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$\mathcal{H}_\infty$ Filter Design for Classes of Nonlinear Systems

by

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To my parents for their love,
endless support and encouragement

Abstract

Estimation of internal states of nonlinear systems has been a wide area of interest in recent years for control design and online processing. According to the difficulty of setting up sensors and also the cost they impose for implementation, estimation of these states would decrease the operation cost of the industrial systems. Nonlinear filter design for two classes of systems known as Lipschitz and one-sided Lipschitz is presented in this thesis. Filter design for Lipschitz nonlinear systems is investigated in discrete-time and one-sided Lipschitz nonlinear systems in continuous-time. One-sided Lipschitz systems represent an extension of the well known class of Lipschitz systems that has been used in the control literature for the past four decades. We present a complete solution of the filtering problem when the noise sources have bounded energy, i.e., we solve the synthesis of the so-called H_∞ filter that minimize the effect of disturbances over the estimates. Our solution will be shown to be robust with respect to parametric and unstructured nonlinear uncertainties. In the case of Lipschitz nonlinear systems, missing information and delayed measurement is modelled and the sufficient condition under which the filter design is asymptotically stable is presented. The problem is then formulated in terms of Linear Matrix Inequalities (LMIs) which can be easily solved using commercial software packages.

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List of Symbols

Symbols	Definition
ρ	One-sided Lipschitz constant
ρ_1	Lipschitz constant
ρ_2	Lipschitz constant
μ	noise attenuation level
w	noise signal
u	input
y	measurement
x	real states
$\hat{x}$	estimated states
L	filter gain
$\gamma(k)$	Bernoulli distribution
$\bar{\gamma}$	mean value of Bernoulli distribution
$\tau(k)$	time delay

List of Abbreviations

LMI	Linear Matrix Inequality
ARE	Algebraic Ricatti Equation

Chapter 1

Introduction

In practical applications, internal states of physical systems play a significant role for the purpose of control, system monitoring, fault detection, etc, therefore having access to instantaneous measures of these states is of crucial importance. Unfortunately, full state information is usually not available because that would require one sensor for each variable which can be very expensive and sometimes impossible to obtain. To reach our goals these unknown quantities must be reconstructed using dynamical systems called observers. Observers accomplish this task using the input and output information of the process combined with the known model of the system. The key work here is to guarantee that under some conditions the estimated states converges to real ones. Observer design for linear time invariant systems is well understood. However, in real application systems show nonlinear characteristics which is more complex and modeled by nonlinear functions. Unfortunately, observer design for nonlinear systems needs much effort and no universal theory has been developed. In many cases, however, some states are available from measurement and only few of them needed to be reconstructed and it is referred to as the reduced-order observer problem. The design problem is to find a dynamical system which is asymptotically stable to guarantee that the estimated states converge to real ones.

Nonlinear observer design is a topic that has attracted constant research over the past four decades. Early results on nonlinear observers considered the class of so-called Lipschitz systems. The seminal work by Thau, [1], presented the first sufficient condition for nonlinear observer (asymptotic) stability. Thau's condition proved to be difficult to use in observer *design* and several researchers contributed to the solution of the observer synthesis problem. Raghavan [2] proposed a method to obtain the observer gain based on solving an algebraic Riccati equation (ARE). Rajamani [3] showed that Raghavan's method fails for some observable (A,C) pairs and proposed an algorithm based on a gradient-based optimization that ensures asymptotic stability. Reference [4] considers a similar problem but uses dynamic observer gains that can provide fast convergence and a larger region of attraction. Additional important recent work on observer design for Lipschitz systems includes [5], [6], [7], [8], as well as extensions to observers for sampled-data systems [9], time-delay systems, [10], unknown input observers [11] and a variety of applications (see for example [7], [12]).

Related to nonlinear observer design is the more involved *filtering* problem, consistent of reducing the effect of noise and disturbance from the estimation error. Nonlinear filtering for Lipschitz systems has focused primarily in noise sources with *bounded energy* and the filtering problem is thus referred to as $\mathcal{H}_\infty$ filtering. Important references on the subject include [4], [13], [14], [15], [16], and [17]. Wear, temperature and other physical phenomena are the main factors causing changing parameters in the real systems. Hence, a proper design procedure must consider this type of uncertainties and changing parameters in the problem formulation to ensure that the estimation does not diverge to unreal values or to infinity. Robust filter design for nonlinear systems in the presence of additive uncertainty and time-varying parameter uncertainties is discussed in [15], [16] and sufficient conditions for stability of the design are presented there.

Lipschitz nonlinear observer design is a well known approach among other design tools introduced for filter design and observer design. Most nonlinear functions satisfy Lipschitz condition in a region around the equilibrium point. One drawback of existing design methods is that they can typically guarantee observer stability for only small Lipschitz constants, thus limiting the operating region to a small neighbourhood of the equilibrium point. Reference [18] deals with this problem by maximizing the Lipschitz constant, in an attempt to enlarge the region of operation.

Development in communication technologies, fast transmission rate has enabled the estimation process to be carried out in a monitoring/control unit away from the plant. However, network failure, package drop out are the most challenging difficulties existent in stability study and filter design. In [19, 20] the missing information is modelled as a Bernoulli distribution for a sensor network application, but it does not consider the time delay existent between the plant and estimator. Motivated by the aforementioned drawbacks, a robust nonlinear filter design that includes both communication time-delays and missing information in measurement is investigated here. Moreover, the measurement equation considered here comprises nonlinearity which is also assumed to satisfy a Lipschitz condition.

Very recently, a more general framework has been proposed based on the use of the so-called *one-sided Lipschitz* continuity condition. One sided-Lipschitz systems provide a generalization of the more familiar notion of Lipschitz systems in the sense that, every Lipschitz system is also one-sided Lipschitz. Moreover, given a Lipschitz system, its one-sided Lipschitz constant is typically smaller than the Lipschitz one, and it can never be larger, thus providing a framework that can provide less restrictive results [21]. Nonlinear observers for one-sided Lipschitz systems were considered in [22, 23]. In these works, sufficient conditions for asymptotic stability of the observers were derived. However, no systematic method for finding the observer gain is proposed. Observer design for one-sided Lipschitz systems was considered in [24], using an algebraic Riccati equation approach, and [21] using linear matrix inequalities (LMIs).

This thesis is organized into two parts: In the first chapter our interest is in the filtering problem for continuous time one-sided Lipschitz systems. As with the previous work on Lipschitz systems, we consider noise sources with *bounded energy* and study the synthesis of $\mathcal{H}_\infty$ filters that minimize the effect of disturbances over the estimates. Our results therefore generalize previous results on observers in that, in the absence of noise or disturbances, our filters provide an asymptotic reconstruction of the state of a one-sided Lipschitz system. Sufficient conditions under which the filter design is asymptotically stable in the presence of uncertainties are also presented. In the second chapter discrete time filter design for Lipschitz nonlinear systems is investigated. As mentioned before, it is assumed that the estimation is carried out in control unit away from the system. So, delayed measurement and failure of the network is modelled and applied in finding the stable filter. Sufficient conditions under which the filter design is asymptotically stable in the presence of uncertainties are also stated.

Chapter 2

Observer Design and Lipschitz Nonlinear Systems

Estimation of internal variables of practical systems plays a significant role in applications such as control, monitoring, fault diagnosis when physical variables of systems can not be measured directly. Practical systems can be modeled by continuous time state space model as presented here,

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + \phi(x, u) \\
y(t) &= Cx(t)
\end{aligned}
\tag{2.1}
$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ are states and outputs of the systems. $\phi(x, u)$ is the nonlinear function describing the nonlinearity in the system. A and C are known matrices with appropriate dimension. A dynamical system, called *observer* recovers the state by applying the known model of the real system along with the input and output information of the system. Observer stability is the main concern in this area which has been attracted many efforts during the last four decades. Asymptotically stable observers guarantee that the estimated variables of the real system asymptotically converge to actual values. Many works on the structure of observers and condition of stability have been done in both linear and nonlinear systems. The most popular structure which is considered by investigators is the so called *Luenberger-like observers*. Finding the proper observer gain which stabilizes the observer and drives the observer state to the actual system state are the main concern in this design problem. Consider the continuous time nonlinear observer below,

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + \phi(\hat{x}, u) + L(y(t) - C\hat{x}(t)) \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned} \tag{2.2}$$

where L is the observer gain and it should be computed in a way to stabilize the observer. If we consider $\phi(x) = 0$, the design problem will be reduced to the linear design problem. Nonlinear observer design problem has remained an open problem for the last several decades. By imposing constraints in the design problem, designers try to find the most appropriate gain for the particular equilibrium point and stabilize the observer around that point. Considering a particular class of nonlinear systems is one of the restriction put in this approach. The most famous of these nonlinear functions is called *Lipschitz* systems described as follows,

$$||\phi(x_1) - \phi(x_2)|| \leq \rho||x_1 - x_2|| \tag{2.3}$$

for all $x_1, x_2 \in \mathfrak{D} \subset \mathbb{R}^n$. A function f satisfying this inequality is said to be a *locally Lipschitz*. ρ is the so-called *Lipschitz constant* which is positive.

Thau did the first work on the analysis of these types of systems [1]. Observer gain L is the only design parameter and also is assumed to be a constant. In his work, no design approach is proposed but just a way for checking the stability of the design by investigating whether or not $A - LC$ is stable and also if corresponding Lyapunov and Riccati equations have a symmetric positive solutions. Several researchers were inspired by Thau's work to develop an algorithm for Luenberger-like observer design. Rajamani shows that the place of both eigenvalues and eigenvectors of $A - LC$ can affect the stability of the observer and it is not a wise decision to put the eigenvalues far from the imaginary axis. To get the good response of the system he tried to minimize the condition number, the ratio of maximum eigenvalue/singular value and minimum eigenvalue/singular value of $A - LC$ and developed a gradient based algorithm to obtain a suitable observer gain for the design problem [3].

So far it was assumed that the observer gain L is constant. Pertew et.al [4] considered a dynamical gain observer design in a new dynamic framework and they suggested the following structure,

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + \phi(\hat{x}, u) + \eta(t) \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned} \tag{2.4}$$

where η has a state space representation. They constructed the dynamic of error and converted the observer design problem to H_∞ design procedure as shown in Fig. 2.1, The design problem can be solved after regularization and they also parameterized

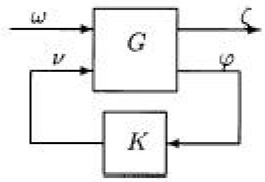


Figure 2.1: H_∞ standard setup

all dynamic observer gain. It is shown that the design procedure is less conservative than some other approaches stated in literature and has the advantage that one can emphasize the performance requirements of the observer over specific frequency ranges. Also results show that the observer has a fast convergence and also ends up with greater Lipschitz constant.

Uncertainty that we encounter in the system modeling is one of the critical factors we should consider in the design problem. Pertew et. al. [11] modeled this uncertainty as a disturbance added to the system and study the stability of the error dynamics by dynamic gain for the observer. Proposed observer suggested by them has the following form,

$$\begin{aligned} \dot{z}(t) &= w_1(t) + w_2(t) + T\phi(\hat{x}) \\ \hat{x}(t) &= z + Hy \end{aligned} \tag{2.5}$$

where w_1, w_2 are dynamical systems and assumed that they have state space representation and T and H are unknown matrices to be calculated based on known matrices of the system [11]. Constructing the error dynamic, the design problem again can be converted to H_∞ design problem mentioned before and solving the problem, the dynamic gain is obtained.

As it can be seen, nonlinear observer design for a class of continuous systems is considered so far and some approaches for the design problem is investigated and a solution for encountering with uncertainty is stated. Nowadays, implementation is carried out by digital systems. To do the design problem in discrete time system we need to discretize the continuous time system to the discrete one and solve the problem. Exact discretization of the system is unavailable, so Euler approximation method has been used by many authors in the literature. It maintains the structure of the original nonlinear model and also it is easy to obtain. Here, we consider the design problem for the class of Lipschitz nonlinear systems with the form as previous one but the only difference is that $\rho_d = T\rho$. Where ρ_d is the discrete time Lipschitz constant.

Abbaszadeh et.al. [9] proposed an LMI approach for the design of the Luenberger like observer for the discretized system. They have shown that by finding the optimal values of the design, we can get a larger Lipschitz constant which leads to the robustness in the design problem. They also extended their approach to a H_∞ filter design problem where an exogenous disturbance which belongs to L_2 spaces added to the nonlinear model. The objective of the design is to stabilize the observer and also reduce the effect of the disturbance on controlled output which is stated as H_∞ filter design. This problem also can be solved again by minimization subject to some inequality which must be hold.

Assoudi et.al [25] considered a more general form of nonlinear systems as below which are uniformly observable and control affine,

$$\begin{aligned} \dot{x} &= f(x(t)) + \textstyle\sum_{k=1}^{m} u_i(t)g_i(x(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{2.6}$$

where f, g and h are smooth functions. It has been shown that there is a transformation that converts the system above to the Lipschitz nonlinear system that we have considered so far. By using Euler approximation, Luenberger based observer can be obtained by this method. They have shown that, under some conditions, exponential observer of the discrete time system is achieved even if Euler approximation can not model the continuous system well.

Arcak et. al. [26] also considered a general form of Luenberger based observer problem and they assume that the compensation part of the observer is a function of input, output and estimated states,

$$\hat{x}(k+1) = F_T^a(\hat{x}(k), u(k)) + l_T(\hat{x}(k), u(k), y(k)) \tag{2.7}$$

in which F_T^a is the approximate discretization and T is the sampling time. To solve the problem, we have to find a Lyapunov function $V(e)$, e is the error dynamic, to satisfy some conditions which leads to semiglobal practical asymptotic convergence of the observer.

Laila et. al. [27] investigated reduced order observer when some of the internal states appear in the output equation. If we separate these two types of variables we have the following form of the nonlinear system,

$$\begin{aligned} \dot{\eta} &= f_1(\eta)\xi \\ \dot{\xi} &= f_2(\eta,\xi) + g(\eta)u \\ y &= \eta \end{aligned} \tag{2.8}$$

where f_1, f_2 and g are smooth functions. After discrtization of the system, the proposed observer is,

$$\begin{aligned} \hat{\xi}^a(k+1) &= f^+(\eta^a(k))\tfrac{\eta^a(k+1)-\eta^a(k)}{T} \\ &+Tf_2(\eta^a(k), f_1^+(\eta^a(k)\tfrac{\eta^a(k+1)-\eta^a(k)}{T}) + Tg(\eta^a(k))u(k) \end{aligned} \tag{2.9}$$

where f^+ is the pseudo inverse of f. They considered stability of the proposed observer in terms of semiglobal asymptotic stability and guaranteed that the error is bounded if some conditions are satisfied.

In the process of modeling real systems we typically encounter model uncertainty. Moreover, time delay often exists in systems, so investigating the effect of these two facts on the design problem is important. Xu [16] and Lu. et al. [12] solved this problem for the discrete time observers. Two groups of undelayed and delayed states along with unstructured uncertainty is used to model the desired problem and it can be seen below,

$$\begin{aligned} x(k+1) &= (A+\Delta A)x(k) + (A_d + \Delta A_d)x(k-d) + B_1 f(k,u(k),x(k),x(k-d)) \\ &+D_1 w_1(k) \\ y(k) &= (C+\Delta C)x(k) + (C_d + \Delta C_d)x(k-d) + B_2 g(k,u(k),x(k),x(k-d)) \\ &+D_2 w_2(k) \end{aligned} \tag{2.10}$$

where $w_1(k)$ and $w_2(k)$ is the exogenous disturbance which belongs to L_2, ΔA, ΔA_d, ΔC and ΔC_d are unknown matrices and also f and g are Lipschitz functions. They have formulated the design procedure using LMI approach and claimed that the solution obtained has robustness against time varying uncertainty imposed to the system.

Throughout this chapter, we provided a brief review on the observer design of Lipschitz nonlinear systems in both continuous and discrete time domain. In the following chapter, a new class of nonlinear system so called *One-Sided Lipschitz* will be presented and the observer design for this type of nonlinear system will be investigated. In the last chapter, filter design for Lipschitz nonlinear system along with time delay, uncertainty and missing information will be presented.

Chapter 3

One-Sided Lipchitz Nonlinear Filter Design

3.1 Problem Statement

Throughout this chapter, $\mathbb{R}$ represent the field of real numbers, $\mathbb{R}^n$ the set of n-tuples of real numbers, and $\mathbb{R}^{n*p}$ represents the set of real matrices with n rows and p columns of real numbers. A matrix $P \in \mathbb{R}^{n*n}$ is positive definite (respectively, positive semi-definite) is for any vector $x \in \mathbb{R}^n$, $x^T Px$ is a positive (respectively, non negative) real number. In this case we write $P > 0$ (respectively, $P \geq 0$). $\mathcal{L}_2[0, \infty]$ denotes the space of Lebesque measurable functions satisfying

$$||u||_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t)\mathrm{d}t} < \infty$$

where $||u||_{\mathcal{L}_2}$ is the $\mathcal{L}_2$ norm of the function u. Consider now the following nonlinear system:

$$\dot{x}(t) = Ax(t) + \phi(x, u) + Bu(t) + Dw(t) \tag{3.1}$$

$$y(t) = Cx(t), \tag{3.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$. w is the noise signal which belongs to $\mathcal{L}_2[0, \infty]$ and $\phi(x, u)$ represents the nonlinearity which satisfies the following one-sided Lipschitz condition:

$$< f(x_1, u^*) - f(x_2, u^*), x_1 - x_2 > \;\; \leq \;\; \rho||x_1 - x_2||^2 \tag{3.3}$$

$$f(x, u) \;\; = \;\; P\phi(x, u) \tag{3.4}$$

for all $x_1, x_2 \in \mathfrak{D} \subset \mathbb{R}^n$, where P is a symmetric positive definite matrix [23]. A function f satisfying (3.24) is said to be a *locally one-sided Lipschitz.* ρ is the so-called *one-sided Lipschitz constant* which can be either positive or negative. If the inequality holds for all $x_1, x_2 \in \mathbb{R}^n$ the function is globally one-sided Lipschitz. Function $\phi(x, u)$ in (3.4) is known to us presenting the nonlinearity in the real system. One of the goals of the design approach here is to find the symmetric positive definite matrix P to satisfy the one-sided Lipschitz condition. So, this matrix which is unknown could be considered one of the design variables in the following Theorems in this chapter.

Consider now a filter with the following form,

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \phi(\hat{x}, u) + Bu(t) + L(y(t) - C\hat{x}(t)) \tag{3.5}$$

$$\hat{y}(t) = C\hat{x}(t). \tag{3.6}$$

We define the estimation error e and the controlled output z as follows:

$$e = x - \hat{x} \tag{3.7}$$

$$z(t) = He(t), \quad H \in \mathbb{R}^{n*n} \tag{3.8}$$

According to (3.7) and (3.5)-(3.6), the error dynamics is given by

$$\dot{e}(t) = (A - LC)e(t) + \phi(x, u) - \phi(\hat{x}, u) + Dw(t). \tag{3.9}$$

Our objective is to find a filter gain L to (i) asymptotically stabilize the error dynamics when $w = 0$, and (ii) minimize the $\mathcal{L}_2$ norm of the controlled output z in the presence of noise, *i.e.* find a minimum μ such that

$$||z||_{\mathcal{L}_2} \leq \mu ||w||_{\mathcal{L}_2} \tag{3.10}$$

Following the approach in [18], our formulation will also maximize the one-sided Lipschitz constant. As shown in Section 3.2, doing so will accomplish two important objectives. In the first place, maximizing the one-sided Lipschitz constant that results in a stable filter will result in a larger region of attraction when the one-sided Lipschitz condition is satisfied only locally. Secondly, maximizing the one-sided Lipschitz constant also provides some desirable stability robustness properties with respect to nonlinear uncertainties, whenever the actual value of the Lipschitz constant is less that the optimized value.

In the following sections, the filter design with three different assumptions will be presented.

1. it is assumed that the nonlinear system has the dynamic equations as described in (3.1), (3.2).
2. additive time-varying uncertainties will be added to the linear part of the nonlinear dynamic system and asymptotic stability of the nonlinear filter is investigated.
3. In the case of having measurement of some system states, reduced-order filter design is carried out to estimate the unknown states.

3.2 Filter Design

We begin this section stating a series of simple Lemmas that will be needed in the proof of our main result.

Lemma 1 *[28] Let D, S and F be real matrices of appropriate dimensions and F satisfying $F^TF \leq I$. Then for any scalar $\epsilon > 0$ and vectors $x, y \in \mathbb{R}^n$, we have*

$$2x^TDFSy \leq \epsilon^{-1}x^TDD^Tx + \epsilon y^TS^TSy.$$

Proof. It is straightforward that for any pair of real numbers a and b we have,

$$ab \leq \frac{a^2}{4} + b^2.$$

The above inequality follows immediately substituting $a = 2\sqrt{\epsilon}x^TDF$, $b = \frac{1}{\sqrt{\epsilon}}y^TS^T$, along with the assumption $F^TF \leq I$. ∎

Lemma 2 **(Schur Complement)** *The following quadratic matrix inequality and LMIs are equivalent,*

$$(a) \quad \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} < 0$$

$$(b) \quad \begin{array}{c} \Phi_{22} < 0 \\ \Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{12}^T < 0 \end{array}$$

Lemma 3 *[29] The matrix inequality given by $X + WW^T < 0$ is equivalent to the LMI*

$$\begin{pmatrix} X & W \\ W^T & -I \end{pmatrix} < 0.$$

Proof. it can be easily verified that

$$\begin{pmatrix} I & W \\ 0 & I \end{pmatrix} \begin{pmatrix} X & W \\ W^T & -I \end{pmatrix} \begin{pmatrix} I & 0 \\ W^T & I \end{pmatrix} = \begin{pmatrix} X + WW^T & 0 \\ 0 & -I \end{pmatrix}$$

We conclude that $\begin{pmatrix} X & W \\ W^T & -I \end{pmatrix}$ is equivalent to $\begin{pmatrix} X + WW^T & 0 \\ 0 & -I \end{pmatrix}$. Using Shur complement $\begin{pmatrix} X + WW^T & 0 \\ 0 & -I \end{pmatrix} < 0$ result in $X + WW^T < 0$. ∎

We now state and prove our first result on filter design. In this result (Theorem 1) we do not seek for optimal noise attenuation. Rather, we assume a known, pre-established *desirable* value μ of the gain condition (3.10) and obtain the observer gain L and positive-definite P that satisfy this gain. This approach has some advantages and disadvantages over optimal solutions. The remarks following Theorem 1 provide additional insight. A different approach which minimizes μ is provided in Theorem 2

Theorem 1 *Consider nonlinear system (3.1)-(3.2) satisfying the one-sided Lipschitz condition (3.3), along with the filter (3.5)-(3.6). The error dynamics is $\mathcal{L}_2$ bounded satisfying the norm condition (3.10) if there exist ϵ >0, matrix $Q > 0$, symmetric positive definite matrix P, matrix G and fixed $\mu > 0$ such that the following LMI optimization problem is feasible,*

$$\begin{aligned} & min(-\rho) \\ & s.t. \begin{bmatrix} H^T H - Q + 2\rho & P\sqrt{\epsilon} & 0 \\ * & -I & 0 \\ * & * & \frac{D^T D}{\epsilon} - \mu^2 I \end{bmatrix} < 0 \end{aligned} \tag{3.11}$$

$$A^T P + PA - GC - C^T G^T + Q < 0 \tag{3.12}$$

After solving the LMI, $L = P^{-1}G$.

Proof: Consider the Lyapunov function $V = e^T P e$. Computing the first derivative of V along the error dynamics we obtain

$$\begin{aligned}\dot{V} &= [e^T(A-LC)^T + (\phi(x,u)-\phi(\hat{x},u))^T + w^T D^T]Pe \\ &\quad + e^T P[(A-LC)e + \phi(x,u) - \phi(\hat{x},u) + Dw] \\ &= e^T[(A-LC)^T P + P(A-LC)]e + 2e^T P(\phi(x)-\phi(\hat{x})) + 2e^T PDw.\end{aligned}$$

Assuming that $(A-LC)^T P + P(A-LC) < -Q$, for some symmetric $Q > 0$, we have

$$A^T P + PA - GC - C^T G^T < -Q \tag{3.13}$$

where $G = PL$. Taking into account of (3.3) and (3.13) we obtain:

$$\dot{V} \leq e^T[(A-LC)^T P + P(A-LC) + 2\rho]e + 2e^T PDw \tag{3.14}$$

$$< e^T[-Q + 2\rho]e + 2e^T PDw \tag{3.15}$$

Using Lemma 1 we have,

$$2e^T PDw \leq \epsilon e^T P^2 e + \epsilon^{-1} w^T D^T Dw. \tag{3.16}$$

Substituting (3.16) into (3.14), the following inequality is obtained,

$$\dot{V} \leq e^T[-Q + 2\rho]e + \epsilon e^T P^2 e + \epsilon^{-1} w^T D^T Dw. \tag{3.17}$$

Now if we define,

$$J = \int_0^\infty (z^T z - \mu^2 w^T w) dt \tag{3.18}$$

then we have that

$$J < \int_0^\infty (z^T z - \mu^2 w^T w + \dot{V}) dt.$$

To guarantee that J ≤ 0, we must have

$$z^T z - \mu^2 w^T w + \dot{V} \leq 0 \tag{3.19}$$

so,

$$z^T z - \mu^2 w^T w + \dot{V} \leq e^T H^T H e - \mu^2 w^T w + e^T[-Q + 2\rho]e \tag{3.20}$$

$$+ \epsilon e^T P^2 e + \epsilon^{-1} w^T D^T Dw \tag{3.21}$$

$$\leq e^T[H^T H - Q + 2\rho + \epsilon P^2]e + w^T[\epsilon^{-1} D^T D - \mu^2 I]w < 0$$

Using Lemma 3 and the fact that $X = H^T H - Q + \rho$ and $W = \sqrt{\epsilon}P$, inequality (3.20) can be converted to the LMI (3.11). As mentioned before, the one-sided Lipschitz constant is maximized in this problem to guaranteed robustness against some uncertainties in the system. Introducing the constant ϵ into the LMI formulation adds some flexibility to the design problem and can help to make the LMI be feasible for a particular choice of μ. ■

Remark 1: Once the matrix P obtained solving the LMIs, the Lyapunov function V and its derivative $\dot{V}$ are, respectively, positive and negative definite for all $x \in \mathbb{R}^n$. Therefore, the only limiting factor in establishing the region of attraction for the filter (3.5)-(3.6) is given by the neighborhood $\mathfrak{D} \subset \mathbb{R}^n$ of the origin where the one-sided Lipschitz condition is satisfied. The following corollary, is therefore an immediate consequence of these observation.

Corollary 1 *Under the assumptions of Theorem 1, the region of attraction for the filter (3.5)-(3.6) is the largest invariant set contained in* $\mathfrak{D} \subset \mathbb{R}^n$.

Remark 2: A mentioned earlier, Theorem 1 maximizes the one-sided Lipschitz constant that can be tolerated in the design. One clear advantage is that maximizing this value leads to a larger region of attraction, as a consequence of Corollary 1. Another important reason, however, is that maximizing ρ brings some important robustness properties to the design. The following proposition clarifies this point. Consider a perturbation of the system (3.1)-(3.2) defined as follows:

$$\dot{x}(t) = Ax(t) + \phi_\Delta(x, u) + Bu(t) + Dw(t) \tag{3.22}$$

$$y(t) = Cx(t), \tag{3.23}$$

$\phi_\Delta(x, u)$ represents the perturbed nonlinearity satisfying,

$$f_\Delta(x, u) = f(x, u) + \Delta f(x, u) \tag{3.24}$$

for all $x_1, x_2 \in \mathfrak{D}$ where $f_\Delta(x, u) = P\phi_\Delta(x, u)$ and Δf unstructured nonlinear uncertainty in $\Phi(x, u)$.

Proposition 1 *Assume that the one-sided Lipschitz constant for the nominal system (3.1)-(3.2) and perturbed one (3.22)-(3.23) are* ρ *and* ρ^**, respectively. The error dynamic is* $\mathcal{L}_2$ *bounded satisfying the norm condition (3.10) if the additive Lipschitz uncertainty* $\Delta\rho$ *defined by*

$$< \Delta f(x_1, u) - \Delta f(x_2, u), x_1 - x_2 >\leq \Delta\rho||x_1 - x_2||^2$$

is such that less than or equal to $\Delta\rho \leq \rho^* - \rho$*.*

Proof. It is straightforward that

$$\begin{aligned} &< f_\Delta(x_1, u) - f_\Delta(x_2, u), x_1 - x_2 >= \\ &\qquad < f(x_1, u) - f(x_2, u), x_1 - x_2 > + < \Delta f(x_1, u) - \Delta f(x_2, u), x_1 - x_2 > \end{aligned}$$

using one-sided Lipschitz inequality conditions, we get

$$< f_\Delta(x_1, u) - f_\Delta(x_2, u), x_1 - x_2 >\leq \rho||x_1 - x_2||^2 + \Delta\rho||x_1 - x_2||^2$$

According to theorem 1 and the fact that ρ^* is the maximum one-sided Lipschitz constant obtained solving the LMIs, we should have $\rho + \Delta\rho \leq \rho^*$. So $\Delta\rho \leq \rho^* - \rho$ shows the maximum uncertainty that can be added to the nominal system without affecting the stability of the error dynamics. ■

Remark 3: Theorem 1 provides a good compromise between a simple observer design (*i.e.* a design that reconstructs the state without while ignoring noise or disturbance action) and an optimal design that renders a filter with optimal noise properties. The advantage of this Theorem is its simplicity: returns a filter with pre-set attenuation properties. Our next result (Theorem 2), given next solves the optimization problem. Our algorithm includes an additional term to the cost function with the purpose of forcing the search algorithm to find lower gains. The price paid is an added dependence on the tuning parameters defined in the theorem. By changing these parameters, optimized values change significantly.

Theorem 2 *Consider again the nonlinear system (3.1)-(3.2) under the same assumptions in Theorem 1 The error dynamics is* $\mathcal{L}_2$ *bounded satisfying the norm condition (3.10) if there exist* ϵ *>0,* $\mu > 0$*,* $\epsilon_0 > 0$*,* $0 < \lambda < 1$*, fixed matrix* $Q > 0$*,* *symmetric positive definite matrix P and matrix G such that the following LMI optimization problem is feasible,*

$$min((1-\lambda)\mu^2 + \epsilon_0 \sum_{i=1}^{N}\sum_{j=1}^{M} w_{ij}g_{ij} - \lambda\rho)$$

$$s.t. \begin{bmatrix} H^T H - Q + 2\rho & P\sqrt{\epsilon} & 0 \\ * & -I & 0 \\ * & * & \frac{D^T D}{\epsilon} - \mu^2 I \end{bmatrix} < 0 \tag{3.25}$$

$$A^T P + PA - GC - C^T G^T + Q < 0 \tag{3.26}$$

where $G=[g_{ij}]$ *and* w_{ij} *are corresponding weights. After solving the LMI,* $L = P^{-1}G$.

Proof. The proof follows the same lines as that of Theorem 1, but here the $\mathcal{L}_2$ gain μ is minimized. The additional terms added to the optimization cost provide additional control overvalues of the resulting filter gains. The parameter λ emphasizes the effect of one-sided Lipschitz constant and the $\mathcal{L}_2$ gain μ norm constant in the cost function. ■

3.3 Filter Design for Systems with Parametric Uncertainties

In this section we consider $\mathcal{H}_\infty$ filter design for systems with additive uncertainty in the linear terms. Consider nonlinear system with uncertainty:

$$\dot{x}(t) = (A + \Delta A)x(t) + \phi(x, u) + Bu(t) + Dw(t) \tag{3.27}$$

$$y(t) = (C + \Delta C)x(t) \tag{3.28}$$

where the time-varying parameter uncertainties are assumed to be of the form,

$$\Delta A(t) = M_1 F(t) N_1 \tag{3.29}$$

$$\Delta C(t) = M_2 F(t) N_2 \tag{3.30}$$

where $F^T(t)F(t) \leq I$. Suppose that the nonlinear system (3.27)-(3.28) satisfies the one-sided Lipschitz condition (3.24), the observer is assumed to have the form of

(3.5)-(3.6), resulting in the error dynamic given by,

$$\dot{e}(t) = (A - LC)e(t) + \phi(x,u) - \phi(\hat{x},u) + (\Delta A + L\Delta C)x(t) + Dw(t) \tag{3.31}$$

Theorem 3 presents a design method to find an observer gain L that stabilizes the error dynamic and satisfies the $\mathcal{L}_2$ norm condition (3.42).

Theorem 3 *Consider nonlinear system (3.27)-(3.28) satisfying the one-sided Lipschitz condition (3.3), along with the filter (3.5)-(3.6). The error dynamics is $\mathcal{L}_2$ bounded satisfying the norm condition (3.10) if there exist $\epsilon > 0$, symmetric matrix Q, symmetric positive definite matrix P and matrix G such that the following LMI optimization problem is solvable,*

$$min(-\rho + \mu^2)$$

$$s.t. \quad \begin{bmatrix} J & P1S & GM_2 & 0 & 0 & 0 \\ * & -I & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & R & P_2M_1 & P_2D \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & \frac{D^TD}{\epsilon} - \mu^2 I \end{bmatrix} < 0 \tag{3.32}$$

$$A^TP_1 + P_1A - GC - C^TG^T + 2\rho + Q < 0 \tag{3.33}$$

where $J = H^TH + A^TP_1 + P_1A - GC - C^TG^T + 2\rho, S = (\epsilon + M_1M_1^T)^{\frac{1}{2}}$and $R = A^TP_2 + P_2A + 2\rho + 2N1^TN_1 + N_2^TN_2$.After solving the LMI, $L = P_1^{-1}G$.

Proof. Consider the Lyapunov function $V = V_1+V_2 = e^TP_1e+x^TP_2x$. Computing the first derivative of V_1 along the error dynamics we get

$$\begin{aligned} \dot{V}_1 &= [e^T(A-LC)^T + (\phi(x)-\phi(\hat{x}))^T + x^T(\Delta A + L\Delta C)^T + w^TD^T]P_1e \\ &\quad +e^TP_1[(A-LC)e + \phi(x) - \phi(\hat{x}) + (\Delta A + L\Delta C)x + Dw] \end{aligned} \tag{3.34}$$

Using one-sided Lipschitz condition (3.24), uncertainty definitions (3.29), (3.29) and the fact that $G = P_1L$ we obtain

$$\begin{aligned} \dot{V}_1 &= e^T[(A-LC)^TP_1 + P_1(A-LC)]e + 2e^TP_1(\phi(x)-\phi(\hat{x})) + 2e^TP_1Dw \\ &\quad +2e^TP_1(\Delta A - L\Delta C)x \le e^T[A^TP + PA - GC - C^TG^T + 2\rho]e \\ &\quad +2e^TP_1Dw + 2e^TP1M_1F(t)N_1x - 2e^TP_1LM_2F(t)N_2x. \end{aligned} \tag{3.35}$$

Using lemma (III.1) we get,

$$2e^TP_1Dw \leq \epsilon e^TP_1^2e + \epsilon^{-1}w^TD^TDw$$
$$2e^TP1M_1F(t)N_1x \leq e^TP_1M_1M_1^TP_1e + x^TN_1^TN_1x$$
$$2e^TP_1LM_2F(t)N_2x \leq e^TGM_2M_2^TG^Te + x^TN_2^TN_2x.$$

Substituting these inequalities into (3.35),

$$\begin{aligned} \dot{V}_1 \leq\; & e^T[A^TP + PA - GC - C^TG^T + 2\rho]e + 2e^TP_1Dw \\ & +2e^TP1M_1F(t)N_1x + 2e^TP_1LM_2F(t)N_2x \\ & \leq e^T[A^TP + PA - GC - C^TG^T + 2\rho]e \\ & +\epsilon e^TP_1^2e + \epsilon^{-1}w^TD^TDw + e^TP_1M_1M_1^TP_1e + x^TN_1^TN_1x \\ & +e^TGM_2M_2^TG^Te + x^TN_2^TN_2x. \end{aligned} \tag{3.36}$$

The first derivative of V_2 is computed as follows,

$$\dot{V}_2 = [x^T(A + \Delta A)^T + \phi(x)^T + w^TD^T]P_2x + x^TP_2[(A + \Delta A)x + \phi(x) + Dw].$$

Using Lemma 1, one-sided Lipschitz condition (3.24) and uncertainty definitions (3.29), (3.30) we get,

$$\begin{aligned} \dot{V}_2 &= x^T[A^TP_2 + P_2A]x + 2x^TP_2\phi(x) + 2x^TP_2M_1FN_1x + 2x^TP_2Dw \\ &\leq x^T[A^TP_2 + P_2A]x + 2\rho x^Tx + x^TP_2M_1M_1^TP_2x + x^TN_1^TN_1x \\ & \quad +2x^TP_2Dw. \end{aligned} \tag{3.37}$$

Adding (3.36), (3.37) we have

$$\begin{aligned} \dot{V}_1 + \dot{V}_2 \leq\; & e^T[A^TP + PA - GC - C^TG^T + 2\rho]e + \epsilon e^TP_1^2e + e^TP_1M_1M_1^TP_1e \\ & +e^TGM_2M_2^TG^Te + x^T[A^TP_2 + P_2A + 2\rho + 2N_1^TN_1 + N_2^TN_2]x \\ & +x^TP_2M_1M_1^TP_2x + 2x^TP_2Dw + \epsilon^{-1}w^TD^TDw. \end{aligned}$$

Using (3.18), (3.19) we obtain,

$$\begin{aligned} z^Tz - \mu^2w^Tw + \dot{V} \leq\; & e^T[H^TH + A^TP + PA - GC - C^TG^T + 2\rho]e \\ & +\epsilon e^TP_1^2e + e^TP_1M_1M_1^TP_1e + e^TGM_2M_2^TG^Te \\ & +x^T[A^TP_2 + P_2A + 2\rho + 2N_1^TN_1 + N_2^TN_2]x \\ & +x^TP_2M_1M_1^TP_2x + 2x^TP_2Dw + \epsilon^{-1}w^TD^TDw \\ & -\mu^2w^Tw < 0. \end{aligned} \tag{3.38}$$

Using Lemma 3 and the fact that $X_{11} = H^T H + A^T P + PA - GC - C^T G^T + 2\rho$, $W_{12} = P_1(\epsilon + M_1 M_1^T)^{\frac{1}{2}}$, $W_{13} = GM_2$, $X_{44} = A^T P_2 + P_2 A + 2\rho + 2N1^T N_1 + N_2^T N_2$ and $W_{54} = P_2 M_1$, we can convert inequality (3.38) into the LMI (3.32). Finally notice that when w(t)=0 we have $A^T P + PA - GC - C^T G^T + 2\rho + Q < 0$ showing asymptotic convergent in the absence of disturbances. ■

3.3.1 Examples

In this section we consider an illustrative example showing the application of both Theorems presented here.

Example 1 *Consider the nonlinear system presented by [30],*

$$\dot{x} = \begin{bmatrix} -10 & 1 & 0 & 0 \\ -48.6 & -1.26 & 48.6 & 0 \\ 0 & 0 & -22 & 1 \\ 19.5 & 0 & -19.5 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3.205sin(x_3) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0.5 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} w$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x$$

w is a summation of truncated high frequency sinusoidal signals with amplitude of 0.1 belonging to $\mathcal{L}_2[0, \infty]$. We define the controlled output as,

$$z = He$$

where $H = 0.25 I_{4\times4}$. Consider now a design based on Theorem 1 in which we seek a filter with a 10db attenuation of disturbances or, equivalently, $\mu \leq 0.3$. Employing Theorem 1 with $\mu = 0.3$ and letting $\epsilon = 20$, we obtain $\rho^* = 0.8231$. In this example, the parameter ϵ was chosen as the smallest integer that makes the LMI solution feasible. The resulting observer gain is

$$L = 10^4. \begin{bmatrix} 2.1582 & -0.0023 & -0.0039 & 0.0025 \\ -0.0016 & 3.2135 & 4.3546 & -0.4719 \end{bmatrix}^T.$$

Convergence of states (with $w = 0$) is depicted in Fig.3.1.

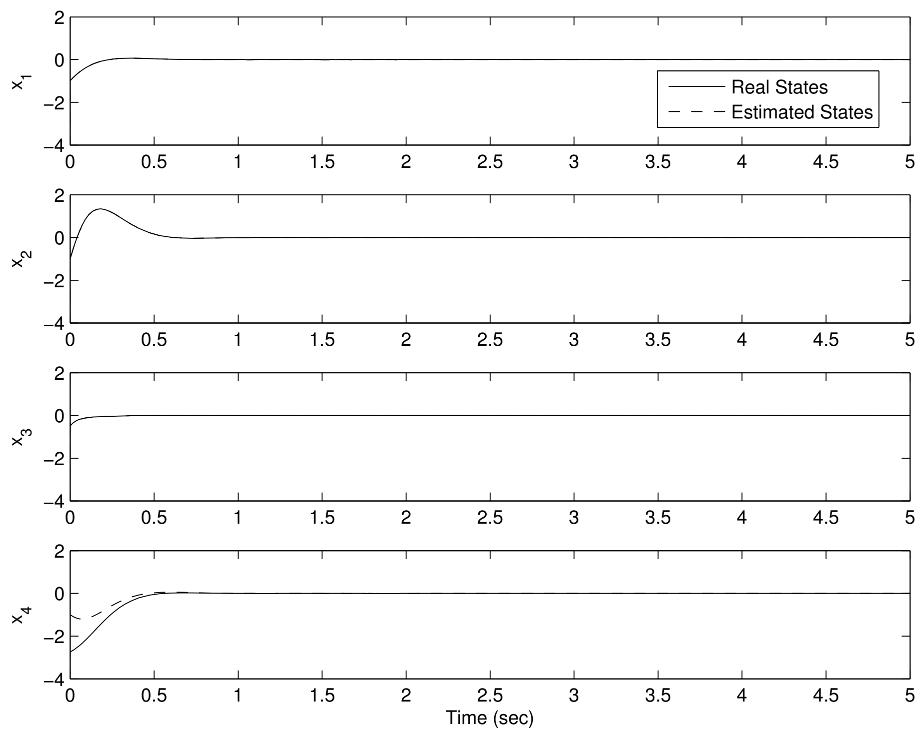


Figure 3.1: Estimated and real states in the case of optimizing ρ

Example 2 *Consider again the same systems used in Example 1. In this example we proceed to simultaneously minimize the effect of disturbance (i.e. by minimizing μ) and maximize the one-side Lipschitz constant ρ using Theorem 2. The tradeoff between these two objectives can be handled with proper selection of the parameter λ. Choosing $\lambda = 0.7$ and with the rest of the design parameters set to $\epsilon = 15$, and $\epsilon_0 = 0$, we complete the design. In the example, matrix Q is assumed to be as $Q = .3I_{4\times 4}$. After solving the LMIs we obtain $\mu^* = .2604, \rho^* = 0.9053$ with the following observer gain:*

$$L = 10^5.\begin{bmatrix} 0.0677 & 0.0014 & -0.0005 & -0.0005 \\ 0 & 4.2974 & -1.0423 & -1.4557 \end{bmatrix}^T.$$

We emphasize that application of Theorem 2 provides enough freedom to the designer to shape the design in the way desired. In our example, disturbance attenuation can be further reduced by taking a smaller value of the design parameter ϵ_0. Doing so, however, would result on larger values of the observer gain.

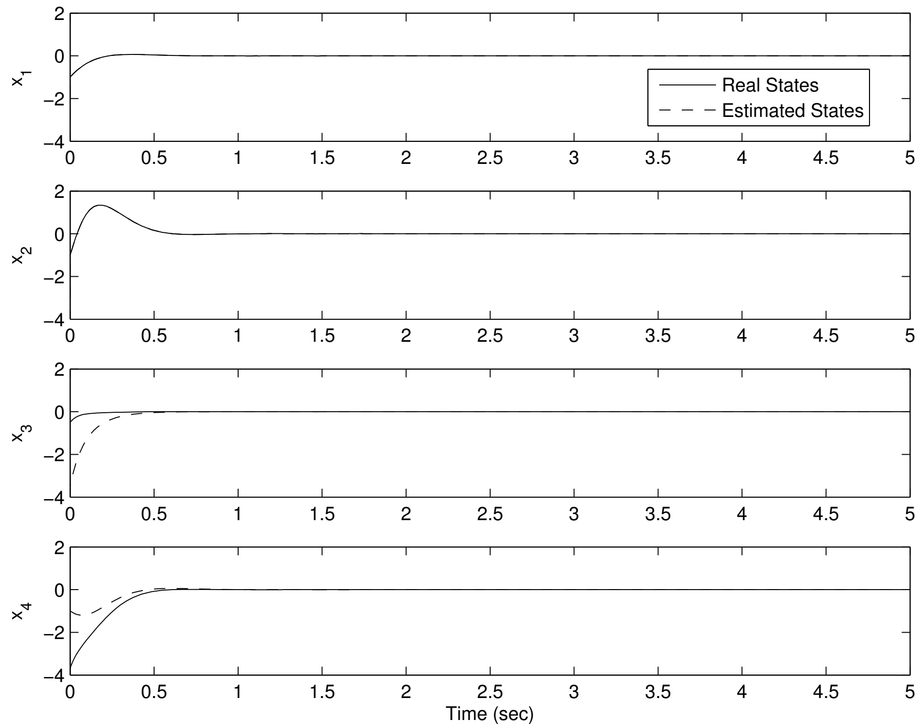


Figure 3.2: Estimated and real states in the case of optimizing ρ and μ

Convergence of the states are depicted in Fig.3.2.

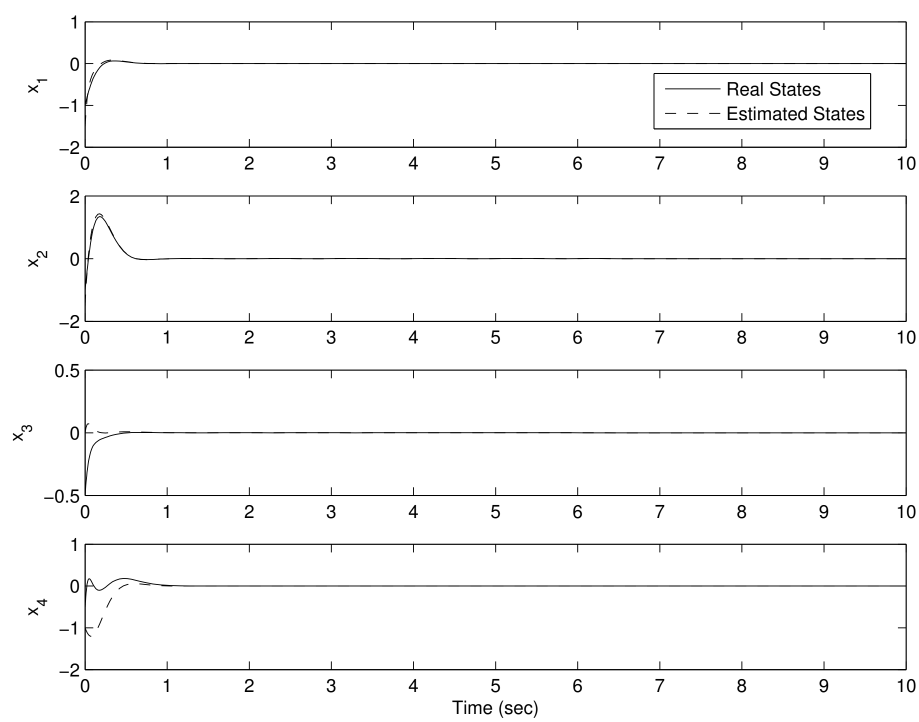


Figure 3.3: Estimated and real states of uncertain filter design

Example 3 *In our final example we consider filter design where there is parametric uncertainty in the system model. To this end we consider a modified version of the system in Example 1 with uncertainties defined as follows,*

$$M_1 = \begin{bmatrix} 2.5 & -0.2 & 0.1 & 0.1 \\ 3.5 & 0.3 & 2.5 & 0.4 \\ 0.2 & -0.3 & -1 & 0.1 \\ 1.5 & 0.5 & 1.5 & 0.1 \end{bmatrix}, M_2 = \begin{bmatrix} 0.1 & 0.5 & -0.7 & 0.2 \\ 0.1 & 0.7 & -0.5 & 0.3 \end{bmatrix}, N_1 = N_2 = .1I_{4\times 4}$$

Solving the problem using Theorem IV.I with design parameters as $H = 0.25I_{4\times 4}$, $\epsilon = 25$, *we obtain* $\rho^* = 0.225$ *and* $\mu^* = .3063$ *and a*

$$L = \begin{bmatrix} 160.535 & -151.1425 & 37.5955 & 456.3471 \\ -135.2764 & 135.4853 & -31.4421 & -390.2546 \end{bmatrix}^T.$$

Convergence of states is depicted in Fig. 3.

3.4 Reduced-Order Filter Design

In this section we consider $\mathcal{H}_\infty$ reduced-order filter design for systems for which some internal states are directly available using the measurement equation. Consider nonlinear system below,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \phi(x,u) + Bu(t) + Dw(t) \\ y(t) &= Cx(t), \end{aligned} \tag{3.39}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$. w is the noise signal which belongs to $\mathcal{L}_2[0,\infty]$ and $\phi(x,u)$ represents the nonlinearity which satisfies the one-sided Lipschitz condition (3.24).

It is assumed that vector x can be partitioned into $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where x_1 are known states and x_2 are unknown ones. Using this fact, the original system can be converted to a dynamic systems below,

$$\begin{aligned} \dot{z}_2(t) &= (LA_{11} + A_{21} - LA_{12}L - A_{22}L)z_1(t) + (LA_{12} + A_{22})z_2(t) + (L\ I_{n-p})\phi(\hat{x},u) \\ y(t) &= z_1(t) \end{aligned}$$

where $z(t) = Tx(t)$. T is the transformation matrix defined as $T = \begin{bmatrix} I_p & 0 \\ L & I_{n-p} \end{bmatrix}$ and matrix A can be described as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$.

Consider the observer with the following form [31],

$$\begin{aligned} \dot{\hat{z}}_2(t) &= (A_{22} + LA_{12})\hat{z}_2(t) + [L(A_{11} - A_{12}L) + A_{21} - A_{22}L]y(t) \\ &\quad + (L\ I_{n-p})\phi(\hat{x}, u) \\ \hat{x}(t) &= \begin{bmatrix} y(t) \\ \hat{z}_2(t) - Ly(t) \end{bmatrix} \end{aligned} \tag{3.40}$$

where $\hat{z}_2$ is the estimated states and L is observer gain should be calculated.

Defining error as $e(t) = z_2(t) - \hat{z}_2(t)$, where z_2 is unknown states, we get

$$\dot{e}(t) = (LA_{12} + A_{22})e(t) + (L\ I_{n-p})(\phi(x, u) - \phi(\hat{x}, u)) + (L\ I_{n-p})Dw(t)$$

Controlled output for error state q is assumed to defined with the form of,

$$q(t) = He(t) \tag{3.41}$$

where H is a known matrix. In this section, we try to find the observer gain L to asymptotically stabilize the error dynamics in the absence of disturbances, and also satisfy the $\mathcal{H}_\infty$ norm of controlled output presented as below to attenuate the effect of noise; *i.e.*,

$$||q||_{\mathcal{L}_2} \leq \mu ||w||_{\mathcal{L}_2} \tag{3.42}$$

Our approach will also attempt to maximize the one-sided Lipschitz constant which will help us guarantee robustness against some nonlinear unstructured uncertainties. The following theorems show the LMI algorithm for finding the desired parameters for the filter.

Theorem 4 *Consider nonlinear system (4.39) satisfying one-sided Lipschitz condition (3.24), the reduced order filter (3.40) is asymptotically stable and also $\mathcal{H}_\infty$ norm (3.42) is satisfied if there is symmetric matrix Q, symmetric positive definite matrix*

P and matrix G such that the following LMI optimization problem is solvable,

$$
\begin{gathered}
min(-\rho+\mu) \\
s.t. \\
\begin{bmatrix} R+2\rho & S\sqrt{\epsilon} & 0 \\ * & -I & 0 \\ * & * & -\mu^2+\frac{D^TD}{\epsilon} \end{bmatrix} < 0 \\
PA_{22}+GA_{12}+A_{12}^TG^T+A_{22}^TP+2\rho+Q<0
\end{gathered}
$$

where $S = P[L\ I_{n-p}]$ *and* $R = H^TH + PA_{22} + GA_{12} + A_{12}^TG^T + A_{22}^TP$. *After solving the LMI,* $L = P^{-1}G$.

Proof. The current system can be transformed to a new one by transformation matrix T,

$$z = Tx$$

where $T = \begin{bmatrix} I_p & 0 \\ L & I_{n-p} \end{bmatrix}$, $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, p is the number of measured outputs, n is the order of the system, z_1 is the known states and z_2 is unknown states.

Using the transformation we get the system,

$$\dot{z} = TAT^{-1}z + T\phi(\begin{bmatrix} y \\ z_2 - Ly \end{bmatrix}) + TDw \tag{3.43}$$

where $T^{-1} = \begin{bmatrix} I_p & 0 \\ -L & I_{n-p} \end{bmatrix}$.

If matrix A is partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and also supposed that $C = (I_p\ 0)$ equation (3.43) can be rewritten as below,

$$
\begin{aligned}
\dot{z}_2 &= (LA_{11}+A_{21}-LA_{12}L-A_{22}L)z_1+(LA_{12}+A_{22})z_2+(L\ I_{n-p})\phi(\hat{x},u) \\
y &= z_1
\end{aligned}
$$

The error dynamic system is defined as $e = z_2 - \hat{z}_2$, so we have

$$\dot{e} = (LA_{12}+A_{22})e + (L\ I_{n-p})(\phi(x)-\phi(\hat{x})) + (L\ I_{n-p})Dw$$

Defining the Lyapunov function as $V = e^T P e$, we get

$$\begin{aligned}
\dot{V} &= [e^T(LA_{12} + A_{22})^T + (\phi(x,u) - \phi(\hat{x},u))^T (L\ I_{n-p})^T \\
&\quad + w^T D^T (L\ I_{n-p})^T] P e + e^T P[(LA_{12} + A_{22})e \\
&\quad + (L\ I_{n-p})(\phi(x,u) - \phi(\hat{x},u)) + (L\ I_{n-p}) D w]
\end{aligned}$$

Equation above result in inequality below,

$$\begin{aligned}
\dot{V} &= e^T[(A_{12}^T L^T + A_{22}^T)P + P(A_{22} + LA_{12})]e \\
&\quad + 2e^T P(L\ I_{n-p})(\phi(x,u) - \phi(\hat{x},u)) + 2e^T P(L\ I_{n-p}) D w \\
&\leq e^T[(A_{12}^T L^T + A_{22}^T)P + P(A_{22} + LA_{12})]e + 2\rho e^T e + 2e^T P(L\ I_{n-p}) D w
\end{aligned}$$

To show the inequality above, assume that $e'^T = [\underbrace{0 \cdots 0}_{p}\, e^T]$, so it can be easily verified that

$$e^T P(L\ I_{n-p})(\phi(x,u) - \phi(\hat{x},u)) =$$
$$e'^T \begin{bmatrix} I_p & 0_{p\times(n-p)} \\ 0_{(n-p)\times p} & P_{(n-p)\times(n-p)} \end{bmatrix} \begin{bmatrix} I_p & 0_{p\times(n-p)} \\ L_{(n-p)\times p} & I_{(n-p)\times(n-p)} \end{bmatrix} (\phi(x,u) - \phi(\hat{x},u))$$

so we get,

$$e^T P(L\ I_{n-p})(\phi(x,u) - \phi(\hat{x},u)) = e'^T \begin{bmatrix} I_p & 0_{p\times(n-p)} \\ PL & P \end{bmatrix} (\phi(x,u) - \phi(\hat{x},u))$$

Using one-sided Lipschitz condition and the fact that $\begin{bmatrix} I_p & 0_{p\times(n-p)} \\ PL & P \end{bmatrix}$ is positive definite, we have

$$e'^T \begin{bmatrix} I_p & 0_{p\times(n-p)} \\ PL & P \end{bmatrix} (\phi(x,u) - \phi(\hat{x},u)) \leq \rho e'^T e' = \rho e^T e$$

Using lemma 1 we get,

$$\begin{aligned}
\dot{V} \leq e^T[(A_{12}^T L^T + A_{22}^T)P + P(A_{22} + LA_{12})2\rho]e \\
+ \epsilon e^T P(L\ I_{n-p})(L\ I_{n-p})^T P e + \epsilon^{-1} w^T D^T D w
\end{aligned}$$

if we define,

$$J = \int_0^\infty (q^T q - \mu^2 w^T w) dt$$

So,

$$J < \int_0^\infty (q^T q - \mu^2 w^T w + \dot{V})dt$$

to guarantee that J ≤ 0, we must have

$$q^T q - \mu w^T w + \dot{V} \leq 0$$

So,

$$\begin{aligned} q^T q - \mu w^T w + \dot{V} \leq e^T [H^T H + (A_{12}^T L^T + A_{22}^T)P + P(A_{22} + LA_{12})2\rho]e \\ +\epsilon e^T P(L\ I_{n-p})(L\ I_{n-p})^T Pe + \epsilon^{-1} w^T (D^T D - \mu^2) w < 0 \end{aligned}$$

Using lemma 3 and fact that $X = H^T H + PA_{22} + GA_{12} + A_{12}^T G^T + A_{22}^T P + 2\rho$, $W = P[L\ I_{n-p}]$ and also $G = PL$ we get the first LMI. The second one also comes up when w(t)=0. To guarantee robustness against uncertainty and also to attenuate the effect of noise on controlled output, maximization on one-sided Lipschitz constant and minimization of $\mathcal{H}_\infty$ constant is carried out here respectively and is presented in cost function. The main role of ϵ in LMI is that it adds some flexibility to the problem by changing that the problem can be feasible. ■

3.4.1 Example

In this section we consider an illustrative example showing the application of Theorem 4 presented above.

Example 4 *Consider nonlinear system described as below,*

$$\dot{x} = \begin{bmatrix} -10 & 1 & 0 & 0 \\ -48.6 & -1.26 & 48.6 & 0 \\ 0 & 0 & -22 & 1 \\ 19.5 & 0 & -19.5 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3.205 sin(x_3) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \\ .5 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} w$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x$$

Two system states x_1 and x_2 are available by measurement equation. Partitioning the nonlinear system in a way described above, estimation of unknown states can be *carried out using Theorem 4. After partitioning matrix A and solving the LMI, we get* $L = \begin{bmatrix} -0.0506 & -9.4257 \\ -0.0001 & -0.0118 \end{bmatrix}$*, For fixed* $\mu = .3$, $\epsilon = 20$ *and* $H = .25I$ *we get optimized* $\rho^* = 0.823$. *Convergence of states are depicted in the following figure.*

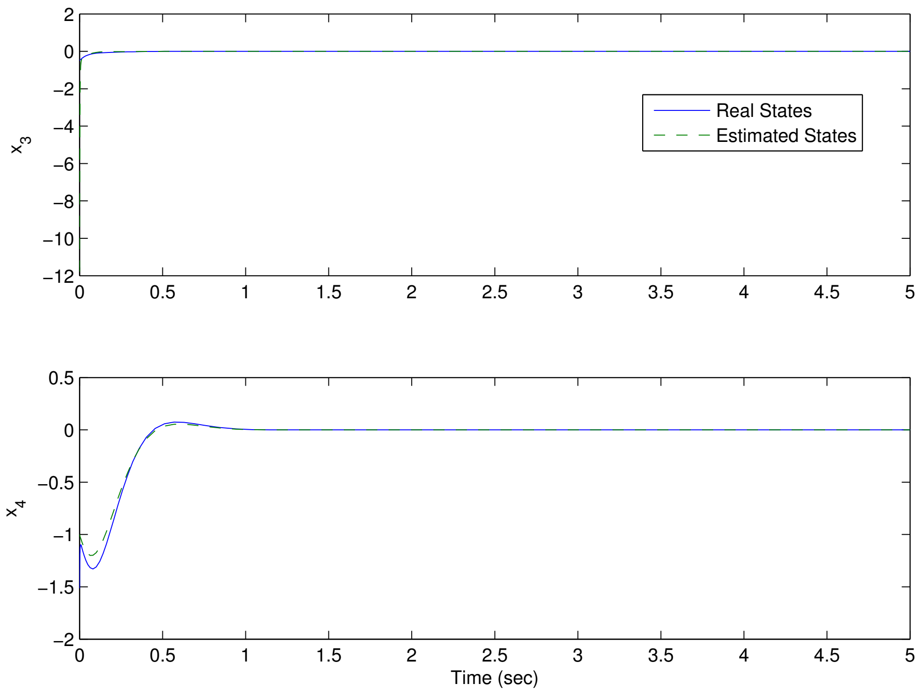


Figure 3.4: State estimation in reduced-order filter design

3.5 Conclusion

A new nonlinear $\mathcal{H}_\infty$ filter design for one-sided Lipschitz systems is proposed. The use of one-sided Lipschitz systems represents a nontrivial, significant extension of similar results existent in the literature for Lipschitz systems. The one-sided Lipschitz assumption is important in that (i) every Lipschitz system is also one sided-Lipschitz, and (ii) the one-sided Lipschitz constant is less than or at most equal to the Lipschitz constant. The combined effect of these two properties guarantees less conservative results than those found using the, more established, theory developed for Lipschitz systems.

Our result maximize the one-sided Lipschitz constant as one of the design goals, a property that provides some robustness properties with respect to nonlinear uncertainties. The problem is formulated in LMI form, which is easily solvable by commercially available software products.

Chapter 4

Discrete-Time Filter Design for a Class of Lipschitz Nonlinear System

4.1 Problem Statement

In this chapter, discrete-time filter design for a class of Lipschitz nonlinear systems is presented. Consider the following nonlinear system,

$$\begin{aligned} x(k+1) &= Ax(k) + f(x(k)) + Dw(k) \\ y(k) &= \gamma(k)(Cx(k) + g(x(k))) \end{aligned} \tag{4.1}$$

in which $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are system states, measurements and w is the noise signal which belongs to $\mathcal{L}_2[0, \infty]$. $f(x(k))$ and $g(x(k))$ represent the nonlinearity which satisfy the following Lipschitz condition:

$$\begin{aligned} |f(x) - f(\hat{x})| &\leq \rho_1 |x - \hat{x}| \\ |g(x) - g(\hat{x})| &\leq \rho_2 |x - \hat{x}| \end{aligned} \tag{4.2}$$

A function satisfying (4.2) is said to be a *locally Lipschitz*. ρ_1, $\rho_2 > 0$ are so-called *Lipschitz constants*. If inequalities above hold for $\forall x, \hat{x} \in \mathbb{R}^n$, nonlinear functions are globally Lipschitz.

As it can be seen in Fig. 4.1 plant information is assumed as the input of the filter containing the model and the measurement of the nonlinear system (4.1). The measurements taken from the plant are sent to the control unit, so this result in an unknown delay, $\tau(k)$, between taking the measurement and receiving it at the control

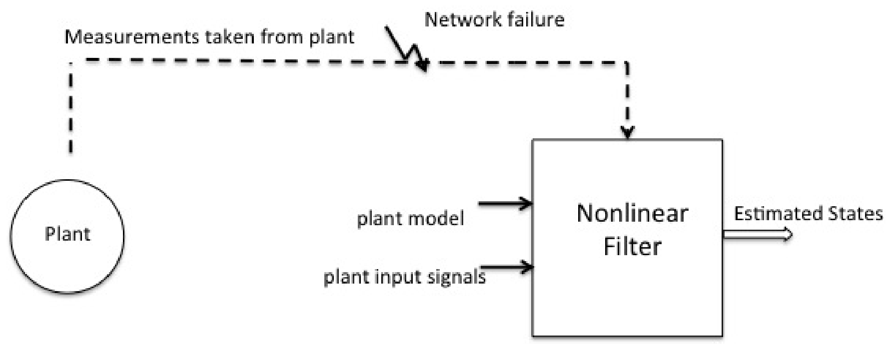


Figure 4.1: State estimation using Lipschitz nonlinear filter

unit which is assumed to be bounded, $\tau_m < \tau(k) < \tau_M$. The information is exchanged using the wireless communication links depicted in Fig. 4.1. This data might be corrupted due to the failure of the communication network, so the measurement is not available at that particular time to the nonlinear filter. Bernoulli distribution $\gamma(k)$ is introduced here to model the missing measurement as below,

$$\begin{aligned} Prob\{\gamma(k)=1\} &= E\{\gamma(k)\} = \bar{\gamma} \\ Prob\{\gamma(k)=0\} &= 1 - E\{\gamma(k)\} = 1-\bar{\gamma} \end{aligned} \tag{4.3}$$

in which the mean value of this random variable $\bar{\gamma}$ is assumed to be known for a particular communication network.

Now consider the Luenberger-like filter having the following form,

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + f(\hat{x}(k)) + L\{y(k-\tau(k)) - \hat{y}(k)\} \\ \hat{y}(k) &= \bar{\gamma}(C\hat{x}(k) + g(\hat{x}(k))) \end{aligned} \tag{4.4}$$

The dynamical error $e(k)$ is formulated as follow,

$$e(k+1) = x(k+1) - \hat{x}(k+1) \tag{4.5}$$

Define the controlled output z as follows,

$$z(k) = He(k) \tag{4.6}$$

Our objective is to find a filter gain L to (i) asymptotically stabilize the error dynamics when $w(k) = 0$, and (ii) minimize the L_2 norm of the controlled output z in the presence of noise, i.e.

$$||z||_{L_2} \leq \mu ||w||_{L_2} \tag{4.7}$$

In the following sections, the filter design with two different assumptions will be presented. Firstly, it is assumed that the nonlinear system has the dynamic equations as described in (4.8). Secondly, additive time-varying uncertainties will be added to the linear part of the nonlinear dynamic system and asymptotic stability of the nonlinear filter will be investigated.

$$\begin{aligned} x(k+1) &= (A+\Delta A)x(k) + f(x(k)) + Dw(k) \\ y(k) &= \gamma(k)((C+\Delta C)x(k) + g(x(k))) \end{aligned} \tag{4.8}$$

in which ΔC, ΔA are time-varying uncertainty matrices and γ is the probability of missing information.

4.2 Filter Design

In this section, nonlinear filter design in absence of uncertainty is presented. Let consider the nonlinear system below,

$$\begin{aligned} x(k+1) &= Ax(k) + f(x(k)) + Dw(k) \\ y(k) &= \gamma(k)(Cx(k) + g(x(k))) \end{aligned} \tag{4.9}$$

We now state and prove our first result on filter design based on nonlinear system above and nonlinear filter (4.2).

Theorem 5 *Consider the nonlinear system (4.1) with given nonlinearity as (4.2), the error dynamic is L_2 bounded satisfying the norm condition (4.7), if there exists $Q > 0$, $\eta, \alpha_1, \alpha_2 > 0$ and positive definite matrix P_1, P_2, P_3 and matrix G such that* *the following LMI is feasible,*

$$\begin{pmatrix} \Psi_1 & 0 & 0 & 0 & \Psi_{15} \\ * & \Psi_2 & 0 & 0 & 0 \\ * & * & \Psi_3 & 0 & \Psi_{35} \\ * & * & * & \Psi_4 & \Psi_{45} \\ * & * & * & * & \Psi_5 \end{pmatrix} < 0$$

$$\begin{pmatrix} I & G^T \\ G & P_1 \end{pmatrix} > 0 \tag{4.10}$$

$$\eta I - P_i > 0 \;\; i = 1, 2, 3$$

where $\alpha_1^{-1} = \eta \rho_1^2$ *and* $\alpha_2^{-1} = \rho_2^2$. Ψ_i *are presented in the following page. When the problem solved, observer gain is obtain using* $L = P_1^{-1} G$.

Proof. Let consider the Lyapunov function as,

$$\begin{aligned} V(k) &= V_1(k) + V_2(k) + V_3(k) + V_4(k) + V_5(k) \\ &= e(k)^T P_1 e(k) + \textstyle\sum_{j=k-\tau(k)}^{k-1} e^T(k) Q e(k) \\ &\quad + \textstyle\sum_{i=k-\tau_M+1}^{k-\tau_m} \sum_{j=i}^{k-1} e^T(k) Q e(k) + x^T(k) P_2 x(k) \\ &\quad + x^T(k - \tau(k) P_3 x(k - \tau(k)) \end{aligned} \tag{4.11}$$

Substituting (4.39), (4.42) into (4.43) we get,

$$\begin{aligned} e(k+1) &= Ae(k) + f(x(k)) - f(\hat{x}(k), u(k)) \\ &\quad - L\{\gamma(k) C x(k - \tau(k)) - \bar{\gamma}(C\hat{x}(k) \\ &\quad + \gamma(k) g(x(k - \tau(k)) + g(\hat{x}(k)))\} + Dw(k) \end{aligned} \tag{4.12}$$

adding and subtracting $\bar{\gamma} LCx(k)$ and $\bar{\gamma} Lg(x)$ we have,

$$\begin{aligned} e(k+1) &= (A - \bar{\gamma} LC) e(k) + f(x(k)) \\ &\quad - f(\hat{x}(k)) - L\{\gamma(k) C x(k - \tau(k)) \\ &\quad + \gamma(k) g(x(k - \tau(k))) - \bar{\gamma} g(x(k)) \\ &\quad + \bar{\gamma}(g(x(k)) - g(\hat{x}(k))) - \bar{\gamma} C x(k)\} + Dw(k) \end{aligned} \tag{4.13}$$

$$\Psi_1 = \begin{pmatrix} H^TH+(1+\tau_M-\tau_m)Q-P_1 & \sqrt{8}\bar{\gamma} & I & A^TP_1-\bar{\gamma}G^TC^T & A^TP_1-\bar{\gamma}G^TC^T & 0 \\ * & -\alpha_2 & 0 & 0 & 0 & 0 \\ * & * & -\frac{\alpha_1}{7} & 0 & 0 & 0 \\ * & * & * & -\frac{7}{41}P_1 & 0 & 0 \\ * & * & * & * & -7P_1 & P_1 \\ * & * & * & * & * & -\frac{\eta}{7} \end{pmatrix}$$

$$\Psi_2 = -Q$$

$$\Psi_3 = \begin{pmatrix} -P_2 & \sqrt{8}\bar{\gamma} & I & A^TP_2 & A^TP_2 & C^TG^T \\ * & -\alpha_2 & 0 & 0 & 0 & 0 \\ * & * & -\frac{\alpha_1}{2} & 0 & 0 & 0 \\ * & * & * & -P_2 & 0 & 0 \\ * & * & * & * & 2P_2-2\eta I & 0 \\ * & * & * & * & * & \frac{1}{7\bar{\gamma}^2}P_1 \end{pmatrix}$$

$$\Psi_4 = \begin{pmatrix} -P_3 & \sqrt{8\bar{\gamma}(1-\bar{\gamma})} & I & A^TP_3 & A^TP_3 & C^TG^T \\ * & -\alpha_2 & 0 & 0 & 0 & 0 \\ * & * & -\frac{\alpha_1}{2} & 0 & 0 & 0 \\ * & * & * & -P_3 & 0 & 0 \\ * & * & * & * & 2P_3-2\eta I & 0 \\ * & * & * & * & * & \frac{1}{7\bar{\gamma}(1-\bar{\gamma})}P_1 \end{pmatrix}$$

$$\Psi_5 = 2(P_2+P_3)+5P_1-\mu^2 I$$

$$\Psi_{15} = \begin{pmatrix} A^TP_1D-\bar{\gamma}C^TG^TD \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Psi_{25} = \begin{pmatrix} 0 \end{pmatrix}$$

$$\Psi_{35} = \begin{pmatrix} \bar{\gamma}C^TG^TD+A^TP_2D \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Psi_{45} = \begin{pmatrix} A^TP_3D \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Computing the expectation of difference of $V_1(k)$ along the trajectory of error, we get

$$
\begin{aligned}
E\{V_1(k+1)-V_1(k)\} \quad = \quad & E\{e^T(k)[(A-\bar{\gamma}LC)^T P_1(A-\bar{\gamma}LC)-P1]e(k) \\
& +2e^T(k)(A-\bar{\gamma}LC)^T P_1 \Delta f_k + 2e^T(k)(A-\bar{\gamma}LC)^T P_1 Dw(k) \\
& -2\gamma(k)e^T(k)(A-\bar{\gamma}LC)^T P_1 LCx(k-\tau(k)) \\
& -2\gamma(k)e^T(k)(A-\bar{\gamma}LC)^T P_1 Lg(x(k-\tau(k))) \\
& -2\bar{\gamma}e^T(k)(A-\bar{\gamma}LC)^T P_1 L\Delta g_k \\
& +2\bar{\gamma}e^T(k)(A-\bar{\gamma}LC)^T P_1 LCx(k) \\
& +2\bar{\gamma}e^T(k)(A-\bar{\gamma}LC)^T P_1 Lg(x(k)) + \Delta f_k^T P_1 \Delta f_k \\
& +2\Delta f_k^T P_1 Dw(k) - 2\gamma(k)\Delta f_k^T P_1 LCx(k-\tau(k))) \\
& -2\gamma(k)\Delta f_k^T P_1 Lg(x(k-\tau(k))) - 2\bar{\gamma}\Delta f_k^T P_1 L\Delta g_k \\
& +2\bar{\gamma}\Delta f_k^T P_1 LCx(k) + 2\bar{\gamma}\Delta f_k^T P_1 Lg(x(k)) \\
& +w^T(k)D^T P_1 Dw(k) - 2\gamma(k)w^T(k)D^T P_1 LC... \\
& ...x(k-\tau(k))) - 2\gamma(k)w^T(k)D^T P_1 Lg(x(k-\tau(k))) \\
& -2\bar{\gamma}w^T(k)D^T P_1 L\Delta g_k + 2\bar{\gamma}w^T(k)D^T P_1 LCx(k) \\
& +2\bar{\gamma}w^T(k)D^T P_1 Lg(x(k)) \\
& +\gamma^2(k)x^T(k-\tau(k))C^T L^T P_1 LCx(k-\tau(k)) \\
& +2\gamma(k)^2 x^T(k-\tau(k))C^T L^T P_1 Lg(x(k-\tau(k))) \\
& +2\gamma(k)\bar{\gamma}x^T(k-\tau(k))C^T L^T P_1 L\Delta g_k \\
& -2\gamma(k)\bar{\gamma}x^T(k-\tau(k))C^T L^T P_1 LCx(k) \\
& -2\gamma(k)\bar{\gamma}x^T(k-\tau(k))C^T L^T P_1 Lg(x(k)) \\
& +\gamma^2(k)g^T(x(k-\tau(k)))L^T P_1 Lg(x(k-\tau(k))) \\
& +2\bar{\gamma}\gamma(k)g^T(x(k-\tau(k)))L^T P_1 L\Delta g_k \\
& -2\bar{\gamma}\gamma(k)g^T(x(k-\tau(k)))L^T P_1 LCx(k) \\
& -2\bar{\gamma}\gamma g^T(x(k-\tau(k)))L^T P_1 Lg(x(k)) \\
& +\bar{\gamma}^2\Delta g_k^T L^T P_1 L\Delta g_k - 2\bar{\gamma}^2\Delta g_k^T L^T P_1 LCx(k) \\
& -2\bar{\gamma}^2\Delta g_k^T L^T P_1 Lg(x(k)) + \bar{\gamma}^2 x^T(k)C^T L^T P_1 LCx(k) \\
& +2\bar{\gamma}^2 x^T(k)C^T L^T P_1 Lg(x(k)) + \bar{\gamma}^2 g^T(x(k))L^T P_1 Lg(x(k))\}
\end{aligned}
\tag{4.14}
$$

in which $\Delta f_k = f(x(k),u(k)) - f(\hat{x}(k),u(k))$ and $\Delta g_k = g(x(k)) - g(\hat{x}(k))$.

Lemma 4 *[15] Let W be the positive definite matrix $W \in \mathbb{R}^{n\times n}$. Then for any vectors $x,y \in \mathbb{R}^n$, we have*

$$2x^T y \leq x^T W^{-1} x + y^T W y.$$

Using Lemma 1 and Lemma 4 and assuming that $P_1 = \eta - W_1 > 0$ and also Lipschitz condition, we get

$$\begin{aligned} &2e^T(k)(A-\bar{\gamma}LC)P_1\Delta f_k + 7\Delta f_k^T(\eta - W_1)\Delta f_k \\ &\leq e^T(k)(A-\bar{\gamma}LC)^T P_1(7\eta - 7P_1)^{-1}P_1(A-\bar{\gamma}LC)e(k) + 7\eta\rho_1^2 e^T(k)e(k) \end{aligned} \tag{4.15}$$

Adding and subtracting $g^T(x(k-\tau(k))g(x(k-\tau(k))$ to the equation below, we have

$$\begin{aligned} g^T(x(k-\tau(k)))L^T P_1 L g(x(k-\tau(k))) &\leq g^T(x(k-\tau(k)))[L^T P_1 L - I]g(x(k-\tau(k))) \\ &+g^T(x(k-\tau(k))g(x(k-\tau(k)) \end{aligned} \tag{4.16}$$

Assuming that $L^T P_1 L - I < 0$ and $G = P_1 L$ we have,

$$L^T P_1 L - I = L^T P_1 P_1^{-1} P_1 L - I < 0 \tag{4.17}$$

Using schur complement, we get

$$\begin{pmatrix} I & G^T \\ G & P_1 \end{pmatrix} > 0 \tag{4.18}$$

The same approach is applied to $g^T(x(k))L^T P_1 L g(x(k))$, $\Delta g_k^T L^T P_1 L \Delta g_k$. Taking into account the above assumptions above we have,

$$\begin{aligned} E\{V_1(k+1) - V_1(k)\} \;\; = \;\; & E\{e^T(k)[-P_1 + \tfrac{41}{7}(A-\bar{\gamma}LC)^T P_1(A-\bar{\gamma}LC) \\ & +(A-\bar{\gamma}LC)^T P_1(7\eta - 7P_1)^{-1}P_1(A-\bar{\gamma}LC) \\ & +\tfrac{1}{7}(A-\bar{\gamma}LC)^T P_1(A-\bar{\gamma}LC) + 8\rho_2^2\bar{\gamma}^2 + 7\eta\rho_1^2]e(k) \\ & +x^T(k)[7\bar{\gamma}^2 C^T L^T P_1 LC + 8\bar{\gamma}^2\rho_2^2]x(k) \\ & +x^T(k-\tau(k))[7\gamma^2(k)C^T L^T P_1 LC + 8\rho_2^2\gamma^2(k)]... \\ & ...x(k-\tau(k)) + 2e^T(k)(A-\bar{\gamma}LC)P_1 Dw(k) \\ & +2\bar{\gamma}w^T(k)D^T P_1 LCx(k) + 5w^T(k)D^T P_1 Dw(k)\} \end{aligned} \tag{4.19}$$

Consider that,

$$(A-\bar{\gamma}LC)^T P_1(7\eta-7P_1)^{-1}P_1(A-\bar{\gamma}LC)+\tfrac{1}{7}(A-\bar{\gamma}LC)^T P_1(A-\bar{\gamma}LC)$$
$$=(A-\bar{\gamma}LC)^T P_1[(7\eta-7P_1)^{-1}+\tfrac{1}{7}P_1^{-1}]P_1(A-\bar{\gamma}LC) \tag{4.20}$$

Assuming that,

$$\begin{aligned}(7\eta-7P_1)^{-1}+\tfrac{1}{7}P_1^{-1} &= (7\eta-7P_1)^{-1}((7\eta-7P_1)\tfrac{1}{7}P_1^{-1}+I)\\ &= (7P_1-7\eta^{-1}P_1^2)^{-1}\end{aligned} \tag{4.21}$$

We have,

$$\begin{aligned}E\{V_1(k+1)-V_1(k)\} \quad = \quad & E\{e^T(k)[-P_1+\tfrac{41}{7}(A-\bar{\gamma}LC)^T P_1(A-\bar{\gamma}LC)\\ &+(A-\bar{\gamma}LC)^T P_1(7P_1-7\eta^{-1}P_1^2)^{-1}P_1\\ &(A-\bar{\gamma}LC)+8\rho_2^2\bar{\gamma}^2+7\eta\rho_1^2]e(k)\\ &+x^T(k)[7\bar{\gamma}^2C^TL^TP_1LC+8\bar{\gamma}^2\rho_2^2]x(k)\\ &+x^T(k-\tau(k))[7\gamma^2(k)C^TL^TP_1LC+8\rho_2^2\gamma^2(k)]...\\ &...x(k-\tau(k))+2e^T(k)(A-\bar{\gamma}LC)P_1Dw(k)\\ &+2\bar{\gamma}w^T(k)D^TP_1LCx(k)+5w^T(k)D^TP_1Dw(k)\}\end{aligned} \tag{4.22}$$

The following equation can be easily verified,

$$E\{V_2(k+1)-V_2(k)\}+E\{V_3(k+1)-V_3(k)\}\leq$$
$$E\{e^T(k)Q_1e(k)-e^T(k-\tau(k))Qe(k-\tau(k))\}+E\{(\tau_M-\tau_m)e^T(k)Qe(k)\} \tag{4.23}$$

the forth part of the Lyapunov function can be analyzed as follows,

$$\begin{aligned}E\{V_4(k+1)-V_4(k)\} \quad = \quad & E\{(Ax(k)+f(x(k))\\ &+Dw(k))^TP_2(Ax(k)+f(x(k))+Dw(k))\\ &-x^T(k)P_2x(k)\}=E\{x^T(k)[A^TP_2A-P_2]x(k)\\ &+2x^T(k)A^TP_2f(x(k))+f^T(x(k))P_2f(x(k))\\ &+2x^T(k)A^TP_2Dw(k)+2w^T(k)B^TP_2f(x(k))\\ &w^T(k)D^TP_2Dw(k)\leq\\ &E\{x^T(k)[A^TP_2A-P_2]x(k)+2x^T(k)A^TP_2f(x(k))\\ &+2f^T(x(k))P_2f(x(k))+2x^T(k)A^TP_2Dw(k)\\ &+2w^T(k)D^TP_2Dw(k)\}\end{aligned} \tag{4.24}$$

Rewriting $P_2 = \eta I - W_2 > 0$ and using lemma 4 we have,

$$\begin{aligned} & 2x^T(k)A^TP_2f(x(k)) + 2f^T(x(k))P_2f(x(k)) \\ & \leq x^T(k)A^TP_2(2W_2)^{-1}P_2Ax(k) + 2\eta f^T(x(k))f(x(k)) \end{aligned} \tag{4.25}$$

Using the Lipschitz condition we have,

$$\begin{aligned} & 2x^T(k)A^TP_2f(x(k)) + 2f^T(x(k))P_2f(x(k)) \\ & \leq x^T(k)[A^TP_2(2\eta - 2P_2)^{-1}P_2A + 2\eta\rho_1^2]x(k) \end{aligned} \tag{4.26}$$

Substituting (4.26) into (4.62) we get,

$$\begin{aligned} E\{V_4(k+1) - V_4(k)\} \leq E\{x^T(k)[-P_2 + A^TP_2A + A^TP_2(2\eta - 2P_2)^{-1}P_2A \\ +2\eta\rho_1^2]x(k) + 2x^T(k)A^TP_2Dw(k) + 2w^T(k)D^TP_2Dw(k)\} \end{aligned} \tag{4.27}$$

Let consider the following delayed systems,

$$x(k+1-\tau(k)) = Ax(k-\tau(k)) + f(x(k-\tau(k))) + Dw_d(k) \tag{4.28}$$

Assuming that

$$\begin{aligned} E\{w_d(k)\} &= E\{w(k)\} \\ E\{w_d^T(k)w_d(k)\} &= E\{w^T(k)w(k)\} \end{aligned} \tag{4.29}$$

The last part of the Lyapunov function can be treated as below,

$$\begin{aligned} E\{V_5(k+1) - V_5(k)\} \;=\; & E\{(Ax(k-\tau(k)) \\ & +f(x(k-\tau(k))) + Dw(k))^TP_3(Ax(k-\tau(k)) \\ & +f(x(k-\tau(k))) + Dw(k)) - x^T(k-\tau(k))P_3... \\ & ...x(k-\tau(k))\} = E\{x^T(k-\tau(k))[A^TP_3A - P_3]... \\ & ...x(k-\tau(k)) + 2x^T(k-\tau(k))A^TP_3f(x(k-\tau(k))) \\ & +f^T(x(k-\tau(k)))P_3f(x(k-\tau(k))) \\ & +2x^T(k-\tau(k))A^TP_3Dw(k) \\ & +2w^T(k)D^TP_3f(x(k-\tau(k))) + w^T(k)D^TP_3Dw(k)\} \\ & = E\{x^T(k-\tau(k))[A^TP_3A - P_3]x(k-\tau(k)) \\ & +2x^T(k-\tau(k))A^TP_3f(x(k-\tau(k))) \\ & +2f^T(x(k-\tau(k)))P_3f(x(k-\tau(k))) \\ & +2x^T(k-\tau(k))A^TP_3Dw(k) + 2w^T(k)D^TP_3Dw(k)\} \end{aligned} \tag{4.30}$$

Assuming $P_3 = \eta I - W_3 > 0$ and using lemma 4 we have,

$$\begin{aligned}
&2x^T(k-\tau(k))A^TP_3f(x(k-\tau(k))) + 2f^T(x(k-\tau(k)))P_3f(x(k-\tau(k))) \leq \\
&x^T(k-\tau(k))A^TP_3(2\eta - 2W_3)^{-1}P_3Ax(k-\tau(k)) \\
&+2\eta_3 f^T(x(k-\tau(k)))f(x(k-\tau(k)))
\end{aligned} \tag{4.31}$$

Using Lipschitz condition we have,

$$\begin{aligned}
&2x^T(k-\tau(k))A^TP_3f(x(k-\tau(k))) + 2f^T(x(k-\tau(k)))P_3f(x(k-\tau(k))) \\
&\leq x^T(k-\tau(k))[A^TP_3(2\eta - 2W_3)^{-1}P_3A + 2\eta\rho_1^2]x(k-\tau(k))
\end{aligned} \tag{4.32}$$

Substituting (4.32) into (4.30) we have,

$$\begin{aligned}
E\{V_5(k+1) - V_5(k)\} \quad = \quad & E\{x^T(k-\tau(k)) \\
& [-P_3 + A^TP_3A + A^TP_3(2\eta_3 - 2W_3)^{-1}P_3A + 2\eta_3\rho_1^2] \\
& x(k-\tau(k)) + 2x^T(k-\tau(k))A^TP_3Dw(k) \\
& +2w^T(k)D^TP_3Dw(k)\}
\end{aligned} \tag{4.33}$$

Now, we define the following cost function,

$$J \triangleq \sum_{k=0}^{\infty}(z_k^Tz_k - \mu^2 w_k^Tw_k) \tag{4.34}$$

where z_k is the controlled output defined as $z(k) = He(k)$.
Adding the ΔV to this equation we have,

$$J \leq \sum_{k=0}^{\infty}(z_k^Tz_k - \mu^2 w_k^Tw_k + \Delta V) \tag{4.35}$$

To guarantee the stability of the error dynamic we make the following assumption,

$$z_k^Tz_k - \mu^2 w_k^Tw_k + \Delta V < 0 \tag{4.36}$$

Taking to account the different part of Lyapunov function and take the expectation we have,

$$\begin{aligned}
\Delta V \;=\;& e^T(k)[H^T H + (1+\tau_M-\tau_m)Q_1 - P_1 + \tfrac{41}{7}(A-\bar{\gamma}LC)^T P_1 P_1 (A-\bar{\gamma}LC) \\
&+(A-\bar{\gamma}LC)^T P_1 (7P_1 - 7\eta_1^{-1}P_1^2)^{-1} P_1 (A-\bar{\gamma}LC) + 8\rho_2^2\bar{\gamma}^2 + 7\eta_1\rho_1^2]e(k) \\
&-e^T(k-\tau(k))Q_1 e(k-\tau(k)) x^T(k)[-P_2 + A^T P_2 A + 2\eta_2\rho_1^2 + 8\bar{\gamma}^2\rho_2^2 \\
&+7\bar{\gamma}^2 C^T L^T P_1 LC + A^T P_2 (2\eta_2 - 2P_2)^{-1} P_2 A]x(k) \\
&+x^T(k-\tau(k))[-P_3 + A^T P_3 A + 8\rho_2^2\bar{\gamma}(1-\bar{\gamma}) + 2\eta_3\rho_1^2 \\
&+A^T P_3 (2\eta_3 - 2P_3)^{-1} P_3 A + 7\bar{\gamma}(1-\bar{\gamma}) C^T L^T P_1 LC]x(k-\tau(k)) \\
&+2e^T(k)(A-\bar{\gamma}LC)P_1 Dw(k) + 2\bar{\gamma}w^T(k)D^T P_1 LCx(k) \\
&+2x^T(k)A^T P_2 Dw(k) + 2x^T(k-\tau(k))^T A^T P_3 Dw(k) \\
&+w^T(k)D^T[2(P_3+P_2) + 5P_1 - \mu^2 I]Dw(k)
\end{aligned} \tag{4.37}$$

Using (4.71) and Schur complement and assuming $\alpha_1^{-1} = \eta\rho_1^2$ and $\alpha_2^{-1} = \rho_2^2$ we have LMI (4.48) and the proof is completed. ■

4.2.1 Example

Example 5 *In this section, state estimation for a continuous time system is carried out using the proposed Lipschitz nonlinear filter described as below,*

$$\begin{aligned}
x(k+1) &= Ax(k) + f(x(k)) + Dw(k) \\
y(k) &= \gamma(k)(Cx(k) + g(x(k)))
\end{aligned} \tag{4.38}$$

in which $A = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}$, $D = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ *and* $g(x(k)) = 0.01sin(x_1)$. *Solving the LMI is needed to initialize some design criteria such as attenuation level* μ *and* H. *These parameters are summarized as follows,*

Design Criteria:

$$\mu = 0.5$$
$$H = 0.25I$$

probability information of the noise, delay and missing data is given. We assume that the measurement data is received at the central control with the missing rate of 80%

means that $\bar{\gamma} = 0.8$ and delay is bounded as $.01 < \tau < .1$. Plus, the noise with zero mean value and variance of 1 affect the process model.

Matlab LMI toolbox is used to implement the LMI (4.48). After solving the design problem using Theorem 5 filter gain is found as,

$$L = \begin{bmatrix} .002 \\ .0021 \end{bmatrix}$$

and the Lipschitz constants obtained as $\rho_1 = 0.0270$ and $\rho_2 = 0.0423$.

To show the accuracy of the filter, the four following cases are considered here:

In the first case, the probability of missing information in the channel assumed to be 0.8 and no delay and noise considered. In the second case, there is a delay in the communication network, but no missing information and no noise is considered. In the third case, noise with variance of 1 affects the performance of the network and in the last one probability of missing information is 0.8, delay is assumed to be in the range mentioned and also noise affects the communication network. 1. $\bar{\gamma} = 0.8$, $\tau = 0$ and $\sigma = 0$

2. $\bar{\gamma} = 1$, $0.01 < \tau < .1$ and $\sigma = 0$

3. $\bar{\gamma} = 1$, $\tau = 0$ and $\sigma = 1$

4. $\bar{\gamma} = 0.8$, $0.01 < \tau < .1$ and $\sigma = 1$

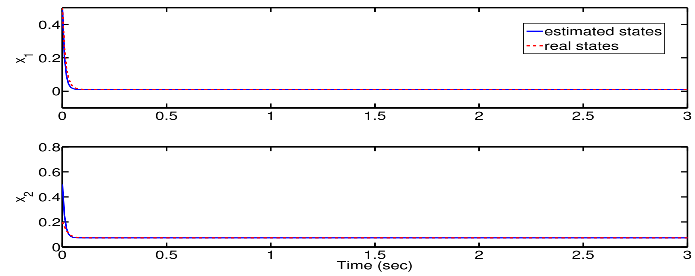
In Fig. 4.2 it is shown that states converge to the real ones for these four cases.

4.3 Filter Design for Systems with Parametric Uncertainties

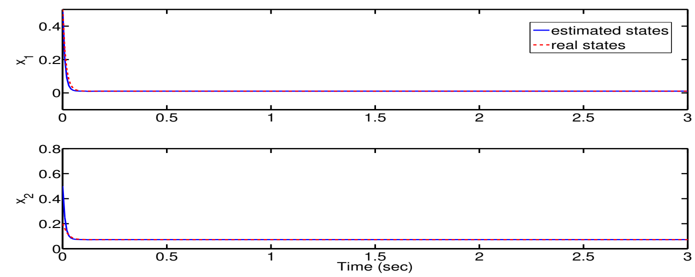
In this section, design of the nonlinear filter for estimating all the system states in the presence of uncertainty is presented. Let's consider the nonlinear system below,

$$\begin{aligned} x(k+1) &= (A + \Delta A)x(k) + f(x(k)) + Dw(k) \\ y(k) &= \gamma(k)((C + \Delta C)x(k) + g(x(k))) \end{aligned} \tag{4.39}$$

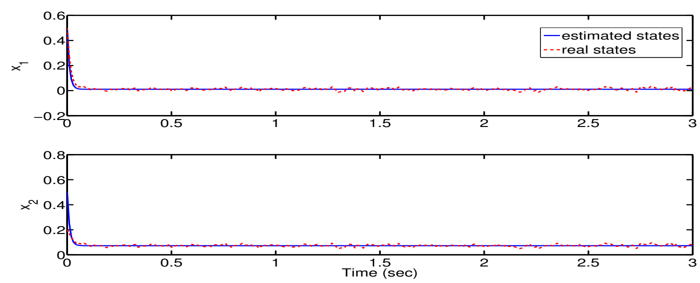
in which $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are system states, measurements and w is the noise signal which belongs to $\mathcal{L}_2[0, \infty]$. f and g are nonlinearities satisfying Lipschitz condition



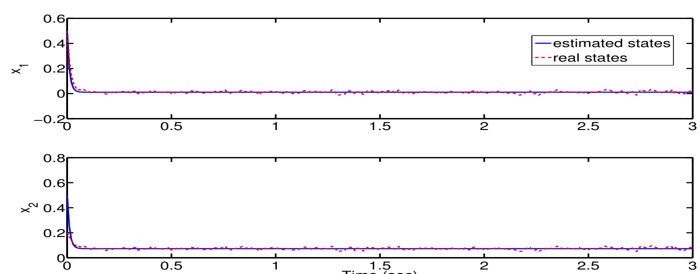
(a) $\bar{\gamma} = 0.8$, $\tau = 0$ and $\sigma = 0$



(b) $\bar{\gamma} = 1$, $0.01 < \tau < .1$ and $\sigma = 0$



(c) $\bar{\gamma} = 1$, $\tau = 0$ and $\sigma = 1$



(d) $\bar{\gamma} = .8$, $0.01 < \tau < .1$ and $\sigma = 1$

Figure 4.2: State estimation of four different cases

(4.2). Time-varying parameter uncertainties are presented by ΔA and ΔC as follows,

$$\Delta A = M_1 F(t) N_1$$
$$\Delta C = M_2 F(t) N_2 \tag{4.40}$$

where M_1, M_2, N_1 and N_1 are known constant matrices with appropriate dimensions and $F(t)$ is an unknown time varying matrix satisfies the following condition,

$$F^T(t)F(t) \leq I \tag{4.41}$$

Assume that the Luenberger-like filter having the following form,

$$\begin{array}{rcl} \hat{x}(k+1) & = & A\hat{x}(k) + f(\hat{x}(k)) + L\{y(k-\tau(k)) - \hat{y}(k)\} \\ \hat{y}(k) & = & \bar{\gamma}(C\hat{x}(k) + g(\hat{x}(k))) \end{array} \tag{4.42}$$

The dynamical error $e(k)$ is formulated as follow,

$$e(k+1) \;\; = \;\; x(k+1) - \hat{x}(k+1) \tag{4.43}$$

Substituting (4.39), (4.42) into (4.43) we get,

$$\begin{array}{rcl} e(k+1) & = & Ae(k) + \Delta Ax(k) + f(x(k)) - f(\hat{x}(k)) - L\{\gamma(k)Cx(k-\tau(k)) \\ & & +\gamma(k)\Delta Cx(k-\tau(k)) + \gamma(k)g(x(k-\tau(k)) - \bar{\gamma}(C\hat{x}(k) + g(\hat{x}(k)))\} \\ & & +Dw(k) \end{array} \tag{4.44}$$

adding and subtracting $L\bar{\gamma}Cx(k)$ and $\bar{\gamma}Lg(x)$ we have,

$$\begin{array}{rcl} e(k+1) & = & (A - \bar{\gamma}LC)e(k) + \Delta Ax(k) + f(x(k)) - f(\hat{x}(k)) \\ & & -L\{\gamma(k)Cx(k-\tau(k)) - \bar{\gamma}Cx(k) + \gamma(k)\Delta Cx(k-\tau(k)) \\ & & +\gamma(k)g(x(k-\tau(k)) - \bar{\gamma}g(x(k)) + \bar{\gamma}(g(x(k)) - g(\hat{x}(k)))\} + Dw(k) \end{array} \tag{4.45}$$

Define the controlled output z as follows,

$$z(k) = He(k) \tag{4.46}$$

Our objective is to find a filter gain L to (i) asymptotically stabilize the error dynamics when $w(k) = 0$, and (ii) minimize the L_2 norm of the controlled output z in the presence of noise, i.e.

$$||z||_{L_2} \leq \mu||w||_{L_2} \tag{4.47}$$

The main result of this section presented as follow,

Theorem 6 *Consider nonlinear system (4.39) with given nonlinearity as (4.2), the error dynamic is L_2 bounded satisfying the norm condition (4.7), if there exists $Q > 0$, $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 > 0$ and $\eta, \alpha_1, \alpha_2 > 0$ and positive definite matrix P_1, P_2, P_3 and matrix G such that the following LMI is feasible,*

$$\begin{pmatrix} \Psi_1 & 0 & 0 & \Psi_{14} \\ * & \Psi_2 & 0 & \Psi_{24} \\ * & * & \Psi_3 & \Psi_{34} \\ * & * & * & \Psi_4 \end{pmatrix} < 0$$

$$\begin{pmatrix} P_1 & GM_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \epsilon_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & P_1 & P_1M_2 & 0 & 0 & 0 & 0 \\ * & * & * & \epsilon_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & P_2 & P_2M_1 & 0 & 0 \\ * & * & * & * & * & \epsilon_3 & 0 & 0 \\ * & * & * & * & * & * & P_3 & P_3M_1 \\ * & * & * & * & * & * & * & \epsilon_4 \end{pmatrix} > 0 \tag{4.48}$$

$$\begin{pmatrix} I & G^T \\ G & \frac{P_1}{10} \end{pmatrix} > 0$$

$$\eta I - P_i > 0 \;\; i = 1, 2, 3$$

where Ψ_i are defined in the following page, in which $G = P_1L$, $\rho_2^2 = \alpha_2^{-1}$ and $\rho_1^2\eta = \alpha_1^{-1}$. After solving the LMIs, filter gain obtained as $L = P_1^{-1}G$.

Proof. Let consider the Lyapunov function as,

$$\begin{aligned} V(k) &= V_1(k) + V_2(k) + V_3(k) + V_4(k) + V_5(k) \\ &= e(k)^T P_1 e(k) + \textstyle\sum_{j=k-\tau(k)}^{k-1} e^T(k)Qe(k) \\ &\quad + \textstyle\sum_{i=k-\tau_M+1}^{k-\tau_m} \sum_{j=i}^{k-1} e^T(k)Qe(k) \\ &\quad + x^T(k)P_2x(k) + x^T(k-\tau(k)P_3x(k-\tau(k)) \end{aligned} \tag{4.50}$$

$$\Psi_1 = \begin{pmatrix} H^TH + (1+\tau_M - \tau_m)Q - P1 & 3I & \bar{\gamma} & A^TP_1 - \bar{\gamma}G^TC^T & 0 & A^TP_1 - \bar{\gamma}G^TC^T & 0 \\ * & -\alpha_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\alpha_1 & 0 & 0 & 0 & 0 \\ * & * & * & -9P_1 & P_1 & 0 & 0 \\ * & * & * & * & -\frac{\eta}{9} & 0 & 0 \\ * & * & * & * & * & -\frac{9}{71}P_1 & 0 \\ * & * & * & * & * & * & -Q \end{pmatrix}$$

$$\Psi_2 = \begin{pmatrix} (10\epsilon_2 + 4\epsilon_3)N_1^TN_1 - P_2 & \bar{\gamma} & \sqrt{3} & A^TP_2 & 0 & C^TG^T & A^TP_2 \\ * & -\alpha_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\alpha_1 & 0 & 0 & 0 & 0 \\ * & * & * & -3P_2 & P_2 & 0 & 0 \\ * & * & * & * & -\frac{\eta}{3} & 0 & 0 \\ * & * & * & * & * & -\frac{P_1}{10\bar{\gamma}^2} & 0 \\ * & * & * & * & * & * & -\frac{3}{5}P_2 \end{pmatrix}$$

$$\Psi_3 = \begin{pmatrix} (4\epsilon_4)N_1^TN_1 + 10\epsilon_1\bar{\gamma}(1-\bar{\gamma})N_2^TN_2 & \sqrt{\bar{\gamma}(1-\bar{\gamma})} & \sqrt{3} & A^TP_3 & 0 & C^TG^T & A^TP_3 \\ * & -\alpha_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\alpha_1 & 0 & 0 & 0 & 0 \\ * & * & * & -3P_3 & P_3 & 0 & 0 \\ * & * & * & * & -\frac{\eta}{3} & 0 & 0 \\ * & * & * & * & * & -\frac{P_1}{10\bar{\gamma}(1-\bar{\gamma})} & 0 \\ * & * & * & * & * & * & -\frac{3}{5}P_3 \end{pmatrix}$$

$$\Psi_4 = 9P_1 + 3(P_2 + P_3) - \mu^2 I$$

$$\Psi_{14} = \begin{pmatrix} A^TP_1D - \bar{\gamma}C^TG^TD \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Psi_{24} = \begin{pmatrix} A^TP_2D \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.49}$$

$$\Psi_{34} = \begin{pmatrix} A^TP_3D \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Computing the expectation of difference of $V_1(k)$ along the trajectory of error and using (4.40), we get

$$
\begin{aligned}
E\{V_1(k+1)-V_1(k)\} \;=\; & E\{e^T(k)[(A-\bar{\gamma}LC)^TP_1(A-\bar{\gamma}LC)-P_1]e(k) \\
& +2e^T(k)(A-\bar{\gamma}LC)^TP_1\Delta f_k+2e^T(k)(A-\bar{\gamma}LC)^T... \\
& ...P_1M_1FN_1x(k)+2e^T(k)(A-\bar{\gamma}LC)^TP_1Dw(k) \\
& -2\gamma(k)e^T(k)(A-\bar{\gamma}LC)^TP_1LCx(k-\tau(k)) \\
& -2\gamma(k)e^T(k)(A-\bar{\gamma}LC)^TP_1LM_2FN_2x(k-\tau(k)) \\
& -2\gamma(k)e^T(k)(A-\bar{\gamma}LC)^TP_1Lg(x(k-\tau(k))) \\
& -2\bar{\gamma}e^T(k)(A-\bar{\gamma}LC)^TP_1L\Delta g_k \\
& +2\bar{\gamma}e^T(k)(A-\bar{\gamma}LC)^TP_1LCx(k) \\
& +2\bar{\gamma}e^T(k)(A-\bar{\gamma}LC)^TP_1Lg(x(k))+\Delta f_k^TP_1\Delta f_k \\
& +2\Delta f_k^TP_1M_1FN_1x(k)+2\Delta f_k^TP_1Dw(k) \\
& -2\gamma(k)\Delta f_k^TP_1LCx(k-\tau(k)))-2\gamma(k)\Delta f_k^TP_1... \\
& ...Lg(x(k-\tau(k)))-2\bar{\gamma}\Delta f_k^TP_1L\Delta g_k \\
& +2\bar{\gamma}\Delta f_k^TP_1LCx(k)+2\bar{\gamma}\Delta f_k^TP_1Lg(x(k)) \\
& -2\gamma(k)\Delta f^TP_1LM_2FN_2x(k-\tau(k)) \\
& +w^T(k)D^TP_1Dw(k)-2\gamma(k)w^T(k)D^TP_1LC... \\
& ...x(k-\tau(k)))-2\gamma(k)w^T(k)D^TP_1Lg(x(k-\tau(k))) \\
& -2\bar{\gamma}w^T(k)D^TP_1L\Delta g_k-2\gamma(k)w_k^T(k)D^TP_1LM_2... \\
& ...FN_2x(k-\tau(k))+2\bar{\gamma}w^T(k)D^TP_1LCx(k) \\
& +2\bar{\gamma}w^T(k)D^TP_1Lg(x(k)) \\
& +2\gamma(k)^2x^T(k-\tau(k))C^TL^TP_1Lg(x(k-\tau(k))) \\
& +2\gamma(k)\bar{\gamma}x^T(k-\tau(k))C^TL^TP_1L\Delta g_k \\
& -2\gamma(k)\bar{\gamma}x^T(k-\tau(k))C^TL^TP_1LCx(k) \\
& -2\gamma(k)\bar{\gamma}x^T(k-\tau(k))C^TL^TP_1Lg(x(k)) \\
& +2\gamma^2(k)x^T(k-\tau(k))C^TL^TP_1LM_2FN_2x(k-\tau(k)) \\
& +\gamma^2(k)x^T(k-\tau(k))N_2^TF^TM_2^TL^TP_1LM_2FN_2... \\
& ...x(k-\tau(k))+2\gamma^2(k)x(k-\tau(k))N_2^TF^TM_2^TL^T...
\end{aligned}
$$

$$
\begin{aligned}
&...P_1Lg(k-\tau(k)) + 2\bar{\gamma}\gamma(k)x^T(k-\tau(k))N_2^TF^T...\\
&...M_2^TL^TP_1L\Delta g_k - 2\bar{\gamma}\gamma(k)x^T(k-\tau(k))N_2^TF^T...\\
&...M_2^TL^TP_1Lg(x(k)) + \gamma^2(k)g^T(x(k-\tau(k)))...\\
&...L^TP_1Lg(x(k-\tau(k))) + 2\bar{\gamma}\gamma(k)g^T(x(k-\tau(k)))...\\
&...L^TP_1L\Delta g_k - 2\bar{\gamma}\gamma(k)g^T(x(k-\tau(k)))L^TP_1LCx(k)\\
&-2\bar{\gamma}\gamma g^T(x(k-\tau(k)))L^TP_1Lg(x(k))\\
&+\bar{\gamma}^2\Delta g_k^TL^TP_1L\Delta g_k - 2\bar{\gamma}^2\Delta g_k^TL^TP_1LCx(k)\\
&-2\bar{\gamma}^2\Delta g_k^TL^TP_1Lg(x(k)) + \bar{\gamma}^2x^T(k)C^TL^TP_1LCx(k)\\
&+2\bar{\gamma}^2x^T(k)C^TL^TP_1Lg(x(k))\\
&+x^T(k)N_1^TF^TM_1^TP_1M_1FN_1x(k)\\
&+2x^T(k)N_1^TF^TM_1^TP_1Dw(k)\\
&+2\bar{\gamma}x^T(k)N_1^TF^TM_1^TP_1LCx(k)\\
&-2\gamma(k)x^T(k)N_1^TF^TM_1^TP_1LCx(k-\tau(k))\\
&-2\gamma(k)x^T(k)N_1^TF^TM_1^TP_1LM_2FN_2x(k-\tau(k))\\
&-2\gamma(k)x^T(k)N_1^TF^TM_1^TP_1Lg(k-\tau(k))...\\
&...-2\bar{\gamma}x^T(k)N_1^TF^TM_1^TP_1L\Delta g_k\\
&+2\bar{\gamma}x^T(k)N_1^TF^TM_1^TP_1Lg(x(k))\\
&-2\gamma(k)\bar{\gamma}x^T(k)C^TL^TP_1LM_2FN_2x(k-\tau(k))\\
&+\bar{\gamma}^2g^T(x(k))L^TP_1Lg(x(k))\}
\end{aligned}
\tag{4.51}
$$

in which $\Delta f_k = f(x(k)) - f(\hat{x}(k))$ and $\Delta g_k = g(x(k)) - g(\hat{x}(k))$. Using Lemma 1 the equation above can be rewritten as below,

$$
\begin{aligned}
E\{V_1(k+1) - V_1(k)\} \quad = \quad & E\{-e^T(k)P_1e(k) + 8e^T(k)(A - \bar{\gamma}LC)^T P_1 ... \\
& ...(A - \bar{\gamma}LC)e(k) + 9\Delta f_k^T P_1 \Delta f_k \\
& +2e^T(k)(A - \bar{\gamma}LC)^T P_1 \Delta f_k \\
& +2e^T(k)(A - \bar{\gamma}LC)^T P_1 Dw(k) \\
& +10\gamma^2(k)x^T(k-\tau(k))C^T L^T P_1 LCx(k-\tau(k)) \\
& +10\bar{\gamma}^2 x^T(k)C^T L^T P_1 LCx(k) \\
& +10\gamma^2(k)x^T(k-\tau(k))N_2^T F^T M_2^T L^T P_1 L M_2 F N_2 ... \\
& ...x(k-\tau(k)) + 10\gamma^2(k)g^T(x(k-\tau(k)))L^T ... \\
& ...P_1 Lg(x(k-\tau(k))) + 10\bar{\gamma}^2 \Delta g_k^T L^T P_1 L \Delta g_k \\
& +10\bar{\gamma}^2 g^T(x(k))L^T P_1 Lg(x(k)) + 9w^T(k)D^T P_1 Dw(k) \\
& +10x^T(k)N_1^T F^T M_1^T P_1 M_1 F N_1 x(k)\}
\end{aligned}
\tag{4.52}
$$

Let assume that,

$$10L^T P_1 L - I < 0 \tag{4.53}$$

Using Lemma 2, we have

$$\begin{bmatrix} I & G^T \\ G & \frac{P_1}{10} \end{bmatrix} > 0 \tag{4.54}$$

Adding and subtracting $\gamma^2(k)g^T(x(k-\tau(k)))$, $\bar{\gamma}^2\Delta g_k^T\Delta g_k$ and $\bar{\gamma}^2 g^T(x(k))g(x(k))$ to (4.52) and using the Lipschitz condition (4.2) and also (4.53), we get

$$\begin{aligned}
E\{V_1(k+1)-V_1(k)\} \;=\;& E\{-e^T(k)P_1e(k)+8e^T(k)(A-\bar{\gamma}LC)^TP_1...\\
&...(A-\bar{\gamma}LC)e(k)+9\Delta f_k^TP_1\Delta f_k\\
&+2e^T(k)(A-\bar{\gamma}LC)^TP_1\Delta f_k\\
&+2e^T(k)(A-\bar{\gamma}LC)^TP_1Dw(k)\\
&+10\gamma^2(k)x^T(k-\tau(k))C^TL^TP_1LCx(k-\tau(k))\\
&+10\bar{\gamma}^2x^T(k)C^TL^TP_1LCx(k)\\
&+10\gamma^2(k)x^T(k-\tau(k))N_2^TF^TM_2^TL^TP_1LM_2FN_2...\\
&...x(k-\tau(k))+\gamma^2(k)\rho_2^2x^T(k-\tau(k))x(k-\tau(k))\\
&+\bar{\gamma}^2\rho_2^2e^T(k)e(k)+\bar{\gamma}^2\rho_2^2x^T(k)x(k)\\
&+9w^T(k)D^TP_1Dw(k)+10x^T(k)N_1^TF^TM_1^TP_1M_1...\\
&...FN_1x(k)\}
\end{aligned} \tag{4.55}$$

Assuming that $P_1=\eta-W_1>0$, using lemma 4 we get

$$\begin{aligned}
&2e^T(k)(A-\bar{\gamma}LC)P_1\Delta f_k+9\Delta f_k^T(\eta-W_1)\Delta f_k\\
&\le e^T(k)(A-\bar{\gamma}LC)^TP_1(9W_1)^{-1}P_1(A-\bar{\gamma}LC)e(k)+9\eta\Delta f_k^T\Delta f_k
\end{aligned} \tag{4.56}$$

Using Lipschitz condition we have,

$$\begin{aligned}
&2e^T(k)(A-\bar{\gamma}LC)P_1\Delta f_k+9\Delta f_k^T(\eta-W_1)\Delta f_k\\
&\le e^T(k)(A-\bar{\gamma}LC)^TP_1(9\eta-9P_1)^{-1}P_1(A-\bar{\gamma}LC)e(k)+9\eta\rho_1^2e^T(k)e(k)
\end{aligned} \tag{4.57}$$

Consider that,

$$\begin{aligned}
&(A-\bar{\gamma}LC)^TP_1(9\eta-9P_1)^{-1}P_1(A-\bar{\gamma}LC)+\tfrac{1}{9}(A-\bar{\gamma}LC)^TP_1(A-\bar{\gamma}LC)\\
&=(A-\bar{\gamma}LC)^TP_1[(9\eta-9P_1)^{-1}+\tfrac{1}{9}P_1^{-1}]P_1(A-\bar{\gamma}LC)
\end{aligned} \tag{4.58}$$

Assuming that,

$$\begin{aligned}
(9\eta-9P_1)^{-1}+\tfrac{1}{9}P_1^{-1} &= (9\eta-9P_1)^{-1}((9\eta-9P_1)\tfrac{1}{9}P_1^{-1}+I)\\
&= (9P_1-9\eta^{-1}P_1^2)^{-1}
\end{aligned} \tag{4.59}$$

Substituting (4.57) into (4.60) and some simplification we have,

$$\begin{aligned}
E\{V_1(k+1)-V_1(k)\} \;=\;& E\{e^T(k)[(A-\bar{\gamma}LC)^TP_1(9P_1-9\eta^{-1}P_1^2)^{-1}P_1...\\
&...(A-\bar{\gamma}LC)-P_1+\tfrac{71}{9}(A-\bar{\gamma}LC)^TP_1(A-\bar{\gamma}LC)\\
&+\bar{\gamma}^2\rho_2^2+9\eta\rho_1^2]e(k)\\
&+2e^T(k)(A-\bar{\gamma}LC)^TP_1Dw(k)\\
&+10\gamma^2(k)x^T(k-\tau(k))C^TL^TP_1LCx(k-\tau(k))\\
&+10\bar{\gamma}^2x^T(k)C^TL^TP_1LCx(k)\\
&+10\gamma^2(k)x^T(k-\tau(k))N_2^TF^TM_2^TL^TP_1LM_2FN_2...\\
&...x(k-\tau(k))+\gamma^2(k)\rho_2^2x^T(k-\tau(k))x(k-\tau(k))\\
&+\bar{\gamma}^2\rho_2^2x^T(k)x(k)+9w^T(k)D^TP_1Dw(k)\\
&+10x^T(k)N_1^TF^TM_1^TP_1M_1FN_1x(k)\}
\end{aligned}\tag{4.60}$$

The following equation can be easily verified,

$$\begin{aligned}
&E\{V_2(k+1)-V_2(k)\}+E\{V_3(k+1)-V_3(k)\}\\
&\le E\{e^T(k)Qe(k)-e^T(k-\tau(k))Qe(k-\tau(k))\}+E\{(\tau_M-\tau_m)e^T(k)Qe(k)\}
\end{aligned}\tag{4.61}$$

the forth part of the Lyapunov function can be analyzed as follows,

$$\begin{aligned}
E\{V_4(k+1)-V_4(k)\} \;=\;& E\{(Ax(k)+\Delta Ax(k)+f(x(k))\\
&+Dw(k))^TP_2(Ax(k)+\Delta Ax(k)+f(x(k))+Dw(k))\\
&-x^T(k)P_2x(k)\}\le E\{2x^T(k)A^TP_2Ax(k)\\
&+4x^T(k)N_1^TF^TM_1^TP_2M_1FN_1x(k)\\
&+3f(x(k))^TP_2f(x(k))+2x^T(k)A^TP_2f(x(k))\\
&+3w^T(k)D^TP_2Dw(k)\\
&+2x^T(k)A^TP_2^TDw(k)-x^T(k)P_2x(k)\}
\end{aligned}\tag{4.62}$$

Using (4.56)-(4.59), we get

$$\begin{aligned}
E\{V_4(k+1)-V_4(k)\} \;=\;& E\{x^T(k)[-P_2+\tfrac{5}{3}A^TP_2A\\
&+A^TP_2(3P_2-3\eta^{-1}P_2^2)^{-1}P_2A+3\eta\rho_1^2]x(k)\\
&+4x^T(k)N_1^TF^TM_1^TP_2M_1FN_1x(k)\\
&+3w^T(k)D^TP_2Dw(k)+2x^T(k)A^TP_2^TDw(k)\}
\end{aligned}\tag{4.63}$$

Let consider the following delayed systems,

$$x(k+1-\tau(k)) = Ax(k-\tau(k)) + \Delta Ax(k-\tau(k)) + f(x(k-\tau(k))) + Dw_d(k) \quad (4.64)$$

The last part of the Lyapunov function can be treated as below,

$$\begin{aligned}
E\{V_5(k+1) - V_5(k)\} \;=\; & E\{(Ax(k-\tau(k)) + \Delta Ax(k-\tau(k)) + f(x(k-\tau(k))) \\
& +Dw_d(k))^T P_3(Ax(k-\tau(k)) + \Delta Ax(k-\tau(k)) \\
& +f(x(k-\tau(k))) + Dw_d(k)) - x^T(k-\tau(k))P_3 ... \\
& ...x(k-\tau(k))\} \leq E\{2x^T(k)A^T P_3 Ax(k) \\
& +4x^T(k)N_1^T F^T M_1^T P_3 M_1 F N_1 x(k) \\
& +3f(x(k))^T P_3 f(x(k)) + 2x^T(k)A^T P_3 f(x(k)) \\
& +3w^T(k)D^T P_3 Dw(k) + 2x^T(k)A^T P_3^T Dw(k) \\
& -x^T(k)P_3 x(k)\}
\end{aligned} \quad (4.65)$$

Using (4.56)-(4.59), we get

$$\begin{aligned}
E\{V_5(k+1) - V_5(k)\} \;=\; & E\{x^T(k)[-P_3 + \tfrac{5}{3}A^T P_3 A \\
& +A^T P_3(3P_3 - 3\eta^{-1}P_3^2)^{-1}P_3 A + 3\eta\rho_1^2]x(k) \\
& +4x^T(k)N_1^T F^T M_1^T P_3 M_1 F N_1 x(k) \\
& +3w^T(k)D^T P_3 Dw(k) + 2x^T(k)A^T P_3^T Dw(k)\}
\end{aligned} \quad (4.66)$$

Lemma 5 *[28] Assume that A, D, E and F are constant matrices, P is positive definite matrix and F satisfying $F^T(t)F(t) \leq I$. Then for $\epsilon > 0$ with $P^{-1} - \epsilon^{-1}DD^T > 0$, we have*

$$(A + DFE)^T P(A + DFE) \leq A^T(P^{-1} - \epsilon^{-1}DD^T)^{-1}A + \epsilon E^T E \quad (4.67)$$

Using this lemma and assuming that $A = 0$ here, we have

$$\begin{gathered}
10\gamma^2(k)x^T(k-\tau(k))N_2^T F^T M_2^T L^T P_1 L M_2 F N_2 x(k-\tau(k)) \\
\leq 10\gamma^2(k)\epsilon_1 x^T(k-\tau(k))x(k-\tau(k)) \\
10x^T(k)N_1^T F^T M_1^T P_1 M_1 F N_1 x(k) \leq 10\epsilon_2 x^T(k)N_1^T N_1 x(k) \\
4x^T(k)N_1^T F^T M_1^T P_2 M_1 F N_1 x(k) \leq 4\epsilon_3 x^T(k)N_1 N_1 x(k) \\
4x^T(k)N_1^T F^T M_1^T P_3 M_1 F N_1 x(k) \leq 4\epsilon_4 x^T(k)N_1 N_1 x(k)
\end{gathered}$$

with the following assumptions,

$$
\begin{aligned}
P_1^{-1} - \epsilon_1^{-1} L M_2 M_2^T L^T > 0 \\
P_1^{-1} - \epsilon_2^{-1} M_1 M_1^T > 0 \\
P_2^{-1} - \epsilon_3^{-1} M_1 M_1^T > 0 \\
P_3^{-1} - \epsilon_4^{-1} M_1 M_1^T > 0
\end{aligned}
\tag{4.68}
$$

Now, we define the following cost function,

$$
J \triangleq \sum_{k=0}^{\infty} (z_k^T z_k - \mu^2 w_k^T w_k) \tag{4.69}
$$

where z_k is the controlled output defined as $z(k) = He(k)$.

Adding the ΔV to this equation we have,

$$
J \leq \sum_{k=0}^{\infty} (z_k^T z_k - \mu^2 w_k^T w_k + \Delta V) \tag{4.70}
$$

To guarantee the stability of the error dynamic we make the following assumption,

$$
z_k^T z_k - \mu^2 w_k^T w_k + \Delta V < 0 \tag{4.71}
$$

Taking to account the different part of Lyapunov function and take the expectation we have,

$$
\begin{aligned}
\Delta V \quad = \quad & E\{e^T(k)[H^T H - P_1 + \tfrac{71}{9}(A - \bar{\gamma} LC)^T P_1 (A - \bar{\gamma} LC) \\
& +(1 - \tau_M - \tau_m) Q_1 + \bar{\gamma}^2 \rho_2^2 + 9\eta \rho_1^2 \\
& +(A - \bar{\gamma} LC)^T P_1 (9P_1 - 9\eta^{-1} P_1^2)^{-1} P_1 (A - \bar{\gamma} LC)] e(k) \\
& -e^T(k - \tau(k)) Q_1 e(k - \tau(k)) \\
& +2e^T(k)(A - \gamma LC)^T P_1 D w(k) + x^T(k)[-P_2 ... \\
& ... + \tfrac{5}{3} A^T P_2 A + 4\epsilon_3 N_1^T N_1 + 3\eta \rho_1^2 + 10\epsilon_2 N_1^T N_1 \\
& +\bar{\gamma}^2 \rho_2^2 + A^T P_2 (3P_2 - 3\eta^{-1} P_2)^{-1} P_2 A \\
& 10\bar{\gamma}^2 C^T L^T P_1 LC] x(k) + 2x^T(k) A^T P_2 D w(k) \\
& +x^T(k - \tau(k))[-P_3 + \tfrac{5}{3} A^T P_3 A + 4\epsilon_4 N_1^T N_1 \\
& +3\eta \rho_1^2 + 10\epsilon_1 \bar{\gamma}(1 - \bar{\gamma}) N_2^T N_2 + \bar{\gamma}(1 - \bar{\gamma}) \rho_2^2 \\
& +A^T P_3 (3P_3 - 3\eta^{-1} P_3)^{-1} P_3 A ... \\
& ... + 10\bar{\gamma}(1 - \bar{\gamma}) C^T L^T P_1 LC] x(k - \tau(k)) \\
& 2x^T(k - \tau(k)) A^T P_3 B w(k) \\
& +w^T(k)[9D^T P_1 D + 3D^T P_2 D + 3D^T P_3 D - \mu^2 I] w(k)\}
\end{aligned}
\tag{4.72}
$$

Using (4.71) and Schur's complement the proof is completed $\blacksquare$

4.3.1 Example

Example 6 *In this section, state estimation for a discrete time system is carried out using the proposed Lipschitz nonlinear filter described as below,*

$$\begin{aligned} x(k+1) &= Ax(k+1) + f(x(k)) \\ y(k) &= \gamma(k)(Cx(k) + g(x(k))) \end{aligned} \tag{4.73}$$

in which $A = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.1 \end{bmatrix}$, $D = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ *and* $g(x(k)) = 0.01sin(x_1)$. *Solving the LMI (4.48) is needed to initialize some design criteria such as attenuation level* μ *and* H *and also tuning parameters like* η *. These parameters are summarized as follows,*

Design Criteria:

$$\mu = 0.5$$
$$H = 0.25I$$

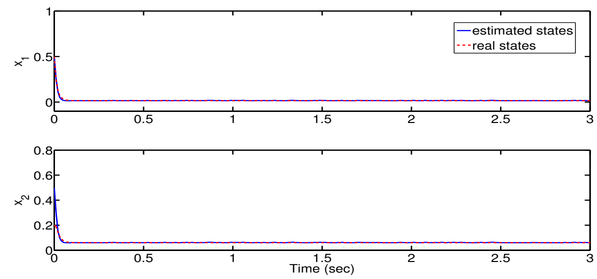
Uncertainty matrices are assumed to be the following,

$$M_1 = 0.05I, N_1 = N_2 = 0.5I, M_2 = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$$

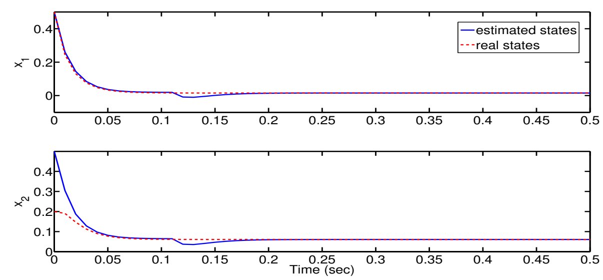
probability information of the noise, delay and missing data is given. We assume that the measurement data is received at the central control with the missing rate of 80% *means that* $\bar{\gamma} = 0.8$ *and delay is bounded as* $.01 < \tau < .1$. *Plus, the noise with zero mean value and variance of 1 affect the process model.*
Matlab LMI toolbox is used to implement the LMI (4.48). After solving the design problem using theorem 6 the filter gain is,

$$L = \begin{bmatrix} 0.0567 \\ 0.0558 \end{bmatrix}$$

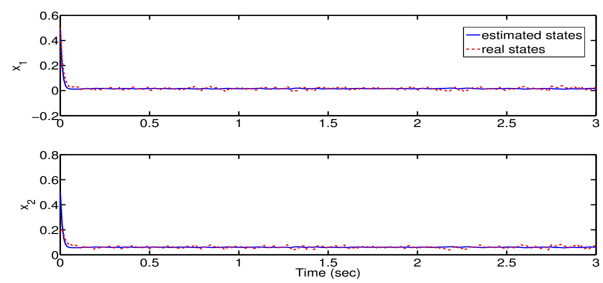
To show the accuracy of the filter, four cases are considered here:
In the first case, the probability of missing information in the channel assumed to be 0.8 and no delay and noise considered. In the second case, there is a delay in the



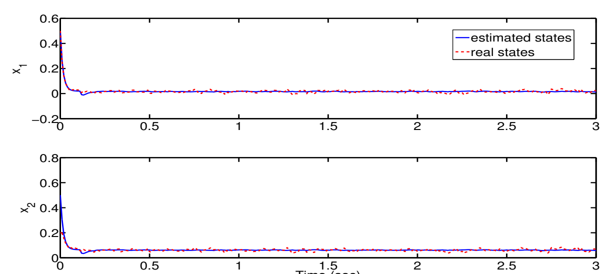
(a) $\bar{\gamma} = 0.8$, $\tau = 0$ and $\sigma = 0$



(b) $\bar{\gamma} = 1$, $0.01 < \tau < .1$ and $\sigma = 0$



(c) $\bar{\gamma} = 1$, $\tau = 0$ and $\sigma = 1$



(d) $\bar{\gamma} = .8$, $0.01 < \tau < .1$ and $\sigma = 1$

Figure 4.3: State estimation of four different cases in presence of uncertainty

communication network, but no missing information occurs and no noise is added. In the third case, noise with a variance of 1 affects the performance of the network. In the final case, probability of missing information is 0.8, delay is assumed to be in *the range mentioned and also noise affects the communication network. 1.* $\bar{\gamma} = 0.8$, $\tau = 0$ *and* $\sigma = 0$

2. $\bar{\gamma} = 1$, $0.01 < \tau < .1$ *and* $\sigma = 0$

3. $\bar{\gamma} = 1$, $\tau = 0$ *and* $\sigma = 1$

4. $\bar{\gamma} = 0.8$, $0.01 < \tau < .1$ *and* $\sigma = 1$

In Fig. 4.3 it is shown that states converge to the real ones for these four cases. As it can be seen in the Fig. 3.3 (b) before 10th Second of the simulation, there is a bias in the estimated states. When the filter receives the measurement from the system, the error converges to the real value.

4.4 Case Study

In this section, fourth order model of a synchronous generator connected to infinite bus is presented [32]. In Fig. 4.4 a power network is shown and as can be seen the entire network can be viewed like an infinite bus by the single synchronous generator since its load and power is much larger compared to the single machine.

This model is described as follows,

$$
\begin{aligned}
\frac{d\delta(t)}{dt} &= \omega_0\omega(t) \\
\frac{d\omega(t)}{dt} &= \frac{1}{2H}[T_m - T_e - D\omega(t)] \\
\frac{dE'_q}{dt} &= \frac{1}{T'_{do}}[-E'_q + (x_d - x'_d)i_d + E_{fd}] \\
\frac{dE'_d}{dt} &= \frac{1}{T'_{qo}}[-E'_d - (x_q - x'_q)i_q]
\end{aligned}
\tag{4.74}
$$

The state vector is defined as $x = [\delta \quad \omega \quad E'_q \quad E'_d]^T$. δ, ω are rotor angle and angular rotor speed and E'_q, E'_d and i_q and i_d are voltage and current magnitudes in d-q frame. T_m, E_{fd} are input mechanical torque and electric field voltage, respectively and T_e is the electrical torque. $\omega_o = 2\pi f$, D, H, T_{do}, T_{qo} are the nominal synchronous speed, damping factor, inertia constant and d-q transient open circuit time constants.

Electrical torque of the synchronous generator is computed as follows,

$$T_e = E'_d i_d + E'_q i_q + (x'_d - x'_q) i_q i_d \tag{4.75}$$

Electrical torque and reactive power P which is the measurement transmitted to central control unit are related through the equation below,

$$T_e = P + I_a^2 R_a \tag{4.76}$$

Neglecting the rotor resistance R_a, $T_e \approx P$. Connecting the synchronous generator to the infinite bus, current magnitudes in the d-q frame are calculated using the formula,

$$\begin{bmatrix} x'_d + Z_I & -(R_a + Z_R) \\ -(R_a + Z_R) & -(x'_q + Z_I) \end{bmatrix} \begin{bmatrix} i_d \\ i_q \end{bmatrix} = \begin{bmatrix} h_1 E_b cos(\delta) + h_2 E_b sin(\delta) - E'_q \\ h_2 E_b cos(\delta) - h_1 E_b sin(\delta) - E'_d \end{bmatrix} \tag{4.77}$$

where E_b is the bus voltage. $Z = Z_R + jZ_I$ is the impedance viewed from the terminal when $E_b = 0$ and $h = h_1 + jh_2$ is the voltage gain at the terminal when armature circuit is open. Assuming the purely inductive impedance, it will be reduced to $Z = Z_I$ and $h = h_1 = 1$.

Taylor series can be applied to the measurement equation as follows,

$$cos(\delta) = 1 - \frac{\delta^2}{2!} + \frac{\delta^4}{4!} - ... \tag{4.78}$$

Using this assumption, the measurement equation can be rewritten into linear part and nonlinear one as below,

$$P = Cx + g(x) \tag{4.79}$$

where $C = \begin{bmatrix} 0 & 0 & h_1 E_b a_{21} + h_2 E_b a_{22} & h_1 E_b a_{11} + h_2 E_b a_{12} \end{bmatrix}$ and g(x) is the nonlinear part of the measurement and $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} x'_d + Z_I & -(R_a + Z_R) \\ -(R_a + Z_R) & -(x'_q + Z_I) \end{bmatrix}^{-1}$.

The measurements taken by local sensors are sampled at each sampling time T and sent to PMU to be transmitted to the central control unit. Reactive power and/or active power are measurements taken as inputs for nonlinear filter to estimate the unknown states as well as field voltage and mechanical torque.

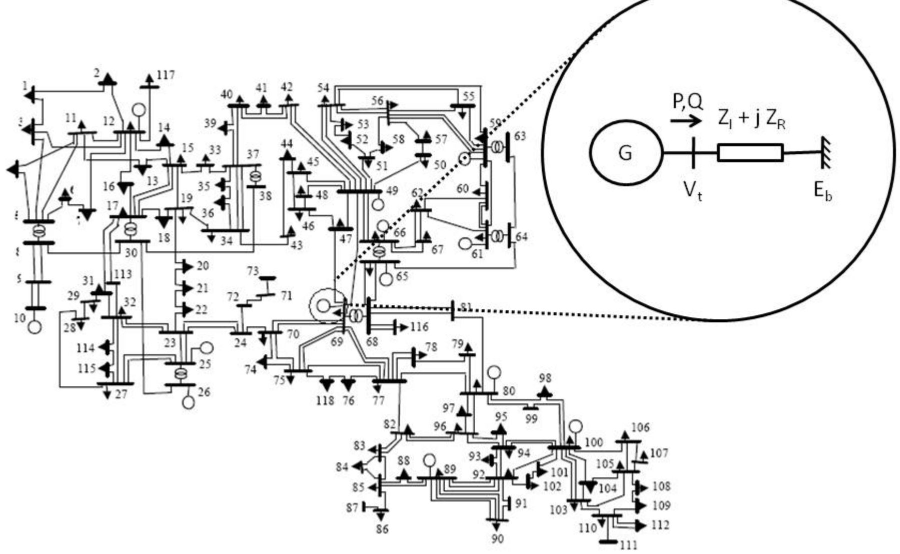


Figure 4.4: Asynchronous generator connected to infinite bus

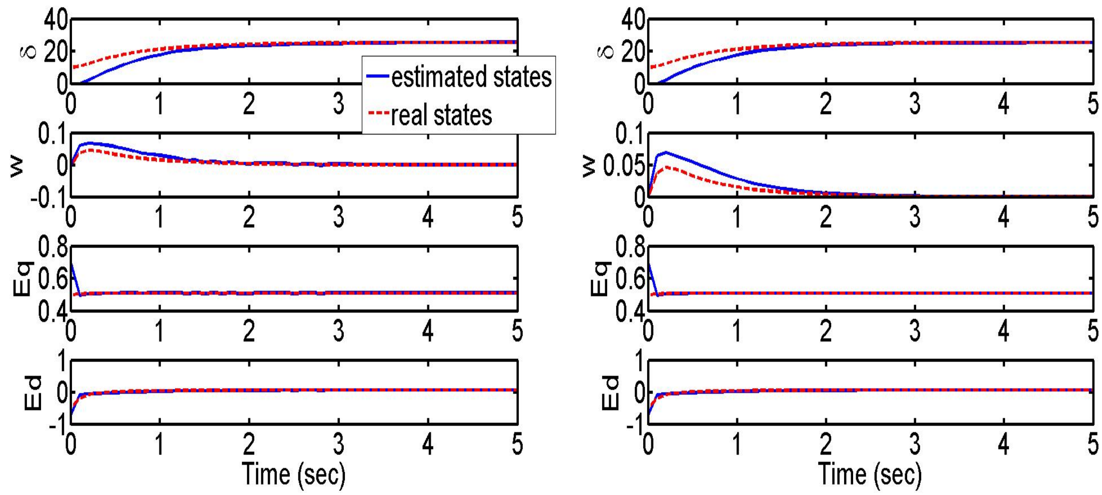
4.4.1 Simulation Results

In this section, state estimation for a synchronous generator connected to infinite bus is carried out using the proposed Lipschitz nonlinear filter. Before solving the problem, we need to modify the system into a reduced one consisting of the last three states since the first state is dependent to the second one. Considering the four order model causes the unobservability problem in the design procedure. Using the states obtained in the third order model, the rotor angle is updated at each sampling time. Using this assumption, we have

$$\begin{aligned} x(k+1) &= (A+\Delta A)x(k) + f(x(k)) + Bw(k) \\ y(k) &= \gamma(k)((C+\Delta C)x(k) + g(x(k))) \end{aligned} \tag{4.80}$$

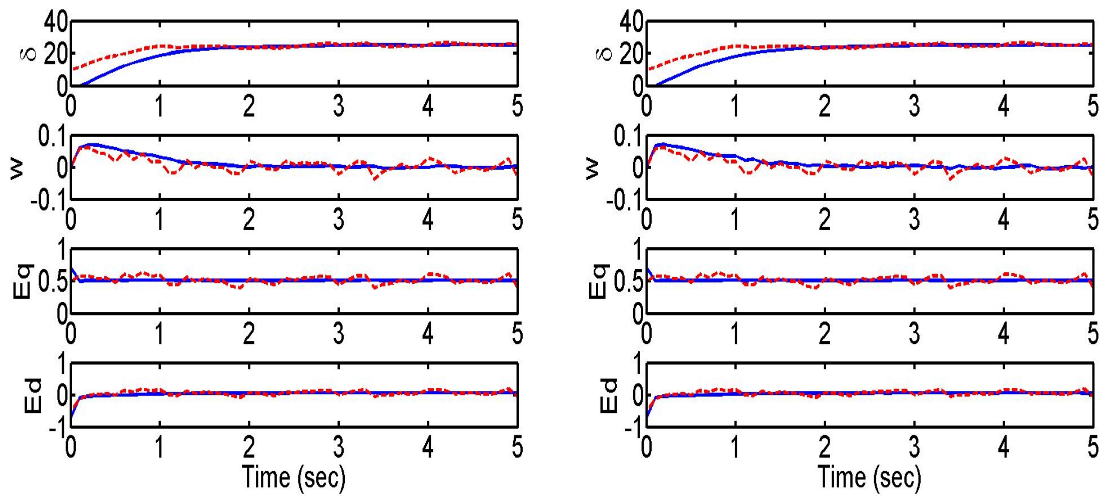
in which $x = [\omega, E'_q, E'_d]^T$. The first equation is used to update the rotor angle at each step of simulation. The design procedure is divided into the following steps:

1. In order to design a filter (4.42), parameters of the generator and also transmission line impedance is needed. The generator used for estimation has the following parameters [32],



(a) $\bar{\gamma} = 0.8$, $\tau = 0$ and $\sigma = 0$

(b) $\bar{\gamma} = 1$, $0.01 < \tau < .1$ and $\sigma = 0$



(c) $\bar{\gamma} = 1$, $\tau = 0$ and $\sigma = 1$

(d) $\bar{\gamma} = .8$, $0.01 < \tau < .1$ and $\sigma = 1$

Figure 4.5: State estimation of four different cases

Generator Model:

$\omega_o = 377$, $H = 1.7$, $D = 25$, $R_a = 0.1$, $T'_{do} = 0.1$, $T'_{qo} = 0.1$, $x_d = .02$, $x_q = 0.01$, $x'_q = 0.05$, $x'_d = 0.015$. In this simulation, we assume that $E_b = 1$, and also the generator inputs are $E_f = 0.5$ and $T_m = 1$.

The transmission line connecting the generator to the bus has the following impedance and gains,

Transmission line:

$$Z_R + jZ_I = j0.192$$
$$h_1 + jh_2 = 1$$

2. Solving the LMI (4.48) is needed to initialize some design criteria such as attenuation level μ and H. These parameters are summarized as follows,

Design Criteria:

$$\mu = 0.3$$
$$H = 0.25I$$

3. In the last step, probability information of the noise, delay and missing data is given. We assume that the measurement data is received at the central control with a missing rate of 80%, which means that $\bar{\gamma} = 0.8$ and delay is bounded as $.01 < \tau < .1$. We also assume that noise with zero mean and a variance of 1 affect the process model.

Matlab LMI toolbox is used to implement the LMI (4.48). After solving the design problem using the theorem 6 filter gain is found as,

$$L = \begin{bmatrix} 0.0057 \\ 0.0048 \\ 0.0085 \end{bmatrix}$$

To show the accuracy of the filter, the four following cases are considered here:

In the first case, the probability of missing information in the channel assumed to be 0.8 and no delay and noise considered. In the second case, there is a delay in the communication network, but no missing information and no noise is considered. In the third case, noise with a variance of 1 affects the performance of the network. In the forth case probability of missing information is 0.8, delay is asumed to be in the range mentioned and also noise affects the communication network.

1. $\bar{\gamma} = 0.8$, $\tau = 0$ and $\sigma = 0$
2. $\bar{\gamma} = 1$, $0.01 < \tau < .1$ and $\sigma = 0$
3. $\bar{\gamma} = 1$, $\tau = 0$ and $\sigma = 1$
4. $\bar{\gamma} = 0.8$, $0.01 < \tau < .1$ and $\sigma = 1$

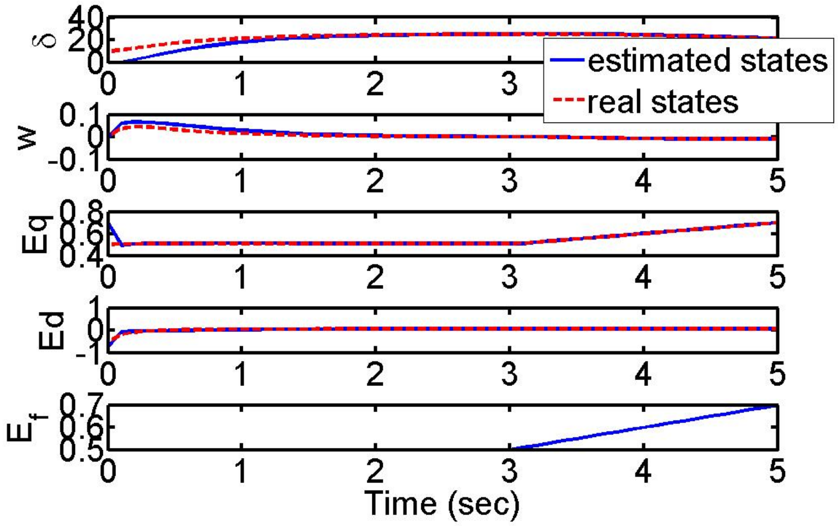


Figure 4.6: State estimation with time-varying E_f

In Fig. 4.5 it is shown that states converge to the real ones for these four cases when the field voltage is fixed.

Convergence of the filter states when the field voltage is time-varying is depicted in Fig. 4.6. As it can be seen, field voltage is fixed for the first 3 seconds and starts increasing from 0.5 with the slope 0f 0.1 in both synchronous generator and filter and it is shown that the filter states track the real ones.

4.5 Conclusion

In this chapter, dynamic state estimation for a class of Lipschitz nonlinear system is presented. Estimating the unknown states of the system is carried out using the measurements available to the filter which come through a wireless channel. The delay in transmitting the data to central units and also packet drop of the information is considered in the proposed method. Simulation results show the efficiency and accuracy of the proposed filter to estimate the unknown states in the case of uncertainty in the linear part of the dynamic system.

Chapter 5

Summary and Conclusions

5.1 Summary and Conclusions

To monitor, control, and diagnose faults in practical applications, having access to instantaneous values of internal states or physical variables play a crucial role. However, due to technical issues and the cost of installing the sensors, part of the information is not available. Hence, these unknown quantities must be estimated and it can be carried out by dynamical systems called observers. Observers utilize the input and output information of the process combined with the known system model to estimate the unknown values. The key work here is to guarantee that under some conditions the estimated states converge to real ones. Filtering is the other topic in the area of estimation which tries to estimate the state of the nonlinear systems in the presence of noise. Different structures and also tools for filter design in the system has been developed. All these try to find a solution which is less conservative and also valid in a large region around an equilibrium point.

In this thesis two types of nonlinear systems for the filter and observer design are considered. first, Lipschitz nonlinear systems which is one of the well known systems is introduced. This type of nonlinearity has been used by many researchers in developing observers and filters. Second, a new general family of nonlinear systems, one-sided Lipschitz systems, has been introduced which can cover a wide range of nonlinearity in systems.

In the first chapter continuous time nonlinear $\mathcal{H}_\infty$ filter design for one-sided Lipschitz systems is proposed. Full order and reduced order filter design is formulated in the form of LMIs. The use of one-sided Lipschitz systems represents a nontrivial, significant extension of similar results existent in the literature for Lipschitz systems. The one-sided Lipschitz assumption is important in that (i) every Lipschitz system is also one sided-Lipschitz, and (ii) the one-sided Lipschitz constant is less than or at most equal to the Lipschitz constant. The combined effect of these two properties guarantees less conservative results than those found using the, more established, theory developed for Lipschitz systems. Our result maximizes the one-sided Lipschitz constant as one of the design goals, a property that provides some robustness properties with respect to nonlinear uncertainties. Moreover, decreasing the effect of noise in the estimation problem is considered by adding an extra term in the cost function presented in this chapter. Simulation results on an unstable nonlinear system show the proficiency of the proposed method in finding the robust solution and decreasing the noise effect.

In the second chapter, we develop a new theorem based on LMI approach for dynamic state estimation for a class of Lipschitz nonlinear system. In the proposed method estimation of the unknown states is carried out using the measurements available to the filter provided by wireless communication to the control unit far from the practical system. The delay in transmitting the data sent to central units and also packet drop of the information which occurs in communication network presented and also considered in the proposed method. The existence of delay in practical system affects the performance of the systems and may cause the instability of real application. Moreover, failure of the communication network for a short period of time may influence the accuracy and reliability of the online application. Considering these two common phenomenon guarantees the stability and reliability of the proposed method. Synchronous generator connected to the infinite bus is a practical system chosen here to show the accuracy and proficiency of the suggested algorithm. Simulation results show that the estimated states are asymptotically converge to the real states which can be utilized for monitoring and control application and also ensure the stability and reliability of the distribution network.

5.2 Suggestions for Future Work

In the One- Sided Lipschitz nonlinear observer design presented in the second chapter, symmetric positive definite matrix P is one of the design variables of the problem. The reason is that the inner product of function $f = P\phi$ and error, e, appear in formulating the Lyapunov function and applying the One-Sided Lipschitz condition as defined in chapter 3 easily reduces the Lyapunov function to a closed form which can be used in forming the LMIs. Function f is an auxiliary nonlinear function while the real nonlinear function existent in the system is ϕ. It would be more practical if just the nonlinear function ϕ is considered in the One-sided Lipschitz condition. However, many works have been done based on defining the One-sided Lipschitz condition as presented in chapter 3. The next step is to find a way of design in which auxiliary function f equal to ϕ and design problem just considers the nonlinearity in the real application. Considering this fact, filter design for both discrete-time and continuous-time One-sided Lipschitz nonlinear systems combined with time delay and missing information can be investigated in this step. Since a wide range of systems can be modelled by this class of nonlinearity, it would be close to the behaviour of the practical application.

Estimation of unknown inputs or disturbances in practical application is one of the topics considered by researchers in the area of Lipschitz nonlinear systems so far. Using adaptive law for the input signal combined with the dynamic system is the common way to estimate the unknown input. Applying this approach for this type of observers can be considered especially for One-Sided Lipschitz nonlinear system. Uncertainty, delay and missing information can be added to the nonlinear model which can be more practical.

Reliability and stability of power distribution network is becoming more challenging since it is getting larger and larger. Having access to all the voltages and currents phasors makes it possible to monitor the stability of the network and take action in critical situation. In the last chapter, state estimation of a synchronous generator connected to the infinite bus was performed. Instead of having the infinite bus, a large distribution network consisting of lots of generators and different kinds of load can be considered. State estimation of each bus in the network can be carried out while interacting with other elements of the network.

In today's network, distribution networks are divided into smaller partitions and state estimation for each part is performed locally in the local control unit. Information provided in substations are transmitted to local control units by communication networks. Uncertainty, delay, network failure and information leakage are common in networks. Dealing with these types of failures and considering them in the communication between substations and control units must be presented in the models provided for state estimation. Moreover, the way in which each partition communicates with the neighbors affects the online estimation of voltages and currents phasors and still is an open area.

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