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Tropical Geometry and Kapranov’s Theorem

by

Guillermo Javier Cuevas Pineda

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Abstract

The tropical variety of a function \( f = \sum c_u x^u \in K[x_1, \ldots, x_n] \) is the set of points \( r \in \mathbb{R}^n \) where the minimum of \( \text{val}(c_u) + \langle r, u \rangle \) is attained at least twice.

We define the initial form of a function with respect to a vector \( r \), \( \text{in}_r(f) \), to be the sum of the monomials of \( f \) with least valuation. Our main focus is on the points \( w \) where the initial form is not a monomial. Such points can be lifted to points \( z \in (K^*)^n \) satisfying \( f(z) = 0 \) and \( \text{val}(z) = w \). This result is known as Kapranov’s Theorem and has been previously proved using geometric properties, field extensions, and different definitions of what an initial form is (including the one given above).

In this thesis we prove Kapranov’s theorem using the aforementioned definition of an initial form and its algebraic properties.

Keywords: Tropical Geometry; Initial forms; Kapranov’s Theorem.
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Introduction

Tropical geometry, a relatively new branch in the realm of Algebraic Geometry, has proved its usefulness by connecting well-known real objects to more complicated topological structures with harder-to-analyze algebraic properties. For instance, Mikhalkin [Mik03] used it to get a new way for computing the number of irreducible curves of degree $d$ and genus $g$ passing through a general set of points $(z_1, \ldots, z_{3d-1+g})$ in the complex projective plane.

On the algebraic side, tropical geometry represents a limit process of an amoeba. Ilia Itenberg et. al. [IMS09] show us the path. By definition, an amoeba is the logarithm of the module of points in the zero set of a function. For a function $f : (\mathbb{C}^*)^n \to \mathbb{C}$,

\[ A(V(f)) := \log V(f) = \left\{ (\log |z_1|, \ldots, \log |z_n|) \in \mathbb{R}^n : f(z) = 0 \right\}. \]

As a second step, allow the coefficients of $f$ to be dependent on one variable $t$, that is, $f \in \mathbb{C}(t)[z_1, \ldots, z_n]$. In this setting the amoeba is defined as

\[ A_t(V(f)) := \log_t V(f_t) = \left\{ (\log_t |z_1|, \ldots, \log_t |z_n|) \in \mathbb{R}^n : f_t(z) = 0 \right\}, \]

where $f_t$ is obtained from $f$ by evaluating at $t$. Moreover, as shown in the same text, if we consider the Hausdorff metric, then a limit exists. Let

\[ A_\infty(V(f_t)) := \lim_{t \to \infty} A_t(V(f_t)). \]
This structure can be generalized to a non-Archimedean field $K$ via the concept of valuation. Having a valuation like the one described in our Definition 1.21 we redefine the amoeba of $f : (K^*)^n \to K$,

$$A_K(V(f)) := \text{val} \left( V(f) \right) = \{ (\text{val}(z_1), \cdots, \text{val}(z_n)) \in \mathbb{R}^n : f(z) = 0 \}$$

It is a version of Viro’s patchworking process (see [IMS09], Theorem 1.4) that $A_\infty = A_K$. $A_K$ is known as a tropical variety. Furthermore, Kapranov showed (see Theorem 4.10) that for every algebraic variety $V$ over a non-Archimedean field $K$ we have $A_K = \text{val}(V(f))$.

Kapranov’s theorem is the main motivation of this work.

Consider $p \in V(f)$, then we have that $w := \text{val}(f(p)) = \infty$. We will prove in 1.23 that in such a case the minimum over the valuation of the monomials $f(z)$ is attained at least twice.

If we drop all the monomials $c_u z^n$ for which $\text{val}(c_u z^n) = \text{val}(c_u) + \langle \text{val}(p), u \rangle$ is not a minimum, we obtain a new polynomial called the initial form of $f$ with respect to $w := \text{val}(p)$. These initial forms are a generalization of the initial forms that can be used for proving Hilbert’s basis theorem (1.9). Therefore, for a function $f = \sum c_u x^u$ we define

$$\text{in}_w^K(f) := \sum_{u : \text{val}(c_u) + \langle w, u \rangle \text{ is minimum}} c_u x^u,$$

and we will be interested in the points where $\text{in}_w^K(f)$ is not a monomial. It is the goal of this thesis to prove Kapranov’s theorem using that definition of initial forms.

The set of points where $\text{in}_w^K(f)$ is not a monomial forms a polyhedral complex. Kapranov used that geometric property and field extensions to prove his theorem (see [EKL04], Theorem 2.1.1). As far as the author is aware, the first person proving Kapranov’s theorem
using the algebraic properties of the initial forms was Bernd Sturmfels in [SS03]. The caveat with his proofs is that he redefines the initial forms to be

\[ \text{in}_w^k(f) := \pi \left( f(t^{w_1}x_1, \ldots, t^{w_n}x_n) \right) \]

(2)

where \( t^w \) is a specific element satisfying \( \text{val}(t^w) = w \), and \( \pi: R_K \to k \) is the projection going from the valuation ring to the residue field. In this thesis we prove Kapranov’s theorem using the initial forms as defined in (1). This has been done in the past. [Dra06] and [OP13] have proved generalizations of this theorem using the definition of initial forms as in (1). Kapranov’s Theorem has also be proved using the initial forms described in (2), [MS13] contains one of those proofs.

This paper is divided in 4 chapters. The first chapter goes over most of the material that is needed for understanding the rest of the text. However, we have no pretensions in saying that the thesis is a self-contained text. In fact, some basic concepts from linear and abstract algebra are assumed.

Chapter 2 explores the initial forms as defined in (2). While that definition might be (arguably) considered unnatural, it has several advantages. In particular, \( \text{in}_w(fg) = \text{in}_w(f)\text{in}_w(g) \), and the proof of Kapranov’s theorem is more algorithmic. The product rule allows a direct definition of what an initial ideal is. It also facilitates some computations. Further, the proof of Kapranov’s theorem presents a constructive way for finding an associated root \( z \) for a \( w \) such that \( \text{in}_w(f) \) is not a monomial.

Chapter 3 constructs the results obtained in Chapter 2, but it uses the initial forms defined in (1). Although not every result from the previous chapter was generalizable, all the main results (that is, the results that enable the proof of Kapranov’s theorem) were. When a result was no longer true under the new definition, a counterexample is presented.

Finally, on Chapter 4 we present the proof of Kapranov’s theorem. We first introduce the concept of a tropical variety as the set of points \( r \in \mathbb{R}^n \) where the minimum of \( \text{val}(c_r) + \langle r, u \rangle \)
is attained at least twice. Then we proof that in such a set, the set of points $r$ where the initial form is not a monomial, and the valuation of the points on the algebraic variety of $f$ coincide.
Chapter 1

Background

This chapter goes over the material that will be needed to construct the tropical objects which are the core of this thesis. Most of this is part of commutative ring theory. In fact, throughout the text, we only work with commutative rings, and so we omit the word commutative. In the end of this chapter we cover some geometric concepts that will also be necessary.

1.1 Semirings and Rings

Let $R$ be a set with at least two distinct elements, in which we have two binary operations, $+, \ast$. We call the first operation a sum and the second one a product.

**Definition 1.1 (Semiring).** $R, +, \ast$ is a semiring if its binary operations satisfy:

+ $1$ $\forall a, b \in R, a + b = b + a.$

+ $2$ There is a neutral element $0 \in R$ such that $\forall a \in R, a + 0 = 0 + a = a.$

+ $3$ The sum is associative, $\forall a, b, c \in R, (a + b) + c = a + (b + c).$

+ $1$ $\forall a, b \in R, a \ast b = b \ast a.$
There is an identity element \( 1 \neq 0 \in R \) such that \( \forall a \in R, a \ast 1 = 1 \ast a = a \).

The product is associative, \( \forall a, b, c \in R, (a \ast b) \ast c = (a \ast b) \ast c \).

The operations are distributive, \( \forall a, b, c \in R, a \ast (b + c) = a \ast b + a \ast c \).

It is customary to drop the product symbol \( \ast \), and so will we do in most of the cases.

We now present the only relevant example for this text.

**Example 1.2 (Tropical Semiring).** Consider the set, \( \mathbb{T} := \mathbb{R} \cup \{\infty\} \), we define the following operations:

\[
\begin{align*}
\forall a, b \in \mathbb{T} & \quad \text{“} a + b \text{”} = \min\{a, b\}. & (1.1) \\
\forall a, b \in \mathbb{T} & \quad \text{“} a \ast b \text{”} = a + b. & (1.2)
\end{align*}
\]

Where the operations in the RHS represent the usual operations in the real numbers. \( \mathbb{T}, “ + ”, “ \ast ” \) is a Tropical Semiring.

We now examine the dequantization process, a process that in the limit leads to the tropical semiring.

**Example 1.3 (Dequantization process, taken from [MZ07]).** Take \( a, b \in \mathbb{R} \cup \{-\infty\}, \) and \( t > e \). We define the operations:

\[
\begin{align*}
+_{t} : a +_{t} b & \quad := \log_{t}(t^{a} + t^{b}) & \quad (1.3) \\
\ast_{t} : a \ast_{t} b & \quad := \log_{t}(t^{a}t^{b}) = a + b & \quad (1.4)
\end{align*}
\]

\(+_{1}\) is trivial. In this semiring \( 0_{t} = -\infty \), so \(+_{2}\) is well defined. Indeed, \( \log(0) = -\infty = 0_{t} \).

Every product property is trivial. The sum is associative because:

\[
(a +_{t} b) +_{t} c = \left( \log_{t}(t^{a} + t^{b}) \right) +_{t} c = \log_{t}(t^{\log_{t}(t^{a} + t^{b}) + t^{c}}) = \log_{t}(t^{a} + t^{b} + t^{c}).
\]
The dequantization process converges in the max-plus algebra that is defined now.

**Definition/Example 1.4 (Max-Plus algebra).** Consider the set, \( T_{\text{max}} := \mathbb{R} \cup \{-\infty\} \), we define the following operations:

\[
\begin{align*}
\forall a, b \in T_{\text{max}} \quad a +_{\text{max}} b &= \max\{a, b\}. \quad (1.5) \\
\forall a, b \in T_{\text{max}} \quad a *_{\text{max}} b &= a + b. \quad (1.6)
\end{align*}
\]

The max-plus algebra defines a semiring.

**Proposition 1.5.** The tropical semiring and the max-plus algebra are isomorphic.

*Proof.* Indeed, as sets, \( \mathbb{T} \) and \( T_{\text{max}} \) are almost identical. Define the isomorphism \( \phi : T_{\text{max}} \rightarrow \mathbb{T} \) via \( \phi(x) = -x \). The product rules remain the same in both semirings, so we only need to check the addition,

\[
\phi(a +_{\text{max}} b) = -\max\{a, b\} = -\min\{-a, -b\} = -\phi(a) + \phi(b)
\]

\[
\square
\]

Depending on the purpose of the paper, or even on the author, the max-plus algebra is called the tropical semiring. Conversely, what we have called in here as tropical semiring would be referred as “min-plus” algebra.

**Proposition 1.6.** The limit of the dequantization process is the max-plus algebra. That is, \( \mathbb{R} \cup \{\infty\}, +, *_{\infty} = T_{\text{max}} \).

*Proof.* If either \( a, b \) or both equal \(-\infty\) the proof is immediate. If \( a = b \) then

\[
\lim_{t \to \infty} \log_t (t^a + t^b) = a + \lim_{t \to \infty} \log_t (2) = a
\]

\[
7
\]
Now assume that \( a < b \), so \( 0 < t^{a-b} < 1 \). From \( \lim_{t \to \infty} \log_t(t^a + t^b) = b + \lim_{t \to \infty} \log_t(t^{a-b} + 1) \) we need to show \( \lim_{t \to \infty} \log_t(t^{a-b} + 1) = 0 \). We bound that limit:

\[
\begin{align*}
0 & \leq t^{a-b} \leq 1 \\
\Rightarrow 1 & \leq 1 + t^{a-b} \leq 2 \\
\Rightarrow 0 & \leq \log_t(1 + t^{a-b}) \leq \log_t(2) \quad \xrightarrow{t \to \infty} 0.
\end{align*}
\]

\[\Box\]

**Definition 1.7 (Ring).** \( R, +, * \) is a ring if it is a semiring with additive inverse:

+4 For every \( a \in R \) there exists \( -a \in R \) such that \( a + (-a) = a - a = 0 \).

The tropical semiring is a semiring because \( \infty \) works as \( +2 \), but it cannot be a ring because for a finite element \( a \), \( \min\{a, b\} \) will always be finite.

### 1.2 Polynomial rings

For \( u \in \mathbb{Z}^n \) we write \( x^u = x_1^{u_1} \cdots x_n^{u_n} \). The **polynomial ring** over a ring \( R \) as

\[
R[x_1, \ldots, x_n] := \left\{ \sum_{\text{finite}} r_u x^u : r \in R, u \in \mathbb{N}^n \right\}.
\]  

(1.7)

If, in addition, we provide our set of variables with multiplicative inverses, the ring is called the **Laurent polynomial ring**, denoted as

\[
R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] := \left\{ \sum_{\text{finite}} r_u x^u : r \in R, u \in \mathbb{Z}^n \right\}.
\]  

(1.8)

**Definition 1.8 (Support of a function).** Let \( f \) be a polynomial function over a ring \( R \). The
support of the function is the set

\[ \text{supp}(f) := \{ u \in \mathbb{Z}^n : r_u \neq 0 \} \]

The following well-known result will be extensively used in the following chapters. We take the definition from [Rot02].

**Theorem 1.9 (Hilbert Basis Theorem).** Let \( R \) be a Noetherian commutative ring. Then \( R[x] \) is also Noetherian.

Although the result is true for any ring, the proof that we present works only for fields, so we will assume that \( R \) is a field, denoted by \( K \). The reason to do that is that our proof will better suit the purpose of the next chapters. A general proof can be found in Rotman [Rot02], as Theorem 6.42. For our proof we will need to define the initial term of a polynomial.

Let

\[ f = \sum_{u \in \mathbb{Z}^n} c_u x^u. \]

We say that \( x^u \preceq x^{u'} \) if \( u \preceq u' \), where the ordering is the lexicographic order, that is, for some \( i, u_i < u'_i \) and for every element \( j < i, u_j = u'_j \). This is a total ordering, and the monomial satisfying \( x^u \succ x^{u'} \forall u' \neq u \) is called the **lexicographic initial term** of \( f \).

**Definition 1.10 (Initial form).** The **initial form** of a function

\[ f = \sum_{u : c_u \neq 0} c_u x^u \in K[x_1, x_2, \cdots, x_n] \]

with respect to the lexicographic order \( \succ \) is

\[ \text{in}(f) := c_u x^u \text{ such that } x^u \succ x^{u'} \forall u' \neq u. \]
Example 1.11. Consider \( f(x, y) = x^3 + 3x^2y^5 + 13xy^2 \).

\[
in(f) = x^3.
\]

We now introduce the concept of initial ideal.

**Definition 1.12 (Initial ideal).** The Lexicographic initial ideal, \( \text{in}(I) \), is defined as:

\[
in(I) := \langle \text{in}(f) : f \in I \rangle.
\] (1.9)

**Proposition 1.13.** Let \( f = \sum_i c_{ui} x^{ui}, \text{in}(f) = c_{u_0} x^{u_0} \) and \( m = dx^v \) be a monomial. Then \( \text{in}(mf) = dc_{u_0} x^{v + u_0} = \text{min}(f) \).

**Proof.** By hypothesis, there exists \( j \in \mathbb{N} \) such that \( u_{0,1} = u_{i,1}, \ldots, u_{0,j-1} = u_{i,j-1}, u_{0,j} > u_{i,j} \) for \( i \neq 0 \). Therefore, \( v_1 + u_{0,1} = v_1 + u_{i,1}, \ldots, v_j - 1 + u_{0,j-1} = v_j - 1 + u_{i,j-1}, v_j + u_{0,j} > v_j + u_{i,j} \). The conclusion follows from there.

We now prove the existence of a division algorithm within this context.

**Proposition 1.14 (Division algorithm).** Let \( I = \langle f_1, \ldots, f_m \rangle \subset S := K[x_1, \ldots, x_n] \), and \( g \in S \). Then, there exists \( a_1, \ldots, a_m \in S \) such that \( g = \sum a_i f_i + r \), where \( r \in S \), \( \text{in}(r) < \text{in}(g) \).

**Proof.** The following algorithm finds the \( a_i \)'s.

1. Set \( g_0 := g, r_0 = 0, i = 0, a_{j,k} = 0 \forall j, k \in \mathbb{N} \).

2. Set \( j = 1 \).

3. If \( \text{in}(f_j) \) divides \( \text{in}(g_i) \) (trivially implying \( \text{in}(f_j) < \text{in}(g_i) \)), set \( a_{j,i} := \sum_{k=0}^i a_{j,k} + \text{in}(g_i)/\text{in}(f_j), r_i = 0, i = i + 1, g_i := g_{i-1} - (\text{in}(g_{i-1})/\text{in}(f_j))f_j \). That eliminates the
leading term of \( g_{i-1} \), thus, \( \text{in}(g_{i-1}) \prec \text{in}(g_i) \). Now go to 2. If \( \text{in}(f_j) \) does not divide \( \text{in}(g_i) \) go to the next step.

4. If \( j < m \), increase it, \( j = j + 1 \) and go to 3. If \( j = m \), we strip \( g \) from its initial term. Set \( i = i + 1 \), \( r_i = \sum_{k=0}^{i-1} r_k + \text{in}(g_i-1), g_i = g_{i-1} - \text{in}(g_{i-1}) \). If \( g_i \neq 0 \) go to 2, otherwise, go to the next step.

5. \( g = \sum_{i,j} a_{j,i} f_j + r_i \). Notice that the process is finite because \( g \) and \( f_j \) contain only finitely many monomials. Moreover, every monomial of \( r = \sum r_i \) is not divisible by any of the initial forms of the \( f_j \), and due to the observation in step 2, \( \text{in}(r) \ll \text{in}(g) \).

\[ \square \]

**Definition 1.15.** A monomial ideal, \( I \), is an ideal that is generated by monomials, \( I = (x^u : u \in \mathbb{N}^n) \).

We use the algorithm to prove Dickson’s Lemma.

**Lemma 1.16 (Dickson’s Lemma).** Every monomial ideal is finitely generated.

**Proof.** The proof is by induction over the number of variables. If \( n = 1 \), by the well ordering principle, there is a minimum \( u \) such that \( x^u \in I \). Then \( \langle x^u \rangle = I \). Now for \( n + 1 \) variables, let \( u = (u_n, u_{n+1}) \in \mathbb{N}^{n+1} \) with \( u_n \in \mathbb{N}^n, x^u \in I \). Let \( A = \{ u_n \in \mathbb{N}^n : \exists x^{(u_n, u_{n+1})} \in I \} \), and \( J := \langle x^v : v \in A \rangle \). By the inductive hypothesis, \( J \) is finitely generated. Each generator of \( J \) matches with generators of \( I \), that is, \( x^v \in J \Rightarrow x^{v_1, \ldots, v_m} \in I \), where \( v_1 \) is the minimal element guaranteed by the well ordering principle.

Let \( v_0 \) be the maximum of \( v_1 \). For each \( i \leq v_0 \) let \( J_i = \langle \{ x^{v_n} : x^{(v_n,i)} \in I \} \rangle \), then \( J_i \) is finitely generated. For each generator \( x^v \) of \( J_i \) add \( x^{(v,i)} \) to the list of generators. Then \( I \) is generated by all the \( x^{(v_n,v_1)} \) and \( x^{(v,i)} \). If \( f = x^{(v_n,v)} \in I \) then it is divisible by one of the \( x^{(v_n,v)} \) if \( v \geq v_0 \). Otherwise \( f \) is divisible by one of the \( x^{(v,i)} \).

\[ \square \]

Using Dickson’s Lemma we finally prove the basis theorem.
Proof of Theorem (1.9). By Dickson’s Lemma, in($I$) is finitely generated. Let $f_1, \cdots, f_m \in I$ be such that $\text{in}(f_i)$ are the generators of $\text{in}(I)$. Our claim is that $I = \langle f_1, \cdots, f_m \rangle$. In fact, for every $g \in I$, if we apply the Division Algorithm, $g = \sum a_i f_i + r$. Since $f_i \in I \Rightarrow r \in I \Rightarrow \text{in}(r) \in \text{in}(I)$. But by construction, no monomial in the remainder is divided by $\text{in}(f_i)$, and because those are the generators, that implies $r = 0$. So, $f$ is finitely generated.

Notice that the ring of polynomials is not complete, that is, its sequences might not converge in it. For example, $1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3, \cdots$ is a sequence that converges in an infinite series, that is, outside of $R[x]$. The completion is known as the ring of formal power series.

Definition 1.17. The ring of formal power series is the completion of a ring with respect to a certain ideal [Eis04]. In our case, it is the completion of $R[x_1, \cdots, x_n]$ with respect to the ideal $I = \langle x_1, \cdots, x_n \rangle$, denoted by $R[[x_1, \cdots, x_n]]$.

Definition 1.18. A homogeneous polynomial of degree $d$ is a member of the set

$$\left\{ \sum_{\text{finite}} r_u x^u : r \in K, u \in \mathbb{Z}^{n+1}, \sum u_i = d \right\}.$$

Notice that if $f$ is a homogeneous function of degree $d$, then, for every scalar $r \in K, f(rx_0, \cdots, rx_n) = r^d f(x_0, \cdots, x_n)$.

Also, the homogeneous polynomials of fixed degree form a vector space, as the sum of polynomials do not affect the degree. If we denote by $S = K[x_0, \cdots, x_n]$, then the vector space of homogeneous functions of degree $d$ is denoted as $S_d$.

A homogeneous ideal is an ideal generated by homogeneous functions. Notice that it can contain non-homogeneous polynomials. The projection of a homogeneous ideal $I \subset K[x_0, \cdots, x_n]$ into the vector space of degree $d$ is denoted by $I_d$. As with any finite vector space, we can work with its dimension.
Definition 1.19 (Hilbert function). The Hilbert Function of a homogeneous ideal $I$ is defined as follows:

\[ H_I(d) := \dim(I_d). \]

This definition will come in handy in the next two chapters when we will prove the equality of two ideals using the following proposition.

Proposition 1.20. Let $I \subset J$ be two homogeneous ideals. $I = J \iff H_I(d) = H_J(d) \forall d \in \mathbb{N}$.

Proof. $(\Rightarrow)$ If $I = J \Rightarrow I_d = J_d \Rightarrow \dim(I_d) = \dim(J_d) \Rightarrow H_I(d) = H_J(d)$.

$(\Leftarrow) I \subset J \Rightarrow I_d \subset J_d \Rightarrow H_I(d) \leq H_J(d)$. Since by hypothesis $H_I(d) = H_J(d)$, then we know that both $I_d, J_d$ are spanned by the same number of elements, and $I_d \subset J_d$ implies that the spanning set of $I_d$ spans a subspace of $J_d$. By the dimension theorem, since they are of same dimension, we have $I_d = J_d$ for every $d$. Finally, we conclude $I = \bigoplus_d I_d = \bigoplus_d J_d = J$.

So far, we have not worked with the coefficients of the polynomials. However, we want to provide them with an algebraic structure that will allow us to generalize the concept of initial forms, and eventually, connect everything with Tropical Geometry.

### 1.3 Valuation rings

Definition 1.21 (Valuation). Let $K$ be a field, and $K^\times$ its set of units. A valuation on $K$ is a function $\text{val} : K \rightarrow \mathbb{R} \cup \{\infty\}$ satisfying the following three axioms

1. $\text{val}(a) = \infty$ if and only if $a = 0$.

2. $\text{val}(ab) = \text{val}(a) + \text{val}(b)$.

3. $\text{val}(a + b) \geq \min\{\text{val}(a), \text{val}(b)\}$ for all $a, b \in K^\times$.

Proposition 1.22. If $\text{val}(a) \neq \text{val}(b) \Rightarrow \text{val}(a + b) = \min\{\text{val}(a), \text{val}(b)\}$. 
Proof. Assume \( \text{val}(a) < \text{val}(b) \). The proof follows immediately from \( \text{val}(a) = \text{val}(a+b-b) \geq \min\{\text{val}(a+b), \text{val}(b)\} \geq \min\{\min\{\text{val}(a), \text{val}(b)\}, \text{val}(b)\} = \text{val}(a) \).

\[
\square
\]

**Corollary 1.23.** If \( \infty = \text{val}(a_1 + \cdots + a_n) > \min\{\text{val}(a_1), \cdots, \text{val}(a_n)\}, n \geq 2 \) and not all the terms are zero, then the minimum is attained by at least two of the summands in the LHS.

**Proof.** In the aim of a contradiction, let us assume the opposite. Without loss of generality, let us assume \( v_1 := \min\{\text{val}(a_1), \cdots, \text{val}(a_n)\} = \text{val}(a_1) \neq \infty \) is the only minimum. Then

\[
v_1 = \min\{\text{val}(a_1), \cdots, \text{val}(a_n)\} < \min\{\text{val}(a_2), \cdots, \text{val}(a_n)\} \leq \text{val}(a_2 + \cdots + a_n).
\]

Then, by the previous proposition \( \text{val}(a_1 + \cdots + a_n) = \infty = \min_i\{\text{val}(a_i)\} = \text{val}(a_1) \), which is a contradiction.

\[
\square
\]

**Example 1.24.** Consider \( \mathbb{C}[[t]] \). For an element \( f = \sum c_u t^u \in \mathbb{C}[[t]] \), we define \( \text{ord}(f) := \min\{u : c_u \neq 0\} \). That is called the order of \( f \). The order is a valuation.

**Proof.** We evaluate the three conditions:

1. If \( \min\{u : c_u \neq 0\} = \infty \), then every \( c_u = 0 \). Therefore, \( f = 0 \). On the other hand, if \( f = 0 \), \( \min\{u : c_u \neq 0\} = \infty \)

2. Let \( f, g \in \mathbb{C}[[t]], \text{ord}(f) = u, \text{ord}(g) = v \) and denote \( f = c_u t^u + \cdots, g = c_v t^v + \cdots \).

Then, \( fg = c_u c_v t^{u+v} + \cdots \). Notice that since \( u, v \) are the minimal terms, no other coefficient with associated exponent \( u + v \) will appear. This proves property 2.

3. If \( f, g \) are as in 2, if \( \text{ord}(f) < \text{ord}(g) \Rightarrow \text{ord}(f + g) = \text{ord}(f) \). Moreover, if \( \text{ord}(f) = \text{ord}(g) \Rightarrow \text{ord}(f + g) \geq \text{ord}(f) \). The strict inequality may happen when \( u = v, c_u = -c_v \). This also exemplifies Proposition 1.22.

\[
\square
\]
The former is an example of a discrete valuation, that is, a valuation that only takes integer values. We will be interested in valuations that are dense in \( \mathbb{R} \). We first see an example of a field with such a valuation.

**Definition 1.25.** The field of Puiseaux series is the algebraic closure of \( \mathbb{C}((t)) \). It can be shown that it is equivalent to:

\[
\mathbb{C}\{t\} := \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n})).
\]

The proof that it is the algebraic closure of \( \mathbb{C}((t)) \) goes beyond the material we want to cover, but a proof can be found in Corollary 13.15 of Eisenbud [Eis04].

**Example 1.26.** \( \mathbb{C}\{t\} \) with \( \text{ord} \) defined as in Example 1.24 forms a dense valuation in \( \mathbb{R} \).

**Proof.** That it forms a valuation is immediate from Example 1.24. If we require the series to have an initial term, then the valuation is completely contained in \( \mathbb{R} \). Observe that \( t^{m/n} = \prod_{i=1}^{m} t^{1/n} \in \mathbb{C}((t^{1/n})) \). Also, for every \( q = \frac{m}{n} \in \mathbb{Q} \), \( q = \text{ord}(t^q) \) showing that \( \text{val}(K^*) \) contains \( \mathbb{Q} \).

\[\square\]

**Definition 1.27.** The local ring \( R_K \) is defined as follows

\[ R_K = \{ x \in K : \text{val}(x) \geq 0 \}. \]

Its maximal ideal is given by

\[ m_K = \{ x \in K : \text{val}(x) > 0 \}. \]

**Example 1.28.** Consider the ring of Laurent polynomials \( \mathbb{C}[x^{\pm 1}] \), with valuation given by \( \text{ord} \).
\[ R_K = \sum a_i x^i, i \geq 0, \]

\[ m_K = \sum a_i x^i, i > 0, \]

\[ R_K / m_K = \sum a_i x^i, i = 0 \cong \mathbb{C}. \]

The quotient field, \( k := R_K / m_K \), is known as residue field.

**Proposition 1.29.** \( R_K \) is integrally closed.

**Proof.** Assume that \( a \in K \) satisfies the following equation:

\[ a^n + c_{n-1}a^{n-1} + \cdots + ca + c_0 = 0 \]

with \( c_i \in R_K \) and \( n \geq 1 \). If \( a \in R_K \) there is nothing to show. Assuming the opposite, then \( a^{-1} \in R_K \), and so \( a^{1-n} \in R_K \). Also,

\[ a + c_{n-1} + \cdots + ca^{2-n} + c_0a^{1-n} = 0 \Rightarrow a = -c_{n-1} - \cdots - c_0a^{1-n} \in R_K \]

\( \square \)

**Proposition 1.30.** Let \( K \) be an algebraically closed field with non-trivial valuation. Then \( k \), the residue field, is also algebraically closed.

**Proof.** Let \( f \in k[x] \). Since \( k \) is a field, without loss of generality assume \( f = \sum a_i x^i \) is monic (so \( a_n = 1 \)). Moreover, \( a_i \in k \) implies that \( a_i = a_i + m \in R_K \). Since the valuation ring is integrally closed, \( g := x^n + \sum_{i=0}^{n-1}(a_i + m)x^i \) has all its roots in \( R_K \), and so, all the roots \( a + m \) are roots for \( f \).

\( \square \)

If we apply the valuation to the non-zero elements of the field, denoted by \( K^* \), we obtain the value group, which we denote by \( \Gamma_{val} := val(K^*) \). It is indeed a group under addition.
It is called divisible if for every $\gamma \in \Gamma_{val}$ and all positive integers $n$, there exists $\gamma'$ such that $n\gamma' = \gamma$.

**Proposition 1.31.** Let $K$ be an algebraically closed field with non-Archimedean valuation. $\Gamma_{val}$ is a subgroup of $(\mathbb{R}, +)$.

Proof. $0 = \text{val}(1) \in \Gamma_{val}$. Let $x = \text{val}(X), y = \text{val}(Y)$, then $\text{val}(XY) = \text{val}(X) + \text{val}(Y) = x + y \in \Gamma_{val}$. Finally, since $K$ is a field, $\gamma \in \Gamma_{val}$ implies the existence of $g \in K$ such that $\text{val}(g) = \gamma$, and $0 = \text{val}(1) = \text{val}(gg^{-1})$ implies that $-\gamma \in \Gamma_{val}$. \hfill \Box

**Proposition 1.32.** Let $K$ be an algebraically closed field with non-Archimedean valuation. Then $\Gamma_{val}$ is a divisible group.

Proof. Let $\gamma = \text{val}(a) \in \Gamma_{val} \Rightarrow \gamma' = \text{val}(a^{1/n})$. We know that $a^{1/n} \in K$ because $K$ is algebraically closed. \hfill \Box

Note that also $K^*$ is a divisible group if $K$ is algebraically closed.

**Proposition 1.33.** Let $K$ be an algebraically closed field with non-trivial valuation and divisible value group. Then $\Gamma_{val}$ is dense in $\mathbb{R}$.

Proof. Let $\gamma = \text{val}(a) \neq 0$. Then for all integers $m, n \neq 0, \frac{m}{n} \gamma = \text{val}(a^{m/n}) \in \Gamma_{val}$. Hence we found a copy of $\mathbb{Q}$ in the value group. Therefore the value group is dense. Moreover, if $1 \in \Gamma_{val}$, then $\mathbb{Q} \subset \Gamma_{val}$. \hfill \Box

We will always assume that $\mathbb{Q} \subset \Gamma_{val}$.

**Proposition 1.34** (Proposition 2.1.15 in [MS13]). Let $K$ be an algebraically closed field with a non-trivial valuation with divisible value group, then there exists a group homomorphism $\varphi : \Gamma_{val} \rightarrow K^*$ such that $\text{val}(\varphi(w)) = w$. 

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Proof. In Proposition 1.33 we found that $\Gamma_{val}$ contains a subgroup isomorphic to $\mathbb{Q}$ in it. We now split the proof in several steps.

1. $\Gamma_{val}$ is torsion free. ("a group $G$ is torsion free, or without torsion, if whenever an element $x$ of $G$ has finite period, then $x$ is the unit element" [Lan93] p.45). Let $n\gamma = 0$ for $\gamma = \text{val}(g) \in \Gamma_{val}$. Therefore, $n\gamma = 0$ for $n\gamma \in \mathbb{R}$. Either $n = 0$, or $\gamma = 0$.

2. $\Gamma_{val}$ is a vector space over $\mathbb{Q}$. We show that it is a vector subspace of $\mathbb{R}$. In Proposition 1.31 we showed that $0 \in \Gamma_{val}$ and that it is closed under addition. Proposition 1.32 shows that it is closed under rational multiplication.

3. Since $\Gamma_{val}$ is (isomorphic to) a $\mathbb{Q}$–vector space, every element $\gamma \in \Gamma_{val}$ can be expressed as finite linear combination of elements in a basis. Let $\{\gamma_b\}_{b \in B}$ be one basis. For every $b \in B$ let $\varphi(b) \in K$ be an element with $\text{val}(\varphi(b)) = b$. Then $\varphi$ extends uniquely to the $\mathbb{Z}$–span of $B$ by $\varphi \left( \sum n_i b_i \right) = \prod \varphi(b_i)^{n_i}$. Since $K^*$ is divisible it is injective and $\varphi$ extends to a group homomorphism. By 1, and because $\text{val}(\varphi(b)) = b$ for all $b \in B$, it follows that $\text{val}(\varphi(x)) = x$ for all $x \in \Gamma_{val}$.

We write $tw$ instead of $\varphi(w)$ where $t = \varphi(1)$.

**Proposition 1.35.** Let $K$ be an algebraically closed field with non-trivial valuation and divisible value group $\Gamma_{val}$. For any $\gamma \in \Gamma_{val}$ there are infinitely many elements in $a \in K$ such that $\text{val}(a) = \gamma$.

**Proof.** Let $\gamma \in \Gamma_{val}$. If there are only finitely many elements $u_i$ such that $\text{val}(u_i) = \gamma$, then the residue field is a finite field because for every $c \in k$ we can find $C \in K$ with $\text{val}(Cu_i) = \gamma$. That is non-sense. The residue field of an algebraically closed field is also algebraically closed, and no finite field satisfies that.

The following proposition will be useful in the last chapter.

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Proposition 1.36. Let $A \in M_{k \times n}(\mathbb{Z}), w \in \mathbb{R}^n, b \in \mathbb{Q}^k$ such that $Aw = b$. Then there exists a sequence of points $\{w_i\} \in \mathbb{Q}^n$ such that $Aw_i = b$.

Proof. The matrix $A$ can be put in the Smith canonical form (See [HO76]), such that $A = X_k \times k D_{k \times n} Y_{n \times n}$, where $X, Y$ are invertible integer matrices, $D = \text{diag}(d_i), d_1 | d_2 | \cdots | d_u, u = \min \{k, n\}$.

To solve $Ax = b$ is the same as solving $DYx = X^{-1}b$ so we may assume $A$ is diagonal. This shows $Ax = b$ has a rational solution if there is a solution at all. Since the rational points are dense in ker $A$ the claim follows. 

\[\square\]

Remark: $\mathbb{Q}$ can be replaced by any divisible subgroup containing $\mathbb{Q}$.

1.4 Geometric background

Consider the unit square in $\mathbb{R}^2$. Algebraically, it can be described as $\{(x, y) \in \mathbb{R}^2 : x \leq 1, -x \leq 0, y \leq 1, -y \leq 0\}$. In short, 

\[
\begin{pmatrix}
1 & 0 \\
-1 & 0 \\
0 & 1 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\leq
\begin{pmatrix}
1 \\
0 \\
1 \\
0
\end{pmatrix}.
\]

Every polyhedral admits a description of this form.

Definition 1.37. A polyhedron is “the intersection of finitely many closed half spaces” ([CLS11], p.190). That is,

\[P := \{x \in \mathbb{R}^n : A_{d \times n}x \leq b_{d \times 1}, a_{i,j}, b_{i,j} \in \mathbb{R}\}. \tag{1.10}\]

Example 1.38. Consider the tetrahedron described by the intersection of the coordinate planes and the plane $x + y + z = 1$. 

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The tetrahedron can be expressed as in (1.10) as:

$$\left\{ (x, y, z) \in \mathbb{R}^3 : \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$  

The oblique face of the tetrahedron satisfies \((1, 1, 1) \cdot (x, y, z) = 1\), while the rest of the inequalities remain inequalities. Formalizing,

**Definition 1.39.** A face of a polyhedron is a component of the boundary of the polyhedron \(P\) as described in (1.10), described by at least one equality,

\[ F := \{ x \in \mathbb{R}^n : \text{there exists } 1 \leq i \leq d \text{ with } A_i \cdot x = b_i, Ax \leq b \}. \]  

We now present the concept of amoeba.

**Definition 1.40 (Amoeba).** Let \( f \in \mathbb{C}[x_1, \cdots, x_n] \) and denote its variety by \( V(f) := \{ z \in (\mathbb{C}^*)^n : f(z) = 0 \} \). The amoeba of the variety is

\[ \mathcal{A}(V(f)) := \{ (-\ln|z_1|, \cdots, -\ln|z_n|) : (z_1, \cdots, z_n) \in V(f) \}. \]  

The previous definition is the same as Definition 1.1 in [IMS09], but we changed the sign. This is due to our definition of valuation.

**Example 1.41.** Consider \( f = z_1 - z_2 - 1 \). It is straightforward to see that \((i, i - 1) \in V(f)\), hence that \((-1, -\sqrt{2}) \in \mathcal{A}(V(f))\).

In this case we can see the complete amoeba in Figure 1.1 (the amoeba was plotted in Maple following a slight modification of what was presented in [AV08]).
Figure 1.1: Amoeba of $f = z_1 - z_2 - 1 \in \mathbb{C}[z_1, z_2]$
Chapter 2

Initial ideals

Throughout this chapter we will analyze the initial form of a polynomial, given a partial order determined by $w \in \mathbb{R}^n$. Moreover, we analyze the case of an initial ideal, which is the ideal generated by the initial forms of the polynomials. We first analyze the homogeneous case and we move to the Laurent polynomials setting after. Given that setting, we prove the existence of a tropical basis, which is the equivalent to a Gröbner basis for the case of Laurent polynomials. This chapter follows closely the structure of sections 2.4 - 2.6 of Maclagan and Sturmfels’ text [MS13]. Many of their proofs involve implicitly or explicitly Proposition 2.29. Whenever possible, we will present a proof that does not use it.

2.1 Preamble

In this chapter we consider $K$ to be a non-Archiimedean, algebraically closed field with a valuation that is dense in $\mathbb{R}$. Let

$$f = \sum_{u \in \mathbb{Z}^n} c_u x^u = \sum_{u \in \mathbb{Z}^n} m_u, w \in \mathbb{R}^n$$
where $f \in K[x_1, \cdots, x_n]$, and $m_u$ represents the monomials of $f$. We say that $m_u \preceq_w m_{u'}$ if $\text{val}(c_u) + \langle w, u \rangle \leq \text{val}(c_{u'}) + \langle w, u' \rangle$. We define

$$W_f(w) := \min_{u \in \mathbb{Z}^n: c_u \neq 0} \{ \text{val}(c_u) + \langle w, u \rangle \}$$

(2.1)

And, if $w$ is fixed, we simply denote it by $W_f$. Similarly, we may have a second partial order $m_u \preceq_w' m_{u'}$ if $\langle w, u \rangle \leq \langle w, u' \rangle$. We define

$$W'_f(w) := \min_{u \in \mathbb{Z}^n: c_u \neq 0} \{ \langle w, u \rangle \}$$

(2.2)

We will actively use $\preceq$ in the rest of the text. On the other hand, $\preceq'$ will seldom be used in this chapter (it becomes more useful on Chapter 3). The terms satisfying $m_u \preceq_w m_{u'} \forall u' \neq u$ are called the initial terms of $f$ with respect to $w$.

We assume that $\mathbb{Q} \subset \Gamma_{val}$, so we can find an element of valuation 1. Moreover, by proposition 1.34 we denote $t := \varphi(1)$, where $\varphi : \Gamma_{val} \to K$ is the splitting. Consider a monomial of $f, m_u = c_u x^u$. We know that $t^{-\text{val}(c_u)}$ has valuation $-\text{val}(c_u)$. Hence, $\text{val}(t^{-\text{val}(c_u)} c_u) = 0 \Rightarrow t^{-\text{val}(c_u)} c_u \in R_K$, the associated coordinate ring. In this chapter we denote by $a_u := \pi(t^{-\text{val}(c_u)} c_u)$, the projection of such an element into the residue field. We are now ready to introduce the concept of initial forms. The definition is a slight modification of what [MS13] presents (p.79).

**Definition 2.1** (initial form). The initial form of a function

$$f = \sum_{u : c_u \neq 0} c_u x^u \in K[x_1, x_2, \cdots, x_n]$$

with respect to a vector $w \in \Gamma_{val}^w$ is

$$\text{in}_w^k(f) := \sum_{u : W_u = W_f} a_u x^u \in k[x_1, x_2, \cdots, x_n]$$

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If $f = 0$, then we define $\text{in}^k_w(f) = 0$.

In the previous definition $k$ is used to emphasize that we are working over the residue field. We need that distinction because we previously worked with another definition of initial forms for proving Hilbert’s Basis Theorem. Still, a third definition of initial forms will be given later. Nevertheless, since this chapter is exclusively devoted to initial forms over the residue field, the superscript $k$ will be omitted.

**Example 2.2.** Consider $f(x, y) = t^{10}x^3 + t^3x^2y + t^3xy^2 \in \mathbb{C}\{\{t\}\}[x, y], w_1 = (2, 0), w_2 = (1, 1)$.

\[
\text{in}_{w_1}(f) = x^3.
\]
\[
\text{in}_{w_2}(f) = x^2y + xy^2.
\]

We now introduce the concept of initial ideal. (p.80) of [MS13] presents that definition for homogeneous ideals: the generalization is straightforward.

**Definition 2.3** (Initial ideal). The initial ideal, $\text{in}_w(I)$, is defined as:

\[
\text{in}_w(I) := \langle \text{in}_w(f) : f \in I \rangle.
\]

(2.3)

By Hilbert’s Basis Theorem (1.9) any polynomial ideal is finitely generated. Now, if we can find a finite set $\{f_i \in I\}$ such that $\{\text{in}_w(f_i)\}$ generate the initial ideal, the former set is called a Gröbner basis.

**Definition 2.4** (Gröbner basis, taken from [Stu95] p.1). A finite set $F := \{f_i \in I\}$ is called a Gröbner basis for $I$ with respect to the partial order $\preceq_w$ if $\text{in}_w(I) = \langle \{\text{in}_w(f_i) : f_i \in F\} \rangle$.

We want to find a Gröbner basis for this new definition of initial ideals. We first restrict ourselves to working with homogeneous ideals and then we present an alternative to Gröbner bases, which will be the tropical bases.
2.2 Initial Homogeneous Ideals

Lemma 2.5. Let \( f = \sum u c_u x^u, m = c_v x^v \in K[x_0, \ldots, x_n] \). Then,

\[
\in_w(mf) = \in_w(m)\in_w(f).
\]

Proof. Observe that \( W_{mf} = \min \{ \val(c_u) + \val(c_v) + \langle w, u + v \rangle \} = \val(c_v) + \langle w, v \rangle + \min \{ \val(c_u) + \langle w, u \rangle \} = W_m + W_f \). Therefore,

\[
\in_w(mf) = \sum_{u:W_u=W_f+W_m} a_u a_v x^{u+v} = a_v x^v \sum_{u:W_u=W_f+W_m} a_u x^u = \in_w(m)\in_w(f).
\]

Propositions 2.6 and 2.9 appear as Proposition 2.4.2 in [MS13]. For the sake of clarity we split them and present some other interesting results.

Proposition 2.6. Let \( w \in \Gamma_{\val}^{n+1}, g \in \in_w(I) \), then we can find \( f \in I \) such that \( \in_w(f) = g \).

Proof. Since \( g \in \in_w(I) \), \( g = \sum_i a_i x^i \in_w(f_i) = \sum_{u:u \neq 0} a_u x^u \), and by Lemma 2.5, we can even assume \( g = \sum_i \in_w(f_i) \). Now consider \( W = \min \{ W_{f_i} \} \), and \( c \in K \) such that \( \val(c) = 1, c^{-1}c = 1 \). We claim that \( f = \sum c^{-W} f_i \in I \) satisfies \( \in_w(f) = g \). It is clear that \( \in_w(c^{-W} f_i) = \in_w(f_i) \). Moreover,

\[
\in_w \left( \sum c^{-W} f_i \right) = \sum_i \in_w(f_i).
\]

It remains to be seen that \( g = \in_w(f) \). For this, it is enough to check that \( W_f = 0 \). Assume the opposite, and choose \( u \in \supp(g) \). Then we must have that \( \val(\sum_i c_{ui}) > \min_i \{ \val(c_{ui}) \} \), which would imply that all elements of minimal valuation cancels each other. As a consequence, \( u \not\in \supp(g) \), which is a contradiction.
Proposition 2.7. Let $f, g \in K[x_0, \cdots, x_n]$ such that $W_f = W_g$, and $\text{in}_w(f) + \text{in}_w(g) \neq 0$. Then $\text{in}_w(f + g) = \text{in}_w(f) + \text{in}_w(g)$.

Proof. Without loss of generality we can assume that $W_f = W_g = 0$ because we can apply the transformation $F = f(t^{w_0}x_0, \cdots, t^{w_n}x_n)$, where we assume that $\mathbb{Q} \subset \Gamma_{val}$ so we can find an element of valuation 1 which, by proposition 1.34 we denote $t := \varphi(1)$, where $\varphi : \Gamma_{val} \to K$ is the splitting. Now, if $a_u x^u$ is a monomial of $\text{in}_0(f)$, that means $c_u = a_u + \text{terms of higher valuation.} \text{in}_0(f) + \text{in}_0(g) \neq 0$ implies $W_{f+g} = 0$. Therefore, every monomial of $\text{in}_0(f + g)$ is of the form $a_u$ for $u$ in the support of $f$ but not in the support of $g$; $a_v$ for $v$ in the support of $g$ but not in the support of $f$; and $a_u + a_v$ for $u = v$ in the support of both $f, g$. Hence, every monomial of $\text{in}_0(f + g)$ is a monomial of $\text{in}_0(f) + \text{in}_0(g)$.

If there were a monomial $a_u x^u$ in $\text{in}_0(f) + \text{in}_0(g)$ but not in $\text{in}_0(f + g)$, that would imply that the terms of support $u$ in $f$ and $g$ cancel one another. Hence, they cancel in $\text{in}_0(f) + \text{in}_0(g)$. The contradiction proves the claim. 

We also have a partial converse to the above proposition.

Proposition 2.8. Let $f, g \in K[x_0, \cdots, x_n]$ be such that $f + g = 0$. Then for any $w \in 1 + \Gamma_{val}$, $\text{in}_w(f) + \text{in}_w(g) = 0$.

Proof. First note that $W_f = W_g$. Since every element $u \in \text{supp}(f)$ is also in $\text{supp}(g)$, if these sets differed we would be able to find $u$ such that $\text{val}(c^f_u) + (w, u) \neq \text{val}(c^g_u) + (w, u) \Rightarrow \text{val}(c^f_u) \neq \text{val}(c^g_u)$, and that would be a contradiction to Corollary 1.23 (the contradiction does not hold if we had three or more functions).

Now that we know $W_f = W_g$, assume that $\text{in}_w(f) + \text{in}_w(g) \neq 0$. Hence, we can find $u \in \text{supp}(\text{in}_w(f)) \cap \text{supp}(\text{in}_w(g))$ such that $a^f_u + a^g_u \neq 0$. However, $a^f_u = \pi\left(t^{-\text{val}(c^f_u)}c^f_u\right) = \pi\left(t^{-\text{val}(c^g_u)}(-1)c^g_u\right) = -a^g_u$, so $\text{in}_w(f) + \text{in}_w(g) = 0$. 


Proposition 2.9. Let $I \subset K[x_0, \ldots, x_n]$. There exists a Gröbner basis for $I$ with respect to $\preceq_w$.

Proof. First we show that $\text{in}_w(I)$ is an ideal. Clearly $0 = \text{in}_w(0) \in \text{in}_w(I)$. If $f, g \in \text{in}_w(I)$, then, by Proposition 2.6 and Proposition 2.7,

$$0 \neq f + g = \text{in}_w(f') + \text{in}_w(g') = \text{in}_w(e^{-W_i} f') + \text{in}_w(e^{-W_i} g') = \text{in}_w(e^{-W_i} f' + e^{-W_i} g')$$

and trivially $e^{-W_i} f' + e^{-W_i} g' \in I$. Finally, for $m = a_w x^n \in k[x_0, \ldots, x_n]$, Lemma 2.5 lets us conclude that $mf = \text{in}_w(mf')$.

By 1.9, we know that $\text{in}_w(I)$ is finitely generated. Let $F := \{ f_i \in \text{in}_w(I) \}$ be the finite set of generators. Then, by Proposition 2.6, we can find a lifting $F' := \{ f'_i : f'_i \in I, \text{in}_w(f'_i) = f_i \}$. $F'$ is the Gröbner basis.

Proposition 2.10. [Lemma 2.4.5 in [MS13]] Let $f \in K[x_0, \ldots, x_n], v, w \in \Gamma^w_{val}$. Then, there exists $\epsilon > 0$ such that for every $\epsilon' \in (0, \epsilon) \cap \mathbb{Q}, \text{in}_{w+\epsilon'v}(f) = \text{in}_v(\text{in}_w(f))$

Proof. First observe that the problem reduces to proving equality of the following sets:

$$\{ u \in \text{supp}(f) : \text{val}(c_u) + \langle w, u \rangle + \epsilon' \langle v, u \rangle \text{ is minimal for all } 0 < \epsilon' < \epsilon \} \quad (2.4)$$

and

$$\{ u \in \text{supp}(f) : \text{val}(c_u) + \langle w, u \rangle \text{ is minimal, and } \langle v, u \rangle \text{ is minimal} \} \quad (2.5)$$

For $\epsilon'$ small enough, take $u \in (2.4)$. Observe that for every other element in the support of the function,

$$\text{val}(c_u) + \langle w, u \rangle + \epsilon' \langle v, u \rangle \leq \text{val}(c_u) + \langle w, u' \rangle + \epsilon' \langle v, u' \rangle.$$

Due to the density of $\mathbb{Q}$, this defines two continuous linear functions on $\epsilon'$. In the limit
\( \epsilon' \to 0^+ \), the inequality will still hold. Thus

\[
\text{val}(c_u) + \langle w, u \rangle \leq \text{val}(c_{u'}) + \langle w, u' \rangle.
\]

That means \( \langle \text{val}(c_u) + \langle w, u \rangle \rangle \) is minimal. Thus, any two elements \( u, u' \in (2.4) \) satisfy \( \text{val}(c_u) + \langle w, u \rangle = \text{val}(c_{u'}) + \langle w, u' \rangle \) and because the inequality holds when multiplying it by any \( \epsilon > 0 \) (in particular, by \( \epsilon' \) as required in (2.4)) that leads to the contradiction \( u \notin (2.4) \). That proves (2.4) \( \subset (2.5) \).

Conversely, take \( u \in (2.5) \), and assume \( u \notin (2.4) \). Then, for every \( \epsilon > 0 \) we can find that for some \( \epsilon' \in (0, \epsilon) \cap \mathbb{Q} \)

\[
\text{val}(c_u) + \langle w, u \rangle + \epsilon' \langle v, u \rangle > \text{val}(c_{u'}) + \langle w, u' \rangle + \epsilon' \langle v, u' \rangle
\]

(2.6)

By continuity, \( \text{val}(c_u) + \langle w, u \rangle \geq \text{val}(c_{u'}) + \langle w, u' \rangle \), and because \( u \in (2.5) \), we have \( \text{val}(c_u) + \langle w, u \rangle = \text{val}(c_{u'}) + \langle w, u' \rangle \). Thus, \( u' \) is also minimal and must satisfy \( \langle v, u \rangle \leq \langle v, u' \rangle \).

That is a contradiction to (2.6). So, for each of the finitely many \( u \in (2.5) \), there is \( \epsilon > 0 \) with the desired property. By taking the minimal \( \epsilon \) we prove the claim.

\( \square \)

Proposition 2.11 is our version of Lemma 2.4.6 in [MS13].

**Proposition 2.11.** Let \( I \subset K[x_0, \cdots, x_n] \) be a homogeneous ideal, and \( w \in \Gamma_{\text{val}}^{n+1} \) fixed. Then there exists \( v \in \Gamma_{\text{val}}^{n+1}, \epsilon > 0 \) such that \( \text{in}_v \text{in}_w(I) \) and \( \text{in}_{w+\epsilon v}(I) \) are monomial ideals.

**Proof.** First, let us define the following monomial ideals:

\[
M_{v'}(w) := \langle \{ m = \text{in}_{v'} \text{in}_w(f) : f \in I, m \text{ is a monomial, } v' \in \Gamma_{\text{val}}^{n+1} \} \rangle
\]

(2.7)
\[ M'_{\nu'}(w) := \{ m = \text{in}_{w + \epsilon' \nu'}(f) : f \in I, m \text{ is a monomial } \nu' \in \Gamma^{n+1}_{\text{val}}, \epsilon' \in \mathbb{Q}^+ \} \quad (2.8) \]

We will take \( M_{\nu}, M_{\nu'} \) to be maximal with respect to inclusion (something that can be done by Hilbert Basis theorem).

First, let us show that \( M_{\nu} = \text{in}_{w, e}(I) \). (\( \subset \)) is clear. For (\( \supset \)), consider \( \text{in}_{w, e}(f) = \sum \tilde{m}_i + \tilde{N}'' \in \text{in}_{w, e}(I) \), where \( \tilde{N}'' \) is comprised by monomials not in \( M_{\nu} \). In the worst case \( f = \sum m_i + M + M' + N + N' + N'' \), where \( W_{m_i}(w) = W_M(w) = W_N(w) = W_{N''}(w) < W_{M'}(w), W_{N'}(w) \), and \( W_{m_i}'(v) = W_{N''}(v) < W_{M'}'(v), W_{N'}'(v) \). By construction, every monomial in \( M_{\nu} \) is also in \( \text{in}_{w, e}(I) \), so we can consider \( \tilde{m}_i = \sum \text{in}_{w, e}(f_i) \)

where \( f_i = m_i + M_i + M'_i + N_i + N_i' \). For each \( i \) we know that \( W_{m_i}(w) = W_M(w) = W_N(w) < W_{M_i'}(w), W_{N_i'}(w) \) and \( W_{m_i}'(v) < W_{M_i'}'(v), W_{N_i'}'(v) \). With that notation we show that it is possible to clear all the monomials in \( M_{\nu} \) and work with \( f' = N'' \). Indeed, let

\[ f' := f - \sum f_i = M - \sum M_i + N - \sum N_i + M' - \sum M'_i + N' - \sum N'_i + N'' \]

By weight considerations, \( \hat{f}' := \text{in}_{w, e}(f') = \tilde{N}'' \). Now choose \( a \in \Gamma^{n+1}_{\text{val}} \) with \( \text{in}_{w, e}(\hat{f}') = \text{in}_{w, e}(f') \) a monomial. By Proposition 2.10, we can find \( \epsilon > 0 \) such that \( \text{in}_{w, e}(f') = \text{in}_{w, e + \epsilon a}(f') \). Moreover, for every \( f_m \in I \) with \( \text{in}_{w, e}(f_m) = m \in M_{\nu} \) and \( \epsilon \in \mathbb{Q}^+ \) small enough \( \text{in}_{w, e + \epsilon a}(f_m) = m \Rightarrow M_{\nu} \subset M_{\nu + \epsilon a} \). That would be a contradiction to the maximality of \( M_{\nu} \), so \( \text{in}_{w, e}(I) \) is a monomial ideal.

We now show \( M_{\nu'} = \text{in}_{w + \epsilon \nu'}(I) \). Using a simplified version of the above argument we can assume \( f' \) containing only monomials not in \( M_{\nu'} \). In this case we consider \( f = \sum m_i + M + N + N'' \), and \( f_i = m_i + M_i + N_i \). By construction, the weights are as follows:

\( W_{m_i}(w + \epsilon v) = W_{N''}(w + \epsilon v) < W_M(w + \epsilon v), W_N(w + \epsilon v) \). And

\[ f' := f - \sum f_i = M - \sum M_i + N - \sum N_i + N'' \]
Select \( v'' \in \Gamma^ {n+1}_ {\text{val}} \) such that \( \text{in}_{w''} \text{in}_{w'' + \epsilon v'} (f') \) is a monomial. Then, there exists \( \epsilon'' \in \mathbb{Q}^+ \) with \( \text{in}_{w'' + \epsilon'' v''} (f') \) a monomial, leading to \( M_{w'' + \epsilon'' v''} \supset M_{v''} \) (for \( \epsilon'' \) small enough), another contradiction.

We want to prove \( M_v(w) = M_v''(w) \). In order to do so, we prove a couple of results regarding the vector spaces that are induced by the homogeneous ideals of fixed degree. For a degree \( d \), \( I_d \) is the homogeneous ideal spanned by the homogeneous functions \( f \in I \subset \mathbb{R} \) of degree \( d \). Also, \( S_K := K[x_0, \cdots, x_n], S_k := k[x_0, \cdots, x_n] \).

Propositions 2.13 and 2.12 are inspired by Lemma 2.4.7 in [MS13].

**Proposition 2.12.** Let \( w \in \Gamma^ {n+1}_ {\text{val}} \) such that \( \text{in}_w(I)_d \) is spanned by monomials. \( B := \{ x^u \in (S_k)_d \setminus \text{in}_w(I)_d \} \) form a \( K \) basis for \( (S_K/I)_d \)

**Proof.** We first show that \( B \) is comprised by linearly independent terms. If \( f = \sum a_u x^u \in I \) is a linear combination of elements in \( B \), then \( \text{in}_w(f) \in \text{in}_w(I) \) is spanned by elements not in \( \text{in}_w(I) \), a monomial ideal. Therefore, \( a_u = 0 \) \( \forall u \). Using the dimension theorem for vector spaces, it also implies \( \dim(I_d) \leq \dim(\text{in}_w(I)_d) \). We will be done once we show that the equality holds. Let us assume the opposite, that is, \( \dim(I_d) < \dim(\text{in}_w(I)_d) \). We will draw a contradiction from it.

Let \( f_i = \sum_j a_{ij} x^{u_j} \in I_d \) such that \( \text{in}_w(f_i) = x^{u_i} \). For \( c_i \in K^* \) suppose \( \sum c_i f_i = 0 \). That implies \( \sum c_i a_{ij} = 0 \) for every \( j \). Take \( j \) such that \( \text{val}(c_j) + \text{val}(a_{ij}) + \langle w, u_j \rangle \) is minimum. Thus,

\[
\text{val}(c_i) + \text{val}(a_{ij}) + \langle w, u_j \rangle > \text{val}(c_i) + \text{val}(a_{ii}) + \langle w, u_i \rangle \geq \text{val}(c_j) + \text{val}(a_{jj}) + \langle w, u_j \rangle
\]

Where the strict inequality is true because \( \text{in}_w(f_i) = x^{u_i} \), and the second one due to the previous assumption. We conclude that \( \text{val}(c_j) + \text{val}(a_{jj}) < \text{val}(c_i) + \text{val}(a_{ij}) \) and, by Proposition 1.22, \( \sum c_i a_{ij} \neq 0 \). That is the contradiction.

\[ \square \]
Proposition 2.13. For a fixed $d \in \mathbb{N}$, and $x^u \in M_v(w)_d$ there exists $f \in \text{in}_w(I)_d$ such that
\[ \text{in}_v(f) = x^u \text{ and } f = x^u + N, \]
where $N$ contains only monomials not in $M_v(w)_d$.

Proof. By Proposition 2.11 $\text{in}_w(I) = M_v(w)$ is a monomial ideal. Hence, the monomials not in $M_v(w)$ form a basis for $S_k/\text{in}_w(I)$. If $x^u \in M_v(w)_d$, there is a linear combination $N$ of monomials of degree $d$ not in $M_v(w)$ such that $f = x^u + N \in \text{in}_w(I)$.

The next result is a part of Corollary 2.4.8 in [MS13] and tells us that even if $\text{in}_w(I)_d$ is not a monomial ideal, it still is spanned by the same number of elements:

Proposition 2.14. For every $w \in \Gamma_{val}^{n+1}$ and every $d \in \mathbb{N}$, we have $\dim_K(S_K/I)_d = \dim_k(S_k/\text{in}_w(I))_d$.

Proof. We split the proof in two cases. First, if $\text{in}_w(I)$ is a monomial ideal, the result follows from Proposition 2.12. That is because in that case, the monomials not in $\text{in}_w(I)_d$ form a $K-$basis for $(S_K/I)_d$.

Second, for $\text{in}_w(I)$ an arbitrary ideal we apply Proposition 2.12 to $\text{in}_w(I)$ and $\text{in}_w(I) = M_v$ to obtain $\dim(S_k/M_v)_d = \dim(S_k/\text{in}_w(I))_d$. Also, the monomials not in $(M_v)_d$ are a linearly independent set in $(S_K/I)_d$. As a consequence, $\dim(S_k/M_v)_d \geq \dim(S_K/I)_d$, which implies
\[ \dim(I_d) \leq \dim(M_v)_d = \dim(\text{in}_w(I)_d) \]

We now have to show that the reverse inequality also holds. We use a contradictory argument for that purpose. Assume that $(M_v)_d = \{ \{m_i\} \}$, such that by Propositions 2.6 and 2.13 we can find $f_i \in I_d$ with $\text{in}_{w+\epsilon v}(f_i) = m_i$ (where $\epsilon = \min\{\epsilon_i\}$ such that $m_i = \text{in}_{w+\epsilon_i}(f_i)$ as seen in Proposition 2.10). By assumption, $\{f_i\}$ is a linearly dependent set, so we can find $c_i \in K$, not all of them zero, such that $\sum c_i f_i = 0$. As in the proof of Proposition 2.12 we get a contradiction. We conclude that $\{f_i\}$ is a linearly independent set proving the proposition.

\[ \square \]
Proposition 2.15 is our version of Corollary 2.4.9 in [MS13].

**Proposition 2.15.** For \( w \in \Gamma_{\text{val}}^{n+1} \) let \( M_v(w) \) as defined in (2.7) be maximal. Then, there is \( \epsilon > 0 \) in \( \mathbb{Q} \) such that for all rational \( 0 < \epsilon' < \epsilon \) we have \( M_{\epsilon'}(w) = M_v(w) \). In particular, \( M_{\epsilon'}(w) \) is maximal.

**Proof.** For \( M_v \), let \( \{f_i\} \subset I \) be its \( K \) Gröbner basis. By Proposition 2.10, for each \( i \) we can find \( \epsilon_i > 0 \) such that \( \text{in}_{w+\epsilon_i v}(f_i) \) is a monomial. Let \( \epsilon = \min \{\epsilon_i\} \). By construction, \( M_v \subset M_{\epsilon'} \subset M_{\epsilon''} \), where \( M_{\epsilon''} \) is some maximal ideal containing \( M_{\epsilon'} \). Notice that for any \( \epsilon' < \epsilon \) we will have \( M_v \subset M_{\epsilon'} \). The equality of the sets now follow from the use of Hilbert functions.

\( \square \)

We will now see that the initial ideal forms a polyhedral. In order to do so, we define first the polyhedral we expect them to be, and then we prove that such geometric figure is actually the support of an initial ideal. For a homogeneous ideal \( I \subset K[x_0, \ldots, x_n] \), the geometric figure (as defined in [MS13] p.85) is

\[
C_I[w] := \{w' \in \Gamma_{\text{val}}^{n+1} : \text{in}_{w'}(I) = \text{in}_w(I)\}
\]  

(2.9)

Propositions 2.16, 2.17, 2.18, 2.19, and 2.21 are inspired by Proposition 2.5.2 of [MS13].

**Proposition 2.16.** If \( \text{in}_w(I) \) is not a monomial ideal, then there exists \( w' \in \Gamma_{\text{val}}^{n+1} \) such that \( \text{in}_{w'}(I) \) is a monomial ideal, and \( C_I[w] \subset \overline{C_I[w']} \).

**Proof.** We pick \( w' := v \), where \( v \) is as constructed in Proposition 2.11. It is clear that \( M_v(w) = \text{in}_v(I) \) is a monomial ideal. Take \( y \in \Gamma_{\text{val}}^{n+1} \) such that \( y \notin C_I[w] \setminus C_I[v] \). By Proposition 2.15, there exists \( \epsilon > 0 \) such that \( M_v(w) = \text{in}_v \text{in}_w(I) = \text{in}_{w+\epsilon v}(I) \), and \( \text{in}_y(I) = \text{in}_v(I) \Rightarrow \text{in}_v \text{in}_y(I) = \text{in}_{y+\epsilon v}(I) = \text{in}_{w+\epsilon v}(I) \). We can take a common \( \epsilon'' \to 0 \), and that will be a contradiction. Therefore, \( C_I[w] \subset \overline{C_I[w']} \).

\( \square \)
Proposition 2.17. Let $w \in \Gamma_{val}^{n+1}$ such that $\text{in}_w(I)$ is a monomial ideal with Gröbner basis $G := \{ f_i := x^{u_i} + \sum c_jx^{u_j} : x^{u_j} \notin \text{in}_w(I) \}$. Then,

$$\overline{C_I(w)} := \{ y \in \mathbb{R}^{n+1} : \langle y, u_i \rangle \leq \text{val}(c_{u_j}) + \langle y, u_j \rangle \}$$

(2.10)

Proof. For \((\subset)\) take $y \in \Gamma_{val}^{n+1} \cap C_I[w]$. Notice that $w \in \Gamma_{val}^{n+1} \cap C_I[w]$. As we observed in Proposition 2.14, $\text{in}_w(I) = \text{in}_y(I) = \langle x^{u_i} \in \text{in}_w(I) \rangle$. By Proposition 2.6, we can find $h_i \in I$ such that $\text{in}_y(h_i) = x^{u_i}$, and Proposition 2.13 tells us that we can even find $h'_i$s of the form $H := \{ h_i := x^{u_i} + \sum c_jx^{u_j} : x^{u_j} \notin \text{in}_y(I) \}$, where $\text{in}_y(I)$ is a monomial ideal. Thus,

$$\langle y, u_i \rangle < \text{val}(c_{u_j}) + \langle y, u_j \rangle$$

(2.11)

Finally, if $y \in \overline{C_I(w)}$, we can find a sequence $\{ y_k \in C_I[w] \} \rightarrow y$. Each element in the sequence satisfies (2.11), and that allows us to conclude that $y$ will satisfy the non-strict inequality.

We now prove \((\supset)\). First, if $y \in \Gamma_{val}^{n+1}$ satisfies the inequalities (2.10), then, the $w$–Gröbner basis looks like $\text{in}_y(f_i) = x^{u_i} + \sum_{j \neq i} c_j x^{u_j}$. Since we know that $\langle w, u_i \rangle < \text{val}(c_{u_j}) + \langle w, u_j \rangle$, we have $\langle w - y, u_i \rangle < \langle w - y, u_j \rangle$. Denote by $\nu := w - y \in \Gamma_{val}^{n+1}$, then for every $\epsilon \in (0, \epsilon'] \cap \mathbb{Q}$ we obtain that $\text{in}_{\nu y}(f_i) = \text{in}_{y + \epsilon y}(f_i) = x^{u_i}$. This immediately implies $\text{in}_{\nu y + \epsilon y}(I) \supset \text{in}_w(I)$, and the equality holds by Proposition 2.12. Proposition 2.14 asserts that $\text{in}_y(I)$ has the same Hilbert function as $\text{in}_{y + \epsilon y}(I)$. We conclude $y \in C_I[w]$.

Now assume $z \in \mathbb{R}^{n+1}$ satisfies the inequalities in (2.10). Notice that $z$ belongs to the $\Gamma_{val}$–rational polytope determined by $\{ y \in \mathbb{R}^{n+1} : \langle y, u_i - u_j \rangle = \text{val}(c_{u_j}) \}$. $w$ itself is an interior point. Therefore, by Proposition 1.36, $z$ can be approached by a sequence of points in $\{ y_i \} \subset \Gamma_{val}^{n+1}$, each satisfying $y_i \in C_I[w]$.

\[ \square \]

Proposition 2.18. Suppose $\text{in}_w(I)$ is a monomial ideal, that is, $M_v(w) = \text{in}_w(I)$. Let $\{ f_i \} \subset I$ be a system of generators of the form $\text{in}_w(f_i) = x^{u_i} + g_i$, where $g_i$ does not involve
any monomial in $M$. Then,

$$C_I[w] := \{ y \in \Gamma^{n+1}_{\text{val}} : \text{in}_y(f_i) = x^{u_i} \}$$  \hspace{1cm} (2.12)

Proof. By Proposition 2.13 we know that a Gröbner basis as the one described does exist. Further, in Proposition 2.17 we learned that $C_I[w]$ is a polyhedron described by (2.10). It is then clear that for all $y \in C_I[w]$ in$_y(f_i) = x^{u_i}$. The converse then follows because all of them have the same Hilbert function.

Proposition 2.19. Let in$_w(I)$, in$_{w'}(I)$ be as in Proposition 2.16. Then $C_I[w]$ is a proper face of $C_I[w']$.

Proof. We first show that $C_I[w] \subset C_I[w']$. Take $y \in C_I[w]$, that is, in$_y(I) = \text{in}_w(I)$. Therefore, some of the inequalities in (2.10) are equalities. We now examine $y$. If every inequality in (2.10) is strict, it follows that $C_I[y] = C_I[w']$, so in$_y(I) = \text{in}_{w'}(I)$, a contradiction. In fact, the equalities must happen for the same terms. If we assume the opposite, we could have situations like

$$\langle w, u_i \rangle = \text{val}(c_{u_j}) + \langle w, u_j \rangle, \quad \langle y, u_i \rangle < \text{val}(c_{u_j}) + \langle y, u_j \rangle.$$  

In that case in$_w(f_i) - \text{in}_w(f_i) = n$, where $n$ is comprised by monomials not in in$_{w'}(I) = M_e(w)$, which is impossible. The same argument invalidates the possibility of having a reverse inequality, that is, $\langle y, u_i \rangle > \text{val}(c_{u_j}) + \langle y, u_j \rangle$. We conclude that $y \in C_I[w']$. By continuity, we must also have $C_I[w] \subset C_I[w']$.

We now show that all the vectors $y \in \Gamma^{n+1}_{\text{val}}$ satisfying the same system of inequalities and equalities as $C_I[w]$ also satisfy in$_y(I) = \text{in}_w(I)$. Take a $w'$–Gröbner basis $\{ f_i \}$ such that in$_w(f_i) = x^{u_i} + N$ with $N \neq 0$, $i \in I'$ and in$_w(f_i) = x^{u_i}$ for $i \in I''$. Every $y$ satisfying the same inequalities trivially satisfies in$_y(I) \supset \text{in}_w(I)$, and the inequality will hold due to having the same Hilbert functions. Notice that every $y \in \mathbb{R}^{n+1}$ satisfying the system will
be a limit point of points in the value group. So it is a face, and it is proper because there is at least one equality.

\[ \square \]

We now prove the finiteness of such geometrical sets.

**Proposition 2.20** (Lemma 2.5.4 in [MS13]). For \( I \subset K[x_0, \cdots, x_n] \) an ideal, there are only finitely many different monomial ideals of the form \( \text{in}_w(I) \) with \( w \in \Gamma^{n+1}_{\text{val}} \).

*Proof.* Assume the opposite. Then, by Lemma 3.23 (which is a result that is independent of the initial forms), applied to the zero ideal, \( \text{in}_w(I) \) would be contained in any other of the infinitely many monomial ideals, we would be able to find an infinite ascending chain of ideals in \( K[x_0, \cdots, x_n] \), a Noetherian ring. \( \square \)

**Proposition 2.21.** Let \( \text{in}_w(I) \), \( \text{in}_{w'}(I) \) be as in Proposition 2.16, and \( x^u \in \text{in}_w(I) \). There exists \( h \in I \) such that \( h = x^u + g \), \( g \) does not involve monomials in \( M_v(w) \) and \( \text{in}_w(h) = x^u \).

*Proof.* By Propositions 2.6, 2.13 and 2.18 we can find \( f \in I \) such that \( \text{in}_w(f) = x^u \), where \( f = x^u + g_1 + g_2 \), where \( g_1 \) contains only monomials in \( M_v(w) = \text{in}_{w'}(I) \), and \( g_2 \) contains no monomials. Let \( m \) be one of the monomials in \( g_1 \). Then, by Proposition 2.12 and the fact that \( M_v = M_v' \), we can find a function \( g'_1 = m + g''_1 \) such that \( \text{in}_w(g'_1) \) is supported by functions in \( m \) and \( g''_1 \) contains no monomials in \( M_v \). Moreover, \( W_f(w) = W_{x^u}(w) < W_m(w) \leq W_{g''_1(w)} \) so \( \text{in}_w(f - g'_1) = x^u \). We now repeat this process to eliminate all the (finitely many) monomials in \( M_v(w) \). \( \square \)

### 2.3 Ideals in the Laurent Polynomial Ring

This section follows section 3.3 closely. When proofs of propositions are true regardless of working in the residue field or not, they are omitted, as Chapter 3 is more relevant to this thesis. The definitions will be stated for the sake of completeness.
Definition 2.22 (Definition 2.2.5 in [MS13]). For \( f \in K[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \), we define its projectivization \( f_{proj} \) by the following homogeneous polynomial

\[
f_{proj} := x^u f \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right)
\]

(2.13)
such that \( u \in \mathbb{Z}^{n+1} \) is such that \( u_i \) is the minimum value for which \( f_{proj} \) has only positive exponents.

Lemma 2.23. \( f_{proj} \) is a homogeneous polynomial.

Proof. Let \( f = \sum_i c_u x^u \), then \( f \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right) = \sum_i c_u x_0^{-u_i} \sum_j u_{ij} x^u \). At this stage, \( f \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right) \) is already “homogeneous of degree 0.” We now multiply times \( x^u \) where \( u \in \mathbb{Z}^{n+1} \) is such that \( u_i \) is the minimum value for which \( f \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right) \) has only positive exponents. And the same \( x^u \) will be multiplied times each monomial of \( f \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right) \), which is already homogeneous. Therefore, \( f_{proj} \) is homogeneous.

\[\square\]

Proposition 2.24. For \( f \in K[x^{\pm}] \), \( w \in \Gamma^n_{val} \), \( \text{in}_{(0,w)}(f_{proj}) = \text{in}_{(0,w)} \left( x^u f \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right) \right) = x^u \text{in}_w \left( f \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right) \right) \).

Proof. See Proposition 3.26

\[\square\]

Proposition 2.25 (Proposition 2.6.2 in [MS13]). Let \( f, f_{proj} \) and \( u \) be as in (2.13). Then,

\[
\text{in}_w(f(x_1, \cdots, x_n)) = \left( x^{-u} \text{in}_{(0,w)}(f_{proj}(x_0, \cdots, x_n)) \right) \bigg|_{x_0=1}.
\]

Proof. See Proposition 3.27

\[\square\]

For \( I \subset K[x^{\pm}] \) an ideal, we now define the Initial Laurent ideal in the way suggested by the previous propositions (and as defined in [MS13] p.94).
Definition 2.26 (Initial Laurent ideal). For $I \subset K[x_0, \cdots, x_n]$, the initial ideal is the ideal generated by the initial terms of the functions in $I$:

$$\operatorname{in}_w(I) := \langle \operatorname{in}_w(f) : f \in I \rangle.$$

The projectivization of such an ideal, $I_{\text{proj}}$, as [MS13] defined it, is the homogeneous ideal:

$$I_{\text{proj}} := \langle \{f_{\text{proj}} : f \in I \} \rangle. \quad (2.14)$$

Proposition 2.27. Let $g \in I_{\text{proj}}$ be a homogeneous polynomial of degree $d$. Then there exists $f \in I$ such that $f_{\text{proj}} = g$.

Proof. See Proposition 3.30.

Proposition 2.28. Let $g \in I_{\text{proj}}$. Then $g(1, x_1, \cdots, x_n) \in I$.

Proof. See Proposition 3.31.

We now present results that cannot be generalized to other initial forms. When taking initial forms over the residue field, the initial form of the product of two functions is the product of initial forms.

Proposition 2.29 (Lemma 2.6.3.3 in [MS13]). Let $w \in \Gamma_{\text{val}}^n$ and $f, g \in K[x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]$. Then $\operatorname{in}_w(fg) = \operatorname{in}_w(f)\operatorname{in}_w(g)$.

Proof. The result is trivial if either $f = 0$ or $g = 0$, so we assume both functions are different than zero (and so their initial forms will also be non-zero). We begin by proving that $W_{fg} = W_f + W_g$. If $f = \sum_i c_{ui} x^{u_i}$, $g = \sum_j c_{uj} x^{u_j}$, then

$$fg = \sum_k \left( \sum_{u_i + u_j = u_k} c_{ui} c_{uj} \right) x^{u_k}.$$
And assume $W_f = \text{val}(c_{u_{i_0}}) + \langle w, u_{i_0} \rangle$, $W_g = \text{val}(c_{u_{j_0}}) + \langle w, u_{j_0} \rangle$. Then, the weight of any other monomial of $fg$ is:

$$weight_w((fg)_k) = \text{val}\left( \sum_{u_i + u_j = u_k} c_{u_i}c_{u_j} \right) + \langle w, u_k \rangle$$

$$\geq \min_{u_i + u_j = u_k} \{ \text{val}(c_{u_i}) + \text{val}(c_{u_j}) \} + \langle w, u_i + u_j \rangle$$

$$= \text{val}(c_{u_{i_0}}) + \text{val}(c_{u_{j_0}}) + \langle w, u_{i_0} + u_{j_0} \rangle \geq W_f + W_g.$$

That is, any non-zero monomial of the product has weight at least $W_f + W_g$. This also proves that the weight of the product of two monomials is the sum of the weights. We now prove that it is the minimum by finding a contradiction when assuming the opposite. If $W_{fg} > W_f + W_g$ for every fixed $u_k = u_i + u_j$ where the minima for $f$ and $g$ is attained, we have

$$\text{val}\left( \sum_{u_i + u_j = u_k} c_{u_i}c_{u_j} \right) > \text{val}(c_{u_{i_0}}) + \text{val}(c_{u_{j_0}}) + \langle w, u_{i_0} + u_{j_0} \rangle = W_f + W_g.$$

Hence, the terms of least valuation are canceled. Using the notation introduced in Definition 2.1 we would obtain $\sum_{u_i + u_j = u_k} a_{u_i} a_{u_j} = 0$. Moreover, we would obtain that result in every case. Thus $\text{in}_w(f) \text{in}_w(g) = 0$, and that is a contradiction as $k[x_1^\pm 1, \cdots, x_n^\pm 1]$ is an integral domain.

Therefore,

$$\text{in}_w(fg) = \sum_{k:W((fg)_k)=W_{fg}} \left( \sum_{u_i + u_j = u_k} a_{u_i} a_{u_j} \right) x^{u_k}$$

$$= \sum_{i,j:W_{f_i} + W_{f_j} = W_f + W_g} \left( \sum_{u_i + u_j = u_k} a_{u_i} a_{u_j} \right) x^{u_i + u_j} = \text{in}_w(f) \text{in}_w(g).$$
Example 2.30. Let \( f(x, y) = tx^2 + xy \), \( g(x, y) = (1 + t)xy + (t^{-1})x^2 \), \( w = (1, 2) \).

Then,
\[
\text{in}_w(f) = x^2 + xy, \quad \text{in}_w(g) = x^2, \quad \text{in}_w(f)\text{in}_w(g) = x^4 + x^3y.
\]

On the other hand,
\[
f g = x^4 + (t^{-1} + t + t^2)x^3y + (1 + t)x^2y^2, \quad \text{in}_w(fg) = x^4 + x^3y.
\]

Counterexample 3.32 shows that this result cannot be extended to a more general setting.

Proposition 2.31. \( \text{in}_w(I) = 1 \iff \text{in}_w(\sqrt{I}) = 1. \)

Proof. (\( \Rightarrow \)) Let \( f \in I \subset \sqrt{I} \) be such that \( \text{in}_w(f) = 1 \Rightarrow f \in \sqrt{I} \Rightarrow \text{in}_w(\sqrt{I}) = 1. \)

(\( \Leftarrow \)) If \( f \in \sqrt{I} \) then \( f^r \in I \) for some \( r \in \mathbb{N} \). Using Proposition 2.29 we also know
\[
\text{in}_w(f) := 1 \Rightarrow \left(\text{in}_w(f)\right)^r = 1 = \left(\text{in}_w(f^r)\right),
\]
so we can find \( g \in I \) such that \( \text{in}_w(g) = 1 \), proving the proposition.

\( \square \)

2.4 Tropical bases

In the absence of Gröbner bases for Laurent polynomials, we have what we call tropical bases.

Definition 2.32 (Tropical basis (Definition 2.6.4 in [MS13])). A set \( F := \{f_1, \ldots, f_n\} \) contained in an ideal \( I = \langle F \rangle \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) is called a finite tropical basis if for every \( w \in \Gamma_{\text{val}}, \text{in}_w(F) := \{\text{in}_w(f_i) : f_i \in F\} \) contains a unit if and only if \( \text{in}_w(I) = 1. \)

Proposition 2.33 (Existence of tropical basis (Theorem 2.6.5 in [MS13])). Every ideal \( I = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) has a finite tropical basis.

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Proof. See Proposition 3.35.

We end this chapter with a discussion of how linear transformations induce transformations in the initial forms. For the following set of characters $X := \{X_1^{a_1} \cdots X_n^{a_n} | a \in \mathbb{Z}^n \}$ we have the diagram:

$$
\begin{array}{ccc}
\mathbb{Z}^n & \xrightarrow{\sim} & X \\
\downarrow{q} & & \downarrow{\varphi} \\
\mathbb{Z}^n & \xrightarrow{\sim} & X
\end{array}
$$

(2.15)

where if $q(z) = Az$ for $A \in GL_n(\mathbb{Z})$, then we obtain the induced map $\varphi(x) = x^{Az}$. This transformation induces another on the level of polynomial functions. That is, we have the following map:

$$
K[x^{\pm 1}] \xrightarrow{\sim} K[x^{\pm 1}]
$$

(2.16)

$$
f(x) = \sum_i p_i x^i \mapsto f^* := f(\varphi(x)) = \sum_i p_i x^{Ai}
$$

Proposition 2.34. With the above notation, let $a \in (K^\times)^n$. If $f(a) = 0$, then $f^*(\varphi^{-1}(a)) = 0$.

Proof. See Proposition 3.36.

Proposition 2.35 (Lemma 2.6.8 in [MS13]). Let $f = \sum_i p_i x^i \in K[x^{\pm 1}]$, $A \in GL_n(\mathbb{Z})$. Consider $f' = \sum_i p_i x^{(A^{-1})^T i}$. If $w \in \Gamma^n_{val}$ is such that $\text{in}_w(f)$ is not a monomial, then $\text{in}_{Aw}(f')$ is not a monomial.

Proof. See Proposition 3.38.

Corollary 2.36 (Corollary 2.6.10 in [MS13]). With $f^*$ an automorphism in the Laurent polynomial ring defined as in (3.17) and with $I' = f^{*-1}(I)$,

$$
1 \in \text{in}_w(I) \Leftrightarrow 1 \in \text{in}_{\varphi(w)}(I').
$$

Proof. See Proposition 3.39.
Chapter 3

Initial ideals revisited

In this chapter we provide a different definition of an initial term. This definition is a direct generalization of that given by Eisenbud [Eis04], but differs from what Sturmfels has proposed as a working definition for the tropical setting (as in [MS13]), which was discussed in the previous chapter. Our ultimate goal is to construct a tropical basis, just as was previously done. For the sake of clarity and comparability, we maintain the same order as in the previous chapter.

3.1 Preamble

In this chapter we consider $K$ to be a non-Archimedean field with a valuation that is dense in $\mathbb{R}$. Just as before, we express a function as the sum of its monomials,

$$f = \sum_{u \in \mathbb{Z}^n} c_u x^u = \sum_{u} m_u.$$
For \( w \in \mathbb{R}^n \) we say that \( m_u \preceq_w m_{u'} \) if \( \text{val}(c_u) + \langle w, u \rangle \leq \text{val}(c_{u'}) + \langle w, u' \rangle \). We denote by

\[
W_f(w) := \min_{u \in \mathbb{Z}^n: c_u \neq 0} \{\text{val}(c_u) + \langle w, u \rangle\} \tag{3.1}
\]

And, if \( w \) is fixed, we simply denote it by \( W_f \). The terms satisfying \( m_u \preceq_w m_{u'} \quad \forall \ u' \neq u \) are called the initial terms of \( f \) with respect to \( w \).

Within this setting, we redefine the initial form as follows:

**Definition 3.1 (Initial form).** The initial form of a function

\[
f = \sum_{u: c_u \neq 0} c_u x^u \in K[x_1, x_2, \ldots, x_n]
\]

with respect to a vector \( w \in \mathbb{R}^n \) is

\[
in^K_w(f) := \sum_{u: W_u = W_f} c_u x^u \in K[x_1, x_2, \ldots, x_n].
\]

Note that the previous definition is almost identical to Definition 2.1 (or the one presented in [MS13] p.74). The only difference is that we do not work with the residue field anymore. Instead, \( K \) is used to emphasize that now both the initial form and the original function lie in the same field. In this chapter, \( in^K_w(f) = in_w(f) \), and when making comparisons the one defined in the previous chapter will be denoted by \( in^K_w(f) \). Observe that \( u \in \text{supp}(in_w(f)) \iff u \in \text{supp}(in^K_w(f)) \).

Now, if we introduce a second partial order for \( w \in \mathbb{R}^n \) we say that \( m_u \preceq'_w m_{u'} \) if \( \langle w, u \rangle \leq \langle w, u' \rangle \). We define

\[
W'_f(w) := \min_{u \in \mathbb{Z}^n: c_u \neq 0} \{\langle w, u \rangle\}.
\]

And, if \( w \) is fixed, we simply denote it by \( W'_f \). The terms satisfying \( m_u \preceq'_w m_{u'} \forall u' \neq u \) are
also called initial terms of $f$ with respect to the weight vector $w$. In this case, we only take into account the degree of the monomial.

**Definition 3.2** (Initial form by the weight vector). The initial form of a function $f = \sum_{u, c_u \neq 0} c_u x^u \in K[x_1, x_2, \ldots, x_n]$ by the weight vector $w \in \mathbb{R}^n$ is

$$d^K_w(f) := \sum_{u : W'_u = W'_f} c_u x^u \in K[x_1, x_2, \ldots, x_n]$$

Notice that these $w$–initial forms are idempotent, that is, $d_w(d_w(f)) = d_w(f)$. Moreover, $u \in \text{supp}(\text{in}_w(f)) \iff u \in \text{supp}(\text{in}_w^K(f)) \iff u \in \text{supp}(d_w(f))$. Also, the analog of Lemma 2.5 holds.

**Lemma 3.3.** Let $f = \sum_u c_u x^u, m = c_v x^v \in K[x_0, \ldots, x_n]$. Then $\text{in}_w(m f) = \text{min}_w(f)$.

**Proof.** This is immediate from observing that for $u \in \text{supp}(f)$

$$\text{if } \text{val}(c_u) + \langle w, u \rangle \leq \text{val}(c_{u'}) + \langle w, u' \rangle \ \forall u' \neq u$$

$$\Rightarrow \text{val}(c_u) + \langle w, u \rangle + \text{val}(c_v) + \langle v, u \rangle \leq \text{val}(c_{u'}) + \langle w, u' \rangle + \text{val}(c_v) + \langle w, v \rangle \ \forall u' \neq u.$$ 

\[\square\]

### 3.2 Initial Homogeneous Ideals

In this section we work exclusively with homogeneous functions and the ideals they generate.

First, we introduce the definition of an initial ideal.

**Definition 3.4** (Initial homogeneous set). For $I \subset K[x_0, \ldots, x_n]$ a homogeneous ideal, the **initial set** is the set

$$\text{in}_w(I) := \{ \text{in}_w(f) : f \text{ is homogeneous in } I \}$$
Notice that, unlike Definition 2.3, this is only a set. When necessary, we will work with the ideal generated by the initial ideal, which we name the initial ideal, and which will be denoted \(\langle \text{in}_w(I) \rangle\). The reason for considering \(\text{in}_w(I)\) only as a set is that we no longer have an equivalent to Proposition 2.6, that is, we can find \(g \in \langle \text{in}_w(I) \rangle\) for which there is no \(f \in I\) such that \(\text{in}_w(f) = g\).

**Counterexample 3.5.** Consider \(f_1 = (t+1)xy+3yz+t^3z^2, f_2 = (t^2-1)xy+yz-3t^{-1}z^2, I = (f_1, f_2), w = (0, 0, 1/2)\). Let \(g \in \langle \text{in}_w(I) \rangle := \text{in}_w(f_1) + \text{in}_w(f_2) = (t^2 + t)xy - 3t^{-1}z^2\). Then there is no \(h \in I\) such that \(\text{in}_w(h) = g\).

**Proof.** Due to the fact that the coefficients of the monomials remain unchanged when taking the initial form, both monomials \(m_1 := (t^2 + t)xy, m_2 := 3t^{-1}z^2\) must be part of such an \(h\). That also means \(W_{m_1} = W_{m_2} = W_h\), but \(W_{m_1} = 1 \neq 0 = W_{m_2}\). Thus, we cannot construct such an element. 

Moreover, the previous example also shows that the initial form is not additive:

\[
\text{in}_w(f_1) + \text{in}_w(f_2) = (t^2 + t)xy - 3t^{-1}z^2 \neq (t^3 - 3t^{-1})z^2 = \text{in}_w(f_1 + f_2),
\]

regardless of the fact that \(W_{f_1} = W_{f_2} = 0\).

Another result that does not follow directly is Proposition 2.10.

**Counterexample 3.6.** Consider \(f = txy + 3z^2, w = (-\frac{1}{2}, -\frac{1}{2}, 0), v = (0, 0, 0)\). We have

\[
\text{in}^k_w(\epsilon v)(f) = \text{in}^k_w(f) = xy + 3z^2 = \text{in}^k_v(xy + 3z^2) = \text{in}^k_v\text{in}^k_w(f),
\]

but

\[
\text{in}^{\epsilon v}_w(f) = \text{in}_w(f) = txy + 3z^2 \neq 3z^2 = \text{in}_v(txy + 3z^2) = \text{in}_v\text{in}_w(f).
\]
However, if we iterate the two initial forms defined in 3.2 and 3.1, we do obtain two equivalents to Propositions 2.10 or 2.4.5 in [MS13].

**Proposition 3.7.** Let $f \in K[x_0, \cdots, x_n], v, w \in \Gamma_{val}^n$. Then there exists $\epsilon > 0$ such that for every $\epsilon' \in (0, \epsilon), d_v(\text{in}_w(f)) = \text{in}_{w+\epsilon'v}(f)$.

**Proof.** We only need to prove the equality of the following sets:

$$\left\{ \begin{array}{l}
u \in \text{supp}(f) : \forall u' \neq u, \\
\text{val}(c_u) + \langle w, u \rangle = \text{val}(c_{u'}) + \langle w, u' \rangle \text{ and } \langle v, u \rangle \leq \langle v, u' \rangle \\
\text{or } \text{val}(c_u) + \langle w, u \rangle < \text{val}(c_{u'}) + \langle w, u' \rangle \end{array} \right\}, \quad (3.2)$$

$$\left\{ \begin{array}{l}
u \in \text{supp}(f) : \forall u' \neq u, \text{there is } \epsilon > 0 \text{ such that } \forall \epsilon' \in (0, \epsilon) \\
\text{val}(c_u) + \langle w + \epsilon'v, u \rangle \leq \text{val}(c_{u'}) + \langle w + \epsilon'v, u' \rangle \end{array} \right\}. \quad (3.3)$$

Take $u \in (3.2)$. If $\text{val}(c_u) + \langle w, u \rangle = \text{val}(c_{u'}) + \langle w, u' \rangle$ and $\langle v, u \rangle \leq \langle v, u' \rangle$, then for every $\epsilon > 0$ we obtain

$$\text{val}(c_u) + \langle w + \epsilon v, u \rangle \leq \text{val}(c_{u'}) + \langle w + \epsilon v, u' \rangle.$$  

Assume now that $\text{val}(c_u) + \langle w, u \rangle < \text{val}(c_{u'}) + \langle w, u' \rangle$. If $\langle v, u \rangle \leq \langle v, u' \rangle$, we reduce to the previous case. Suppose instead $\langle v, u \rangle > \langle v, u' \rangle$. Let

$$\delta := \text{val}(c_{u'}) - \text{val}(c_u) + \langle w, u' - u \rangle > 0.$$  

There exists $\epsilon > 0$ such that $0 < \epsilon \langle v, u - u' \rangle < \delta \Rightarrow \delta - \epsilon \langle v, u - u' \rangle > 0$. Therefore, for every $\epsilon' \in (0, \epsilon),

$$\text{val}(c_{u'}) - \text{val}(c_u) + \langle w, u' - u \rangle + \langle \epsilon'v, u' - u \rangle > 0$$

It immediately follows that $u \in (3.3)$. For the other inclusion, let $u \in (3.3), u' \in \text{supp}(f).$
By continuity we have

$$\lim_{\epsilon \to 0^+} \text{val}(c_u) + (w + \epsilon v, u) \leq \lim_{\epsilon \to 0^+} \text{val}(c_{u'}) + (w + \epsilon v, u'),$$

that is, $\text{val}(c_u) + (w, u) \leq \text{val}(c_{u'}) + (w, u')$. For each $u'$ for which a strict inequality holds, there is nothing to do. If equality holds, then from $\text{val}(c_u) + (w + \epsilon v, u) \leq \text{val}(c_{u'}) + (w + \epsilon v, u')$ it follows that $\epsilon (v, u) \leq \epsilon (v, u') \Rightarrow (v, u) \leq (v, u')$.

\[\square\]

**Corollary 3.8** (to the previous proof). Let $f \in K[x_0, \cdots, x_n], v, w \in \Gamma^\nu_{\text{val}}$. Then there exists $\epsilon > 0$ such that for every $\epsilon' \in (0, \epsilon), d_v(d_w(f)) = d_{w+\epsilon' v}(f)$.

**Proof.** Just as before, we need to prove the equality of the following sets:

\[
\begin{align*}
\left\{ u \in \text{supp}(f) : \forall u' \neq u, \quad (w, u) = (w, u') \text{ and } (v, u) \leq (v, u') \right\} \quad (3.4) \\
\left\{ u \in \text{supp}(f) : \forall u' \neq u, \quad (w, u) < (w, u') \right\}
\end{align*}
\]

\[
\begin{align*}
\left\{ u \in \text{supp}(f) : \forall u' \neq u, \quad (w + \epsilon v, u) \leq (w + \epsilon v, u') \right\} \quad (3.5)
\end{align*}
\]

We now mimic the proof of Proposition 3.7, for the special case where the valuation is trivial.

Take $u \in (3.4)$. If $(w, u) = (w, u')$ and $(v, u) \leq (v, u')$, then for every $\epsilon > 0$ we obtain

$$(w + \epsilon v, u) \leq (w + \epsilon v, u').$$

Assume now that $(w, u) < (w, u')$. We reduce to the case $(v, u) > (v, u')$. Let

$$\delta := (w, u' - u) > 0.$$ 

Then there exists $\epsilon > 0$ such that $0 < \epsilon (v, u - u') < \delta \Rightarrow \delta - \epsilon > 0$. From this we
conclude that

\[(w, u' - u) + (\epsilon v, u' - u) > 0.\]

It immediately follows that \(u \in (3.5)\). For the other contention, let \(u \in (3.5), u' \in \text{supp}(f)\). By continuity we have

\[
\lim_{\epsilon \to 0^+} \langle w + \epsilon v, u \rangle \leq \lim_{\epsilon \to 0^+} \langle w + \epsilon v, u' \rangle.
\]

That is, \(\langle w, u \rangle \leq \langle w, u' \rangle\). For each \(u'\) where a strict inequality holds, there is nothing to do. If equality holds, then from \(\langle w + \epsilon v, u \rangle \leq \langle w + \epsilon v, u' \rangle\) it follows that \(\epsilon \langle v, u \rangle \leq \epsilon \langle v, u' \rangle \Rightarrow \langle v, u \rangle \leq \langle v, u' \rangle\).

\[
\square
\]

For a function \(f \in I\), if \(\text{in}_w(f)\) is not a monomial (that is, it contains more than one monomial), we know that \(\text{val}(c_u) + \langle w, u \rangle = \text{val}(c_{u'}) + \langle w, u' \rangle \leq \cdots\). An equality like that is “almost impossible” to maintain if we move \(w\) a little bit. That is, it is almost sure that \(\text{in}_{w+\epsilon v}(f) = \text{d}_v \text{in}_w(f)\) is going to be a monomial. Now, for arbitraries \(w, v' \in \mathbb{R}^{n+1}\) define the set

\[
M_{v'} := \{ m \in \text{d}_v \text{in}_w(I) : m \text{ is a monomial} \}.
\]

In particular, we are interested in \(v\) such that, for a fixed \(w\), \(\langle M_v \rangle\) is maximal with respect to contention. The existence of such a \(v\) is guaranteed because \(K[x_0, \cdots, x_n]\) is Noetherian. It turns out that every monomial in this ideal is generated by an element in \(I\).

**Lemma 3.9.** For every monomial \(m \in \langle M_v \rangle\), there exists \(f \in I\) such that \(m = d_v \text{in}_w(f)\)

**Proof.** Since \(m\) is a monomial, it follows that

\[
m = c_u x^u m', m' \in M_v \Rightarrow m = c_u x^u d_v \text{in}_w(f_{m'}) = d_v \text{in}_w(c_u x^u f_{m'})
\]
Proposition 3.10. \( \langle M_v \rangle = \langle d_v \text{in}_w(I) \rangle \).

Proof. For \((\subseteq)\), if \( f \in \langle M_v \rangle \)

\[
f = \sum m_i = \sum d_v \text{in}_w(f_i) \in \langle d_v \text{in}_w(I) \rangle
\]

For \((\supseteq)\), let \( f \in I \) such that \( d_v \text{in}_w(f) = \sum m_i + N = \sum d_v \text{in}_w(f_i) + N \), where \( m_i \) are monomials in \( M_v \) and \( N \) contains only monomials not in \( M_v \). The lift, \( f = (\sum m_i) + M + M' + N + N' + N'' \) satisfies \( \text{in}_w(f) = \sum m_i + M + N + N'' \), \( d_v \text{in}_w(f) = \sum m_i + N \).

For each monomial \( m_i \in M_v \) we can find \( f_i = m_i + M_i + M_i' + N_i + N_i' \in I \) such that \( \text{in}_w(f_i) = m_i + M_i + N_i \) and \( d_v \text{in}_w(f_i) = m_i \) (where \( m_i, M_i, M_i' \) contain monomials in \( M_v \), and \( N_i, N_i' \) do not). In terms of the weight, \( W_{f_i}(w) = W_{m_i}(w) = W_f(w) = W_M(w) = W_N(w) = W_{N'}(w) = W_{M'}(w) = W_{M'}(w), W_{M'}(w), W_{N'}(w) \). And, \( W'_{\text{in}_w(f_i)}(v) = W'_{N'}(v) = W'_{m_i}(v) < W'_{M'}(v), W'_{N'}(w), W'_{N'}(v) \). Therefore,

\[
f - \sum_i f_i = M - \sum M_i + M' - \sum M_i' + N + N'' - \sum N_i + N' - \sum N_i'
\]

\[
\Rightarrow \text{in}_w \left( f - \sum_i f_i \right) = M - \sum M_i + N - \sum N_i + N''
\]

\[
\Rightarrow d_v \text{in}_w \left( f - \sum_i f_i \right) = \tilde{N} \text{ where } \tilde{N} \text{ has support in } N.
\]

Hence, \( f' := f - \sum f_i \in I \) contains only monomials in \( N \). Pick \( v' \in \Gamma_{val}^{n+1} \) such that \( d_v d_v \text{in}_w(f') \) is a monomial. Observe that for every \( g \in I \) such that \( d_v \text{in}_w(g) \) is a monomial,
then $d_v' d_v \text{in}_w(g)$ is also a monomial, so, for some $\epsilon > 0$ such that $d_v' d_v \text{in}_w(g) = d_{v+\epsilon v} \text{in}_w(g)$ for some generators for $M_v$, 

$$d_v' d_v \text{in}_w(f') = d_{v+\epsilon v} \text{in}_w(f') \in M_{v+\epsilon v} \supset M_v$$

By the maximality of $M_v$, $\langle M_{v+\epsilon v} \rangle = \langle M_v \rangle$. Therefore, $N = \emptyset$, and so $f' \in \langle M_v \rangle$.

Lastly, let $f' \in \langle d_v \text{in}_w(I) \rangle$. Then, $f' = \sum c_u x^{u_i} d_v \text{in}_w(f_i) = \sum d_v \text{in}_w(c_u x^{u_i} f_i)$. The previous result guarantees that $d_v \text{in}_w(c_u x^{u_i} f_i) \in \langle M_v \rangle \Rightarrow f' \in \langle M_v \rangle$. \qed

Proposition 3.7 establishes that $d_v \text{in}_w(f) = \text{in}_w + \epsilon v(f)$, and so we would like to prove the same result for the ideals $\langle d_v \text{in}_w(I) \rangle, \langle \text{in}_w + \epsilon v(I) \rangle$. We already fixed $v$ for $\langle M_v \rangle$ to be maximal. We now denote by

$$M'_v := \left\{ m \in \text{in}_w + \epsilon' v(I) : m \text{ is a monomial, } w, v \in \mathbb{R}^{n+1}, \epsilon' \in \mathbb{R}^+ \right\}. \quad (3.7)$$

And pick $(v, \epsilon)$ such that $\langle M'_v \rangle$ is maximal, for a fixed $w$. We first prove a result that is similar to Proposition 3.10

**Proposition 3.11.** $\langle M'_v \rangle = \langle \text{in}_w + \epsilon v(I) \rangle$.

**Proof.** For $(\subset)$, let $f \in \langle M'_v \rangle$

$$f = \sum m_i = \sum \text{in}_w + \epsilon v(f_i) \in \langle \text{in}_w + \epsilon v(I) \rangle$$

For $(\supset)$, let $f' \in \text{in}_w + \epsilon v(I)$ contain only monomials not in $M'_v$. For that matter assume as before that $f = \sum m_i + M + N + N'$ such that $\text{in}_w + \epsilon v(f) = \sum m_i + N = \sum \text{in}_w + \epsilon v(f_i) + N$, with $f_i = m_i + M_i + N_i$ for $M_i \in \langle M'_v \rangle$ and $N_i$ not in $\langle M'_v \rangle$. Using the same argument, $W_f(w+\epsilon v) = W_{m_i}(w+\epsilon v) = W_N(w+\epsilon v) < W_M(w+\epsilon v), W_{M_i}(w+\epsilon v), W_{N_i}(w+\epsilon v), W_{N'}(w+\epsilon v)$. It follows that
\[ f - \sum_i f_i = M - \sum_i M_i + N - \sum_i N_i + N' \]
\[ \Rightarrow \text{in}_{w+\epsilon v} \left( f - \sum_i f_i \right) = \tilde{N} \text{ where } \tilde{N} \text{ has support in } N. \]

Just as constructed in Proposition 3.10, take \( f' = f - \sum_i f_i \). Now choose \( v' \in \mathbb{R}^{n+1} \) such that \( d_{v'} \text{in}_{w+\epsilon v'}(f) \) is a monomial not in \( M_{v'} \). Then, for \( \epsilon' \) small enough we have,
\[ d_{v'}(f') = d_{v'} \text{in}_{w+\epsilon v}(f) = \text{in}_{w+\epsilon v+\epsilon' v'}(f) = \text{in}_{w+\epsilon v+\epsilon' v'}(I) \subset \langle M_{v+\epsilon' v'} \rangle, \]
where \( \epsilon' \) is small enough for making \( \text{in}_{w+\epsilon v+\epsilon' v'}(g) = \text{in}_{w+\epsilon v}(g) \), and the \( \text{in}_{w+\epsilon v}(g) \) are monomials generating \( M_{v'} \). Hence, \( \langle M_{v'} \rangle \subset \langle M_{v+\epsilon' v'} \rangle \) and that would be a contradiction to the maximality of \( \langle M_{v'} \rangle \).

This method guarantees that no monomial outside of \( M_{v'} \) is in \( \text{in}_{w+\epsilon v}(f) \), thus every monomial of \( f' \) is an element of \( \langle M_{v'} \rangle \), and so \( f' \in \langle M_{v'} \rangle \).

Finally, let \( f' \in \langle \text{in}_{w+\epsilon v}(I) \rangle \). Then, \( f' = \sum c_u x^u \text{in}_{w+\epsilon v}(f_i) = \sum \text{in}_{w+\epsilon v}(c_u x^u f_i) \). The previous result guarantees that \( \text{in}_{w+\epsilon v}(c_u x^u f_i) \in \langle M_{v'} \rangle \Rightarrow f' \in \langle M_{v'} \rangle \).

We introduce the notation
\[ S_K := K[x_0, \ldots, x_n] \tag{3.8} \]

Note that for every \( w \in \Gamma_{val}^{n+1} \), we obtain one \( M_v \), which initially is dependent of \( w \). We now prove that the codimension of \( \langle M_v \rangle \) and \( I \) are the same. 3.12 and 3.14 are equivalent to 2.12, generalizing 2.4.7 of [MS13].

**Proposition 3.12.** Suppose that \( \langle \text{in}_w(I) \rangle \) is generated by the monomials in \( \text{in}_w(I) \). Then \( \langle M_v(w) \rangle = \langle \text{in}_w(I) \rangle \). For \( d \in \mathbb{N} \), the monomials of degree \( d \) that are not in \( M_v \) form a basis for \( (S_K/I)_d \).

**Proof.** We first proof the linear independence of \( G := \{ x^u \notin M_v : |u| = d \} \). Assume there
exists \( a_u \in K \) such that \( \sum a_u x^u \in I_d \). Then, \( d_v \text{in}_w(\sum a_u x^u) \) is not a monomial, as if it were, it would be on \( M_v \). Moreover, \( d_v \text{in}_w(\sum a_u x^u) \not\in \langle M_v \rangle \), as it only contains monomials not in \( M_v \) and so it cannot be generated by monomials in \( M_v \). However, by proposition 3.10

\[ d_v \text{in}_w(\sum a_u x^u) \in d_v \text{in}_w(I) \subset \langle d_v \text{in}_w(I) \rangle = \langle M_v \rangle \]

Which is a clear contradiction, unless every \( a_u = 0 \). That proves the linear independence. It also shows \( \dim(\langle M_v \rangle d) \geq \dim(I_d) \).

In the aim of a contradiction, assume \( \dim(\langle M_v \rangle_d) = \dim(\langle \text{in}_w(I) \rangle_d) > \dim I_d \). Therefore, we can find a linearly independent set \( \{ m_i \} = \{ x^{u_i} \} = \{ \text{in}_w(f_i) \} \subset \text{in}_w(I)_d \) such that there exist \( c_i \in K^* \) with \( \sum c_i f_i = 0 \). For this proposition we denote \( f_i = \sum_j a_{ij} x^{u_j} \) where some \( a_{ij} \) may be equal to 0, but for every \( i \) there is at least one \( j \) with \( a_{ij} \neq 0 \), and \( a_{ii} = 1 \). For each \( i \) we have

\[ \text{val}(c_i) = \text{val}(c_i) + \text{val}(a_{ii}) \]

And, because \( x^{u_i} = \text{in}_w(f_i) \), we also know that for every \( j \neq i \), \( \text{val}(a_{ij}) + \langle w, u_j \rangle > \langle w, u_i \rangle \).

Therefore,

\[ \text{val}(c_i) + \langle w, u_i \rangle < \text{val}(c_i) + \text{val}(a_{ij}) + \langle w, u_j \rangle \]

Now select \( u_j \in \mathbb{N}^{n+1} \) in the support of the functions, such that \( \text{val}(c_j) + \langle w, u_j \rangle \leq \text{val}(c_j) + \langle w, u_j \rangle \forall u_j \neq u_j \). Our assumption implies \( \sum_i c_i a_{ij} = 0 \), and

\[ \text{val}(c_j) + \langle w, u_j \rangle \leq \text{val}(c_i) + \langle w, u_i \rangle < \text{val}(c_i) + \text{val}(a_{ij}) + \langle w, u_j \rangle \]

That implies \( \text{val}(c_j) = \text{val}(c_j) + \text{val}(a_{ij}) < \text{val}(c_i) + \text{val}(a_{ij}) \), thus we have a coefficient of least valuation that will not cancel with any other element. That implies \( \sum_i c_i a_{ij} \neq 0 \), and that is our desired contradiction.

\[ \square \]

Both \( \langle M_v \rangle \), \( \langle M'_v \rangle \) are monomial ideals, and we can find functions \( f_i \in I \) that only contain
one monomial in $M_v$. This useful result is proved now. Proposition 3.13 and Corollary 3.15 are the equivalent results to 2.13 in the previous chapter (and 2.4.7 in [MS13]).

**Proposition 3.13.** Let $m = d_1 \in_w (f) \in M_v$ be a monomial of degree $d$. Then we can find $f \in I_d$ such that $f = m + N$ where $N$ contains only monomials of degree $d$ that are not in $M_v$.

**Proof.** This is now a linear algebra problem. In Proposition 3.12 we found that the functions \{f_i\} $\subset I_d$ such that $\in_w (f_i) = x^{u_i} \in M_v$ are linearly independent. We first “triangularize the matrix”. Let $f_i = \sum_j c_{ij} x^{u_j} + N_i$, with $c_{ii} = 1$. The set \{f_1, f_2 - c_{21} f_1, \ldots, f_n - c_{n1}\} is still linearly independent. Indeed, if

$$a_1 f_1 + \sum_{i=2}^n a_i (f_i - c_{i1} f_1) = \left( a_1 - \sum_{i=2}^n a_i c_{i1} \right) f_1 + \sum_{i=2}^n a_i f_i = 0$$

Then, $a_2 = \cdots = a_n = 0$, so $a_1 = 0$. Observe that $\in_w (f_i - c_{i1} f_1) = (1 - c_{i1} c_{11}) x^{u_i}$. We exemplify this looking at the first two polynomials. Let $f_1 = x^{u_1} + c_2 x^{u_2} + c_3 x^{u_3} + R_1, f_2 = d_1 x^{u_1} + x^{u_2} + d_3 x^{u_3} + R_2$, and let us look at

$$f_2 - d_1 f_1 = (1 - c_2 d_1) x^{u_2} + (d_3 - c_3 d_1) x^{u_3} + R_2 - d_1 R_1$$

First note that $\langle w, u_1 \rangle < \text{val}(c_2) + \langle w, u_2 \rangle < \text{val}(c_2) + \text{val}(d_1) + \langle w, u_1 \rangle$. Therefore, $0 < \text{val}(c_2 d_1)$, and we conclude (using Proposition 1.22) that $\text{val}(1 - c_2 d_1) = 0$. Now assume $d_3 - d_1 c_3 \neq 0$ (if all the coefficients of the form $d_j - c_j d_1$ were zero there would be nothing to show). Observe that $\text{val}(d_3 - c_3 d_1) + \langle w, u_3 \rangle \geq \min\{\text{val}(d_3), \text{val}(c_3 d_1)\} + \langle w, u_3 \rangle$. If $\min\{\text{val}(d_3), \text{val}(c_3 d_1)\} = \text{val}(d_3)$ we obtain

$$\text{val}(d_3 - c_3 d_1) + \langle w, u_3 \rangle \geq \text{val}(d_3) + \langle w, u_3 \rangle > \langle w, u_2 \rangle = \text{val}(1 - c_2 d_1) + \langle w, u_2 \rangle$$

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And, if \( \min\{\text{val}(d_3), \text{val}(c_3d_1)\} = \text{val}(c_3d_1) \) we obtain

\[
\text{val}(d_3 - c_3d_1) + \langle w, u_3 \rangle \geq \text{val}(d_1) + \text{val}(c_3) + \langle w, u_3 \rangle > \text{val}(d_1) + \langle w, u_1 \rangle
\]

\[
> \langle w, u_2 \rangle = \text{val}(1 - c_2d_1) + \langle w, u_2 \rangle
\]

Therefore, \( \hat{f}_1 := f_1, \hat{f}_i := \frac{1}{1 - c_ii} (f_i - c_if_i) \) form a linearly independent set in the vector space \( I_d \), such that \( \text{in}_w(\hat{f}_i) = x^{u_i} \). Further, for \( i > 1, \hat{f}_i \) does not contain monomials with support in \( u_1 \). We now repeat the process until we have a “triangular matrix”. Once we have it, we diagonalize it. Assume \( f_{n-1} = x^{u_{n-1}} + c_n x^{u_n} + N_{n-1}, f_n = x^{u_n} + N_n \).

\[
f_{n-1} - c_nf_n = x^{u_{n-1}} + N_{n-1} - c_nN_n \in I_d
\]

Let \( \bar{n} = c_{\bar{n}}x^{u_{\bar{n}}} \) be one of the monomials in \( N_{n-1} - c_nN_n \). If it is only a monomial in \( N_{n-1} \), then we know that \( \langle w, u_{n-1} \rangle < \text{val}(c_{\bar{n}}) + \langle w, u_{\bar{n}} \rangle \). If it is only a monomial in \( N_n \), then \( \langle w, u_n \rangle < \text{val}(c_{\bar{n}}) + \langle w, u_{\bar{n}} \rangle \), and so, \( \langle w, u_{n-1} \rangle < \text{val}(c_{\bar{n}}) + \langle w, u_n \rangle \). Finally, if it is a monomial in both, then \( \text{val}(c_{\bar{n}-1} - c_n c_{\bar{n}}) + \langle w, u_{\bar{n}} \rangle \geq \min\{\text{val}(c_{\bar{n}-1}), \text{val}(c_n c_{\bar{n}})\} + \langle w, u_{\bar{n}} \rangle \). Note that we assumed \( c_{\bar{n}-1} - c_n c_{\bar{n}} \neq 0 \). If \( \min\{\text{val}(c_{\bar{n}-1}), \text{val}(c_n c_{\bar{n}})\} = \text{val}(c_{\bar{n}-1}) \), then

\[
\text{val}(c_{\bar{n}-1} - c_n c_{\bar{n}}) + \langle w, u_{\bar{n}} \rangle \geq \text{val}(c_{\bar{n}-1}) + \langle w, u_{\bar{n}} \rangle > \langle w, u_{n-1} \rangle
\]

And, if \( \min\{\text{val}(c_{\bar{n}-1}), \text{val}(c_n c_{\bar{n}})\} = \text{val}(c_n c_{\bar{n}}) \),

\[
\text{val}(c_{\bar{n}-1} - c_n c_{\bar{n}}) + \langle w, u_{\bar{n}} \rangle \geq \text{val}(c_{\bar{n}}) + \text{val}(c_n) + \langle w, u_{\bar{n}} \rangle
\]

\[
> \text{val}(c_{\bar{n}}) + \langle w, u_{\bar{n}} \rangle > \langle w, u_{n-1} \rangle
\]

Therefore, \( \text{in}_w(f_{n-1} - c_nf_n) = x^{u_{n-1}} \). We complete the process until we get a “diagonal matrix,” and that finishes the proof. 

\[ \square \]
Corollary 3.14 (to the proof of Proposition 3.12). For every $w \in \Gamma_{\text{val}}^{n+1}$ and $d \in \mathbb{N}$, the monomials of degree $d$ that are not in $M^*_{\epsilon}$ form a basis for $(S_K/I)_d$.

Proof. The proof follows the same steps as in Proposition 3.12. We first proof the linear independence of $G^* := \{x^u \notin M^*_{\epsilon} : |u| = d\}$. If there exists $a_u \in K$ such that $\sum a_u x^u \in I_d$, by proposition 3.11

$$\text{in}_{w+ev}(\sum a_u x^u) \in \text{in}_{w+ev}(I) \subset (\text{in}_{w+ev}(I)) = \langle M^*_{\epsilon} \rangle$$

A clear contradiction, unless every $a_u = 0$. That proves the linear independence. It also shows $\dim((M^*_{\epsilon})_d) \geq \dim(I_d)$. We will again assume that $\dim((M^*_{\epsilon})_d) > \dim(I_d)$ in order to get a contradiction.

Would that be the case, we could find $x^{u_1} = \text{in}_{w+ev}(f_1) \in M^*_{\epsilon}$ linearly independent, such that $\{f_i\} \subset I_d$ is a linearly dependent set. Hence, there are coefficients $c_i \in K^*$ such that $\sum c_i f_i = 0$. We denote $f_i = \sum_j a_{ij} x^{u_j}$ with $a_{ii} = 1$. Also, select $\hat{i}$ such that $\text{val}(c_{\hat{i}}) + \langle w + ev, u_{\hat{i}} \rangle \leq \text{val}(c_i) + \langle w + ev, u_i \rangle$ for all $i \neq \hat{i}$.

We know that $\sum_j c_j a_{ji} = 0$, and for $j \neq \hat{i}$,

$$\text{val}(c_j) + \text{val}(a_{j\hat{i}}) + \langle w + ev, u_{\hat{i}} \rangle > \text{val}(c_j) + \langle w + ev, u_j \rangle$$

due to the fact that $\text{in}_{w+ev}(f_j) = x^{u_j}$. And by the property of $\hat{i}$,

$$\text{val}(c_j) + \text{val}(a_{j\hat{i}}) + \langle w + ev, u_{\hat{i}} \rangle > \text{val}(c_j) + \langle w + ev, u_j \rangle \geq \text{val}(c_j) + \langle w + ev, u_j \rangle$$

which implies that for all $j \neq \hat{i}$, $\text{val}(c_{\hat{i}}) = \text{val}(c_i) + \text{val}(a_{ii}) < \text{val}(c_j) + \text{val}(a_{j\hat{i}})$, and so, $\sum_j c_j a_{ji}$ can never be zero. Therefore, $\dim((S_K/I)_d) = |G^*|$.

\[\square\]

Corollary 3.15. Let $m = \text{in}_{w+ev}(f) \in M^*_{\epsilon}$ of degree $d$. Then we can find $f \in I_d$ such that $f = m + N$ where $N$ contains only monomials of degree $d$ that are not in $M^*_{\epsilon}$. 

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Proof. The proof is identical to the one in Lemma 3.13. All we have to do is to change $w$ by $w + \epsilon v$.

The equality $\langle M_v \rangle = \langle M^\epsilon_v \rangle$ holds, as we show now. It generalizes 2.15 (or 2.4.5 in [MS13]) to the new definition of initial forms.

**Corollary 3.16.** $\langle M_v \rangle = \langle M^\epsilon_v \rangle$, that is, there is $\epsilon > 0$ such that $M^\epsilon_v = M_v$ and $M^\epsilon_v$ is maximal.

*Proof.* Note that for small enough $\epsilon$, $M_v \subset M^\epsilon_v$. As in the proof of Proposition 3.12,

$$\dim(\langle M_v \rangle_d) \geq \dim(I_d)$$

because the monomials not in $M_v$ are linearly independent mod $I$. Let $M^\epsilon_v \subset M^\epsilon_v$, with the latter maximal. By Corollary 3.15, $\langle M^\epsilon_v \rangle$ has the same Hilbert functions as $I$. Hence,

$$\langle M_v \rangle_d = \langle M^\epsilon_v \rangle \Rightarrow \langle M_v \rangle = \langle M^\epsilon_v \rangle.$$

With $M^\epsilon_v$ maximal. Since $\langle M_v \rangle = \langle M^\epsilon_v \rangle$ are monomial ideals, then the monomials in them must be equal. We conclude $M_v = M^\epsilon_v$.

Finally, we tie all the previous results in the following corollary (the result is a partial generalization of 2.14 and 2.4.8 in [MS13] for in$^K$).

**Corollary 3.17.** For every $d \in \mathbb{N},$

$$\dim((S_K/I)_d) = \dim((S_K/\langle M_v \rangle)_d) = \dim((S_K/\langle M^\epsilon_v \rangle)_d).$$

*Proof.* Proposition 3.12 shows that $\dim((S_K/I)_d) = |G| = \dim((S_K)_d) - \dim(\langle M_v \rangle)_d) = \dim((S_K/\langle M_v \rangle)_d)$. Corollary 3.14 shows $\dim((S_K/I)_d) = |G^\epsilon| = \dim((S_K)_d) - \dim(\langle M^\epsilon_v \rangle)_d) = \dim((S_K/\langle M^\epsilon_v \rangle)_d)$. **Therefore**, the equality holds.

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\[
\dim((S_K/\langle M_\sigma \rangle_d). \\
\]

Despite the fact that \( M_v \) depends of each \( w \in \mathbb{R}^{n+1} \), there are not so many of those. In fact, the same set of monomials will work for a number of \( w \) within certain neighborhood. In order to see that, we introduce the following sets:

\[
C_I(w) := \{ w' \in \mathbb{R}^{n+1} : \langle \text{in}_w(I) \rangle = \langle \text{in}_{w'}(I) \rangle \}. \tag{3.9}
\]

We now generalize the result given in 3.13.

**Lemma 3.18.** Let \( x^u \in \text{in}_w(I) \). Let \( M_v \) be maximal as before. Then we can find \( f \in I \) such that \( f = m + N \) where \( N \) contains only monomials that are not in \( M_v \).

**Proof.** By definition \( x^u = \text{in}_w(f) \) for some \( f \). Let \( f = x^u + g_1 + g_2 \) where \( g_1 \) involves no monomial in \( M \) and \( g_2 \) involves only monomials in \( M = M_v \). Let \( m \) be a monomial in \( g_2 \). Then by the above lemma \( m = \text{in}_{w+cv}(h) \) where \( h = m + R \) and \( R \) involves only monomials not in \( M_v \). Moreover, \( W_R(w) \geq W_m(w) > W_f(m) \). So \( f' := f - h \) will still have \( \text{in}_w(f') = x^u \), but the number of monomials in \( M \) strictly decreased. Repeating if necessary we arrive at an \( f \) as desired.

\]

By the previous lemma we guarantee the existence of \( f_i := x^{u_i} + N \in I \), where \( N \) is a set of monomials that are not in \( M_v \).

The points \( w \in \mathbb{R}^{n+1} \) such that \( \text{in}_w(I) \) is the generator of the monomial ideal will be special to us.

**Definition 3.19.** Let \( w \in \mathbb{R}^{n+1} \) such that \( \langle \text{in}_w(I) \rangle \) is generated by the monomials in \( \text{in}_w(I) \). \( w \) is called a generic point.

With that definition we prove the following lemma.
Lemma 3.20. Suppose \( w \) is generic and let \( M \subset \text{in}_w(I) \) be the list of monomials. Then \( M \) is maximal: If for \( x \in \Gamma_{\text{val}}^{n+1} \) we have \( M \subset \text{in}_x(I) \), then \( M \) is the list of all monomials in \( \text{in}_x(I) \).

Proof. First observe that \( \langle M \rangle \) has the same Hilbert function as \( I \) where \( H \) is the Hilbert function, as defined in Definition 1.19. Indeed, by Proposition 3.12 \( H_{\langle M \rangle} = H_I \) because \( \langle \text{in}_w(I) \rangle = \langle M_v(w) \rangle = \langle M \rangle \).

Assume \( M \subset \text{in}_x(I) \). Select \( v \in \mathbb{R}^{n+1} \) such that \( M_v(x) \) is maximal. Again,

\[
H_{\langle M_v(x) \rangle} = H_I = H_{\langle M \rangle}
\]

So, \( \langle M \rangle = \langle M_v(x) \rangle \), and the latter is generated by the monomials of \( \text{in}_x(I) \).

\[ \square \]

Corollary 3.21. With the notation as in Lemma 3.20, if \( M \subset \text{in}_x(I) \), then \( x \) is generic and \( x \in C_I(w) \).

Proof. Since \( M \) is generated by monomials, then for every \( v \in \mathbb{R}^{n+1} \) with \( M_v(x) \) maximal we have that \( \langle d_v \text{in}_w(I) \rangle = \langle M \rangle \). If \( x \) is not generic, then we can find \( f \in I \) such that \( \text{in}_x(f) \notin \langle M \rangle \). As usual, let \( f = m + m' + N + N' \), where \( N \) is comprised by monomials not generated by \( M \), \( m \in \langle M \rangle \), and \( W_f(x) = W_m(x) - W_{m'}(x), W_{N'}(x) \). We know \( m = \sum m_i \). Let \( f_i \in I \) such that \( f_i = m_i + N_i \) and \( \text{in}_x(f_i) = m_i \), that is, \( W_{f_i} = W_{m_i}(x) = W_m(x) < W_{N_i} \). Those functions exist because by assumption \( M \subset \text{in}_x(I) \). Thus, \( f' := f - \sum f_i = m' + N + N' - \sum N_i \). Due to weighting considerations, \( \text{in}_x(f') \neq 0 \), and in fact, it is supported by \( N \). Hence, we can find \( v \in \Gamma_{\text{val}}^{n+1} \) that will single out one monomial supported in \( N \). We now have the contradictory statement \( d_v \text{in}_x(f') \notin \langle M \rangle \).

\[ \square \]

We now show that the \( C_I(w) \) for a generic \( w \) form a polyhedral complex. We did the same with the old definition of initial terms in Proposition 2.17. The result is equivalent to a part of Proposition 2.5.2 in [MS13].

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Proposition 3.22. Let $w \in \mathbb{R}^{n+1}$ generic with Gröbner basis

$$\{f_i = x^{u_i} + N_i : \text{supp}(N_i) \cap \text{supp}(M) = \emptyset\}$$

Then, for every $u_j \in \text{supp}(N_i)$,

$$C_I(w) := \left\{ y \in \mathbb{R}^{n+1} : \langle y, u_i \rangle < \text{val}(c_{u_j}) + \langle y, u_j \rangle \forall u_j \in \text{supp}(N_i) \right\}$$ (3.10)

Proof. For $(\subset)$ let $x \in \mathbb{R}^{n+1}$. Notice that $\text{in}_x(f_i) = x^{u_i}$. Indeed, by assumption, $\langle \text{in}_x(I) \rangle = \langle \text{in}_w(I) \rangle = \langle M \rangle$. That implies $\text{in}_x(f_i) = x^{u_i} + N_i'$. But then, $\text{in}_w(f_i) - \text{in}_x(f_i) \in \langle \text{in}_w(I) \rangle = \langle M \rangle$, and so we must have $N_i' = 0$. Therefore,

$$\langle x, u_i \rangle < \text{val}(c_{u_j}) + \langle x, u_j \rangle$$ (3.11)

That completes the proof of $(\subset)$.

For $(\supset)$ consider $x \in \mathbb{R}^{n+1}$ satisfying the inequalities of (3.10). Therefore, $M \subset \text{in}_x(I)$. That implies $x$ is generic, and so $\langle \text{in}_x(I) \rangle = \langle \text{in}_w(I) \rangle$.

As we have seen, every $w$ satisfying the inequalities of (3.10) generates the same monomial ideal $\langle M_w \rangle$. If we prove the existence of only finitely many $M_w$’s, we will have proven the existence of finitely many polyhedrons, and the union of all of them will form the polyhedral complex. First observe that for all $w \in \Gamma_{\text{val}}^{n+1}$, by Corollary 3.17, $\dim(\langle \text{in}_w(I) \rangle_d) = \dim(\langle M_w(w) \rangle_d)$ remains constant. Therefore, for every $\langle M_w \rangle$, we know that

$$\forall d \in \mathbb{N}, \forall v, w, v', w' \in \Gamma_{\text{val}}^{n+1}, \langle M_v \rangle, \langle M_w \rangle \text{ maximal satisfying Proposition 3.10}$$

$$H_{\langle M_v(w) \rangle}(d) = H_{\langle M_v(w') \rangle}(d)$$ (3.12)
In order to prove the finiteness of these monomial ideals, we prove that if there were an infinite number of them, then we would be able to construct an infinite ascending chain.

**Lemma 3.23.** If a monomial ideal $M$ is contained in infinitely many monomial ideals $\{M_i\}_{i \in I}$, such that $H_M \neq H_{M_i} \forall i, j \in I$, then there exists $M'$ such that $M \subsetneq M' \subsetneq M_i$ for infinitely many $i \in I' \subset I$.

**Proof.** First, since $M \subset M_i$, and they have different Hilbert functions, it follows that $M \subsetneq M_i$. Let $d \in \mathbb{N}$ be the minimum number such that $H_M(d) < H_{M_i}(d)$. Using the notation introduced in (3.8), let $N_d := \{x^{u_1}, \ldots, x^{u_r}\} \subset (S_K)_d \setminus M_d$. Observe that $1 < |N_d|$, or else, for every $i \in I$, $\dim((M_i)_d) = \dim((S_K)_d)$. That would imply that for every $d'' \geq d, H_{M_i}(d'') = H_{(S_K)_d}(d'')$, and for every $d' < d, H_{M_i}(d') = H_{M}(d')$, so for every $i, j \in I, (M_i)_{d'} = (M_j)_{d'}, (M_i)_{d''} = (M_j)_{d''}$, and that is a contradiction to the fact that $I$ is an infinite set. Therefore, $M \subsetneq M_i := \langle M, x^{u_k} \rangle, x^{u_k} \in N_d$.

$I$ index an infinite family, and $N_d$ is a finite collection of monomials that are not in $M$, but since $H_M(d) < H_{M_i}(d)$, at least one of the elements in $N_d$ will belong to infinitely many of the monomial ideals indexed by $I$. That is, there exist $x^{u_k}$ such that $M'_k \subseteq M_i$, for $i \in I' := \{i \in I : x^{u_k} \in (M_i)_d\}$. Set $M' := M'_k$. We can repeat the same process for $M'$, proving the result.

The next proposition is equivalent to 2.20 and appeared as Lemma 2.5.4 in [MS13].

**Proposition 3.24.** Let $w, v \in \Gamma_{val}^{n+1}$ and $\langle M_v \rangle$ as in Proposition 3.10. Then, there are only finitely many different monomial ideals $\langle M_v \rangle$’s.

**Proof.** We get a contradiction if we assume the opposite. That is, if there are infinitely many monomial ideals $\langle M_w \rangle$, then we can construct an infinite ascending chain in the Noetherian ring $S_K$. In order to do so, observe that $\langle 0 \rangle \subsetneq \langle M_v \rangle$, and apply Lemma 3.23.

\[\square\]
So far we have worked with homogeneous ideals, in a homogeneous ring. We now move on to general ideals in the Laurent polynomial ring.

3.3 Ideals in the Laurent polynomial ring

The main idea of this section is to find the equivalent of a Gröbner basis for an ideal in a Laurent ring. First, observe that for \( f \in K[x^\pm], \) \( \text{in}_w(f) = \text{in}_{(0,w)}(x_0^m f(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0})) \), for some \( m \in \mathbb{N} \). We now formalize these concepts.

**Definition 3.25** (Definition 2.2.5 in [MS13]). For \( f \in K[x_1^\pm, \cdots, x_n^\pm] \), we define its projectivization \( f_{\text{proj}} \) by the following homogeneous polynomial

\[
f_{\text{proj}} := x^u f \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right)
\]

such that \( u \in \mathbb{Z}^{n+1} \) is such that \( u_i \) is the minimum value for which \( f_{\text{proj}} \) has only positive exponents.

The new definition of initial forms do not affect the results of Propositions 2.24 and 2.25, which proof is presented here (they form part of Proposition 2.6.2 in [MS13]).

**Proposition 3.26.** For \( f \in K[x^\pm], \) \( w \in \Gamma_{val} \), \( \text{in}_{(0,w)}(f_{\text{proj}}) = \text{in}_{(0,w)} \left( x^u f \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right) \right) = x^u \text{in}_w \left( f \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right) \right). \)

**Proof.** By Lemma 3.3, we know

\[
\text{in}_{(0,w)} \left( x^u f \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right) \right) = x^u \text{in}_{(0,w)} \left( f \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right) \right) = x^u \text{in}_w \left( f \left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right) \right).
\]

The last equality holds because \( f \) does not have an \( x_0 \) component. \( \square \)

In the opposite direction we have:
Proposition 3.27. Let \( f, f_{\text{proj}} \) as in (3.13).

\[
in_w(f(x_1, \cdots, x_n)) = (x^{-u} \in_{(0,w)}(f_{\text{proj}}(x_0, \cdots, x_n)))\bigg|_{x_0=1}
\]

Proof. We just have to apply the previous proposition:

\[
(x^{-u} \in_{(0,w)}(f_{\text{proj}}(x_0, \cdots, x_n)))\bigg|_{x_0=1} = (x^{-u} x^u \in_w \left(f\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right)\right))\bigg|_{x_0=1} = \in_w(f).
\]

\(\square\)

Example 3.28. Let \( f(x_1, x_2) = tx_1 + x_1 x_2^{-1}, w = (1, -1) \).

Observe that \( f\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) = t x_1 x_2^{-1} + x_1 x_2^{-1} \), so \( u = (1, 0, 1) \in \mathbb{Z}^3 \) is such that

\[
x^u f = tx_1 x_2 + x_0 x_1 =: f_{\text{proj}}
\]

Then, \( \in_{(0,w)}(f_{\text{proj}}) = \in_{0, -1}(tx_1 x_2 + x_0 x_1) = tx_1 x_2 + x_0 x_1 \), just as in Proposition 3.26.

Also, \( \in_w(f) = tx_1 + x_1 x_2^{-1} = x^{(-1, 0, -1)} t \frac{x_1}{x_0} + x_1 x_2^{-1} \bigg|_{x_0=1} = (x^{-u} \in_{(0,w)}(f_{\text{proj}})) \), as seen on Proposition 3.27. \(\square\)

For \( I \subset K[x^\pm] \) an ideal, we now define the Initial Laurent set and the Initial Laurent ideal in the way suggested by the previous propositions.

Definition 3.29 (Initial Laurent set). For \( I \subset K[x_0, \cdots, x_n] \), the initial set is the set

\[
in_w(I) := \{ \in_w(f) : f \in I \}
\]

We name the Initial Laurent ideal, to the ideal generated by the initial set, denoted by \((\in_w(I))\). The projectivization of such an ideal, \( I_{\text{proj}} \) is the homogeneous ideal:

\[
I_{\text{proj}} := \langle \{ f_{\text{proj}} : f \in I \} \rangle \tag{3.14}
\]
Propositions 3.30 and 3.31 are the equivalent of Propositions 2.27 and 2.28, and are presented as Proposition 2.6.2 in [MS13].

**Proposition 3.30.** Let \( g \in I_{proj} \) be a homogeneous polynomial of degree \( d \). Then there exists \( f \in I \) such that \( f_{proj} = g \).

**Proof.** As \( g \in \langle \{f_{proj} : f \in I\} \rangle \), \( g = \sum f^{(i)}_{proj} (x_0, \cdots, x_n) \). Moreover,

\[
g = \sum x^{u_i} f_i (x_1, \cdots, x_n) \quad (3.15)
\]

By the equivalence relation, \( g(x_0, \cdots, x_n) = g(1, \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}) \). So,

\[
g = g \left( 1, \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right) = \frac{1}{x_0} \sum x^{(u_i, 1, \cdots, u_i, n)} f_i (x_1, \cdots, x_n)
\]

\( f := \sum x^{(u_i, 1, \cdots, u_i, n)} f_i (x_0, \cdots, c_n) \in I \) contains only positive exponents, and so, trivially, \( f_{proj} = g \).

\( \square \)

**Proposition 3.31.** Let \( g \in I_{proj} \). Then \( g(1, x_1, \cdots, x_n) \in I \).

**Proof.** From (3.15), \( g(x_0, \cdots, x_n) = \sum x^{u_i} f_i (x_1, \cdots, x_n) \), which implies

\[
g(1, x_1, \cdots, x_n) = \sum f^{(i)}_{proj} (x_1, \cdots, x_n) \in I
\]

\( \square \)

Just as we have found counterexamples to some properties of \( \text{in}_w^k (f) \) for \( f \) homogeneous, we now find that Proposition 2.29 does not follow.

**Counterexample 3.32.** Let \( f(x_1, x_2) = tx_1 + x_1 x_2^{-1}, g(x_1, x_2) = t^{-1} x_2^{-1} + x_1^2 + t^3, w = \)
(1, −1). Then,

\[
\text{in}_w(f)\text{in}_w(g) = (tx_1 + x_1x_2^{-1})(t^{-1}x_2^{-1}) = x_1x_2^{-1} + t^{-1}x_1x_2^{-2}
\]

\[
\text{in}_w(fg) = \text{in}_w\left( (t^3 + 1)x_1x_2^{-1} + \cdots \right) = (t^3 + 1)x_1x_2^{-1}
\]

In fact, the best we can get is the following,

**Proposition 3.33.** Let \( f, g \in K[x^\pm] \), \( W_f \) as defined in (3.1). Then, \( W_{fg} = W_f + W_g \).

**Proof.** This proposition is independent from the initial forms of the functions. Therefore, the argument is the same as the one used in Proposition 2.29.

\[
\square
\]

### 3.4 Tropical bases

As we saw in Definition 2.32 and Proposition 2.33, in the absence of Gröbner bases for Laurent polynomials, we have tropical bases. Recall that for almost every \( w \in \Gamma_{\text{val}} \), \( \text{in}_w(f) \) will be a monomial, which is the case of the Laurent polynomial ring is equivalent to being a unit. If \( I = \langle f_1, \cdots, f_n \rangle \), and for some \( i \), \( \text{in}_w(f_i) \) is a monomial, we would like to say that \( 1 \in \text{in}_w(I) \). A set \( F := \{ f_1, \cdots, f_n \} \) satisfying that is called a tropical basis.

**Definition 3.34** (Tropical basis, adapted from Definition 2.6.4 of [MS13]). A set \( F := \{ f_1, \cdots, f_n \} \) contained in an ideal \( I = \langle F \rangle \subset K[x_1^\pm, \cdots, x_n^\pm] \) is called a finite Tropical basis if for every \( w \in \Gamma_{\text{val}} \), \( \text{in}_w(F) \) contains a monomial \( \iff \text{in}_w(I) \) contains a monomial .

The term finite refers to the cardinality of \( F \), and is the only interesting case (\( I \) itself is an infinite tropical basis). The term tropical will be discussed in the next chapter. We now prove the existence of a tropical basis for every ideal in the Laurent polynomial ring.
Proposition 3.35 (Existence of tropical basis (Theorem 2.6.5 in [MS13])). Every ideal
\[ I = K[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \] has a finite tropical basis.

Proof. If \( \text{in}_w(I) \) contains a monomial, then we know there exists \( f \in I \) such that \( \text{in}_w(f) = 1 \), and thus, \( \text{in}_{(0,w)}(f_{\text{proj}}) = m \), a monomial.

Let \( \langle M_1 \rangle, \cdots, \langle M_N \rangle \) be the finitely many monomial ideals as in Proposition 3.24, from which we extract a representative \( w_i \in \Gamma_{\text{val}}^{n+1} \). Also, let \( F_{i1}, \cdots, F_{iL_i} \) be the finitely many faces of \( C_I(w_i) \). Let \( F_0 := \{ f_i \} \subset I \) so that \( \text{in}_{(0,w)}(F_0) \) generates \( \langle M_i \rangle \) and \( F_0 \) is a generating set of \( I \).

First, if \( (0,w) \) is not in a face, it is a generic point. Thus, \( \text{in}_{(0,w)}(F_0) \) contains a monomial.

Otherwise \( (0,w) \) is in some face \( F_j \), associated to \( C_I(w_0) \), for \( w_0 \in \Gamma_{\text{val}}^{n+1} \) a generic point (so \( w_0 \in C_I(w_i) \)). Therefore, there exists \( g \in I_{\text{proj}} \) such that \( g = m + N \), for \( N \) not supported in \( M_i \). Hence, \( \text{in}_{w_i}(g) = m \) and we can add \( g(1,x_1,\cdots,x_n) \in I \) to \( F_0 \) and \( \text{in}_{(0,w)}(g) = m \). For every \( x \in F_i \), \( \text{in}_x(g) = m \), proving that \( m \) and \( g \) depend on \( F_i \) and not in the particular choice of \( w \) because \( \text{in}_y(g) = m \) for all \( y \in C_I(w_i) \), and so the resulting inequalities are strict along every interior point of a face. Adding all the finitely many \( g \)'s for all faces we obtain the tropical basis.

\[ \square \]

Since the polynomials have support in a finite subset of \( \mathbb{Z}^n \), we would like to see how they get affected by a linear transformation in \( \mathbb{Z}^n \). First, consider \( X := \{ X_1^a \cdots X_n^a \mid a \in \mathbb{Z}^n \} \) a set of characters. Then we have the following diagram:

\[
\begin{array}{ccc}
\mathbb{Z}^n & \xrightarrow{\sim} & X \\
\downarrow q & & \downarrow \varphi \\
\mathbb{Z}^n & \xrightarrow{\sim} & X
\end{array}
\]

where if \( q(z) = Az \) for \( A \in GL_n(\mathbb{Z}) \), then we obtain the induced map \( \varphi(x) = x^{Az} \). This transformation induces another one in the level of polynomial functions. That is, we have
the following map:

\[
K[x^\pm 1] \xrightarrow{f} K[x^\pm 1]
\]  

(3.17)

\[
f(x) = \sum_i p_i x^i \quad f^* := f(\varphi(x)) = \sum_i p_i x^{Ai}
\]

The next proposition is equivalent to Proposition 2.34

**Proposition 3.36.** With the above notation, let \( a \in (K^\times)^n \). If \( f(a) = 0 \Rightarrow f^*(\varphi^{-1}(a)) = 0 \).

**Proof.**

\[
f^*(\varphi^{-1}(a)) = f(\varphi(\varphi^{-1}(a))) = f(a) = 0.
\]

\(\square\)

The above proposition has a simple proof. However, one has to be careful when applying \( \varphi^{-1} \), as \( \varphi \) is a map acting in the set of characters. Let us illustrate it with an example.

**Example 3.37.** Consider \( f(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 + x_4^3 - 3 x_1^{-1} x_4 + 1 \), \( q(z_1, z_2, z_3, z_4) = (z_1, 2z_1 + z_2, 3z_1 + 4z_2 + z_3, 5z_1 + 6z_2 + 7z_3 + z_4) \), and \( f(2, 3, -1, 2) = 0 \).

First, we express \( q \) as a matrix:

\[
q(z) = Az = \begin{pmatrix}
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
3 & 4 & 1 & 0 \\
5 & 6 & 7 & 1
\end{pmatrix} z
\]

Observe that \( q(1, 1, 1, 0)^T = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 18 \end{pmatrix} \), and in a similar fashion
we obtain $q(0,0,0,3)^T = (0,0,0,3)^T$, $q(-1,0,0,1)^T = (-1,-2,-3,-4)^T$. Thus,

$$f^* = x_1x_2^3x_3^3x_4^{18} + x_3^3 - 3x_1^{-1}x_2^{-2}x_3^{-3}x_4^{-4} + 1$$

We calculate $A^{-1}$ and obtain

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 5 & -4 & 1 & 0 \\ -28 & 22 & -7 & 1 \end{pmatrix}$$

That translates in the set of characters as the transform

$$\varphi^{-1}(X) = \left( \varphi^{-1}(X_1), \varphi^{-1}(X_2), \varphi^{-1}(X_3), \varphi^{-1}(X_4) \right)$$

$$= \left( A^{-1}e_1, A^{-1}e_2, A^{-1}e_3, A^{-1}e_4 \right)$$

$$= \left( X_1X_2^{-2}X_3^3X_4^{-28}, X_2X_3^{-4}X_4^{22}, X_3X_4^{-7}, X_4 \right)$$

Therefore, $\varphi^{-1}(2,3,-1,2) = (-3^2 \cdot 2^{-27}, 3 \cdot 2^{22}, -2^{-7}, 2)$. It is easy to verify that

$$f^* \left( -3^2 \cdot 2^{-27}, 3 \cdot 2^{22}, -2^{-7}, 2 \right) = 0.$$

The following is the equivalent to Proposition 2.35.

**Proposition 3.38.** Let $f = \sum_i p_i x^i \in K[x^\pm 1]$, $A \in \text{GL}_n(\mathbb{Z})$. Consider $f' = \sum_i p_i x^{(A^{-1})^T i}$. If $w \in \Gamma_{\text{val}}^n$ is such that $\text{in}_w(f)$ is not a monomial, then $\text{in}_{Aw}(f')$ is not a monomial.

**Proof.** Indeed, if $w \in T_j$ it happens that $\text{val}(p_i) + \langle i, w \rangle = W_j$ for more than one monomial.
Furthermore,

\[ \text{val}(p_i) + \langle i, w \rangle = \text{val}(p_i) + \langle i, A^{-1}Aw \rangle = \text{val}(p_i) + \langle (A^{-1})^T i, Aw \rangle = W_f \]

that is, the weight value remains the same. Therefore, if the minimum weight \( W_f \) was attained at least twice in \( f \), then that same weight is still the minimum and is still attained twice in \( f' \).

The next corollary is the equivalent to 2.36 and Corollary 2.6.10 in [MS13].

**Corollary 3.39.** With \( f^* \) an automorphism in the Laurent polynomial ring defined as in (3.17), and \( I' = f^{-1}(I) \),

\[ 1 \in \text{in}_w(I) \iff 1 \in \text{in}_{\varphi(w)}(I') \]

**Proof.** By Proposition 3.35, we know that \( I \) and \( I' \) contain a tropical basis, that is, if the ideals generate the whole ring, one of the elements in the tropical basis will contain a monomial. Now we only have to apply Proposition 3.38 twice. \( \square \)
Chapter 4

Tropical Geometry and Kapranov’s Theorem

This chapter covers Tropical Geometry, a fairly recent branch of algebraic geometry. While it deals with real finite dimensional spaces, it connects with general varieties via the Kapranov’s Theorem, as we will see. We first recall some concepts from Chapter 1.

4.1 The Tropical Semiring

Consider \( a, b \in \mathbb{R} \) such that \( 0 < a < b \). Define the operations \( \oplus_t, \otimes_t \) as follows:

\[
\begin{align*}
    a \oplus_t b &= \log_t (t^a + t^b) = b + \log_t \left( 1 + t^{a-b} \right) \\
    a \otimes_t b &= \log_t t^a t^b = a + b
\end{align*}
\]  

(4.1)

(4.2)

The process of passing from the real numbers to this new structure is known as the Maslov dequantization process, as can be seen in [MZ07]. From (4.1) we observe that \( \log_t (1 + t^{a-b}) \leq \log_t (2) \xrightarrow{t \to \infty} 0 \). In the first chapter we saw that this limiting process leads to the max-plus...
algebra, that is isomorphic to the tropical semiring. Due to that isomorphism, and only in this section shall we define the following operations:

\[
\begin{align*}
  a \oplus_b \infty & := \text{“}a + b\text{”} = \max\{a, b\} \\
  a \otimes_b \infty & := \text{“}ab\text{”} = a + b
\end{align*}
\]  

(4.3)  

(4.4)

4.1.1 Tropicalization of a field

Since the tropical operations do not depend on the logarithmic rules anymore, we can define the tropical semifield as \(\mathbb{T} = (\mathbb{R}, \text{“} + \text{”}, \text{“} \cdot \text{”})\) with the operations described in (4.3) and (4.4). In that sense, \(\mathbb{T}\) only lacks an additive inverse to become a field. For a field \(K\) with non-Archimedean valuation \(\text{val} : K^\times \to \mathbb{R}\), we define the tropicalization of its set of units \(K^\times\) as follows:

\[
a \in K^\times \mapsto a' = (\text{val}(a))
\]

Thus, for \(a, b \in k^\times\), the operations look like:

\[
\begin{align*}
  \text{“}a + b\text{”} & = \max\{a', b'\} \\
  \text{“}ab\text{”} & = a' + b'
\end{align*}
\]  

(4.5)  

(4.6)

Moreover, for a field \(F\) with real norm, we can define its tropicalization by taking logarithm of the norm of its set of units:

\[
a \in F^\times \mapsto a' = \log \|a\|
\]
4.1.2 Tropicalization of a polynomial

Consider $f \in K[x]$ be a polynomial with coefficients in the field $K$. Also, consider $A \subset \mathbb{Z}^n$ be the finite support of the polynomial, then the tropicalization of $f$ is given by:

$$\text{"}f\text{"}(w) = \sum_{u \in A} c_u x^u = \max_{u \in A} \{ \text{val}(a_u) + \langle w, u \rangle \}, w \in \mathbb{R}^n$$

If we denote the $k$–zero set of the polynomial by $V(f)$, and assume that $x_0 \in V(f)$, then we know that $\frac{1}{f}$ is not defined in $x_0$. We would like a similar definition for the tropical variety. From (4.6), we notice that the tropical multiplicative inverse of a number is given by its negative valuation. However, when $\text{"}f\text{"}$ is defined by two or more different terms, the fact of having more than one equation defining it becomes an ambiguity does not allow a clear definition of what a multiplicative inverse should be. That will be considered a tropical variety. We state the formal definition now.

**Definition 4.1** (Tropical variety [Mik03]). For a tropical polynomial in $n$ variables we define its variety $T_f \subset \mathbb{R}^n$ as the set of points where the piecewise linear function $\text{"}f\text{"}$ is not smooth. In other words, $T_f$ is the corner locus of $f$.

**Example 4.2.** Consider the polynomial $f = x^3 + 6x^2 + 6x + 4 \in \mathbb{R}[x]$. It can be verified that for $\text{"}f\text{"} = \max\{1 + 3x, 6 + 2x, 6 + x, 4\}$, $\text{"}f(5)\text{"}$, $\text{"}f(0)\text{"}$, $\text{"}f(-2)\text{"}$ are points where an ambiguity -coming from the fact that two monomials hit its maximum at the same time-exists. Thus, $T_f = \{-2, 0, 5\}$.

We now introduce the concept of tropical graph. In classical geometry, for a given function $f \in k[x_1, \cdots, x_n]$, its graph is defined as $\gamma_f = \{(y, x) \in k \times k^n | y = f(x)\}$, or equivalently, $\gamma_f = V(y - f(x))$. In the tropical world we obtain a similar result.
Figure 4.1: Graph of $T_f$, $f^\top = \max\{1 + 3x, 6 + 2x, 6 + x, 4\}$

**Definition 4.3 (Tropical graph).** The Tropical graph of a function $f$ is defined as:

$$\Gamma = T((y+f(x)^\top))$$

Notice that we changed the sign because the tropical semiring lacks from additive inverse [Mik03]. We use this definition in Figure 4.1 where the graph of the previous example can be seen. From the figure we can observe that every vertex corresponds to a point in the tropical variety. Also, every edge corresponds to a set of points where only one monomial hits its maximum. Moreover, if we remove all the vertical edges, we are left with $\Gamma = T_f$.

If we take a field $K$ with non-Archimedean valuation, then by (4.5), we have to work with the negative of a valuation. This can be fixed by taking min instead of max. However, by doing so, our graphs will look the opposite they are supposed to do. For example, Figure 4.1 would be a $180^\circ$ rotation of what it is supposed to look like. Due to the fact that in this
text our emphasis is on the algebraic part and not in the geometric part, we will take min as our standard definition.

4.2 Kapranov’s Theorem

**Definition 4.4** (Tropical semiring, taken from [MS13] p.8). Let $M$ be a commutative monoid with a total order. Its tropicalization, $\mathbb{T}(M)$, is a semiring with the following operations:

\[
\begin{align*}
\text{"}a + b\text{"} & = \min \{a, b\} \quad (4.7) \\
\text{"}ab\text{"} & = a + b \quad (4.8)
\end{align*}
\]

In consequence, the definition of tropical variety also changes.

**Definition 4.5** (Tropical variety, taken from [MS13] p.18). For a polynomial in $n$ variables we define its variety $T_f \subset \mathbb{R}^n$ as the set of points where the piecewise linear function

\[
\forall f := \min_{u \in A} \{ \text{val}(c_u) + \langle w, u \rangle \}, w \in \mathbb{R}^n \text{ is not smooth.}
\]

At the end of this section we prove the connection between this definition and Definition 3.1. We first introduce the following characterization for the tropical variety of a function.

**Proposition 4.6.**

\[
T_f := \left\{ w \in \mathbb{R}^n : \min_{u \in \text{supp}(f)} \{ \text{val}(c_u) + \langle w, u \rangle \text{ is attained at least twice} \} \right\}
\]

**Proof.** Let $w \in \mathbb{R}^n$ such that the minimum is attained at least twice. First note that if the minimum is attained at least twice at $w$ we can find two distinct neighborhoods $N_i, N_j$ in which \text{"}$f”(w + \epsilon_i), “f”(w + \epsilon_j)$\text{"} are described by different formulas. Indeed, for a double equality, if val$_{c_{u_i}} + \langle w, u_i \rangle = \text{val}_{c_{u_j}} + \langle w, u_j \rangle$, and $k$ is such that $u_{ik} < u_{jk}$, then for $\epsilon = (0, \ldots, 0, \epsilon_k, 0 \cdots, 0), \epsilon_k > 0$ we know that \text{“}$f”(w + \epsilon)$ and \text{“}$f”(w - \epsilon)$ are described by
different formulas. In each of those neighborhoods the partial derivatives will be different, and so \( f \) is not smooth at \( w \), thus \( w \in T_f \). Now, if \( w \in T_f \), then it is a point where \( f \) is not smooth. Since \( f \) is the minimum of linear functions, it is always differentiable, and thus smooth, unless the minimum is attained at least twice.

Consider \( I \) be an ideal, \( I \subset K[x^\pm] \). The tropical variety over \( I \) is defined as

**Definition 4.7** (Tropical variety, taken from Definition 3.2.1 in [MS13]). A tropical algebraic variety of an ideal \( I \) in the Laurent polynomial ring is given by

\[
T(I) = \bigcap_{f \in I} T_f \subset \mathbb{R}^n
\]

We now work with an illustrative, short example.

**Example 4.8.** Consider \( f(x, y) = txy + x^{-1} \).

Note that for \( w = (-1, 1) \in T_f \), \( \text{in}_w^K(f) = f \). For that specific \( w \), we can find \( x = t^{-1} \), \( y = -t \) such that \( \text{val}(x, y) = w \), and \( f(x, y) = 0 \). If we extend that result, for every \( w \in \mathbb{R} \), \( (w, -2w - 1) \in T_f \), and if additionally \( w \in \Gamma_{\text{val}}^2 \), \( \text{in}_{(w, -2w-1)}(f) = f \). In fact, \( T_f = \{(x, y) \in \mathbb{R}^2 : y = -2x - 1\} \). For each one of those points we can find \( a \in \mathbb{C} \) such that \( x = at^w \), \( y = -\frac{1}{a^2}t^{-2w-1} \) is in the zero set of \( f \).

We would like to generalize that previous result. That result is known as Kapranov’s theorem. We first prove a trivial statement that was discussed in the previous chapter.

**Proposition 4.9.** Let \( w \in \Gamma_{\text{val}}^n \). \( u \in \text{supp}(\text{in}_w^K(f)) \iff u \in \text{supp}(\text{in}_w^K(f)) \).

**Proof.** Let \( f = \sum p_u x^n \).

\[
u \in \text{supp}(\text{in}_w^K(f)) \iff \text{val}(p_u) + \langle w, u \rangle = W_f \iff u \in \text{supp}(\text{in}_w^K(f))\]
Theorem 4.10 (Kapranov’s Theorem (taken from Theorem 3.1.3 in [MS13]).) Fix a Laurent polynomial \( f = \sum_{u \in A \subset \mathbb{Z}^n} p_u x^u \in K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \), for \( K \) an algebraically closed field with non trivial valuation, value group dense in \( \mathbb{R} \), and residue field \( k \). The following three sets coincide:

1. \( T_f \subset \mathbb{R}^n \).
2. the closure in \( \mathbb{R}^n \) of the set \( \{ w \in \Gamma^{\mathbb{Z}}_\text{val} : \text{in}^K_w(f) \text{ contains more than one monomial} \} \).
3. the closure of the set \( \{ (\text{val}(x_1), \ldots, \text{val}(x_n)) : x \in V(f) \} \).

In addition, if \( w = \text{val}(x), x \in (K^\times)^n, f(x) = 0, n > 1 \), then \( U_w = \{ x' \in V(f) : \text{val}(x') = w \} \) is an infinite subset of the hypersurface \( V(f) \).

Proof.

(1) \( \subset \) (2)

For a vector \( w = (w_1, \ldots, w_n) \in T_f \), assume without lose of generality \( \text{val}(p_1) + u_{1,1}w_1 + \cdots + u_{1,n}w_n = \text{val}(p_2) + u_{2,1}w_1 + \cdots + u_{2,n}w_n \leq \cdots \). Observe that \( W_{f_\nu}(w) = \text{val}(p_1) + u_{i,1}w_1 + \cdots + u_{i,n}w_n \), so \( W_f = \text{val}(p_1) + u_{1,1}w_1 + \cdots + u_{1,n}w_n \). Therefore, \( \text{in}_w(f) \) contains more than one monomial. It remains to see that \( w \in \Gamma^{\mathbb{Z}}_\text{val} \). For this part of the proof we rearrange the monomials by increasing weight, obtaining:

\[
a_{u_1} + \langle w, u_1 \rangle = \cdots = a_{u_k} + \langle w, u_k \rangle < a_{u_{k+1}} + \langle w, u_{k+1} \rangle \leq \cdots \leq a_{u_m} + \langle w, u_m \rangle \tag{4.9}
\]

where \( a_{u_j} = \text{val}(p_j) \).

\(^1\)There are many proofs of this theorem and the Fundamental Theorem (which is the generalization of this theorem for the case where we work with a polynomial ideal). In [MS13] the proofs take the initial forms over the residue field. In other papers, such as [Dra06] or [OP13], they take the initial forms over \( K \) just as we did in the previous chapter. In short, this theorem has been previously proved by other authors using the definitions of Chapter 3, just as we do now.
We also introduce the following integer matrices: 
\[ A := \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix}, B := \begin{pmatrix} u_1 \\ \vdots \\ u_k \\ u_{k+1} \\ \vdots \\ u_m \end{pmatrix} \]
and the following real numbers: 
\[ c_i := a_{u_i} + \langle w, u_i \rangle, d_i := c_1 - a_{u_i}. \]
Also, let us introduce the column vectors 
\[ c_i := (c_1, \cdots, c_i), d_i := (d_1, \cdots, d_i). \]
We will be particularly interested in \( c^k, d^k, \) and \( d^m. \) Using this notation we will see that \( w \in \Gamma^n_{val}. \) We split the proof into 3 cases.

First, assume that the system \( Aw = d^k \) has only one solution, namely, \( w, \) and assume \( k + 1 \geq n. \) That system induces a second linear system:
\[
\langle u_1 - u_2, w \rangle = a_{u_2} - a_{u_1} \\
\vdots \\
\langle u_1 - u_k, w \rangle = a_{u_k} - a_{u_1}
\]
leading to 
\[ A'w := \begin{pmatrix} u_1 - u_2 \\ \vdots \\ u_1 - u_k \end{pmatrix} w = \begin{pmatrix} a_{u_2} - a_{u_1} \\ \vdots \\ a_{u_k} - a_{u_1} \end{pmatrix} =: b'. \]

Then, if we substitute \( \mathbb{Q} \) with \( \Gamma_{val}, \) Proposition 1.36 allows us to conclude that \( w \in \Gamma^n_{val}. \)

Now assume that the system \( Aw = d^k \) has only one solution, namely, \( w, \) and \( k = n. \)
Notice that in this case \( A \) has to be an invertible matrix, as otherwise at least one row could be deleted (via row reduction), leading to infinitely many solutions, and we are assuming a unique solution. Consider a sequence of elements in the value group \( \{x_t\} \to c_1 = \cdots = c_k, \epsilon_t := c_1 - x_t. \)
Notice that for \( d_{i,t} := d_i - \epsilon_t \) we obtain a sequence \( \{A^{-1}d_{i,t}^k\} \) in the value group that converges to \( w. \) Moreover, for all \( t, c_{1,t} = \cdots = c_{k,t} := d_{i,t} + a_{u_i}, \) by continuity, for \( \epsilon \) small enough, we will also obtain \( c_{i,t} = \cdots = c_{k,t} < c_{k+1,t} \leq \cdots \leq c_{m,t}. \) Thus, \( \{A^{-1}d_{i,t}^k\} \) represents a sequence of vectors in the value group, converging to \( w, \) such that \( \text{in}_{A^{-1}d_{i,t}^k}(f) \) contains more than one monomial. Therefore, \( w \in \Gamma^n_{val}. \)

In case the system has infinitely many solutions. Then the reduced system \( A'w = b' \) and its associated reduced homogeneous system \( A'w = 0 \) also have infinitely many solutions.
Moreover, since $b' \in \Gamma^{k-1}_{val}$, we can find a vector $w' \in \Gamma^n_{val}$ such that $A'w' = b'$. The system $A'w' = 0$ also has infinitely many solutions, which are spanned by some vectors $v_1, \cdots, v_r \in \Gamma^n_{val}$. Hence, any solution of the non-homogeneous system $A'w = b'$ can be written as the sum of a particular solution and a linear combination of solutions of the homogeneous system. In particular, $w = w' + t_1v_1 + \cdots + t_rv_r, t_i \in \mathbb{R}$. Notice that since $w'$ is a solution for the reduced system,

$$
\begin{pmatrix}
  u_1 - u_2 \\
  \vdots \\
  u_1 - u_k
\end{pmatrix} w' =
\begin{pmatrix}
  a_{u_2} - a_{u_1} \\
  \vdots \\
  a_{u_k} - a_{u_1}
\end{pmatrix} \Rightarrow
\begin{pmatrix}
  a_{u_1} + \langle u_1, w' \rangle = a_{u_2} + \langle u_2, w' \rangle \\
  \vdots \\
  a_{u_1} + \langle u_1, w' \rangle = a_{u_k} + \langle u_k, w' \rangle
\end{pmatrix}.
$$

And, for all $t_i \in \mathbb{R}, a_{u_1} + \langle u_1, w' + t_1v_1 + \cdots + t_rv_r \rangle$ are equal, implying that $\text{in}_{w' + t_1v_1 + \cdots + t_rv_r}(f)$ contains more than one monomial if $w' + t_1v_1 + \cdots + t_rv_r$ is close to $w$. By the density of the value group, we can take $t_i \in \mathbb{Q}$ with $w' + t_1v_1 + \cdots + t_rv_r$ arbitrarily close to $w$. Therefore, $w \in \Gamma^n_{val}$.

$$
(2) \subset (1)
$$

Now take $w \in G := \{ w' \in \Gamma^n_{val} : \text{in}_{w'}(f) \text{ contains more than one monomial} \}$. Pick a sequence $\{w_i\} \to w$ with $w_i \in \Gamma^n_{val}$ and such that $\text{in}_{w_i}(f)$ contains more than one monomial. There is a subsequence such that $\text{in}_{w_i}(f)$ contains the same monomials. Hence, $\text{in}_w(f)$ contains those monomials as well.

$$
(3) \subset (1)
$$

By $(1) = (2)$ we are trying to prove the equality between two closed sets. Take $x \in V(f)$. Therefore, $f(x) = 0 \Rightarrow \text{val}(f(x)) = \infty$. By our definition of valuation,

$$
\infty = \text{val}(f(x)) = \text{val}\left( \sum_{u \in \mathbb{Z}^n} p_u x^u \right) \geq \min_u \left\{ a_u + \langle u, \text{val}(x) \rangle \right\}.
$$
The zero polynomial satisfies the theorem trivially. Now, if we assume \( f \neq 0 \) we can assume \( \infty > \{a_u + \langle u, \text{val}(x) \rangle \} \) for every \( u \) in the support of the function. From Corollary 1.23 it follows that the minimum is attained at least twice. Therefore, \( \text{val}(x) \in T_f \).

(1) \( \subset \) (3)

Now we assume that \( \text{in}_w^K(f) \) is not a monomial. The proof is done by induction over the number of variables.

First, assume \( f \in K[x^{\pm 1}] \). If \( f = \sum p_u x^u = \prod (x - \alpha_i) \). At least one of the terms \( \text{in}_w^K(x - \alpha_i) \) is not a monomial. Indeed, if all of them were monomials, then for each factor, either \( w \) or \( \text{val}(\alpha_i) \) is smaller, thus a term \( \prod \alpha_i x^v \) has minimal weight. Then the monomial \( (\prod \alpha_i + \sum \cdots) x^v \) has minimum weight and thus \( \text{in}_w(f) \) is a monomial, a contradiction. So we conclude that \( w = \text{val}(\alpha_i) \) for at least one \( i \), implying \( \text{val}(x) = \text{val}(\alpha_i) = w \).

Now assume \( f \in K[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}] \). Since \( \text{in}_w(f) = p_u x^u + p_v x^v + \cdots \) we know for at least one \( i, u_i \neq v_i \). Without loss of generality, assume \( u_1 \neq v_1 \). We now introduce the following functions:

\[
\begin{align*}
  h_j(x_{n+1}) & \text{ such that } \text{in}_w(f) \geq \sum_{j=1}^{n} h_j(x_{n+1})m_j, \\
  g_j'(x_{n+1}) & \text{ such that } f = \sum_{j=1}^{N} g_j'(x_{n+1})m_j,
\end{align*}
\]

where the \( m_j \)'s are monomials in the rest of the variables. Observe that by writing \( \text{in}_w(f) \) in this way, it is not a monomial in \( x_1, \ldots, x_n \), that is, \( n \geq 2 \). Moreover, we also re-express \( f \) as a function of its monomials in the first \( n \) variables by introducing

\[
\begin{align*}
  g_j'(x_{n+1}) & \text{ such that } f = \sum_{j=1}^{N} g_j'(x_{n+1})m_j,
\end{align*}
\]

where we still have \( N \geq 2 \). Furthermore, \( h_j \) is always contained in \( g_j' \), that is, the monomials of \( h_j \) are always monomials of \( g_j' \). We now pick a \( y \in K \) such that \( \text{val}(y) = w_{n+1} \), with the additional constrains \( h_j(y) \neq 0, g_j'(y) \neq 0, g_j(y) := g_j'(y) - h_j(y) \neq 0 \) whenever the
polynomials are not zero. This can be done by Proposition 1.35. Now define,

\[ g(x_1, \cdots, x_n) := \sum_{j} (h_j(y) + g_j(y)) m_j = f(x_1, \cdots, x_n, y) \]

Define \( w' = (w_1, \cdots, w_n) \). We analyze \( \text{in}_{w'}(g) \) with the goal of showing that \( \text{in}_{w'}(g) \) is not a monomial. We first see that \( W_f(w) = W_g(w') \). To do so, we unfold the notation. Let \( m_j = x^{w_j} \) (all the constants are assumed to be taken in either \( h_j \) or \( g_j \)), \( h_j = \sum c_{ij} x_{n+1}^i, g_j = \sum d_{ij} x_{n+1}^i \). Note that the support of \( h_j \) is disjoint to that of \( g_j \) because if they overlapped, then the overlap would be part of \( \text{in}_w(f) \). Also observe that if \( h_j \neq 0 \) then for every \( i, j \) and \( k \), such that \( c_{ij} \neq 0 \),

\[ \text{val}(c_{ij}) + i w_{n+1} = \text{val}(c'_{ij}) + i' w_{n+1} + 1 < \text{val}(d_{kj}) + k w_{n+1}. \]

Given that \( \text{val}(y) = w_{n+1} \),

\[ \text{val}(c_{ij}) + i \text{val}(y) + \langle w', u_j \rangle = \text{val}(c'_{ij}) + i' \text{val}(y) + \langle w', u_j \rangle < \text{val}(d_{kj}) + k \text{val}(y) + \langle w', u_j \rangle. \]

It immediately follows that \( W_f(w) = W_g(w') \). Thus, every monomial of \( \text{in}_w(f) \) will become a monomial of \( \text{in}_{w'}(g) \). So, if \( h_j m_j \) is a monomial of \( \text{in}_w(f) \), then \( h_j(y)m_j \) is a monomial of \( \text{in}_{w'}(g) \). We conclude

\[ \text{in}_{w'}(g) = \sum_{j} (h_j(y) + g_j(y)) m_j, \]

and \( \text{in}_{w'}(g) \) is not a monomial. We now apply the inductive hypothesis for finding \( Y := \)
\((y_1, \cdots, y_n) \in K^n\) such that \((\text{val}(y_1), \cdots, \text{val}(y_n)) = w'\) and \(Y \in V(g)\). Therefore,
\[
g(Y) = \sum_{j=1}^{N} \left(h_j(y) + g_j(y)\right) m_j = f(Y, y) = 0
\]
and \((Y, y) \in V(f)\) as required. This finishes the proof.

\[\square\]

**Corollary 4.11** (Kapranov’s Theorem). Fix a Laurent polynomial \(f = \sum_{u \in A \subseteq \mathbb{Z}^n} p_u x_u \in K[x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]\), for \(K\) an algebraically closed field with non trivial valuation, value group dense in \(\mathbb{R}\), and residue field \(k\). The following three sets coincide:

1. \(T_f \subset \mathbb{R}^n\).

2. the closure in \(\mathbb{R}^n\) of the set \(\{w \in \Gamma^n_{\text{val}} : \text{in}^k_{\text{val}}(f) \text{ contains more than one monomial}\}\).

3. the closure of the set \(\{(\text{val}(x_1), \cdots, \text{val}(x_n)) : x \in V(f)\}\).

In addition, if \(w = \text{val}(x), x \in (K^*)^n, f(x) = 0, n > 1\), then \(U_w = \{x' \in V(f) : \text{val}(x') = w\}\) is an infinite subset of the hypersurface \(V(f)\).

**Proof.** The result is immediate from Kapranov’s Theorem (4.10), after applying Proposition 4.9. Further, most of the proof does not change at all. The part that does change is \((1) \subset (3)\). We present the proof of that part in the spirit of the proof presented by Sturmfels [MS13] as part of his Theorem 3.1.3. That part of the proof is Proposition 3.1.5.

**Proposition 4.12** ([MS13] Proposition 3.1.5). Let \(f = \sum_{u \in \mathbb{Z}^n} p_u x_u \in K[x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]\), for \(K\) an algebraically closed field with non trivial valuation, value group \(\Gamma_{\text{val}}\) dense in \(\mathbb{R}\), and residue field \(k\). Let \(b \in V\left(\text{in}_w(f)\right)\). Then there exists \(z \in (K^*)^n\) such that \(f(z) = 0, \text{val}(z) = w\), and the image of \(t^{-w}z\) on the residue field is \(b\).

**Proof.** The proof is done by induction over \(n\). In this proof \(k\) will be omitted from \(\text{in}^k_w(f)\). For \(n = 1\) we express the function as \(f = \sum_{u \in \mathbb{Z}} p_u x_u\). Moreover, by Proposition 2.5, we can
consider \( u \in \mathbb{N} \). That will allow us to use the fundamental theorem of algebra. Thus,

\[
f = \sum_{i \in \mathbb{N}} p_i x^i = \prod_u (c_u + d_u x), \quad \text{where} \quad c_u, d_u \in K.
\]

Take \( w \in T_f \). Then \( \text{in}_w(f) \) contains more than one monomial, so \( \text{in}_w(f) = \prod_u (c_u + d_u x) \) contains more than one monomial. Then, for some fixed \( u \in \mathbb{N} \), \( \text{in}_w(c_u + d_u x) = t^{-\alpha_u} c_u + t^{-\beta_u} d_u x \), that is \( \text{val}(c_u) = \text{val}(d_u) + w \). Fix \( z := -\frac{\alpha_u}{\beta_u} \in K \). Then \( \text{val}(z) = w \) and \( z \in V(f) \), as we wanted.

We now assume that \( (1) \subset (3) \) for all \( 1 \leq m < n \). Take \( w \in T_f \). Consider \( b \in V(\text{in}_w(f)) \in (K^\times)^n \). The proof splits into two cases depending on whether \( \text{in}_w(f)(y_1, \ldots, y_{n-1}, b_n) \) is identically zero (for any values \( y_i \in k \)) or not.

Consider first that \( \text{in}_w(f)(y_1, \ldots, y_{n-1}, b_n) \) is not identically zero. Take \( z_n \in K \) such that \( \text{val}(z_n) = w_n \in \Gamma_{\text{val}} \) and the image of \( t^{-w_n} z_n \in K \) over the residue field is \( b_n \). Note that such an element exist by the surjection between \( K \) and its residue field \( k \). For instance, in the field of Puiseaux series, with the order as valuation we can consider \( z_n = b_n t^{w_n} \). Now, define the following function:

\[
g(x_1, \ldots, x_{n-1}) := f(x_1, \ldots, x_{n-1}, z_n) \in K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]
\]

We prove that for every \( w \in T_f \), \( w' \in T_g \), where \( w' := (w_1, \ldots, w_{n-1}) \). For this we prove that for every \( w \in T_f \), \( W_{f_u}(w) = W_{g_u}(w) \), where \( u \in \mathbb{Z}^n \) is in the support of both functions and \( f_u, g_u \) are monomials of \( f \). Indeed,

\[
W_{g_u}(w) = W_{p_u x^u z_n^{w_n}}(w') = \text{val}(p_u z_n^{w_n}) + \sum_{i=1}^{n-i} u_i w_i = \text{val}(p_u) + \sum_{i=1}^{n-i} u_i w_i,
\]

\[
W_{f_u}(w) = W_{p_u x^u}(w) = \text{val}(p_u) + \sum_{i=1}^{n-i} u_i w_i + u_n w_n.
\]

It follows that \( \text{in}_w(f) = \text{in}_w(g) \), thus \( \text{in}_w(f)(b) = \text{in}_w(g)(b') = 0 \). Therefore, \( g \) is a polyno-
mial in \(n - 1\) variables satisfying the hypotheses of the proposition. By induction, we can find \(z' \in (K^\times)^{n-1}\) such that \(g(z') = 0 = f(z', z_n)\).

Notice that the assumption of not being identically zero is used when we work with \(g\), as we assume that \(g \neq 0\), and that is why each monomial has the same weight. Finally, if \(\text{in}_w(f)(y_1, \cdots, y_{n-1}, b_n)\) is identically zero, we find a linear automorphism where this does not occur, and apply the previous case. We proceed as follows. Take \(w \in T_f, v \not\in T_f\). We show that \(\text{in}_{w+\epsilon v}(f)\) contains only one monomial.

Arrange the monomials of \(f\) as in (4.9). Since \(w \in T_f, k \geq 2\). Additionally, we assume \(u_i \neq u_j \ \forall i \neq j\). As for \(v \not\in T_f\), we also require that it satisfy \(v \perp (u_1 - u_2)\). In the aim of a contradiction, assume \(\text{in}_{w+\epsilon v}(f)\) contains more than one monomial. Then the following inequalities are satisfied:

\[
a_{u_i} + \langle w, u_i \rangle + \epsilon \langle v, u_i \rangle = a_{u_j} + \langle w, u_j \rangle + \epsilon \langle v, u_j \rangle \leq \cdots
\]

For \(\epsilon\) small enough we can assume \(i = 1, j = 2\). That implies \(\langle v, u_1 \rangle = \langle v, u_2 \rangle\), in contradiction to the fact that \(v \perp (u_1 - u_2)\). We can make such a choice of \(v\), as otherwise, if for every \(v \perp (u_1 - u_2)\) we had \(v \in T_f\), we could pick \(v = u_1, 2u_1\), leading to the contradiction \(u_2 = u_1\).

Now consider the linear transformation \(\phi : \mathbb{Z}^n \rightarrow \mathbb{Z}^n\), given by \(\phi(z) = (A^{-1})^T z, A \in GL_n(\mathbb{Z})\) such that \(Ae_1 = e_1\). The proof for the existence of \(\phi\) can be found in [MS13] p.70. We use the notation introduced in propositions 2.5, 2.34, and 2.35. First notice that by 2.35, for a fixed \(u \in \mathbb{Z}^n\),

\[
W_{f_u^*}(Aw) = W_{f_u}(w).
\]
Thus, if we define $f_{in} := \text{in}_w(f)$, it is straightforward to see that

$$f_{in}^* = \sum_{u:\text{weight}_w(f_u)=W} a_u x^{(A^{-1})^Tu} = \sum_{u:\text{weight}_{A_w}(f_u^*)=W} a_u x^{(A^{-1})^Tu} = (f_{in})^*.$$ 

Therefore, by proposition 2.34, $f_{in}(b) = 0 \Rightarrow f_{in}^*(\phi^{-1}(b)) = 0$.

We prove that $f_{in}^*(\phi^{-1}(x_1, \cdots, x_{n-1}, b_n))$ is not identically zero. To do so, we use the fact that $\text{in}_{w+\epsilon v}(f)$ contains one monomial. By Proposition 2.15, it follows that $\text{in}_v(\text{in}_w(f))$ contains only one monomial, which is the same as saying that $\min_{u:a_u \langle w, u \rangle=W} \{\langle v, u \rangle\}$ contains only one element. Notice that

$$\min_{u:a_u \langle w, u \rangle=W} \{\langle v, u \rangle\} = \min_{u:a_u \langle w, u \rangle=W} \{(A^{-1}e_1, u)\} = \min_{u:a_u \langle w, u \rangle=W} \{(e_1, (A^{-1})^Tu)\},$$

meaning that $\text{in}_{\phi(w+\epsilon v)}(f^*)$ contains only one monomial. But

$$\text{in}_{\phi(w)}(f)(\phi^{-1}(x_1, \cdots, x_{n-1}, b_n)) = \lim_{\epsilon \to 0} \text{in}_{\phi(w)+\epsilon e_1}(f) = \lim_{\epsilon \to 0} \text{in}_{e_1} \left( \text{in}_{\phi(w)}(f)(\phi^{-1}(x_1, \cdots, x_{n-1}, b_n)) \right) = \text{in}_{e_1} \left( \text{in}_{\phi(w)}(f)(\phi^{-1}(x_1, \cdots, x_{n-1}, b_n)) \right).$$

where the latter is only one monomial and, since $b_n \neq 0$, that monomial is not identically zero. We now can apply the previous case.

That finishes the alternative proof.

We now present some examples of the applications of the previous theorem. For the case where we have only one variable, the methods described by Theorem 4.10 and by Corollary
4.11 are very similar, but the computations for the latter are easier. That is the reason for the following example.

**Example 4.13.** Consider the function

\[
f(x) = (5t^3 + 5t^{11})x^3 + (10t^2 + t^3 - 5t^4 + 10t^{10} + t^{11} - 5t^{12})x^2
\]

\[
+ (10 + t - 3t^2 - 20t^3 - t^4 + 2t^{10} - 10t^{11} - t^{12})x
\]

\[
+ 2 - 10t - t^2 - 2t^3 - 2t^{11} - 2tx^{-1}.
\]

For \( w \in \mathbb{R} \), we calculate \( W \) for each monomial, leading to the following result:

\[
W_{f_i}(w) = 3 + 3w, 1 + 2w, w, 0, 1 - w.
\]

Any two of those weights coincide and have a minimum for \( w \in \{-2, -1, 0, 1\} \). Therefore, \( T_f = \{-2, -1, 0, 1\} \). Moreover, for those values \( \text{in}_w(f) \) contains more than one monomial. We now find the associated values \( x \in V(f) \). In order to do that, we factor \( f = (x - t)(5x + 1)(tx + 2)(x^{-1} + t^2 + t^{10}) \). In order to apply the procedure described by the theorem, we must work with the polynomial \( f' = xf = (x - t)(5x + 1)(tx + 2)((t^2 + t^{10})x + 1) \). The associated points of the variety are:

\[
\text{in}_{-2}^k(f') = (x)(x)(x)((t^2 + t^{10})x + 1) \quad \Rightarrow x = -\frac{1}{t^2 + t^{10}} \in K
\]

\[
\text{in}_{-1}^k(f') = (x)(5x)(tx + 2)(1) \quad \Rightarrow x = -\frac{2}{t} \in K
\]

\[
\text{in}_0^k(f') = (x)(5x + 1)(2)(1) \quad \Rightarrow x = -\frac{1}{5} \in K
\]

\[
\text{in}_1^k(f') = (x - t)(1)(2)(1) \quad \Rightarrow x = t \in K.
\]

We now present a couple of examples that use the proof of Theorem 4.10.
Example 4.14. Consider the function
\[ f(x, y) = \left(1 + t + t^2\right)x + \left(5 + \sqrt{t}\right)xy + t^4xy^2 + t^6x^2 + t^{-1}y^3. \]

We first tropicalize it:
\[ "f" = \min_w \{ w_1, w_1 + w_2, 4 + w_1 + 2w_2, 6 + 2w_1, -1 + 3w_2 \}, w \in \mathbb{R}^2 \]

From that, we calculate \( T_f \) as the union of the following parametrized lines:
\[
T_f = \begin{cases} 
(t, 0) & \text{where } w_1 = w_1 + w_2 \\ (-6, t) & 0 \leq t \text{ where } w_1 = 6 + 2w_1 \\ (-1 + 3t, t) & 0 \leq t \text{ where } w_1 = -1 + 3w_2 \\ (t - 6, t) & -4 \leq t \leq 0 \text{ where } w_1 + w_2 = 6 + 2w_1 \\ (-1 + 2t, t) & -4 \leq t \leq 0 \text{ where } w_1 + w_2 = -1 + 3w_2 \\ (t, -4) & -10 \leq t \leq -9 \text{ where } w_1 + w_2 = 4 + w_1 + 2w_2 \\ (2t - 2, t) & t \leq -4 \text{ where } 4 + w_1 + 2w_2 = 6 + 2w_1 \\ (-5 + t, t) & t \leq -4 \text{ where } 4 + w_1 + 2w_2 = -1 + 3w_2 \end{cases} \]

By the previous theorem, \( \{ w \in \Gamma_{\text{val}} : \text{in}_w(f) \text{ contains more than one monomial} \} \subset \mathbb{R}^2 \) and \( T_f \) coincide. The set of points of \( T_f \) can be seen in Figure 4.2.

Moreover, for each point \( w \in T_f \) we can find a point \( x \in V(f) \) such that \( x \in \mathcal{K}, \text{val}(x) = w \). We find such a point for \( w = (-1, 0) \in T_f \). We find that
\[
\text{in}_w^K(f) = (1 + t + t^2)x + (5 + \sqrt{t})xy + t^{-1}y^3 = t^{-1}y^3 + [(1 + t + t^2) + (5 + \sqrt{t})y]x
\]

Observe that \( y = 1 \) will satisfy \( \text{val}(y) = 0, h_j \neq 0, g_j \neq 0 \). Now, let \( g := f(x, 1) = \)
Figure 4.2: Graph of $T_f, f(x, y) = (1 + t + t^2) x + (5 + \sqrt{t}) xy + t^4 xy^2 + t^6 x^2 + t^{-1} y^3$
\( t^6 x^2 + (6 + \sqrt{t} + t + t^2 + t^4)x + t^{-1} \). As we saw in the proof, \( \text{in}^K_w (g) \) is not a monomial. Indeed,

\[ \text{in}^K_1 (g) = (6 + \sqrt{t} + t^2 + t^4)x + t^{-1}. \]

We now apply the base case of the induction. First we factor \( g \),

\[
\begin{align*}
g &= \left( t^3 x + \frac{(6 + \sqrt{t} + t^2 + t^4) + \sqrt{(6 + \sqrt{t} + t^2 + t^4)^2 - 4t^5}}{2t^3} \right) \\
&\times \left( t^3 x + \frac{(6 + \sqrt{t} + t^2 + t^4) - \sqrt{(6 + \sqrt{t} + t^2 + t^4)^2 - 4t^5}}{2t^3} \right).
\end{align*}
\]

We want to show that for at least one of the factors, its initial form still has both terms. First, we know that \( \text{val}(t^3) + (-1)(1) = 2 \). And, after using Mathematica, we find the power series expansion for the coefficients:

\[
\begin{align*}
\frac{(6 + \sqrt{t} + t^2 + t^4) + \sqrt{(6 + \sqrt{t} + t^2 + t^4)^2 - 4t^5}}{2t^3} &= \frac{6}{t^3} + \frac{1}{t^{5/2}} + \cdots \\
\frac{(6 + \sqrt{t} + t^2 + t^4) - \sqrt{(6 + \sqrt{t} + t^2 + t^4)^2 - 4t^5}}{2t^3} &= \frac{t^2}{6} - \frac{t^{5/2}}{36} + \cdots
\end{align*}
\]

Therefore, the root we are looking for is \( x = -\frac{1}{6t^3} + \frac{1}{36t^{5/2}} + \cdots \), which satisfies \( \text{val}(x) = -1 \) and so \( f(x, 1) = 0 \), as required.

We now present an easier-to-compute example in more variables.

**Example 4.15.** Consider \( f(x, y, z) = xyz + xy^2 z + xyz^2 \), \( w = (1, 0, 1) \).

Observe that \( \text{in}^K_w (f) = (xy)z + (xy^2)z \). We now find \( z_0 \in K \) such that its valuation is 1 and it does not cancel the fact that we do not have a monomial. Say, \( z_0 = t \).

\[ g := (t + t^2)xy + txy^2. \]
And, in\(^{(1,0)}_K (g) = g\). In this case, \(g\) is not a monomial because the exponents of \(y\) are different, so we must keep \(y\) for the final step. We now need to choose \(x_0\) such that val\((x_0) = 1\) and where it won’t eliminate any of the monomials. Just as before, set \(x_0 = t\).

\[ h := (t^2 + t^3)y + t^2 y^2. \]

Again, in\(^{(0)}_K (h) = h\). In this step we apply the base of the induction. For that, we need to factor \(h\):

\[ h = y\left(t^2 y + (t^2 + t^3)\right). \]

Notice that we cannot have \(y = 0\) because it would not satisfy val\((y) = 0\), so we must set \(y = -1 - t\). Just to verify,

\[ f(t, -1 - t, t) = -t^2(1 + t) + t^2(1 + t)^2 - t^3(1 + t) = -t^2 - t^3 + 2t^3 + t^4 - t^3 - t^4 = 0. \]

Therefore, \((t, -1 - t, t) \in V(f)\) and val\((t, -1 - t, t) = w\).

Note that the procedure used for \(k\)–initial forms, and the one used for \(K\)–initial forms differ. We now present examples for the former.

**Example 4.16.** Consider

\[
\begin{align*}
    f(x_1, x_2, x_3, x_4) &= t^2 x_1 x_2^2 + t^3 x_2 x_3 x_4 + (4t + 3t^4)x_3 x_4 - 4tx_2 x_4 + (-3t^2 - t^3)x_1 x_4 \\
    &\quad - t^3 x_2 x_3 - (4t^2 + 3t^5)x_3 + 4t^2 x_2 + 3t^3 x_1,
\end{align*}
\]

\[ w = (-1, 0, 0, -1) \in T_f. \] We will find a point \(z \in V(f)\) such that val\((z) = w\). Notice that in\(^k_w (f) = 4x_3 x_4 - 4x_2 x_4 - 3x_1 x_4\), and in\(^k_w (f)(4, 1, 4, 1) = 0\) but in\(^k_w (f)(x_1, x_2, x_3, 1) = 4x_3 - 4x_2 - 3x_1\) is not identically zero. Thus, we find \(z_4\) such that val\((z_4) = -1\) with image
$1 \in k$. Fix $z_4 = t^{-1}$. Let

$$g(x_1, x_2, x_3) = f(x_1, x_2, x_3, t^{-1}) = \left(-3t - t^2 + 3t^3\right) x_1 + \left(-4 + 4t^2\right) x_2$$

$$+ \left(4 - 4t^2 + 3t^3 - 3t^5\right) x_3 + t^2 x_1 x_2 + \left(t^2 - t^3\right) x_2 x_3,$$

so $\text{in}_{(-1,0,0)}^k(g) = -3x_1 - 4x_2 + 4x_3, \text{in}_{(-1,0,0)}^k(g)(4,1,4) = 0, \text{in}_{(-1,0,0)}^k(g)(x_1, x_2, 4) = -3x_1 - 4x_2 + 16$ not identically zero. We now have to find $z_3$ such that $\text{val}(z_3) = 0$ with image $4 \in k$. Fix $z_3 = 4$. Let

$$h(x_1, x_2) = g(x_1, x_2, 4) = f(x_1, x_2, 4, t^{-1}) =$$

$$\left(-3t - t^2 + 3t^3\right) x_1 + \left(-4 + 8t^2 - 4t^3\right) x_2 + t^2 x_1 x_2^2 + \left(16 - 16t^2 + 12t^3 - 12t^5\right),$$

$\text{in}_{(-1,0)}^k(h) = -3x_1 - 4x_2 + 16, \text{in}_{(-1,0)}^k(h)(4,1) = 0, \text{in}_{(-1,0)}^k(h)(x_1,1) = -3x_1 + 12$ is not identically zero. We now have to find $z_2$ such that $\text{val}(z_2) = 0$ with image $1 \in k$. Fix $z_2 = 1$. Let

$$p(x_1) = h(x_1, 1) = g(x_1, 1, 4) = f(x_1, 1, 4, t^{-1}) =$$

$$\left(-3t + 3t^3\right) x_1 + \left(12 - 8t^2 + 8t^3 - 12t^5\right)$$

$\text{in}_{(-1)}^k(p) = p$. We are now in the case $n = 1$, for which we have to find $z_1$ such that $\text{val}(z_1) = -1$ with image $4 \in k$. Moreover, has to be a root for $p$. This is the only case where we have only one option. In all the previous cases we made one choice out of infinitely many available ones. The only choice is $x_1 = \frac{12 - 8t^2 + 8t^3 - 12t^5}{3t - 3t^3}$. To see that the image under the residue field is 4, we calculate the series expansion for $x_1$, obtaining $x_1 = 4t^{-1} + \frac{4}{3}t + t^2 + \cdots$.

Therefore, $w = (-1,0,0,-1) \in T_f$ is associated with

$$z = \left(\frac{12 - 8t^2 + 8t^3 - 12t^5}{3t - 3t^3}, 1, 4, \frac{1}{t}\right) \in V(f),$$

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where \( \text{val}(z) = w \). In general, \( w \) is associated with points lying in the algebraic variety with valuation \( w \). In the next example we illustrate the case where \( \text{in}_w(f) \) is identically zero.

**Example 4.17.** Consider

\[
f(x_1, x_2, x_3) = x_1 x_2 x_3 + 2t^2 x_3 - t^2 x_1 x_2 - 2t^4 + t^5 x_3.
\]

Notice that \( w = (1, 1, 2) \in T_f \), \( \text{in}_w^k(f) = x_1 x_2 x_3 + 2x_3 - x_1 x_2 - 2 \), \( \text{in}_w^k(f)(-1, 1, 1) = 0 \), and \( \text{in}_w^k(f)(x_1, x_2, 1) \equiv 0 \). Thus, we have to find a transformation. In order to do so, we observe that \( \text{in}_w^k(f)(-1, 1, 1) \) is not orthogonal to the subtraction of two vectors in the support of \( f \). Finally, define

\[
A := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \in GL_n(\mathbb{Z}),
\]

which satisfies \( Av = e_1, Aw = (4, -1, 2) \). Define

\[
\phi(z) = (A^{-1})^T z = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 0 \\ 2 & -1 & -1 \end{pmatrix} z.
\]

Under that transformation we obtain:

\[
f^* = f(\phi) = x_1 + (2t^2 + t^5) x_1 x_3^{-1} - t^2 x_3 - 2t^4.
\]

Note that

\[
\text{in}_w^k(f^*) = x_1 + 2x_3^{-1} - x_3 - 2 = (\text{in}_w(f))^*.
\]

For \( b = (-1, 1, 1) \in V(\text{in}_w f) \), \( \text{in}_w^k(f^*(\phi^{-1}(b)) = \text{in}_w^k(f^*(-1, 1, -1) = 0 \), but

\[
f^*(x_1, x_2, -1) = -x_1 - 1 \not\equiv 0.
\]

Indeed, for \( (b_1, b_2, b_3) \in V(\text{in}_w^k(f)) \) we have that \( (b_1b_2b_3, b_2b_3^{-1}, b_1b_2) \in V(\text{in}_w^k(f^*)) \). We now apply the general procedure to \( f^* \).

In the first step we need to find \( z_3 \) such that \( \text{val}(z_3) = 2 = Aw_3 \) with image in the residue
field $-1 = b_3$. Pick $z_3 = -t^2$. Define $g(x_1, x_2) := f^*(x_1, x_2, -t^2) = (-1 - t^3)x_1 - t^4$. Then \( \text{in}^k_{(4, -1)}(g) = -x_1 - 1, \text{in}^k_{(4, -1)}(g)(-1, 1) = 0 \) and so we need to find $z_2$ such that $\text{val}(z_2) = -1 = Aw_2$ with image in the residue field $1 = b_2$. We propose $z_2 = t^{-1}$. Finally, define $h(x_1) = g(x_1, t^{-1}) = (-1 - t^3)x_1 - t^4$. The only choice for $z_1$ is $z_1 = \frac{-t^4}{t^3 + 1} = -t^4 + t^7 - t^{10} + \cdots$.

As we can observe, $\text{val}(z_1) = 4 = Aw_4$ and the image in the residue field is $-1 = b_4$. So, $b^* \left( \frac{-t^4}{t^3 + 1}, t^{-1}, -t^2 \right) \in V(f^*)$.

We need to find the associated root in $f$. That will be given by $\phi(b^*)$, that is, by applying $b^*$ to $(A^{-1})^T$ column-wise. In this case,

\[
 b := \left( \begin{array}{c}
 \left( \frac{-t^4}{t^3 + 1} \right)^{-1} (t^{-1})^{-1} (-t^2)^2, \\
 \left( \frac{-t^4}{t^3 + 1} \right)^{-1} (t^{-1}) (-t^2)^{-1}, \\
 \left( \frac{-t^4}{t^3 + 1} \right) (-t^2)^{-1} \\
 \left( t(1 + t^3), \frac{t^5}{1 + t^3}, \frac{t^2}{1 + t^3} \right)
 \end{array} \right) \in V(f).
\]

### 4.3 Tropical Varieties

We previously defined a tropical variety in the Laurent polynomial ring by

\[
 T(I) = \bigcap_{f \in I} T_f \subset \mathbb{R}^n.
\]

Moreover, in the previous sections we talked about tropical bases. The name (tropical variety) comes from the fact that $T(I)$ can also be obtained by the finite intersection of functions in the tropical basis.

**Proposition 4.18.** Let $I \subset K[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$ be a polynomial ideal with tropical basis $F$, as defined in Definition 2.32. Then

\[
 T(I) = \bigcap_{f \in F} T_f. \tag{4.10}
\]

**Proof.** First, by Proposition 2.33, a tropical basis for $I$ can be found. If $w \in T(I) = \bigcap_{f \in I} T_f$, then trivially it is also in $\bigcap_{f \in F} T_f$. 

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The other inclusion will be proved by contradiction. For $w \in \Gamma_{val}^n$, assume that $w \in \bigcap_{f \in F} T_f$, that is, $\text{in}_w(f)$ is not a monomial for every $f \in F$, but $w \notin T(I)$. Thus, there exists $g \in I$ such that $w \notin T_g \Rightarrow \text{in}_w(g)$ is a monomial (as the minimum is attained only once). Moreover, $\text{in}_w(I) = 1$, and by Definition 2.32 that implies $\text{in}_w(F)$ contains a monomial. That last part is a contradiction.

For $w \in \mathbb{R}^n$, by Kapranov’s theorem we know there is a sequence $\{w_i\} \to w$ in which every element satisfies that $\text{in}_{w_i}(f)$ is not a monomial. Therefore, for every $f \in I$, the minimum (as described in Proposition 4.6) is attained at least twice. By continuity that property still holds in the limit, so $w \in T(I)$.

Finally, we prove the existence of a connection between the initial ideal and the tropical ideal.

**Theorem 4.19.** Let $I \subset K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, and $X := V(I)$ its classical variety in the torus $(K^\times)^n$. The following sets are equivalent:

1. $T(I)$ as defined in definition 4.7.
2. The closure in $\mathbb{R}^n$ of the set of all vectors $w \in \Gamma_{val}^n$ for which $\text{in}_w(I)$ does not contain a monomial.

**Proof.**

(1) $\subset$ (2) First observe that the set of units of the ring $K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ consists in the set of non-zero monomials.

Take a point $w \in T(I)$. Then, for every $f \in I$, we know that $\text{in}_w(f)$ is not a monomial, or equivalently for every $f = \sum c_i^f x^{u_i} \in I$, $a_i^f + \langle u_i^f, w \rangle = a_2^f + \langle u_2^f, w \rangle \leq \cdots$, where $a_i^f = \text{val}(c_i^f)$. Therefore, for every $f \in I$ we know that $\langle u_1^f - u_2^f, w \rangle = a_2^f - a_1^f \in \Gamma_{val}$. As a
result, we obtain the infinite system

\[
A^T w = \begin{pmatrix}
u_1 f_1 - u_1 f_1 \\
u_1 f_1 - u_2 f_1 \\
\vdots \\
u_1 f_2 - u_1 f_2 \\
\vdots \\
u_1 f_k - u_1 f_k \\
\end{pmatrix} w = \gamma_{\infty \times 1} \in \Gamma_{val}
\]

(4.11)

where \(k\) represents the last of the terms for which \(a_1^f + \langle u_1^f, w \rangle = a_2^f + \langle u_2^f, w \rangle = \cdots = a_k^f + \langle u_k^f, w \rangle < \cdots\) and the \(f_i\)'s are the elements of \(I\). Since \(w \in \mathbb{R}^n\) is a solution to that system, there are up to \(n\) linearly independent rows in \(A\). By Proposition 4.18 it is enough to consider the finite system \(B'w\), where \(B'\) is the system that is formed by taking the functions that are part of the tropical basis. Moreover, let us select the maximum amount of linearly independent rows from \(B'\) which will form the matrix \(B\).

If we are able to select \(n\), then \(B\) is an invertible matrix and \(w = B^{-1} \gamma_{n \times 1} \in \Gamma_{val}^n\). If we can only select \(m < n\) linearly independent rows, then \(Bw = XDYw = \gamma\), where \(X,Y \in GL_n(\mathbb{Z}), D_{m \times n} = diag(d_i)\). Define \(D_{m \times m}^{-1} = diag(d_i^{-1})\). We can solve \(DYw' = X^{-1} \gamma\) with \(w \in \Gamma_{val}^n\) and hence approach \(w\) by solutions in \(\Gamma_{val}^n\) (since, as we showed in the first chapter, the rational solutions are dense in the kernel of \(B\)).

\((2) \subset (1)\)

We first consider \(w \in \Gamma_{val}^n\). If \(\text{in}_{w_1}(I)\) does not contain a monomial for every \(f \in I\) we have \(\text{in}_{w_1}(f)\) is not a monomial, and so, the minimum is attained at least twice. Therefore, \(w \in T_f \forall f \in I \Rightarrow w \in T(I)\).

If \(w \in \mathbb{R}^n\) and there exists a sequence of points in the value group \(\{w_i\} \rightarrow w\) such that \(\text{in}_{w_i}(I)\) contains no monomial, then we know that for each \(f \in I\), \(\text{in}_{w_i}(f)\) is not a monomial. That implies that the minimum is attained at least twice, and by continuity, in the limit the minimum is still attained at least twice. We conclude again \(w \in T_f \forall f \in I \Rightarrow w \in T(I)\). 

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