypergroup. Thus, $\phi(L_{g*N} f) = L_g(\phi f)$ for $f \in L_\infty(K/N)$ and $g \in K$. Similarly, $\phi(R_{g*N} f) = R_g(\phi f)$ for $f \in L_\infty(K/N)$ and $g \in K$. Let m be an inner invariant mean on $L_\infty(K)$ and define $m_1(f) = m(\phi f)$, $f \in L_\infty(K/N)$. Then m_1 is a mean on $L_\infty(K/N)$. In addition, for $f \in L_\infty(K/N)$ and $g \in K$

$$m_1(L_{g*N} f) = m(\phi(L_{g*N} f)) = m(L_g \phi f) = m(R_g \phi f) = m(\phi(R_{g*N} f)) = m_1(R_{g*N} f).$$

$\square$

4.3 Hahn-Banach extension and monotone extension properties

Definition 4.3.1. *Monotone extension property: If X is a subspace of a partially ordered linear space Y, p is a positive-homogeneous, subadditive (sublinear) function from Y to $\mathbb{R}$, f is an additive monotonic function from X to $\mathbb{R}$ such that $f(x) \leq p(x)$, $x \in X$, then there exists an additive monotonic extension F of f defined from Y to $\mathbb{R}$ such that $F(y) \leq p(y)$, for all $y \in Y$.*

It is the purpose of this section to provide a hypergroup version of Hahn-Banach extension property and monotone extension property by which amenable hypergroups can be characterized.

Let E be a partially ordered Banach space over $\mathbb{R}$. An element $1 \in E$ is called a topological order unit if for each $f \in E$ there exists $\lambda > 0$ such that $-\lambda 1 \leq f \leq \lambda 1$ and the set $\{f \in E \mid 1 \leq f \leq 1\}$ is a neighbourhood of E and a proper subspace I of E is said to be a proper ideal if $[0, f] \subseteq I$, for each $f \in E$. Moreover, a separately continuous linear representation $\mathscr{T} = \{T_g \mid g \in K\}$ of K on E is positive if $T_g f \geq 0$ for all $g \in K$ and $f \geq 0$. $\mathscr{T}$ is normalized if $T_g 1 = 1$ for all $g \in K$.

Definition 4.3.2. *(Riesz, 1928) A vector lattice E is defined to be a vector space endowed with a partial order, $\leq$, that for any $x, y, z \in E$, satisfies:*

1. *$x \leq y$ implies $x + z \leq y + z$.*
2. *For any scalar $0 \leq a$, $x \leq y$ implies $ax \leq ay$.*
3. *For any pair of vectors $x, y \in E$ there exists a supremum in E with respect to the partial order of the lattice structure ($\leq$).*

Definition 4.3.3. *A Banach lattice E is a vector lattice that is at the same time a Banach space with a norm which satisfies the monotonicity condition, i.e, $x \leq y$ implies $||x|| \leq ||y||$, for all $x, y \in E$.*

Theorem 4.3.1. *The following are equivalent:*

1. *$RUC(K)$ has a right invariant mean.*

2. *Let $\{T_g \in B(E) \mid g \in K\}$ be a separately continuous representation of K on a Banach space E and let F be a closed T_K-invariant subspace of E. Let p be a continuous seminorm on E such that $p(T_g x) \leq p(x)$ for all $x \in E$ and $g \in K$ and Φ be a continuous linear functional on F such that $|\Phi(x)| \leq p(x)$ and $\Phi(T_g x) = \Phi(x)$ for $g \in K$ and $x \in F$. Then there is a continuous linear functional $\tilde{\Phi}$ on E such that*

   (a) $\tilde{\Phi}|_F \equiv \Phi$.

   (b) $|\tilde{\Phi}(x)| \leq p(x)$ for each $x \in E$.

   (c) $\tilde{\Phi}(T_g x) = \tilde{\Phi}(x)$ for $g \in K$ and $x \in E$.

3. *For any positive normalized separately continuous linear representation $\mathscr{T}$ of K on a partially ordered real Banach space E with a topological order unit 1, if F is a closed $\mathscr{T}$-invariant subspace of E containing 1, and Φ is a $\mathscr{T}$-invariant monotonic linear functional on F, then there exists a $\mathscr{T}$-invariant monotonic linear functional $\tilde{\Phi}$ on E extending Φ.*

4. *For any positive normalized separately continuous linear representation $\mathscr{T}$ of K on a partially ordered real Banach space E with a topological order unit 1, E contains a maximal proper $\mathscr{T}$-invariant ideal.*

5. *Whenever $\mathscr{T} = \{T_g \in B(E) \mid g \in K\}$ is a separately continuous representation of K on a real Banach space E and let F be a closed $\mathscr{T}$-invariant subspace of E and p is a continuous sublinear map on E such that $p(T_g x) \leq p(x)$ for all $x \in E$ and $g \in K$. If Φ is a continuous $\mathscr{T}$-invariant linear functional on F such that $\Phi(x) \leq p(x)$ for $x \in F$, then there is a continuous $\mathscr{T}$-invariant extension $\tilde{\Phi}$ of Φ to E such that $\tilde{\Phi}(x) \leq p(x)$ for each $x \in E$.*

Proof. $1 \Rightarrow 2$: By Hahn-Banach extension theorem there is a continuous linear functional Φ_1 on E such that $|\Phi_1(x)| \leq p(x)$ for each $x \in E$ and $\Phi_1|_F \equiv \Phi$. For each $f \in E$ define a continuous bounded function $h_{\Phi_1,f}$ on K via $h_{\Phi_1,f}(g) = \Phi_1(T_g f)$. Let $\{g_\alpha\}$ be a net in K converging to e. Then

$$\begin{aligned}
||R_{g_\alpha} h_{\Phi_1,f} - h_{\Phi_1,f}|| &= \sup_{g\in K} |R_{g_\alpha} h_{\Phi_1,f}(g) - h_{\Phi_1,f}(g)| \\
&= \sup_{g\in K} |\textstyle\int \Phi_1(T_u f) d\delta_g * \delta_{g_\alpha}(u) - \Phi_1(T_g f)| \\
&= \sup_{g\in K} |\Phi_1(T_g T_{g_\alpha} f) + \Phi_1(-T_g f)| \\
&\leq \sup_{g\in K} p(T_g T_{g_\alpha} f - T_g f) \\
&\leq p(T_{g_\alpha} f - f) \to 0,
\end{aligned}$$

since $\Phi_1 \in E^*$. Hence, $h_{\Phi_1,f} \in RUC(K)$ ([64], Remark 2.3). Let m be a right invariant mean on $RUC(K)$ and let $\tilde{\Phi}(f) = m(h_{\Phi_1,f})$, for $f \in E$. Then $\tilde{\Phi}|_F \equiv \Phi$ since $h_{\Phi_1,f}(g) = \Phi_1(T_g f) = \Phi(f)$, for $f \in F$. Furthermore, $|\tilde{\Phi}(f)| \leq \sup_{g\in K} |\Phi_1(T_g f)| \leq p(f)$, for $f \in E$ and

$$h_{\Phi_1,T_g f}(k) = \Phi_1(T_k T_g f) = \int \Phi_1(T_u f) d\delta_k * \delta_g(u) = \int h_{\Phi_1,f}(u) d\delta_k * \delta_g(u) = R_g h_{\Phi_1,f}(k).$$

Thus,

$$\tilde{\Phi}(T_g f) = m(h_{\Phi_1,T_g f}) = m(R_g h_{\Phi_1,f}) = m(h_{\Phi_1,f}) = \tilde{\Phi}(f).$$

$2 \Rightarrow 1$: Let $E = RUC(K)$, $F = \mathbb{C}.1$ and consider the continuous representation $\{R_g \mid g \in K\}$ of K on $RUC(K)$. Define a seminorm p on E by $p(f) = ||f||$. Then $p(R_g f) \leq p(f)$, for $f \in E$ and $g \in K$. In addition, δ_a is a left invariant mean on F for a given $a \in K$ with $|\delta_a(f)| \leq p(f)$. Therefore, there is some $m \in RUC(K)^*$ such that $m|_F \equiv \delta_a$, $m(f) \leq ||f||$ and $m(R_g f) = m(f)$, for $f \in E$ and $g \in K$. Then m is a right invariant mean on $RUC(K)$ because $m(1) = \delta_a(1) = 1 = ||m||$.

For all other parts we refer to ([40], Theorem 2) and a similar argument as above. □

Let $CB_{\mathbb{R}}(K)$ denote all bounded continuous real-valued functions on K and $UC_{\mathbb{R}}(K)$ ($RUC_{\mathbb{R}}(K)$) denote all functions in $CB_{\mathbb{R}}(K)$ which are (right) uniformly continuous. It is easy to see that $UC_{\mathbb{R}}(K)$ and $RUC_{\mathbb{R}}(K)$ are norm-closed translation invariant subspace of $CB_{\mathbb{R}}(K)$ containing constants. However, in contrast to the group case, $RUC_{\mathbb{R}}(K)$ need not be a Banach lattice in general. The following result is a consequence of Theorem 4.3.1 and the proof of ([40], Theorem 1).

Remark 4.3.1. *Let K be a hypergroup such that $RUC_{\mathbb{R}}(K)$ is a Banach lattice. Then the following are equivalent:*

1. *$RUC(K)$ has a right invariant mean.*

2. *For any linear action $\mathscr{T}$ of K on a Banach space E, if U is a $\mathscr{T}$-invariant open convex subset of E containing a $\mathscr{T}$-invariant element, and M is a $\mathscr{T}$-invariant subspace of E which does not meet U, then there exists a closed $\mathscr{T}$-invariant hyperplane H of E such that H contains M and H does not meet U.*

3. *For any contractive action $\mathscr{T} = \{T_g \in B(E) \mid g \in K\}$ of K on a Hausdorff Banach space E, any two points in $\{f \in E \mid T_g f = f, \ \ \forall g \in K\}$ can be separated by a continuous $\mathscr{T}$-invariant linear functional on E.*

Example 4.3.1.

1. *Let K be a hypergroup such that the maximal subgroup $G(K)$ is open. Then $RUC_{\mathbb{R}}(K)$ is a Banach lattice.*

2. *Let $K = H \vee J$ be the hypergroup join of a compact hypergroup H and a discrete hypergroup J with $H \cap J = \{e\}$. Then $RUC_{\mathbb{R}}(K) = CB_{\mathbb{R}}(K)$ is a Banach lattice.*

Proof. To see 1, let $f, h \in RUC_{\mathbb{R}}(K)$ and $\{g_\alpha\}$ be a net in K converging to e. Then $g_\alpha \in G(K)$, for some α_0 and all $\alpha \geq \alpha_0$ since $G(K)$ is open. Thus, $R_{g_\alpha}(f \vee h) = R_{g_\alpha} f \vee R_{g_\alpha} h$ for $\alpha \geq \alpha_0$. Therefore, the mapping $g \mapsto (R_g f, R_g h) \mapsto R_g f \vee R_g h$ from K to $CB_{\mathbb{R}}(K)$ is continuous at e and hence $f \vee h \in RUC_{\mathbb{R}}(K)$. $\square$

Next we use Theorem 4.3.1 to prove that $UC(K) = LUC(K) \cap RUC(K)$ has an invariant mean, for any commutative hypergroup K.

Corollary 4.3.2. *Let K be a commutative hypergroup. Then $UC(K)$ has an invariant mean.*

Proof. Let $\mathscr{T} = \{T_g \in B(E) \mid g \in K\}$ be a separately continuous representation of K on a real Banach space E and let F be a closed $\mathscr{T}$-invariant subspace of E. Let p be a continuous sublinear map on E such that $p(T_g x) \leq p(x)$ for all $x \in E$ and $g \in K$ and ϕ be a continuous $\mathscr{T}$-invariant linear functional on F such that $\phi(x) \leq p(x)$ for $x \in F$. Define a representation $\{T_\mu \in B(E) \mid \mu \in M_1^c(K)\}$ of $M_1^c(K)$, the probability measures with compact support on K, on E via $T_\mu x = \int T_g x d\mu(g)$. Then $T_{\mu * \nu} = T_\mu T_\nu$, for $\mu, \nu \in M_1^c(K)$. In addition, $p(T_\mu x) = p(\int T_g x d\mu(g)) \leq \int p(T_g x) d\mu(g) \leq p(x)$.

Define a real valued function q on E via $q(x) = \inf\{\frac{1}{m} p(T_{\mu_1} x + ... + T_{\mu_m} x)\}$, where the inf is taken over all finite collections of probability measures with compact support $\{\mu_1, ..., \mu_m\}$ on K. Then $q(x) \leq p(x)$ for $x \in E$ since for each $m \in \mathbb{N}$,

$$\tfrac{1}{m} p(T_{\mu_1} x + ... + T_{\mu_m} x) \quad \leq \tfrac{1}{m}\left[p(T_{\mu_1} x) + ... + p(T_{\mu_m} x)\right] \leq p(x).$$

Moreover, q is sublinear. In fact for $m \in \mathbb{N}$, $\alpha \in \mathbb{R}^+$ and $x \in E$,

$$\tfrac{1}{m} p(T_{\mu_1}(\alpha x) + ... + T_{\mu_m}(\alpha x)) = \tfrac{1}{m} \alpha p(T_{\mu_1} x + ... + T_{\mu_m} x).$$

Thus, $q(\alpha x) = \alpha q(x)$ for $\alpha \in \mathbb{R}^+$ and $x \in E$. To see that $q(x+y) \leq q(x) + q(y)$, let $x, y \in E$ and $\epsilon > 0$ be given. Choose probability measures $\mu_1, ..., \mu_m, \nu_1, ..., \nu_n$ on K with compact support such that $\frac{1}{m} p(T_{\mu_1} x + ... + T_{\mu_m} x) \leq q(x) + \epsilon$, and $\frac{1}{n} p(T_{\nu_1} x + ... + T_{\nu_n} x) \leq q(y) + \epsilon$. Consider the set

$\mathcal{K} = \{\nu_j * \mu_i \mid j = 1, ..., n,\ i = 1, ..., m\}$. Then

$$\begin{aligned}
\tfrac{1}{nm} p\Big[\textstyle\sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i} x \Big] &= \tfrac{1}{nm} p\big[\textstyle\sum_{j=1}^n T_{\nu_j} (\sum_{i=1}^m T_{\mu_i} x)\big] \\
&\leq \tfrac{1}{nm} \textstyle\sum_{j=1}^n p\big[T_{\nu_j} (\sum_{i=1}^m T_{\mu_i} x)\big] \\
&\leq \tfrac{1}{nm} \textstyle\sum_{j=1}^n p\big[\sum_{i=1}^m T_{\mu_i} x\big] \\
&= \tfrac{1}{m} p\big[\textstyle\sum_{i=1}^m T_{\mu_i} x\big] \\
&\leq q(x) + \epsilon,
\end{aligned}$$

and similarly, $\frac{1}{nm} p\Big[\sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i} y \Big] \leq q(y) + \epsilon$. Hence,

$$\begin{aligned}
& \tfrac{1}{nm} p\Big[\textstyle\sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i} (x + y) \Big] \\
&= \tfrac{1}{nm} p\Big[\textstyle\sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i} x + \sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i} y \Big] \\
&\leq \tfrac{1}{nm} p\Big[\textstyle\sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i} x \Big] + \tfrac{1}{nm} p\Big[\textstyle\sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i} y \Big] \\
&\leq q(x) + q(y) + 2\epsilon.
\end{aligned}$$

Therefore, $q(x + y) \leq q(x) + q(y)$. For $\mu \in M_1^c(K)$, $x \in E$ and $m \in \mathbb{N}$,

$$\begin{aligned}
& \tfrac{1}{m} p(T_{\mu_1} T_\mu x + ... + T_{\mu_m} T_\mu x) \\
&= \tfrac{1}{m} p(T_\mu T_{\mu_1} x + ... + T_\mu T_{\mu_m} x) \\
&\leq \tfrac{1}{m} p(T_{\mu_1} x + ... + T_{\mu_m} x).
\end{aligned}$$

Hence, $q(T_\mu x) \leq q(x)$. Furthermore, for each $m \in \mathbb{N}$

$$\tfrac{1}{m} p(T_{\mu_1} x + ... + T_{\mu_m} x) \quad \leq \tfrac{1}{m} [p(T_{\mu_1} x) + ... + p(T_{\mu_m} x)] \leq p(x).$$

Thus, $q(x) \leq p(x)$. By Hahn-Banach extension theorem there is a continuous linear functional $\tilde{\phi}$ on E such that $\tilde{\phi}(x) \leq q(x)$ for each $x \in E$ and $\tilde{\phi}|_F \equiv \phi$.

For $x \in E$, $n \in \mathbb{N}$ and $\mu \in M_1^c(K)$

$$
\begin{aligned}
&q(x - T_\mu x) \\
&\leq \tfrac{1}{n+1} p\Big[\Big(T_e(x - T_\mu x) + T_\mu(x - T_\mu x) + T_\mu T_\mu(x - T_\mu x) + \ldots + \underbrace{T_\mu T_\mu \ldots T_\mu}_{\text{n times}}(x - T_\mu x)\Big)\Big] \\
&= \tfrac{1}{n+1} p(x + \underbrace{T_\mu T_\mu \ldots T_\mu}_{\text{n+1 times}}(-x)) \leq \tfrac{1}{n+1}[p(x) + p(-x)] \to 0.
\end{aligned}
$$

Therefore, $\tilde{\phi}(x - T_\mu x) \leq q(x - T_\mu x) \leq 0$. Since $\tilde{\phi}$ is linear By replacing x by $-x$, one has $\tilde{\phi}(T_\mu x) = \tilde{\phi}(x)$. In particular, $\tilde{\phi}(T_g x) = \tilde{\phi}(x)$ for $g \in K$ and $x \in E$. Therefore, $UC(K)$ has an invariant mean (Theorem 4.3.1). □

4.4 Weak*-invariant complemented subspaces of $L_\infty(K)$

Let X be a weak*-closed left translation invariant, invariant complemented subspace of $L_\infty(K)$. Then this section provides a connection between X being invariantly complemented in $L_\infty(K)$ by a weak*-weak*-continuous projection and the behaviour of $X \cap C_0(K)$.

Definition 4.4.1. *A closed left translation invariant complemented subspace Y of $L_\infty(K)$ is called invariant subspace, if there is a continuous projection P from $L_\infty(K)$ onto Y commuting with left translations. If Y is weak*-closed and P is weak*-weak*-continuous, then we say that Y is weak*-invariant complemented subspace of $L_\infty(K)$.*

Theorem 4.4.1. *Let X be a weak*-closed, left translation invariant, invariant complemented subspace of $L_\infty(K)$. Then $X \cap C_0(K)$ is weak* dense in X if and only if there exists a weak*-weak*-continuous projection Q from $L_\infty(K)$ onto X commuting with left translations.*

Proof. Let P be a continuous projection from $L_\infty(K)$ onto X commuting with left translations. We first observe that $P(LUC(K)) \subseteq LUC(K)$. In fact if

$f \in LUC(K)$ and $\{g_\alpha\}$ is a net in K such that $g_\alpha \to g \in K$, then

$$||L_{g_\alpha}Pf - L_gPf|| = ||P(L_{g_\alpha}f - L_gf)|| \leq ||P||\ ||L_{g_\alpha}f - L_gf|| \to 0.$$

Thus, $P|_{C_0(K)}$ is a bounded operator from $C_0(K)$ into $CB(K)$. Define a bounded linear functional on $C_0(K)$ by $\psi_1(f) := (P\check{f})(e)$. Let $\mu \in M(K)$ be such that $(Pf)(e) = \int \check{f}(x)d\mu(x)$, for each $f \in C_0(K)$. Then for $x \in K$ and $f \in C_0(K)$,

$$(Pf)(x) = L_xPf(e) = PL_xf(e) = \int L_xf(\check{y})d\mu(y) = f * \mu(x).$$

Hence, $P(f) = f * \mu$, for $f \in C_0(K)$. Define an operator $T : L_1(K) \to L_1(K)$ via $T(h) := h * \check{\mu}$. Then $Q = T^*$ is weak*-weak*-continuous and $< Qf, h >=< f, h * \check{\mu} >=< f * \mu, h >$, for $h \in L_1(K)$ and $f \in C_0(K)$. Thus, $Q(f) = f * \mu$ for $f \in C_0(K)$. In addition, Q commutes with left translations on $L_\infty(K)$, since for $h \in L_1(K)$ and $f \in L_\infty(K)$

$$< QL_xf, h >=< L_xf, h * \check{\mu} > \ =< f, (L_{\check{x}}h) * \check{\mu} >=< Q(f), L_{\check{x}}h >=< L_xQ(f), h >$$

We will show that Q is a projection. For $f \in C_0(K) \cap X$, and $h \in L_1(K)$,

$$\begin{aligned} < f * \mu, h > \ &= [(f * \mu) * \check{h}](e) \\ &= [f * (h * \check{\mu})\check{}\,](e) \\ &= [(h * \check{\mu}) * \check{f}](e) \\ &= \int (h * \check{\mu})(x)\check{f}(\check{x})dx \\ &=< f, h * \check{\mu} > . \end{aligned}$$

Hence,

$$< Q(f), h > \ =< f, h * \check{\mu} >=< f * \mu, h >=< P(f), h >=< f, h > .$$

If $X \cap C_0(K)$ is weak* dense in X, let $\{f_\alpha\}$ be a net in $X \cap C_0(K)$ such that $f_\alpha \to f$ in the weak*-topology of $L_\infty(K)$. Then, $Q(f) = f$ since Q is weak*-continuous.

Moreover, for $f \in C_0(K)$ and $h \in X^{\perp}$,

$$< Q(f), h > \;\; =< f, h * \check{\mu} >=< f * \mu, h >=< P(f), h >= 0.$$

Thus, $< Q(f), h >= 0$, for each $f \in L_\infty(K)$ and $h \in X^{\perp}$, since $C_0(K)$ is weak*-dense in $L_\infty(K)$. i.e. $Q(f) \in X$.

Conversely, if Q is a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations, then there exists some $\mu \in M(K)$ such that $Q^*|_{L_1(K)}(h) = h * \mu$, for $h \in L_1(K)$ ([7], Theorem 1.6.24). Hence, for $f \in C_0(K)$ we have $Q(f) = f * \check{\mu}$ which is in $C_0(K) \cap X$ ([7], Theorem 1.2.16, iv). Then $C_0(K) \cap X$ is weak*-dense in $X = \{Q(f) \mid f \in L_\infty(K)\}$ since $C_0(K)$ is weak*-dense in $L_\infty(K)$ and Q is weak*-weak*-continuous. □

As a direct consequence of Theorem 4.4.1 we have the following result:

Corollary 4.4.2. *Let K be a compact hypergroup and let X be a weak*-closed left translation invariant subspace of $L_\infty(K)$. Then X is invariantly complemented if and only if there is a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations.*

Corollary 4.4.3. *Let K be a compact hypergroup and let X be a left translation invariant W^*-subalgebra of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB. Then X is the range of a weak*-weak*-continuous projection commuting with left translations.*

Proof. This follows from ([68], Corollary 3.13, Lemma 3.9) and Theorem 4.4.1.

□

Corollary 4.4.4. *The following are equivalent:*

1. *K is compact.*

2. *K is amenable and for every weak*-closed left translation invariant, invariant complemented subspace X of $L_\infty(K)$, there exists a weak*-weak*-*

continuous projection from $L_\infty(K)$ onto X commuting with left translations.

Proof. If K is compact, then item 2 follows from ([68], Lemma 3.9), Theorem 4.4.1 and ([64], Example 3.3) since $C_0(K) = LUC(K)$ and $LUC(K) \cap X$ is weak*-dense in X. Conversely, consider the one-dimensional subspace $X = \mathbb{C}.1$. Then X is a weak*-closed left translation invariant, invariant complemented subspace of $L_\infty(K)$, since K is amenable. If P is a weak*-weak*-continuous projection from $L_\infty(K)$ onto $\mathbb{C}.1$ commuting with left translations, then there is some $\phi \in L_1(K)$ such that $P(f) = \delta_\phi(f)$ for $f \in L_\infty(K)$. Hence, $\delta_\phi(1) = 1$ and $< \delta_\phi, L_g f >=< \delta_\phi, f >$. i.e., $L_g\phi = \phi$, for $g \in K$. In particular, $L_g\phi(e) = \phi(g) = \phi(e)$, for all $g \in K$. Therefore, $1 = \delta_\phi(1) = \int_K \phi(g) d\lambda(g) = \phi(e)\lambda(K)$ which means that K is compact. $\square$

Chapter 5

Final remarks and open problems

5.1 Remarks related to chapter 2

A subspace X of $M(K)$ is said to be $C_0(K)$-invariant if $X.C_0(K) \subseteq X$, where, $d(\mu.\phi)(g) = \phi(g)d\mu(g)$. Similar to locally compact groups, we have the following characterization of non-zero weak*-closed, $C_0(K)$-invariant $*$-subalgebras of $M(K)$ (for the group case see [3], Theorem 3.1).

Remark 5.1.1. *X is a non-zero weak*-closed, $C_0(K)$-invariant $*$-subalgebra of $M(K)$ if and only if $X = \{\mu \in M(K) \mid supp(\mu) \subseteq N\}$, for some closed subhypergroup N of K, where $N = \overline{\bigcup_{\mu\in X} supp(\mu)}$.*

Remark 5.1.2. *X is a left translation invariant strictly closed, C^*-subalgebra of $CB(K)$ possessing the local translation property TB if and only if there exists a unique closed Weil subhypergroup N such that X is given by $X = \{f \in CB(K) \mid R_g f = R_k f, \quad \forall g \in k * N, \quad k \in K\}$.*

Furthermore, N is normal if and only if X is inversion invariant and translation invariant.

Proof. The proof is very similar to that of Theorem 2.2.5. □

By a similar method as in Lemma 2.3.2 we observe that:

Remark 5.1.3. *Let K be an amenable hypergroup. Then for any reflexive Banach space X and any jointly weak*-weak*-continuous representation $\{T_g \in B(X^*) \mid g \in K\}$, there exists a semigroup $\mathscr{S}$ of operators in $B(X^*)$ such that each $P \in \mathscr{S}$ commutes with any continuous linear operator T from X^* into X^* which commutes with $\{T_g \in B(X^*) \mid g \in K\}$ and for each $f \in X^*$, $\overline{co(T_K f)} \cap \{f \in X^* \mid T_g f = f, \ \ \forall g \in K\}$ is the set of all Pf such that $P \in \mathscr{S}$ and P is a continuous projection from X^* onto F.*

Corollary 5.1.1. *Let N be a closed Weil subhypergroup of an amenable hypergroup K. Then there exists a semigroup $\mathscr{S} = \{P_m \in B(L_p(K)) \mid m \in Mean(LUC(K))\}$ of operators on $L_p(K)$ ($1 < p < \infty$) such that for each $f \in L_p(K)$, $\overline{co(R_N f)} \cap \{f \in L_p(K) \mid R_n f = f, \ \ \forall n \in N\}$ is the set of all Pf such that $P \in \mathscr{S}$ is a continuous projection from $L_p(K)$ onto $\{f \in L_p(K) \mid R_n f = f, \ \ \forall n \in N\}$ commuting with left translations.*

Proof. This follows from Remarks 2.2.6 and 5.1.3. □

Open problem 1. *Does any left translation invariant W^*-subalgebra of $L_\infty(K)$ contain the constant functions? This statement is true when K is a locally compact group ([42], Lemma 3.1).*

The answer is affirmative if K is a compact hypergroup.

Remark 5.1.4. *Let K be a compact hypergroup and let X be a non-zero weak*-closed left translation invariant subspace of $L_\infty(K)$. Then X contain the constant functions. In fact let $f \in X$ be non-zero. Then $1 * f \in X$ is a non-zero constant function in X since $L_1(K) * X \subseteq X$.*

Open problem 2. *Provide an example of a Weil subhypergroup without the translation property TB.*

Open problem 3. *Let N be a closed, normal Weil subhypergroup of K such that $WAP(K/N)$ and $WAP(N)$ both have invariant means, does $WAP(K)$ possess an invariant mean? Is a similar statement true for $AP(K)$?*

5.2 Problems related to chapter 3

Open problem 4. *Let K be an arbitrary hypergroup such that $WAP(K)^*$ has weak*-normal structure. Can we conclude that K is finite?*

Open problem 5. *If $AP(K)$ is finite dimensional, then $AP(K)^*$ has weak*-normal structure. Is the converse true?*

Open problem 6. *Can amenability of K be characterized by any common fixed point property for non-expansive mappings?*

Open problem 7. *Can we characterize hypergroups possessing an invariant mean on the space $WAP(K)(AP(K))$ with any common fixed point property for non-expansive mappings?*

Open problem 8. *By a similar argument as in the proof of Remark 3.3.1 (Theorem 3.3.4) one can show that if for any separately (weakly) continuous, (quasi weakly) equicontinuous and nonexpansive representation $\{T_g \mid g \in K\}$ of K on a compact convex subset Y of a separated locally convex topological vector space (E, τ), Y has a common fixed point for K, then $AP(K)$ ($WAP(K)$) has an invariant mean. Is the converse true?*

Open problem 9. *Let K be an amenable hypergroup and let X be a weak*-closed left translation invariant subspace of $L_\infty(K)$, which is contained and complemented in $LUC(K)$. Is X invariantly complemented in $LUC(K)$?*

Open problem 10. *Let K be an amenable hypergroup and X be a weak*-closed left translation invariant, complemented subspace of $L_\infty(K)$. Is X invariantly complemented in $L_\infty(K)$? We do not know the answer even if K is commutative.*

Open problem 11. *Let K be an amenable hypergroup and let m be an invariant mean on $WAP(K)$. Is $AP(K)$ complemented in $WAP(K)$?*

It is known that $WAP(G) = AP(G) \oplus \{f \in WAP(G) \mid m_1(|f|) = 0\}$, where G is a locally compact group and m_1 is an invariant mean on $WAP(G)$.

However, by ([75], p, 72) if K a hypergroup arising from the Jacobi polynomial $p^{(1/2,1/2)}(x)$, then there is a non-zero element $h^* \in C_0(K) \cap AP(K)$. Hence, $m(|h^*|) = 0$ ([65], Remark 3.4.2). Therefore, $AP(K) \cap \{f \in WAP(K) \mid m(|f|) = 0\} \neq \{0\}$

Open problem 12. *Let K be an amenable hypergroup such that $C_0(K) \cap AP(K) = \{0\}$ and let m be an invariant mean on $WAP(K)$. Do we have that $AP(K) \cap \{f \in WAP(K) \mid m(|f|) = 0\} = \{0\}$?*

5.3 Problems related to chapter 4

Open problem 13. *Is there any non-inner amenable hypergroup K such that $Z(L_1(K))$ is non-trivial?*

Open problem 14. *Let K be a hypergroup such that $L_1(K)$ has a positive non-trivial center. Is there a compact neighbourhood V of the identity with $\Delta(g)R_g\chi_V = L_g\chi_V$?*

Open problem 15. *Let K be a connected, inner amenable hypergroup. Is K amenable?*

We say that a hypergroup K is topologically inner amenable if there exists a mean m on $L_\infty(K)$ such that $m((\Delta\phi)\check{} * f) = m(f * \check{\phi})$ for any positive norm one element ϕ in $L_1(K)$ and any $f \in L_\infty(K)$. It is easy to see that any inner invariant mean on $UC(K)$ is topologically inner invariant since for a positive norm one element ϕ in $L_1(K)$ and $f \in LUK(K)$

$$\begin{aligned} m(f * \check{\phi}) &= \int < m, R_g f\phi(g) > d\lambda(g) \\ &= \int < m, L_g f\phi(g) > d\lambda(g) \\ &=< m, \int L_g f\phi(g) d\lambda(g) > \\ &=< m, \int L_g f\phi(g)\Delta(g) d\check{\lambda}(g) > \\ &= m((\Delta\phi)\check{} * f). \end{aligned}$$

However, on the space $L_\infty(K)$ the relation between topological inner invariant means and inner invariant means is not clear.

Open problem 16. *Let m be a topological inner invariant mean on $L_\infty(K)$. Is m also an inner invariant mean?*

Open problem 17. *Let K be an inner amenable hypergroup. Is there any topological inner invariant mean on $L_\infty(K)$?*

Open problem 18. *Let K be an inner amenable hypergroup. Does K satisfy central Reiter's condition P_1? (see ([57], Remark) for the group case).*

5.4 Other remarks and problems

Let A be a closed translation invariant subalgebra of $L_\infty(K)$ containing constant functions. In what follows we provide an equivalent condition for A to possess a multiplicative left invariant mean. This equivalence is given in terms of a fixed point property which is a generalization of Mitchell fixed point theorem ([58], Theorem 1).

Definition 5.4.1. *Let A be a closed translation-invariant subalgebra of $L_\infty(K)$ containing constant functions. Let E be a separated locally convex topological vector space and Y be a compact subset of E. Let X be the space of all probability measures on Y. Let $\mathscr{T} = \{T_g \mid g \in K\}$ be a continuous representation of K on X. Suppose that $B := \{y \in Y \mid T_g y \in Y, \ \forall g \in K\} \neq \emptyset$ and for each $y \in B$, define $h_{y,\phi}(g) = \phi(T_g y)$, for $g \in K$ and $\phi \in CB(Y)$. It is easy to see that $h_{y,\phi}$ is continuous and $||h_{y,\phi}|| \leq ||\phi||$. Therefore, $h_y : \phi \mapsto h_{y,\phi}$ is a bounded linear operator from $CB(Y)$ into $CB(K)$. Let $Y_1 := \{y \in B \mid h_y(CB(Y)) \subseteq A\}$.*

The family $\mathscr{T}$ is an $E-E$-representation of K, A on X if $B \neq \emptyset$ and $Y_1 \neq \emptyset$,

Definition 5.4.2. *The pair K, A has the common fixed point property on compacta with respect to $E-E$-representations if, for each compact subset Y of a separated locally convex topological vector space E and for each $E-E$-representation of K, A on X, there is in Y a common fixed point of the family $\mathscr{T}$.*

Remark 5.4.1. *Let A be a closed translation-invariant subalgebra of $L_\infty(K)$ containing constant functions. Then the following are equivalent:*

1. *A has a multiplicative left invariant mean.*
2. *The pair K, A has the common fixed point property on compacta with respect to $E-E$-representations.*

Proof. Let $\mathscr{T}$ be an $E-E$-representation of K, A on X. Then there exists an element $y \in Y$ such that $h_y(CB(Y)) \subseteq A$ and $T_g y \in Y$ for all $g \in K$. Let h_y^* be the adjoint of h_y and let m be a multiplicative left invariant mean on A. Then $< h_y^* m, 1 >= 1$, where 1 is the constant 1 function on Y. Also $h_y(f_1 f_2) = (h_{y,f_1})(h_{y,f_2})$, for $f_1, f_2 \in CB(Y)$ and $g \in K$. In addition, since m is multiplicative, $h_y^* m$ is a nonzero multiplicative linear functional on $CB(Y)$ and $< h_y^*(m), \bar{h} >= \overline{< h_y^*(m), h >}$, Thus, there exists an element $x_y \in Y$ such that $f(x_y) =< h_y^* m, f >=< m, h_{y,f} >$, for all $f \in CB(Y)$.

For each $g \in K$, define a map $\Psi_g : E^* \to CB(Y)$ via $(\Psi_g f)(z) =< f, T_g z >$, for $f \in E^*$, $z \in Y$. Then $h_{y,\Psi_g f} = L_g[h_{y,f}]$ since $f \in E^*$. Hence,

$$< f, x_y >= m(h_{y,f}) = m(L_g h_{y,f}) = m(h_{y,\Psi_g f}) = h_{y,\Psi_g f}(x_y) =< f, h_{y,\Psi_g f} >$$

Thus, $T_g x_y = x_y$, for each $g \in K$ since m is left translation invariant and E^* separates point of E.

Conversely, let $E = A^*$ and Y be the set of all multiplicative means on A. Then $X = Mean(A)$. Define $(g, m) \mapsto L_g^* m$ from $K \times Mean(A)$ into $Mean(A)$, where $Mean(A)$ has the weak*-topology of A^*. Then $\mathscr{T} = \{L_g^* \mid g \in K\}$ is a separately continuous representation of K on X. We note that each $\phi \in CB(Y)$ corresponds to an element $f_\phi \in A$ such that $\phi(m) = m(f_\phi)$, for $m \in Y$. Let $P(K) = \{g \in K \mid \delta_k * \delta_g$ is a point mass measuse, δ_{kg}, $\forall k \in K\}$, $g \in P(K)$ and $k \in K$. Then

$$\delta_{g_{L_K}} \phi(k) = \phi(L_k^* \delta_g) = \phi(\delta_{kg}) = \delta_{kg}(f_\phi) = R_g f_\phi(k).$$

Hence, $\delta_{g_{L_K}} \phi \in A$, since A is right translation invariant. i.e, $\delta_{g_{L_K}}(CB(Y)) \subseteq$

A, for $g \in P(K)$. Thus, $\mathscr{T}$ is an $E{-}E$-representation of $K,\ A$ on X. Therefore, there is some $m_0 \in Y$ such that $L_g^* m_0 = m_0$, for all $g \in K$. $\square$

Let T be a bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$. Then T commutes with convolution from the left if $T(\phi * f) = \phi * T(f)$, for all $\phi \in L_1(K)$ and $f \in L_\infty(K)$. The following can be proved by a similar argument as in ([44], Theorem 2).

Remark 5.4.2. *The following are equivalent:*

1. *K is compact.*

2. *Any bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ which commutes with convolution from the left is weak*-weak* continuous.*

Using bounded approximate identity of $L_1(K)$, one can show that any bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ which commutes with convolution from the left also commutes with left translations. However, the converse is not true in general. For instance, if K is a direct product $G \times J$ of any locally compact non-discrete group G which is amenable as a discrete group and a finite hypergroup J, then for any left invariant mean m on $L_\infty(K)$ which is not topological left invariant, the operator $T(f) := m(f).1$ commutes with left translations but not with convolutions from the left. It is important to note that in contrast to the group case, there is a class of compact commutative hypergroups for which any bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ commuting with convolution is weak*-weak* continuous:

Example 5.4.1. *Fix $0 < a \leq \frac{1}{2}$ and let H_a be the hypergroup on $\mathbb{Z}_+ \cup \{\infty\}$ given by $\delta_m * \delta_n = \delta_{min(n,m)}$, for $m \neq n \in \mathbb{Z}_+$, $\delta_\infty * \delta_m = \delta_m * \delta_\infty = \delta_m$ and $\delta_n * \delta_n = \frac{1-2a}{1-a}\delta_n + \sum_{k=n+1}^{\infty} a^k \delta_k$ ([16]). Then any bounded linear operator from $L_\infty(H_a)$ into $L_\infty(H_a)$ commuting with translations is weak*-weak* continuous.*

Proof. Let T be a bounded linear operator from $L_\infty(H_a)$ into $L_\infty(H_a)$ commuting with translations. For each $\phi \in L_1(K)$ and $n \in \mathbb{Z}_+$ define a function ϕ_n on K which coincide with ϕ on $\{0, 1, ..., n\}$ and zero otherwise. Then $||\phi_n - \phi||_1 \to$

0. In addition, for each $f \in L_\infty(K)$ we have $||T(\phi_n * f) - T(\phi * f)|| \to 0$ and $||\phi_n * Tf - \phi * Tf|| \to 0$ ([30], 6.2 C). For each $f \in L_\infty(K)$

$$\begin{aligned} T(\phi_n * f) &= T(\textstyle\sum_{k=0}^n \phi_n(k)(1-a)a^k L_{\check{k}} f) \\ &= \textstyle\sum_{k=0}^n \phi_n(k)(1-a)a^k T(L_{\check{k}} f) \\ &= \textstyle\sum_{k=0}^n \phi_n(k)(1-a)a^k L_{\check{k}} Tf \\ &= \phi_n * Tf \end{aligned}$$

we have that $T(\phi * f) = \phi * Tf$. Now the result follows from Remark 5.4.2. □

Open problem 19. *Let K be a compact hypergroup such that $L_\infty(K)$ has a unique left invariant mean. Let T be a bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ which commutes with left translations. Can we conclude that T commutes with convolution from the left?*

As a consequence of Remark 5.4.2 and a similar argument as in ([47], Proposition 1) we have the following:

Corollary 5.4.1. *Let K be a compact hypergroup. The following are equivalent:*

1. *Any bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ which commutes with left translations is weak*-weak*-continuous.*

2. *Any bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ which commutes with left translations also commutes with convolution from the left.*

Each of items 1 or 2 implies that $L_\infty(K)$ has a unique left invariant mean.

Remark 5.4.3. *Let T be a representation of K on a finite dimensional vector space V. Then there is an inner product $<,>_1$ on V such that T is unitary.*

Proof. Let $<,>$ be an inner product on V and define an inner product $<,>_1$

on V by $< v_1, v_2 >_1 := \int\int < T_g v_1, T_k v_2 > d\check{\lambda}(g) d\check{\lambda}(k)$. Then for $k_0 \in K$

$$\begin{aligned}
< T_{k_0} v_1, T_{k_0} v_2 >_1 &= \int\int < T_g T_{k_0} v_1, T_k T_{k_0} v_2 > d\check{\lambda}(g) d\check{\lambda}(k) \\
&= \int\int\int\int < T_x v_1, T_y v_2 > d\delta_g * \delta_{k_0}(x) d\delta_k * \delta_{k_0}(y) d\check{\lambda}(g) d\check{\lambda}(k) \\
&= \int\int < T_x v_1, T_y v_2 > d\check{\lambda} * \delta_{k_0}(x) d\check{\lambda} * \delta_{k_0}(y) \\
&= \int\int < T_x v_1, T_y v_2 > d\check{\lambda}(x) d\check{\lambda}(y) \\
&= < v_1, v_2 >_1 .
\end{aligned}$$

$\square$

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