# Imperfect Hedging in Defaultable Markets and Insurance Applications

by

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# Abstract

In this thesis, we study the impact of random times to model and manage unpredictable risk events in the financial models. First, as a generalization of the classical Neyman-Pearson lemma, we show how to minimize the probability of type-II-error when the null hypothesis, alternative and the significance level all are revealed to us randomly. This randomness arises some measurability requirements that we have dealt with them by using a measurable selection argument. Then, we consider a regime-switching financial model which is subject to a default time satisfying the so-called the density hypothesis. For this model, we present a Girsanov type result and an explicit representation for the problem of superhedging. In both cases, the desired representation is decomposed into an after-default and a global before-default decomposition. Another problem consists in minimizing the expected shortfall risk for defaultable securities under initial capital constraint. The underlying model is exposed to multiple independent default times satisfying the intensity hypothesis. We illustrate the results by numerical examples and the applications to Guaranteed Minimum Maturity Benefit (GMMB) equity-linked life insurance contracts. Finally, we construct a framework to consider a Guaranteed Minimum Death Benefit (GMDB) equity-linked life insurance contract as a Bermudan option. Under an initial capital constraint, we provide closed-form solutions for the quantile hedging problem of a GMDB contract with a constant guarantee.

# Preface

I, Amir Nosrati, declare that this thesis titled, "Imperfect Hedging in Defaultable Markets and Insurance Applications" and the work presented in it are my own. I confirm that:

- Chapter 4 of this thesis has been published as Alexander Melnikov and Amir Nosrati, "Efficient hedging for defaultable securities and its application to equity-linked life insurance contracts", International Journal of Theoretical and Applied Finance, Vol. 18, No. 7, 2015. I was responsible for the literature review and the proofs of the results as well as the manuscript composition. Professor Alexander Melnikov assisted with the arguments and contributed to manuscript edits. Professor Alexander Melnikov was the supervisory author and was involved with concept formation and manuscript composition.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Dedicated to

My Parents

*"We have not succeeded in answering all our problems. The answers we have found only serve to raise a whole set of new questions. In some ways we feel we are as confused as ever, but we believe we are confused on a higher level and about more important things."*

Posted outside the mathematics reading room,
Tromsø University

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# Chapter 1

# Introduction

The main focus of this thesis is to study random times and their applications in modelling and managing the risk induced by unpredictable events in financial and insurance markets. It is formed of five chapters; the first chapter is the introduction and the other four chapters investigate rather independent problems in statistical test theory, default times, shortfall minimization, and minimum guarantee equity-linked life insurance contracts.

Chapter 1 provides a brief introduction about the main problems and results presented in Chapters 2, 3, 4, and 5. In this chapter, we outline the main techniques and ideas utilized in the thesis. This is organized in separate sections with each section introducing one chapter of the thesis.

In Chapter 2, we want to generalize the classical Neyman-Pearson lemma to the case that the null hypothesis and alternative are selected randomly by two random times $\tau_1$ and $\tau_2$. Let $\big(\Omega, (\mathcal{F}_t)_{t\in[0,T]} \subseteq \mathcal{G}\big)$ be a filtered measurable space. The classical Neyman-Pearson lemma evaluates a null hypothesis corresponding to a probability measure $Q$ on $(\Omega, \mathcal{F}_T)$ against an alternative hypothesis corresponding to a probability measure $P$ on $(\Omega, \mathcal{F}_T)$. In this case, both $P$ and $Q$ are known at time $t = 0$ and the testing problem is performed at $t = 0$ for $P$ and $Q$ defined on $(\Omega, \mathcal{F}_T)$. In contrast, we consider a setting that $P$ and $Q$ are determined stochastically. To do so, suppose $\tau_1$ and $\tau_2$ to be two random times which are not necessarily $\mathbb{F}$-stopping times. For $\omega \in \Omega$, let $\tau_1(\omega) = s \in [0, T)$ and $\tau_2(\omega) = t \in (s, T]$. The random pair $(s, t)$ reveals the probability measures ${}^sP^t$ and ${}^sQ^t$ defined on $(\Omega, \mathcal{F}_t)$ where $\tau_1(\omega) = s$ indicates that we evaluate ${}^sP^t$ against ${}^sQ^t$ conditioned on $\mathcal{F}_s$. To be precise, for a given $\tilde{x} \in [0, 1]$, we solve the following problem:

$$\operatorname*{ess\,sup}_{\varphi \in \mathcal{R}_t} E^{{}^sP^t}\big[\varphi \,\big|\, \mathcal{F}_s\big]$$

subject to the constraint

$$E^{{}^sQ^t}\big[\varphi \,\big|\, \mathcal{F}_s\big] \leq \tilde{x}\,,$$

where $\mathcal{R}_t := \big\{\varphi : \Omega \longrightarrow [0, 1] \;\big|\; \varphi \text{ is } \mathcal{F}_t\text{-measurable}\big\}$.

Chapter 3 deals with a regime-switching financial model which the jump to a new regime occurs at a random time $\tau$. This random time satisfies the so-called density hypothesis, i.e. there exists a conditional density process for the survival process associated to $\tau$. This model was considered by Jiao and Pham [20] to study the utility maximization problem for the case of a CRRA utility function. To avoid the complexity of the dynamic programming in these types of problems, we provide explicit representations for the probability martingale measures and the superhedging problem. Both representations are given in terms of after-default and before-default decompositions in complete markets. We decide to investigate the applications of our results in the future research.

In Chapter 4, we work on efficient hedging problem for defaultable securities with multiple default times and non-zero recovery rates. First, we convert the efficient hedging problem into a Neyman-Pearson problem with composite hypothesis against a simple alternative. Then we apply the non-smooth convex duality to provide a solution in the framework of a defaultable Black-Scholes model. Moreover, in the case of zero recovery rates, we find a closed form solution for the problem. The original problem is formulated in a filtration enlarged by geometric Brownian motion and the multiple independent default times. Our results give us an algorithm to reduce the efficient hedging problem into a similar optimization problem in the default-free Black-Scholes model. As an application, it is shown how to use such type of results in pricing equity-linked life insurance contracts. In addition, we demonstrate the results by some numerical examples.

In the last chapter, Chapter 5, we study a Guaranteed Minimum Death Benefit (GMDB) equity-linked life insurance contract. Under an initial capital constraint, we want to maximize the probability of a successful hedge for a GMDB contract with a constant guarantee. In other words, we solve the quantile hedging problem for a GMDB contract. In the first step, we consider the client's death time, $T(x)$, as a random time. Then, using the progressively enlargement of filtrations, the filtration generated by the underlying equity is enlarged by $T(x)$. This allows us to treat the GMDB contract as an American option with a finite set of permitted exercise dates. Applying the superhedging approach from Schweizer [48], we provide a simple method to hedge the GMDB contract in this framework. Moreover, the max-min problem corresponding to the quantile hedging problem is converted into a straightforward quantile hedging problem for a put option in a complete market.

## 1.1 Neyman-Pearson Lemma for Randomly Selected Hypotheses

In statistical test theory, as a statistical inference method, the main concern is to evaluate two nonempty complementary classes of hypotheses $H_0$ and $H_1$. The statistician, after observing the sample, must test $H_0$ (known as the *null hypothesis*) against $H_1$ (known as the *alternative*). In other words, s/he must decide between the following two options:

- Accepting $H_0$; the null hypothesis is true for the observed sample (rejecting $H_1$).
- Rejecting $H_1$; the alternative hypothesis is false for the sample (accepting $H_0$).

This decision procedure is called a hypothesis test. In performing a test for such a problem, two types of errors might occur:

1. Rejecting $H_0$ when it is true (Type-I-error).
2. Accepting $H_0$ when it is false (Type-II-error).

In general, it is not possible to minimize the probability of these two errors simultaneously. However, we can fix a threshold $\tilde{\alpha} \in (0, 1)$ called the *significance level* to control the probability of type-I-error (*size*) of the acceptable tests. Then, considering this constraint, the optimal test is defined as a test with the size less than or equal to $\alpha$ and the minimum probability of type-II-error. For a given test, one minus the probability of type-II-error is called the *power* of the test which is equal to the probability of *rejecting* $H_0$ when it is *false*. If a class of hypothesis (or alternative) consists of only a single element it is called simple, otherwise it is called composite. For more details about this theory, see for instance Ferguson [13] and Lehmann and Romano [32].

Suppose that $P$ and $Q$ are two probability measures on the measurable space $(\Omega, \mathcal{F})$. Let $R = \frac{1}{2}(P+Q)$, $H_0 = \frac{dQ}{dR}$ and $H_1 = \frac{dP}{dR}$. For the testing problem $Q$ versus $P$, the randomized tests are given in terms of $\mathcal{F}$-measurable random variables $\varphi : \Omega \longrightarrow [0,1]$. For $\omega \in \Omega$, $\varphi(\omega)$ $\big($resp. $1-\varphi(\omega)\big)$ is the conditional probability of rejecting (resp. accepting) $Q$ given $\omega$. Taking into account the probability measure $Q$ on $(\Omega, \mathcal{F})$, then the size of randomized test $\varphi$ is $E^Q[\varphi] = \int_\Omega \varphi(\omega) Q(d\omega)$. Similarly the power of $\varphi$ is given by $E^P[\varphi] = \int_\Omega \varphi(\omega) P(d\omega)$. For a fixed significance level $\tilde{\alpha} \in (0,1)$, define the randomized test $\tilde{\varphi}$ as follows

$$\tilde{\varphi} := 1_{\left\{\frac{dP}{dQ} > \tilde{a}\right\}} + \tilde{\gamma}\, 1_{\left\{\frac{dP}{dQ} = \tilde{a}\right\}} \quad \text{for some constant} \quad \tilde{a} > 0,$$

where $\tilde{\gamma} := \dfrac{\tilde{\alpha} - Q(\frac{dP}{dQ} > \tilde{a})}{Q(\frac{dP}{dQ} = \tilde{a})}$ if $Q(\frac{dP}{dQ} = \tilde{a}) \neq 0$, and $\tilde{\gamma}$ equal to zero otherwise. Moreover, the constant $\tilde{a}$ is computed from the constraint $E^Q[\tilde{\varphi}] = \tilde{\alpha}$. By the classical Neyman-Pearson lemma, it is well known that $\tilde{\varphi}$ has the maximal power on the significance level $\tilde{\alpha}$.

Cvitanić and Karatzas [8] and Rudloff and Karatzas [47] studied the above problem with two families of probability measures $\{Q_i\}_{i\in I}$ and $\{P_j\}_{j\in J}$ considered as composite null hypothesis and alternative, respectively. In Cvitanić and Karatzas [8], first the set of null hypotheses denisities is enlarged. Then techniques of non-smooth convex analysis along with a theorem by Kolmós are applied to find a dual solution and an algorithm for computing the optimal test. By contrast, Rudloff and Karatzas [47] use Fenchel duality and avoid enlarging the set of densities. Under some compactness assumptions, strong duality and existence of a dual solution are obtained simultaneously.

Follmer and Leukert [14] and Follmer and Leukert [15] take a Neyman-Pearson lemma approach to minimize the risk of shortfall in a financial model. In a complete market, to provide explicit solutions, they reduced the original problem to a problem of testing a simple null hypothesis against a simple alternative. Nakano [42] and Melnikov and Nosrati [36] investigated the efficient hedging problem in defaultable markets, this leads to testing a composite null hypothesis versus a simple alternative. They adapted the techniques of generalized Neyman-Pearson lemma from Cvitanić and Karatzas [8] to obtain explicit solutions in these types of incomplete financial models.

In Chapter 2, we consider a filtered probability space $\big(\Omega, (\mathcal{F}_t)_{t\in[0,T]} \subseteq \mathcal{G}\big)$ with two nonnegative $\mathcal{G}$-random variables $\tau_1$ and $\tau_2$ known as $\mathcal{G}$-random times. We assume that probability measures ${}^sP^t$ and ${}^sQ^t$, defined on $(\Omega, \mathcal{F}_t)$, are revealed to us by the random instants $\tau_1 = s \in [0, T)$ and $\tau_2 = t \in (s, T]$. Taking ${}^sQ^t$ and ${}^sP^t$ as the simple null hypothesis and alternative, respectively, we want to solve the problem of testing ${}^sQ^t$ against ${}^sP^t$ conditioned on the $\sigma$-field $\mathcal{F}_s$. In the case of the classical Neyman-Pearson lemma, the only available information regarding $\omega \in \Omega$ is the fact that $\omega$ belongs to $\Omega$, i.e. the $\sigma$-field $\mathcal{F}_0 = \{\emptyset, \Omega\}$. However, in Chapter 2, we make the decisions at time $s$ and the available information is $\mathcal{F}_s$. Thus we take the corresponding conditional expectations $E^{{}^sP^t}\big[\varphi \,\big|\, \mathcal{F}_s\big]$ and $E^{{}^sQ^t}\big[\varphi \,\big|\, \mathcal{F}_s\big]$ as the power and the size of the randomized test $\varphi$. To deal with the arising measurability requirements, we exploit a measurable selection argument to determine an $\mathcal{F}_s \times \mathcal{B}(\mathbb{R}_+)$-measurable random variable $\tilde{a}_s$ similar to the constant $\tilde{a}$. Combining this with the classical Neyman-Pearson lemma, we provide a closed form for the optimal test with maximal power on the given significance level. We are mainly motivated by the application of this result to efficient hedging problems for regime switching financial models under default density hypothesis. Chapter 2 is structured as follows:

In Section 2.1, we formulate the problem and present the main results and their proofs. Section 2.2 illustrates the theory with several examples originated from mathematical finance and insurance. In the appendix, we recall a measurable selection theorem used in the proof of Theorem 2.1.

## 1.2 Girsanov Theorem and Superhedging in a Default Density Framework

Motivated by Jiao and Pham [20] and Karoui et al. [25], in Chapter 3 we study a Black-Scholes regime switching model where the rate of return and the volatility of the model jump to a new regime at a random time $\tau$.

$\tau$ can be interpreted as the default time of a counterparty which induces a jump in the price of the underlying risky asset $(S_t)_{t\in[0,T]}$. Let $(\Omega, \mathcal{G}, P)$ be a complete probability space. We assume the following representation for $S$ on $(\Omega, \mathcal{G}, P)$:

$$S_t = S_t^{\mathbb{F}} 1_{\{t<\tau\}} + S_t^d(\tau) 1_{\{t\geq\tau\}}, \quad t \in [0,T], \tag{1.1}$$

where $(S_t^{\mathbb{F}})_{t\in[0,T]}$ and $(S_t^d(\tau))_{t\in[0,T]}$ are governed by geometric Brownian motion models. Before the default occurrence, i.e. on $\{t < \tau\}$ we have $S_t = S_t^{\mathbb{F}}$ with

$$dS_t^{\mathbb{F}} = S_t^{\mathbb{F}}(\mu_t^{\mathbb{F}} dt + \sigma_t^{\mathbb{F}} dW_t), \qquad S_0^{\mathbb{F}} = S_0 > 0, \quad t \in [0,T].$$

After the default, on $\{t \geq \tau\}$, the processes $\mu^{\mathbb{F}}$ and $\sigma^{\mathbb{F}}$ switch to $\left(\mu_t^d(\theta)\right)_{t\in[\theta,T]}$ and $\left(\sigma_t^d(\theta)\right)_{t\in[\theta,T]}$ respectively with $\theta = \tau$ denoting time of the default. In this case, $S_t = S_t^d(\theta)$ with

$$\begin{cases} dS_t^d(\theta) = S_t^d(\theta)\big(\mu_t^d(\theta)dt + \sigma_t^d(\theta)dW_t\big) \quad , \, t \in (\theta, T] \\ \\ S_\theta^d(\theta) = S_\theta^{\mathbb{F}}(1-\gamma_\theta) \end{cases}$$

The process $(\gamma_t)_{t\in[0,T]}$, satisfying $-\infty < \gamma_t < -1$ for all $t \in [0,T]$, represents the size of the jumps.

Let $\mathbb{F}$ be the filtration generated by $\big(S_t^{\mathbb{F}}\big)_{t\in[0,T]}$, and $\mathbb{H} := \big(\sigma(\tau \wedge t)\big)_{t\in[0,T]}$. Then by progressive enlargement of the filtrations define $\mathbb{G} := \mathbb{F} \vee \mathbb{H}$. We assume that there exists a family of positive $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}_+)$-measurable functions $\alpha_t(\theta)$ such that

$$P\big(\tau \in d\theta | \mathcal{F}_t\big) = \alpha_t(\theta)d\theta.$$

This assumption is well-known as the *density hypothesis* for the random time $\tau$.

Considering the above framework, Jiao and Pham [20] studied the utility maximization problem in the enlarged filtration $\mathbb{G}$ for the CRRA function $U(x) = \dfrac{x^p}{p}$, $p < 1$, $p \neq 0$, $x > 0$. They decomposed the problem into two subproblems: an after-default utility maximization problem, and a global before-default optimization problem. The after-default problem can be solved by a standard duality approach. But the global before-default problem is more challenging, Jiao and Pham [20] used a dynamic programming approach to characterize the optimal solution in terms of backward stochastic differential equations (BSDE). Although their approach is very interesting, it heavily relies on their choice of utility function. In particular, the global before-default problem becomes too complicated when the utility function is not a CRRA function.

To find an easier way to deal with these types of problems for the financial model (1.1), in Chapter 3, we are looking for an explicit representation for the Radon-Nikodým density of probability martingale measures of $(S_t)_{t\in[0,T]}$ with respect to $(\mathbb{G}, P)$. In fact, by having these representations we can use the classical techniques of utility maximization in incomplete markets. We point out that decomposing the problem into after-default and global before-default problem transfers the complexity of the problem to the second problem. The global-before default problem turns into a stochastic control problem which it is in general difficult to find a closed-form solution. In Chapter 3, we only work on the structure of probability martingale measure and the superhedging problem as the first step; the application of the results will be investigated in our future research.

In the case of infinite time horizon, Karoui et al. [25] give a characterization of $\mathbb{G}$-martingales in terms of $\mathbb{F}$-martingales. We adapt their techniques to find a similar result for the case of finite time horizon. This result enables us to determine the desired representation for the probability martingale measures. As is expected, the after-default part of Radon-Nikodým derivatives is given by the standard Girsanov theorem. However, the global before-default part satisfies a different SDE which, in addition to the rate of return and volatility, incorporates the size of jumps $(\gamma_t)_{t\in[0,T]}$ into the equation.

Using our characterization for the probability martingale measures, we solve the superhedging problem for a general $\mathcal{G}_T$-measurable contingent claim on the probability space $(\Omega, \mathcal{G}, P)$. Again, we decompose the superhedging problem into two parts:

1. An after-default perfect hedge by investing in $\left(S_t^d(\theta)\right)_{t\in[\theta,T]}$ for $\theta \in [0, T]$.

2. A global before-default perfect hedge by investing in $\left(S_t^{\mathbb{F}}\right)_{t\in[0,T]}$.

## 1.3 Efficient Hedging for Defaultable Securities and its Applications to Life Insurance

It is known that in a complete market, starting with a large enough initial capital, there is a perfect hedge for every contingent claim. However, if the market is incomplete the initial cost of superhedging (see El Karoui and Quenez [12] or Karatzas [22]) is too high. As we know, defaultable markets usually turn into incomplete markets. In fact the default time which is represented by a random time can not be hedged by investing in the available assets in the market. This issue makes superhedging too expensive in defaultable markets. Therefore, we are forced to introduce new measures of risk and start investing with a smaller initial capital than the superhedging cost. But *high cost of superhedging* is not the only reason that makes the efficient hedging interesting. It is true that the perfect hedge or superhedge eliminates risk but it eliminates opportunities too. There are financial institutions that seek out risk, financial institutions as insurance companies expose themselves intentionally to risk and exploit risk to generate value.

In the framework of Black-Scholes models with a differentiable *loss function* $l$, we consider the efficient hedging problem as the following minimization problem:

$$\min_{\substack{\pi \in \mathcal{A}^{\mathbb{G}}(v) \\ v \leq \tilde{u}}} E\Big[l\big((H_\delta - V_T^{v,\pi})^+\big)\Big] \tag{1.2}$$

where $H$ is a default-free contingent claim, $H_\delta = H \prod_{i=1}^{n} \big(1_{\{\tau_i > T\}} + \delta_i 1_{\{\tau_i \leq T\}}\big)$ defaultable with recovery rates $\delta_i$'s, and $\tau_i$'s represent the default times. Also, $\tilde{u}$ is the available initial capital, and $V_T^{v,\pi}$ the terminal value of value process corresponding to the admissible strategy $\pi$.

In the context of general incomplete markets, Follmer and Leukert [15] studied the minimization of the expectation of shortfall risk weighted by a general loss function $l$, problem (1.2), in a general semimartingale setting. They emphasized the Neyman-Pearson lemma and provided an explicit solution for differentiable loss functions in complete markets. Nakano [42] adapted the convex duality approach introduced by Cvitanić [7] and Cvitanić and Karatzas [8] for defaultable claims with a single default time. Then for a linear loss function $l(x) = x$, recovery rate $\delta = 0$ and under some assumptions on the solution of the dual problem, $\tilde{\varphi}$, it was shown that there exists a solution for the partial hedging problem in defaultable markets.

In contrast to Nakano [42] that only shows the existence of the solution, we work with more general loss functions (not necessarily linear) and there is no restriction on $\tilde{\varphi}$. Moreover, if the recovery rates are zero, our approach provides an explicit solution for problem (1.2) with multiple default times. For a given initial capital, we find a relation between the minimum value of shortfall risk in the defaultable market and its corresponding value in the default-free market. Chapter 4 is organized as follows:

In Section 4.1, we introduce our financial model. We also recall some definitions and notations regarding default times and incomplete markets. Section 4.2 presents the formulation of the problem. The first step to solve problem (1.2) is to notice that this problem is clearly a *dynamic optimization problem* with respect to time. Follmer and Leukert [15] proved that (1.2) is equivalent to a *static optimization problem* and there exists a solution to the static problem. Our idea is to reduce problem (1.2) to this static problem, then using Gateaux derivative we find a *max-min problem* for testing a *composite hypothesis* against a *simple alternative*. The results of Cvitanić and Karatzas [8] on *generalized Neyman-Pearson lemma* give us a representation for the solution ($\tilde{\varphi}$) of the *max-min problem*. Then the optional decomposition of the modified claim $\tilde{\varphi}H_\delta$ provides the optimal solution to problem (1.2). Furthermore, we show that, for the zero recovery rates, the efficient hedging problem in the defaultable market can be reduced to the study of problem in the default-free market.

To demonstrate our results, in Section 4.3, we apply them to equity-linked life insurance contract. Equity-linked life insurance contract is a well-developing area of theory and applications now. We consider these contracts from point view of their pricing using efficient hedging techniques. See the book of Hardy [17] as a good reference for such contracts; and Melnikov and Romaniuk [38] as one of the first papers with insurance applications of efficient hedging techniques. To our knowledge, the problem of pricing of equity-linked life insurance contracts has not been studied yet in defaultable markets, and we are going to adapt these techniques to this case. We know that the equity-linked life insurance contracts usually are long-term contracts with maturity $T = 15,\ 20$ or 25, so it is reasonable to take into account the default possibility of insurance company during the life of the contract. By solving problem (1.2), we can find a competitive premium to offer to the insured for the accepted level of risk by the insurer. Also we are interested to calculate the corresponding shortfall risk for a given premium as the available initial capital. Finally, we illustrate our method with numerical results and compare the efficient hedging problem in a default-free market with the analogous problem in the presence of default. For the reader's convenience, the results of Follmer and Leukert [15]; and Cvitanić and Karatzas [8] are summarized in the Appendices.

## 1.4 Bermudan Options and their Connections to GMDB Contracts

To provide both investment opportunities and the mortality protections, insurance companies have designed equity-linked life insurance contracts. This type of life insurance contracts became popular in the United Kingdom in the late 1960's through to the late 1970's. Gradually equity-linked life contracts were introduced in the countries where the UK insurance companies were influential such as Australia, South Africa, and the United states. In the late 1990's, segregated fund contracts as a type of equity-linked life insurance became available in Canada. Segregated fund structure is usually a complex combination of guaranteed values upon the death or survival of the client during the term of the contract. See Hardy [17], Aase and Persson [1] and Ekern and Persson [11] for a detailed and comprehensive study of the equity-linked life insurance contracts.

Two sources of randomness are involved in equity-linked life insurance contracts: the mortality risk of the insurer and the financial risk associated to the underlying equity. On one hand, the insurer sells a large number of contracts to different clients, and on the other hand, the survival of the insureds and the financial risk are highly independent. By these two features, traditionally the strong law of large numbers is utilized to estimate the total number of claims at the maturity. Then this mean value of the total claims, which is still exposed to the financial risk, is hedged by using a dynamic-hedging approach. This method was introduced by Brennan and Schwartz [5] to price and hedge guaranteed minimum maturity benefit equity-linked life insurance contracts. After diversifying the mortality risk by the strong law of large numbers, they decomposed the benefit into a constant guarantee and a call option. The constant value is hedged by investing in a risk free bond, and Black-Scholes formula provides a perfect hedge for the embedded call option.

In Chapter 5, we study the Guaranteed Minimum Death Benefit (GMDB) life insurance contracts where the benefit is paid upon the insured's death over the term of the contract. The payoff process of a GMDB contract with a constant guarantee is given by

$$U_t := \text{Max}(K, S_t), \qquad \text{for } t \in R := \{1, 2, ..., T\}, \tag{1.3}$$

where $K > 0$ is the constant amount of guarantee and $(S_t)_{t\in[0,T]}$ is the price process of the underlying asset. The finite set $R$ is a suitable subset of $[0, T]$, for instance months.

Inspired by Schweizer [48] and the techniques of enlargement of filtrations, we construct a framework which allows us to view the GMDB contract as an American option with the predetermined finite exercise dates $R$. These types of American options are known as Bermudan options. In this setting, one can also compare the GMDB contract (1.3) with the Option Based Portfolio Insurance (OBPI) dynamic hedging introduced by Leland and Rubinstein [33]. Let us denote the filtration generated by $S$ by $\mathbb{F}$ and the filtration generated by the client's lifetime by $\mathbb{H}$. To make the exercise date of the GMDB contract a stopping time, we progressively enlarge $\mathbb{F}$ with $\mathbb{H}$ and denote the enlarged filtration by $\mathbb{G}$.

Assume $\tilde{v}_0 > 0$ and a $\mathbb{G}$-predictable $S$-integrable process $(\pi_t)_{t\in[0,T]}$ to be given. Define the corresponding value process as follows:

$$V_t^{\tilde{v}_0,\pi} := \tilde{v}_0 + \int_0^t \pi_s dS_s, \qquad P\text{-a.s., for all } t \in [0, T].$$

We represent the set of all process $\pi$ satisfying

$$V_t^{\tilde{v}_0,\pi} \geq 0 \qquad P\text{-a.s. for all } t \in [0,T]$$

by $\mathcal{A}^{\mathbb{G}}(\tilde{v}_0)$.

Using the independency assumption, as described above, we provide an explicit form for the Radon-Nikodým density of the probability martingale measures of $S$ with respect to the new filtration $\mathbb{G}$.

The superhedging method and the techniques of Schweizer [48] are adapted to price the embedded Bermudan option in the GMDB contract. From the independency and the Radon-Nikodým density representation, we show that superhedging value process of the GMDB contract is equal to the perfect hedging of the European option $\text{Max}(K, S_T)$. In addition, by the separate account design of GMDB, the actual liability of the insurance company becomes $(K - S_T)^+$, i.e. the shortfall in the case that the guarantee $K$ matures in-the-money.

Our main aim, in Chapter 5, is to solve the quantile hedging problem for the Bermudan option $(U_t)_{t\in R}$. More precisely, max-min problem (1.4) is solved and the optimal trading strategy that achieves the maximal value is determined. Let $\mathcal{S}_{0,T}(R)$ be the set of all $\mathbb{G}$-stopping with values in $R$, then we investigate the following problem:

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}}(\tilde{v}_0)} \left( \inf_{\tau\in\mathcal{S}_{0,T}(R)} P\big(V_\tau^{\tilde{v}_0,\pi} \geq U_\tau\big)\right). \tag{1.4}$$

We prove that for any $\pi \in \mathcal{A}^{\mathbb{G}}(\tilde{v}_0)$ the worst scenario always occurs at $\tau \equiv T$, i.e.

$$\inf_{\tau\in\mathcal{S}_{0,T}(R)} P\big(V_\tau^{\tilde{v}_0,\pi} \geq U_\tau\big) = P\big(V_T^{\tilde{v}_0,\pi} \geq U_T\big). \tag{1.5}$$

This result simplifies (1.4) significantly, without (1.5) we need to find a saddle point for the objective function $P\big(V_\tau^{\tilde{v}_0,\pi} \geq U_\tau\big)$. However, the existence of a saddle point is not always guaranteed, in particular for a stochastic dynamic problem such as (1.4). Aguilar [2] studied the quantile hedging problem for American options in a general semimartingale setting. He reformulated the problem as a hypothesis testing problem and applied a convex duality method similar to Cvitanić [7] and Cvitanić and Karatzas [8], but he only achieved an upper bound for this problem. In a Black-Scholes framework, we solve (1.4) for its optimal value with equality.

At the end, we show that the optimal trading strategy $\tilde{\pi}$ belongs to $\mathcal{A}^{\mathbb{F}}(\tilde{v}_0)$, this helps us to give an explicit representation for the maximal probability of success and its optimal trading strategy.

# Chapter 2

# Conditional Neyman-Pearson Lemma for Randomly Selected Hypothesis and Alternative

## 2.1 Problem formulation and the main results

For a fixed $T > 0$, let $\big(\Omega, \mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]} \subset \mathcal{G}\big)$ be a complete measurable space equipped with the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ such that $\mathcal{F}_t \subseteq \mathcal{G}$ for any $t \in [0,T]$. We consider the families of probability measures $\big\{({}^sP^t)_{t\in(s,T]}\big\}_{s\in[0,T)}$ and $\big\{({}^sQ^t)_{t\in(s,T]}\big\}_{s\in[0,T)}$ where, for $0 \leq s < t \leq T$, both ${}^sP^t$ and ${}^sQ^t$ are probability measures on the complete measurable space $(\Omega, \mathcal{F}_t)$.

For any $t \in [0,T]$, let us define $\mathcal{R}_t := \big\{\varphi : \Omega \longrightarrow [0,1] \,\big|\, \varphi \text{ is } \mathcal{F}_t\text{-measurable}\big\}$. In this chapter, we consider a setting that ${}^sP^t$, ${}^sQ^t$ and $\mathcal{F}_t$ are randomly selected according to two $\mathcal{G}$-random times (positive $\mathcal{G}$-measurable random variables) $\tau_1$ and $\tau_2$ such that $\tau_1 = s$ and $\tau_2 = t$ with $s < t$. We point out that $\tau_1$ and $\tau_2$ are not necessarily $\mathbb{F}$-stopping times. For a given $\tilde{x} \in [0,1]$ and these random choices of $s$ and $t$, our main goal is to solve the following maximization problem:

$$\operatorname*{ess\,sup}_{\varphi\in\mathcal{R}_t} E^{{}^sP^t}\big[\varphi \,\big|\, \mathcal{F}_s\big] \tag{2.1}$$

subject to the constraint

$$E^{{}^sQ^t}\big[\varphi \,\big|\, \mathcal{F}_s\big] \leq \tilde{x} \tag{2.2}$$

In the following, Theorem 2.1, Theorem 2.2 and Theorem 2.3 together fully characterize the solutions to problem (2.1)-(2.2). Consider the $\mathcal{G}$-random times $\tau_1$ and $\tau_2$. At $\tau_1 = s$ and $\tau_2 = t$, with $0 \leq s < t \leq T$, let us introduce ${}^sR^t := \frac{1}{2}\big({}^sP^t + {}^sQ^t\big)$ and

$$\frac{d\,{}^sP^t}{d\,{}^sQ^t} := \begin{cases} \dfrac{d\,{}^sP^t}{d\,{}^sR^t} \cdot \left(\dfrac{d\,{}^sQ^t}{d\,{}^sR^t}\right)^{-1} & ;\ \text{on } \left\{\dfrac{d\,{}^sQ^t}{d\,{}^sR^t} \neq 0\right\} \\ \\ +\infty & ;\ \text{otherwise.} \end{cases} \tag{2.3}$$

For a given positive $\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+)$-measurable random variable $(\omega, x) \longrightarrow \tilde{a}_s(\omega, x)$ define:

$$\tilde{\varphi}(s,t) := 1_{\left\{\frac{d\,{}^sP^t}{d\,{}^sQ^t} > \tilde{a}_s\right\}} + \tilde{\gamma}_s 1_{\left\{\frac{d\,{}^sP^t}{d\,{}^sQ^t} = \tilde{a}_s\right\}} \qquad {}^tR^s\text{-a.s.}, \tag{2.4}$$

where

$$\tilde{\gamma}_s := \begin{cases} 0 & ;\ {}^sQ^t\Big(\frac{d\,{}^sP^t}{d\,{}^sQ^t} = \tilde{a}_s \,\Big|\, \mathcal{F}_s\Big) = 0 \\ \\ \dfrac{\tilde{x} - {}^sQ^t\Big(\frac{d\,{}^sP^t}{d\,{}^sQ^t} > \tilde{a}_s \,\Big|\, \mathcal{F}_s\Big)}{{}^sQ^t\Big(\frac{d\,{}^sP^T}{d\,{}^sQ^t} = \tilde{a}_s \,\Big|\, \mathcal{F}_s\Big)} & ;\ \text{otherwise.} \end{cases} \tag{2.5}$$

**Theorem 2.1.** *For any $\tilde{x} \in [0,1]$, there exists a positive $\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+)$-measurable random variable $(\omega, x) \longrightarrow \tilde{a}_s(\omega, x)$ such that $\tilde{\varphi}(s,t)$, defined by (2.3)-(2.5),* *satisfies the constraint:*

$$E^{{}^sQ^t}\big[\tilde{\varphi}(s,t)\,\big|\,\mathcal{F}_s\big] = \tilde{x}, \quad {}^tR^s\text{-}a.s. \tag{2.6}$$

*Proof.* Our main idea is to extend the proof of the classical Neyman-Pearson lemma and combine it with a measurable selection argument. For any $\omega \in \Omega$ and $a \in \mathbb{R}$, let us define:

$$X_s(\omega, a) := {}^sQ^t\Big(\frac{d\,{}^sP^t}{d\,{}^sQ^t} > a\,\Big|\,\mathcal{F}_s\Big)(\omega) = 1 - {}^sQ^t\Big(\frac{d\,{}^sP^t}{d\,{}^sQ^t} \le a\,\Big|\,\mathcal{F}_s\Big)(\omega). \tag{2.7}$$

By the properties of a distribution function, $X_s(\omega, \cdot)$ is a non-increasing right continuous function. Define

$$\tilde{a}_s(\omega, \tilde{x}) := \inf\Big\{a \ge 0 \,\Big|\, {}^sQ^t\Big(\frac{d\,{}^sP^t}{d\,{}^sQ^t} > a\,\Big|\,\mathcal{F}_s\Big)(\omega) \le \tilde{x}\Big\}. \tag{2.8}$$

Since $g_s(\omega, \tilde{x}) = {}^sQ^t\big(\frac{d\,{}^sP^t}{d\,{}^sQ^t} > a\,\big|\,\mathcal{F}_s\big)(\omega) - \tilde{x}$, as a function of $(\omega, \tilde{x}) \in \Omega \times \mathbb{R}_+$, is $\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+)$-measurable, we can show that $Graph\big(\tilde{a}_s(\cdot,\cdot)\big) \in \mathcal{F}_s \times \mathcal{B}(\mathbb{R}_+)$. Therefore, there exists an $\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+)$-measurable selection $\tilde{a}_s : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ for $\tilde{a}_s(\cdot,\cdot)$ such that $\tilde{a}_s(\omega) := \tilde{a}_s\big(\omega, \tilde{x}\big)$ for all $\omega \in \Omega$. For the reader's convenience we recall the Aumann's measurable selection theorem in Appendix C.

It is easy to see that

$${}^sQ^t\Big(\frac{d\,{}^sP^t}{d\,{}^sQ^t} = \tilde{a}_s(\omega)\,\Big|\,\mathcal{F}_s\Big)(\omega) = X_s(\omega, \tilde{a}_s(\omega)-) - X_s(\omega, \tilde{a}_s(\omega)). \tag{2.9}$$

Now let us consider $\tilde{\varphi}(s,t)$, constructed as in (2.4). If

$$X_s(\omega, \tilde{a}_s(\omega)-) \neq X_s(\omega, \tilde{a}_s(\omega))$$

then from (2.5) and (2.9) we get

$$E^{{}^sQ^t}\big[\tilde{\varphi}(s,t)\,\big|\,\mathcal{F}_s\big](\omega) = {}^sQ^t\big(\frac{d\,{}^sP^t}{d\,{}^sQ^t} > \tilde{a}_s(\omega)\,\big|\,\mathcal{F}_s\big)(\omega)$$
$$+\Big(\frac{\tilde{x} - {}^sQ^t\big(\frac{d\,{}^sP^t}{d\,{}^sQ^t} > \tilde{a}_s(\omega)\,\big|\,\mathcal{F}_s\big)(\omega)}{{}^sQ^t\big(\frac{d\,{}^sP^T}{d\,{}^sQ^t} = \tilde{a}_s(\omega)\,\big|\,\mathcal{F}_s\big)(\omega)}\Big)\,{}^sQ^t\big(\frac{d\,{}^sP^t}{d\,{}^sQ^t} = \tilde{a}_s(\omega)\,\big|\,\mathcal{F}_s\big)(\omega)$$
$$= \tilde{x}.$$

In the case that $X_s(\omega, \tilde{a}_s(\omega)-) = X_s(\omega, \tilde{a}_s(\omega))$, the function $X_s(\omega, \cdot)$ is continuous at $\tilde{a}_s(\omega)$. This with $X_s(\omega, \cdot)$ being decreasing and definition (2.8) implies that

$$ {}^sQ^t\big(\frac{d\,{}^sP^t}{d\,{}^sQ^t} > \tilde{a}_s(\omega)\,\big|\,\mathcal{F}_s\big)(\omega) = \tilde{x}. \tag{2.10}$$

From (2.5), in this case, one can see that $\tilde{\gamma}_s(\omega) = 0$. This with (2.10) shows that again

$$E^{{}^sQ^t}\big[\tilde{\varphi}(s,t)\,\big|\,\mathcal{F}_s\big](\omega) = \tilde{x}. \tag{2.11}$$

This completes the proof. □

**Theorem 2.2.** *Under the same assumptions as in Theorem 2.1, $\tilde{\varphi}(s,t) \in \mathcal{R}_t$ defined by (2.4)-(2.5) solves the maximization problem (2.1)-(2.2).*

*Proof.* Using the Lebesgue decomposition of ${}^sQ^t$ with respect to ${}^sP^t$, one gets

$$ {}^sP^t(A) = \int_A \frac{d\,{}^sP^t}{d\,{}^sQ^t}\,d\,{}^sQ^t + {}^sP^t\Big(A \cap \Big\{\frac{d\,{}^sQ^t}{d\,{}^sR^t} = 0\Big\}\Big) \tag{2.12}$$

for all $A \in \mathcal{F}_s$, see Föllmer and Schied [16]. Notice that $\frac{d\,{}^sP^t}{d\,{}^sQ^t}$ is defined by (2.3) and ${}^sQ^t\big(\frac{d\,{}^sQ^t}{d\,{}^sR^t} = 0\big) = 0$.

Consider $\varphi \in \mathcal{R}_t$ satisfying the constraint condition

$$E^{{}^sQ^t}\big[\varphi\,\big|\,\mathcal{F}_s\big] \le \tilde{x} = E^{{}^sQ^t}\big[\tilde{\varphi}(s,t)\,\big|\,\mathcal{F}_s\big], \tag{2.13}$$

where $\tilde{\varphi}(s,t)$ is determined from Theorem 2.1. Using decomposition (2.12), for any $A \in \mathcal{F}_s$ we have:

$$\begin{aligned}
\int_A E^{{}^sP^t}\big[\tilde{\varphi}(s,t)\,\big|\,\mathcal{F}_s\big] - E^{{}^sP^t}\big[\varphi\,\big|\,\mathcal{F}_s\big]\, d\,{}^sP^t &= \int_A E^{{}^sP^t}\big[\tilde{\varphi}(s,t) - \varphi\,\big|\,\mathcal{F}_s\big]\, d\,{}^sP^t \\
&= \int_A \tilde{\varphi}(s,t) - \varphi\, d\,{}^sP^t \\
&= \int_A \big(\tilde{\varphi}(s,t) - \varphi\big)\frac{d\,{}^sP^t}{d\,{}^sQ^t}\, d\,{}^sQ^t \\
&+ \int_A \big(\tilde{\varphi}(s,t) - \varphi\big)1_{\left\{\frac{d\,{}^sQ^t}{d\,{}^sR^t}=0\right\}}\, d\,{}^sP^t \\
&\geq \int_A \tilde{a}_s\big(\tilde{\varphi}(s,t) - \varphi\big)\, d\,{}^sQ^t \\
&= \int_A \tilde{a}_s E^{{}^sQ^t}\big[\tilde{\varphi}(s,t) - \varphi\,\big|\,\mathcal{F}_s\big]\, d\,{}^sQ^t \geq 0.
\end{aligned} \tag{2.14}$$

To get the inequality, notice that:

- $\tilde{\varphi}(s,t) = 1 \geq \varphi \quad {}^sR^t\text{-a.s. on } \left\{\frac{d\,{}^sP^t}{d\,{}^sQ^t} > \tilde{a}_s\right\} \cup \left\{\frac{d\,{}^sQ^t}{d\,{}^sR^t} = 0\right\}$
- $\big(\tilde{\varphi}(s,t) - \varphi\big)\left(\frac{d\,{}^sP^t}{d\,{}^sQ^t} - \tilde{a}_s\right) = 0 \quad {}^sR^t\text{-a.s. on } \left\{\frac{d\,{}^sP^t}{d\,{}^sQ^t} = \tilde{a}_s\right\}$
- $\tilde{\varphi}(s,t) = 0 \leq \varphi \quad {}^sR^t\text{-a.s. on } \left\{\frac{d\,{}^sP^t}{d\,{}^sQ^t} \leq \tilde{a}_s\right\}$

Therefore, in any case, we can see that

$$\big(\tilde{\varphi}(s,t) - \varphi\big)\left(\frac{d\,{}^sP^t}{d\,{}^sQ^t} - \tilde{a}_s\right) \geq 0, \qquad {}^sR^t\text{-a.s.} \tag{2.15}$$

In the last equality of (2.14), we used the fact that $\tilde{a}_s \in \mathcal{F}_s$. Now, (2.14) means that for any arbitrary $A \in \mathcal{F}_s$:

$$\int_A E^{{}^sP^t}\big[\tilde{\varphi}(s,t)\,\big|\,\mathcal{F}_s\big] - E^{{}^sP^t}\big[\varphi\,\big|\,\mathcal{F}_s\big]\, d\,{}^sP^t \geq 0,$$

which is equivalent to

$$E^{{}^sP^t}\big[\varphi \,\big|\, \mathcal{F}_s\big] \leq E^{{}^sP^t}\big[\tilde{\varphi}(s,t) \,\big|\, \mathcal{F}_s\big], \qquad {}^sP^t\text{-a.s.} \tag{2.16}$$

This inequality with (2.13) proves that $\tilde{\varphi}(s,t)$ is a solution to problem (2.1)-(2.2). □

**Theorem 2.3.** *For $\tilde{x} \in [0,1]$, and $s < t$ given as in Theorem 2.1, suppose $\tilde{\psi}(s,t) \in \mathcal{R}_t$ satisfies the condition*

$$E^{{}^sQ^t}\big[\tilde{\psi}(s,t) \,\big|\, \mathcal{F}_s\big] = \tilde{x}, \qquad {}^sR^t\text{-}a.s. \tag{2.17}$$

*and it also solves problem (2.1)-(2.2). Then*

$$\tilde{\psi}(s,t) = \tilde{\varphi}(s,t), \qquad {}^sR^t\text{-}a.s. \tag{2.18}$$

*where $\tilde{\varphi}(s,t)$ is the random variable given by (2.4)-(2.5).*

*Proof.* Since $\tilde{\psi}(s,t)$ satisfies the constraint (2.2) and $\tilde{\varphi}(s,t)$ is a solution to problem (2.1)-(2.2), we have

$$E^{{}^sP^t}\big[\tilde{\psi}(s,t) \,\big|\, \mathcal{F}_s\big] \leq E^{{}^sP^t}\big[\tilde{\varphi}(s,t) \,\big|\, \mathcal{F}_s\big], \qquad {}^sP^t\text{-a.s.} \tag{2.19}$$

With a similar argument, it is easy to see the reverse inequality which together with (2.19) yields to $E^{{}^sP^t}\big[\tilde{\psi}(s,t) \,\big|\, \mathcal{F}_s\big] = E^{{}^sP^t}\big[\tilde{\varphi}(s,t) \,\big|\, \mathcal{F}_s\big]$, ${}^sP^t$-a.s.

Now, by applying the Lebesgue decomposition (2.12) one more time, we have

$$0 = \int_A E^{{}^sP^t}\big[\tilde\varphi(s,t)\,\big|\,\mathcal{F}_s\big] - E^{{}^sP^t}\big[\tilde\psi(s,t)\,\big|\,\mathcal{F}_s\big]\,d\,{}^sP^t$$
$$= \int_A \big(\tilde\varphi(s,t)-\tilde\psi(s,t)\big)\frac{d\,{}^sP^t}{d\,{}^sQ^t}\,d\,{}^sQ^t + \int_A \big(\tilde\varphi(s,t)-\tilde\psi(s,t)\big)1_{\left\{\frac{d\,{}^sQ^t}{d\,{}^sR^t}=0\right\}}\,d\,{}^sP^t \tag{2.20}$$

for all $A \in \mathcal{F}_s$. From (2.6), (2.17) and $\tilde a_s \in \mathcal{F}_s$:

$$\int_A \tilde a_s\big(\tilde\varphi(s,t)-\tilde\psi(s,t)\big)\,d\,{}^sQ^t = \int_A \tilde a_s E^{{}^sQ^t}\big[\tilde\varphi(s,t)-\tilde\psi(s,t)\,\big|\,\mathcal{F}_s\big]\,d\,{}^sQ^t = 0\,, \tag{2.21}$$

where $\tilde a_s$ is defined as in Thereom 2.1. By combining(2.20) and (2.21), we can write

$$\int_A \big(\tilde\varphi(s,t)-\tilde\psi(s,t)\big)\big(\frac{d\,{}^sP^t}{d\,{}^sQ^t}-\tilde a_s\big)d\,{}^sQ^t+\int_A \big(\tilde\varphi(s,t)-\tilde\psi(s,t)\big)1_{\left\{\frac{d\,{}^sQ^t}{d\,{}^sR^t}=0\right\}}d\,{}^sP^t = 0. \tag{2.22}$$

Using (2.15) and ${}^sQ^t \ll {}^sR^t$, we find that

$$\big(\tilde\varphi(s,t)-\tilde\psi(s,t)\big)\big(\frac{d\,{}^sP^t}{d\,{}^sQ^t}-\tilde a_s\big) \geq 0 \quad {}^sQ^t\text{-a.s.}$$

A similar argument and ${}^sP^t \ll {}^sR^t$ imply

$$\big(\tilde\varphi(s,t)-\tilde\psi(s,t)\big)1_{\left\{\frac{d\,{}^sQ^t}{d\,{}^sR^t}=0\right\}} \geq 0 \quad {}^sP^t\text{-a.s.}$$

Therefore, we have $\tilde\psi(s,t) = \tilde\varphi(s,t)\ \ {}^sQ^t$-a.s. on $\big\{\frac{d\,{}^sP^t}{d\,{}^sQ^t} \neq \tilde a_s\big\}$, and

$$\tilde\psi(s,t) = \tilde\varphi(s,t), \qquad {}^sP^t\text{-a.s., on } \Big\{\frac{d\,{}^sQ^t}{d\,{}^sR^t}=0\Big\}. \tag{2.23}$$

In other words, by the definition of $\tilde{\varphi}(s,t)$ we can write

$$\tilde{\psi}(s,t) = 1_{\left\{\frac{d\,{}^sP^t}{d\,{}^sQ^t} > \tilde{a}_s\right\}} + \bar{\gamma}_s 1_{\left\{\frac{d\,{}^sP^t}{d\,{}^sQ^t} = \tilde{a}_s\right\}}, \qquad {}^sQ^t\text{-a.s.,} \tag{2.24}$$

for some nonnegative $\mathcal{F}_s$-measurable random variable $\bar{\gamma}_s$. Since $E^{{}^sQ^t}\big[\tilde{\psi}(s,t)\big|\mathcal{F}_s\big] = \tilde{x}$ ${}^sQ^t$-a.s., we can choose $\bar{\gamma}_s$ such that $\bar{\gamma}_s = \tilde{\gamma}_s$ ${}^sQ^t$-a.s. This with (2.24) and the definition of $\tilde{\varphi}(s,t)$ prove that

$$\tilde{\psi}(s,t) = \tilde{\varphi}(s,t), \qquad {}^sQ^t\text{-a.s.} \tag{2.25}$$

By (2.3), it is clear that ${}^sP^t \ll {}^sQ^t$ on $\left\{\frac{d\,{}^sQ^t}{d\,{}^sR^t} \neq 0\right\}$. From this, (2.23) and (2.25):

$$\tilde{\psi}(s,t) = \tilde{\varphi}(s,t), \qquad {}^sP^t\text{-a.s.} \tag{2.26}$$

Finally, by combining (2.25) and (2.26) with the definition of ${}^sR^t$, equation (2.18) is concluded. □

As a particular case, let us consider the following modification of problem (2.1)-(2.2):

$$\operatorname*{ess\,sup}_{A\in\mathcal{F}_t} {}^sP^t\big(A \,\big|\, \mathcal{F}_s\big) \tag{2.27}$$

under the constraint

$${}^sQ^t\big(A \,\big|\, \mathcal{F}_s\big) \leq \tilde{x} \tag{2.28}$$

**Corollary 2.4.** *Let $\tilde{a}_s \in \mathcal{F}_s$ be given as in the proof of Theorem 2.1. Also, assume that there exists an $\mathcal{F}_t$-measurable set $\tilde{A}(s,t)$ such that*

$$\tilde{A}(s,t) = \left\{\frac{d\,{}^sP^t}{d\,{}^sQ^t} > \tilde{a}_s\right\}, \qquad {}^sR^t\text{-a.s.} \tag{2.29}$$

*and it satisfies the constraint*

$$ {}^sQ^t\big(\tilde{A}(s,t)\,\big|\,\mathcal{F}_s\big) = \tilde{x}, \qquad {}^sR^t\text{-}a.s. \tag{2.30} $$

*Then $\tilde{A}(s,t)$ is a solution to problem (2.27)-(2.28). Moreover, if $\hat{A}(s,t)$ is any solution to problem (2.27)-(2.28) satisfying condition (2.30) then*

$$ \hat{A}(s,t) = \tilde{A}(s,t), \qquad {}^sR^t\text{-}a.s. \tag{2.31} $$

*Proof.* Similar to the arguments of the proof of Theorem 2.2, we can verify that $\tilde{A}(s,t)$ given by (2.29) and (2.30) solves problem (2.27)-(2.28). Equation (2.31) is proved analogous to Theorem 2.3.

□

Notice that, on the contrary to Theorem 2.1, for any $\tilde{x} \in [0,1]$, the existence of $\tilde{A}(s,t) \in \mathcal{F}_t$ in the form of (2.29) and satisfying (2.30) is *not* guaranteed.

**Remark 2.5.** *If $\tau_1 \equiv 0$ and $\tau_2 \equiv T$ then our results coincide with the classical Neyman-Pearson lemma. The case of $\tau_1 \equiv 0$ and $\tau_2$ as an $\mathbb{F}$-stopping time can be used to study the problem of efficient hedging of American contingent claims in financial models. Similarly, the results of the general case with the $\mathcal{G}$-random times $\tau_1$ and $\tau_2$ potentially can be explioted to solve the quantile hedging problem (or risk minimization problem) in financial markets with successive defaults $\tau_1$ and $\tau_2$.*

**Remark 2.6.** *We point out that the maximization problem (2.1)-(2.2) depends on the random pair $s < t$, i.e. the optimal set as a measurable random variable needs to satisfy some measurability requirements depending on both $s$ and $t$. For example, assume that $\tau_1$ represents a default time (with an $\mathbb{F}$-conditional density process $\big(\alpha(s,t)\big)_{s,t\geq 0}$ in a financial model. At the default occurrence* *time $\tau_1 = s < T$, following Follmer and Leukert [14], we can formulate the quantile hedging problem with the initial time $\tau_1 = s$ and the maturity time $\tau_2 \equiv T$. The corresponding problem depends on the available initial capital (significance level), ${}^sP^t$ (the physical probability measure) and ${}^sQ^t$ (the equivalent martingale probability measure), which all of these components depend on $\tau_1 = s$ (and the default density $\alpha(s,T)$). Therefore, to determine the optimal test and the optimal trading strategy we need to deal with some measurability requirements depending on both $\tau_1$ and $\tau_2$. In the next section, we explain this point with several examples from mathematical finance and insurance.*

## 2.2 Examples

In this section, we demonstrate the results of Section 2.1 with some explicit examples for $\tau_1$, $\tau_2$, $\left\{({}^sP^t)_{t\in(s,T]}\right\}_{s\in[0,T)}$ and $\left\{({}^sQ^t)_{t\in(s,T]}\right\}_{s\in[0,T)}$.

Let us consider the probability space $\left(\Omega, (\mathcal{F}_t)_{t\in[0,T]} \subset \mathcal{G}, P\right)$ such that the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ is generated by a $P$-standard Brownian motion $(W_t)_{t\in[0,T]}$. In addition, assume that $\tau_2 \equiv T$ and the $\mathcal{G}$-random time $\tau_1$ admits an $\mathbb{F}$-density $\alpha$, i.e.

$$P(\tau_1 \in ds|\mathcal{F}_t) = \alpha_t(s)ds, \qquad t \in [0,T], \tag{2.32}$$

where $(\omega, s) \longrightarrow \alpha_t(s)(\omega)$ is a positive $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function. For any $s \geq 0$, the process $\left(\alpha_t(s)\right)_{t\in[0,T]}$ is a $(P,\mathbb{F})$-martingale, see Karoui et al. [25] for more details.

Keeping in mind the Girsanov theorem, we introduce the family of uniformly bounded stochastic processes $\left\{\left(\Theta_t(s)\right)_{s\leq t\leq T}, \ s \in [0,T]\right\}$ where $\Theta_t(s)$

is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable for all $t \in [0,T]$. Set

$$Z_t(s) := \exp\Big(\int_s^t \Theta_u(s)\, dW_u - \frac{1}{2}\int_s^t \Theta_u^2(s)\, du\Big), \quad \text{for } 0 \le s < t \le T. \tag{2.33}$$

It is easy to check that, for a fixed $s \in [0,T]$, $\big(Z_t(s)\big)_{s\le t\le T}$ is a $\big(P, (\mathcal{F}_t)_{s\le t\le T}\big)$-martingale with $Z_s(s) = 1$. Using the martingale property of $\alpha$, for each $s \in [0,T]$ we define the following probability measures on $(\Omega, \mathcal{F}_T)$:

$$\frac{d\,{}^sP^t}{d\,P} := \frac{\alpha_T(s)}{E\big[\alpha_T(s)\,|\,\mathcal{F}_s\big]} = \frac{\alpha_T(s)}{\alpha_s(s)} \quad \text{and} \quad \frac{d\,{}^sQ^t}{d\,P} := Z_T(s). \tag{2.34}$$

**Example 2.1.** *For a given $\tilde{x} \in [0,1]$ and the probability measures defined as in (2.34) with $\tau_1$, $\tau_2$ as above, we utilize Corollary 3.13 to find an explicit solution to problem (2.27)-(2.28). By taking into account (2.3), at $\tau_1 = s \in [0,T)$ set $\tilde{A}(s,T)$ introduced by (2.29) is given by*

$$\begin{aligned}\tilde{A}(s,T) &= \Big\{\frac{d\,{}^sP^T}{d\,{}^sQ^T} > \tilde{a}_s\Big\} = \Big\{\frac{d\,{}^sP^T}{d\,P} > \tilde{a}_s\,\frac{d\,{}^sQ^T}{d\,P}\Big\} \\ &= \Big\{\frac{\alpha_T(s)}{\alpha_s(s)} > \tilde{a}_s\, Z_T(s)\Big\}.\end{aligned} \tag{2.35}$$

*From (2.30), $\tilde{a}_s \in \mathcal{F}_s$ can be determined by:*

$${}^sQ^T\big(\tilde{A}(s,T)\big|\mathcal{F}_s\big) = \tilde{x}. \tag{2.36}$$

*By ${}^sP^T \approx P$ and ${}^sQ^T \ll P$, we get*

$$\frac{d\,{}^sQ^T}{d\,{}^sR^T} = \frac{d\,{}^sQ^T}{d\,P}\cdot\frac{d\,P}{d\,{}^sP^T}\cdot\frac{d\,{}^sP^T}{d\,{}^sR^T},$$

*this easily implies $\Big\{\frac{d\,{}^sQ^T}{d\,{}^sR^T} = 0\Big\} = \Big\{\frac{d\,{}^sP^T}{d\,{}^sR^T} = 0\Big\}$. In addition, using the definition of conditional expectation, it is straightforward to see that ${}^sP^T\Big(\frac{d\,{}^sP^T}{d\,{}^sR^T} =$*

$0 \,|\, \mathcal{F}_s\Big) = 0$ *and* ${}^s Q^T\Big(\dfrac{d\,{}^sQ^T}{d\,{}^sR^T} = 0 \,|\, \mathcal{F}_s\Big) = 0$. *In fact, we have used this argument to get the second equality in* (2.35) *and show that this new representation for* $\tilde{A}(s,T)$ *satisfies* (2.36).

*To find a more explicit form for* $\tilde{a}_s$, *in the remainder of this example, let us consider* $\tau_1$ *to be independent of the Brownian motion* $(W_t)_{t\in[0,T]}$. *In addition, assume that for some* $\lambda > 0$ *constant*

$$P(\tau_1 \in ds \,|\, \mathcal{F}_t) = P(\tau_1 \in ds) = \lambda e^{-\lambda s}\, ds, \quad \text{for } t \in [0,T].$$

*In this case,* $\alpha_t(s)$ *is only a deterministic function of* $s$, *i.e.* $\alpha_T(s) = \alpha_s(s)$ *and we have* $P(\tau_1 \le s \,|\, \mathcal{F}_s) = P(\tau_1 \le s) = 1 - e^{-\lambda s}$. *Therefore, for all* $s \in [0,T]$, *the probability measure* ${}^sP^T$ *is equal to* $P$ *and* $\tilde{A}(s,T)$ *simplifies to*

$$\tilde{A}(s,T) = \Big\{ Z_T(s) < \frac{1}{\tilde{a}_s} \Big\}. \tag{2.37}$$

*Moreover, we suppose that for any* $s \in [0,T]$

$$\big(\Theta_t(s)\big)_{s \le t \le T} \equiv Ks + 1,$$

*for some constant* $K > 0$. *Recall that, by Girsanov's theorem,* $\big(W_t - \Theta_t(s)\big)_{s\le t\le T}$ *is a* $\big({}^sQ^T, (\mathcal{F}_t)_{s\le t\le T}\big)$*-standard Brownian motion. Thus* (2.36) *becomes*

$$\begin{aligned} \tilde{x} = {}& {}^sQ^T\Big( W_T - W_s < \frac{-\ln \tilde{a}_s + \frac{1}{2}(Ks+1)^2\,(T-s)}{Ks+1} \,\Big|\, \mathcal{F}_s \Big) \\ = {}& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tilde{d}_s} \exp\Big(\frac{-y^2}{2}\Big)\, dy = \Phi(\tilde{d}_s), \end{aligned} \tag{2.38}$$

*where* $\tilde{d}_s := \dfrac{-\ln \tilde{a}_s + \frac{1}{2}(Ks+1)^2\,(T-s)}{(Ks+1)\sqrt{T-s}}$ *and* $\Phi$ *is the standard normal distribution function. Now the constant* $\tilde{d}_s$, *as a priori* $\tilde{a}_s$, *can be determined from* *equation* (2.38).

**Example 2.2.** *Let* $\big(\Omega, \mathcal{F}_T, \mathbb{F}, P\big) := \big(\Omega_1 \times \Omega_2, \mathcal{F}^1_T \times \mathcal{F}^2_T, \mathbb{F}_1 \times \mathbb{F}_2, P_1 \times P_2\big)$ *where* $\big(\Omega_1, \mathcal{F}^1_T, \mathbb{F}_1, P_1\big)$ *is the probability space described in Example 2.1, and* $\big(\Omega_2, \mathcal{F}^2_T, \mathbb{F}_2, P_2\big)$ *is a probability space as given below.*

*Suppose that* $T_i : \Omega_2 \longrightarrow \mathbb{R}^+$, *for* $i = 1, ..., n$, *is a positive* $\mathcal{F}^2_T$*-measurable random variable defined on* $\big(\Omega_2, \mathcal{F}^2_T, \mathbb{F}_2 := (\mathcal{F}^2_t)_{t\in[0,T]}, P_2\big)$ *such that the* $T_i$*'s are i.i.d with* $\mathcal{F}^2_t := \sigma\big(T_i \le t\,;\, i = 1, ..., n\big)$ *for* $t \in [0, T]$. *Considering this setting, define process* $\big(N_t\big)_{t\in[0,T]}$ *as the following:*

$$N_t := \sum_{i=1}^{n} 1_{\{T_i \le t\}}\,. \tag{2.39}$$

*Let us also introduce* $p(t) := P_2(T_i > t)$, *for all* $t \in [0, T]$, *with the convention that* $p(0) \equiv 1$. *Now, take* $\tau_1 \equiv s$ *for some* $s \in [0, T]$ *and* $\tau_2 \equiv T$. *We define the probability measures* ${}^sP^T$ *and* ${}^sQ^T$ *on* $\big(\Omega, \mathcal{F}_T\big)$ *by*

$$\frac{d\,{}^sP^T}{d\,P} := \frac{n - N_T}{E^{P_2}\big[n - N_T|\mathcal{F}^2_s\big]} \quad \text{and} \quad \frac{d\,{}^sQ^T}{d\,P} := Z_T(s)\cdot\frac{n - N_T}{E^{P_2}\big[n - N_T|\mathcal{F}^2_s\big]}\,, \tag{2.40}$$

*where* $Z_T(s)$ *is given by equation* (2.33) *and* $E^{P_2}\big[n - N_T|\mathcal{F}^2_s\big]$ *means the conditional expectation of* $n - N_T$ *w.r.t the probability measure* $P_2$ *and the* $\sigma$*-field* $\mathcal{F}^2_s$. *Notice that* ${}^sP^T$ *and* ${}^sQ^T$ *depend on the constant* $n$, *but for simplicity of the notations we omit* $n$ *in the left side of the definitions of* ${}^sP^T$ *and* ${}^sQ^T$ *in* (2.40).

*For a given* $\tilde{x} \in (0, 1)$, *we find the optimal solution to the corresponding problem* (2.27)-(2.28). *Using Corollary 3.13, we get*

$$\begin{aligned}\tilde{A}(s, T) &= \Big\{\frac{d\,{}^sP^T}{d\,P} > \tilde{a}_s\,\frac{d\,{}^sQ^T}{d\,P}\Big\}\\ &= \Big\{Z_T(s) < \frac{1}{\tilde{a}_s}\Big\},\end{aligned} \tag{2.41}$$

*such that* $\tilde{a}_s \in \mathcal{F}_s$ *is determined from*

$$\begin{aligned}\tilde{x} &= {}^sQ^T\big(\tilde{A}(s,T)\,|\,\mathcal{F}_s\big)\\ &= {}^sQ_1^T\Big(Z_T(s) < \frac{1}{\tilde{a}_s}\,|\,\mathcal{F}_s^1\Big)\cdot\frac{E^{P_2}\big[n-N_T|\mathcal{F}_s^2\big]}{E^{P_2}\big[n-N_T|\mathcal{F}_s^2\big]},\end{aligned} \tag{2.42}$$

*and* $\dfrac{d\,{}^sQ_1^T}{d\,P} := Z_T(s)$ *is defined on* $\big(\Omega_1, \mathcal{F}_T^1\big)$*. Therefore,* $\tilde{a}_s$ *is given by a similar calculation as in equation* (2.38)*.*

In the above example, $\dfrac{d\,{}^sP^T}{d\,P}$ is simplified in (2.41)-(2.42) and it does not have any impact on the size of the optimal test $\tilde{A}(s,T)$. But the power of $\tilde{A}(s,T)$ is weighted by $\dfrac{d\,{}^sP^T}{d\,P}$ depending on $\tau_1$ and $\tau_2$. This framework is used to study pure endowment life insurance contracts linked to an equity independent of the clients lifetime, see for instance Melnikov [35].

**Example 2.3.** *Let* $\Pi = (\Pi_t)_{t\geq 0}$ *be a Poisson process with the intensity* $\lambda > 0$ *on the probability space* $(\Omega, \mathbb{F}, P)$ *where the filtration* $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ *is generated by* $\Pi$*. The Poisson process* $(\Pi_t)_{t\geq 0}$ *has jumps of size only* $+1$ *and it is constant between two jumps. For* $m \in \mathbb{N}$*, define*

$$T_m := \inf\{t \geq 0 : \Pi_t = m\}. \tag{2.43}$$

*It is easy to see that* $\{T_m \leq s\} = \{\Pi_s \geq m\}$ *for any* $s \geq 0$ *and* $m \geq 1$*. Thus, for* $T > 0$ *fixed and* $s < T$*, by the properties of* $\Pi$*, we get*

$$\begin{aligned}P(T_m \leq T\,|\,\mathcal{F}_s) &= P(\Pi_T \geq m\,|\,\mathcal{F}_s)\\ &= P(\Pi_T \geq m\,|\,\Pi_s)\\ &= P(\Pi_T - \Pi_s \geq m - \Pi_s\,|\,\Pi_s)\\ &= \sum_{k=(m-\Pi_s)^+}^{+\infty} e^{-\lambda(T-s)}\,\frac{\big(\lambda(T-s)\big)^k}{k!}.\end{aligned} \tag{2.44}$$

*Let $\tau_1 = T_m$ for a fixed $m \in \mathbb{N}$, and $\tau_2 \equiv T$. If $s = T_m(\omega_0) < T$ for some $\omega_0 \in \Omega$ we introduce the probability measures ${}^sP^T$ and ${}^sQ^T$ on $(\Omega, \mathcal{F}_T)$ as follows:*

$$\frac{d\,{}^sP^T}{d\,P} := \frac{1}{P\big(T_m \le T \,|\, \mathcal{F}_s\big)(\omega_0)} \cdot 1_{\{T_m \le T\}}, \tag{2.45}$$

*and*

$$\frac{d\,{}^sQ^T}{d\,P} := \exp\big\{(\lambda - \lambda^*)(T - s) + (\ln\lambda^* - \ln\lambda)(\Pi_T - \Pi_s)\big\}, \tag{2.46}$$

*where the constant $\lambda^* > \lambda$ is the intensity of $(\Pi_t)_{t \ge 0}$ with respect to ${}^0Q^T$. The probability measure ${}^0Q^T$ is obtained from (2.46) with $s = 0$. Using Theorem 2.1, the solution $\tilde{\varphi}(s,T) \in \mathcal{R}_T$ is given by*

$$\tilde{\varphi}(s,T) = 1_{\left\{\frac{d\,{}^sP^T}{d\,P} > \tilde{a}_s(\omega_0)\cdot\frac{d\,{}^sQ^T}{d\,P}\right\}} + \tilde{\gamma}_s(\omega_0) 1_{\left\{\frac{d\,{}^sP^T}{d\,P} = \tilde{a}_s(\omega_0)\cdot\frac{d\,{}^sQ^T}{d\,P}\right\}}, \tag{2.47}$$

*for a random variable $\tilde{a}_s \in \mathcal{F}_s$, and $\tilde{\gamma}_s$ is determined from:*

$$\tilde{\gamma}_s(\omega_0) = \frac{\tilde{x} - {}^sQ^T\big(\frac{d\,{}^sP^T}{d\,P} > \tilde{a}_s(\omega_0)\cdot\frac{d\,{}^sQ^T}{d\,P} \,|\, \mathcal{F}_s\big)(\omega_0)}{{}^sQ^T\big(\frac{d\,{}^sP^T}{d\,P} = \tilde{a}_s(\omega_0)\cdot\frac{d\,{}^sQ^T}{d\,P} \,|\, \mathcal{F}_s\big)(\omega_0)} \tag{2.48}$$

*if ${}^sQ^T\big(\frac{d\,{}^sP^T}{d\,P} = \tilde{a}_s(\omega_0)\cdot\frac{d\,{}^sQ^T}{d\,P} \,|\, \mathcal{F}_s\big)(\omega_0) \neq 0$, and $\tilde{\gamma}_s(\omega_0) = 0$ otherwise.*

*Again, the optimal solution $\tilde{\varphi}(s,T)$ satisfies the constraint condition*

$$E^{{}^sQ^T}\big[\tilde{\varphi}(s,T) \,\big|\, \mathcal{F}_s\big](\omega_0) = \tilde{x}. \tag{2.49}$$

*Notice that, henceforth, we drop $\omega_0$ in our calculations. Let*

$$\tilde{b}_s := \frac{-\ln\big(\tilde{a}_s P(T_m \le T \,|\, \mathcal{F}_s)\big) + (\lambda^* - \lambda)(T - s)}{\ln(\frac{\lambda^*}{\lambda})}, \tag{2.50}$$

*with* $P(T_m \le T \,|\, \mathcal{F}_s)$ *given by* (2.44).

*Since* $(\Pi_u - \Pi_s)_{s \le u \le T}$ *is an* $\big((\mathcal{F}_u)_{s\le u\le T},\, {}^sQ^T\big)$*-Poisson process with the intensity* $\lambda^* > 0$*, condition* (2.49) *becomes*

$$\begin{aligned}\tilde{x} =&\ {}^sQ^T\big(\{\Pi_T - \Pi_s < \tilde{b}_s\} \cap \{T_m \le T\} \,|\, \mathcal{F}_s\big)\\ &+ \tilde{\gamma}_s \cdot {}^sQ^T\big(\{\Pi_T - \Pi_s = \tilde{b}_s\} \cap \{T_m \le T\} \,|\, \mathcal{F}_s\big)\\ =&\ {}^sQ^T\big(\{\Pi_T - \Pi_s < \tilde{b}_s\} \cap \{\Pi_T \ge m\} \,|\, \mathcal{F}_s\big)\\ &+ \tilde{\gamma}_s \cdot {}^sQ^T\big(\{\Pi_T = \tilde{b}_s + \Pi_s\} \cap \{\Pi_T \ge m\} \,|\, \mathcal{F}_s\big)\\ =&\ {}^sQ^T\big(m - \Pi_s \le \Pi_T - \Pi_s \le \tilde{b}_s - 1 \,|\, \mathcal{F}_s\big)\\ &+ \tilde{\gamma}_s \cdot {}^sQ^T\big(\{\Pi_T - \Pi_s = \tilde{b}_s\} \cap \{m \le \tilde{b}_s + \Pi_s\} \,|\, \mathcal{F}_s\big).\end{aligned} \tag{2.51}$$

*By* (2.8) *and* (2.50)*, and the fact that* $\Pi$ *takes only nonnegative integer values, we suppose that* $\tilde{b}_s$ *is a nonnegative integer. In fact, we assume*

$$\tilde{b}_s = \inf\Big\{b \in \mathbb{Z}^+ \cup \{0\} \,:\, {}^sQ^T\Big(\{\Pi_T - \Pi_s < b\} \cap \{\Pi_T \ge m\} \,\Big|\, \mathcal{F}_s\Big) \le \tilde{x}\Big\}. \tag{2.52}$$

*Taking into account the distribution of* $(\Pi_u - \Pi_s)_{s\le u\le T}$ *under the probability measure* ${}^sQ^T$*, equation* (2.51) *simplifies to*

$$\begin{aligned}\tilde{x} =& \sum_{k=(m-\Pi_s)^+}^{\tilde{b}_s - 1} e^{-\lambda^*(T-s)} \frac{\big(\lambda^*(T-s)\big)^k}{k!}\\ &+ \tilde{\gamma}_s\, e^{-\lambda^*(T-s)} \frac{\big(\lambda^*(T-s)\big)^{\tilde{b}_s}}{\tilde{b}_s!}\, 1_{\{m \le \tilde{b}_s + \Pi_s\}},\end{aligned} \tag{2.53}$$

*and $\tilde{\gamma}_s$ is given as follows:*

$$\tilde{\gamma}_s = \begin{cases} 0 & ; \ \left\{m > \tilde{b}_s + \Pi_s\right\} \\ \\ \dfrac{\tilde{x} - \displaystyle\sum_{k=(m-\Pi_s)^+}^{\tilde{b}_s-1} e^{-\lambda^*(T-s)} \frac{\left(\lambda^*(T-s)\right)^k}{k!}}{e^{-\lambda^*(T-s)} \frac{\left(\lambda^*(T-s)\right)^{\tilde{b}_s}}{\tilde{b}_s!}} & ; \ otherwise \end{cases} \tag{2.54}$$

*For the applications of Poisson process and its induced probability measures, such as (2.46), to modeling and pricing of contingent claims in financial markets see Melnikov et al. [40].*

**Example 2.4.** *Let $W$ be a one-dimensional standard Brownian motion on the probability space $(\Omega, \mathbb{F} \subset \mathcal{G}, P)$ where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by $W$. Similar to Example (2.1), we consider $\mathcal{G}$-random time $\tau_1$ with the following $\mathbb{F}$-conditional density*

$$\alpha_t^1(s) = \lambda_1 e^{-\lambda_1 s}, \qquad \text{for some } \lambda_1 > 0,$$

*for all $t \geq 0$. Let $(\Pi_t)_{t\geq 0}$ be a Poisson process independent of $W$ with the intensity $\lambda_2 > 0$. For $a \neq 0$, take*

$$\tau(a) := \inf\{u \geq 0 : \ W_u = a\} \quad \text{and} \quad \xi := \inf\{u \geq 0 : \ \Pi_u \geq 1\}. \tag{2.55}$$

*Define $\mathcal{G}$-random time $\tau_2$ as follows:*

$$\tau_2 := \tau(a) \wedge \xi$$

*From Jiao and Li [19], the $\mathbb{F}$-conditional density of $\tau_2$ is given by*

$$\alpha_t^2(s) = \lambda_2 e^{-\lambda_2 s}\Big[1_{\{s\le t\}}1_{\{\tau(a)>s\}} + 1_{\{s>t\}}1_{\{\tau(a)>t\}}\mathrm{erf}\Big(\frac{W_t - a}{\sqrt{2(s-t)}}\Big)\Big], \qquad (2.56)$$

*for all $s,t\ge 0$, with erf$(z) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z e^{-u^2}\,du$. By the independence assumption between $\tau_1$, $W$ and $\Pi$, the joint $\mathbb{F}$-conditional density of $(\tau_1,\tau_2)$ is the product of $\alpha^1$ and $\alpha^2$ and we denote it by $\alpha_t(s_1,s_2)$:*

$$\alpha_t(s_1,s_2) = \alpha_t^1(s_1)\alpha_t^2(s_2), \qquad \textit{for all } s_1,s_2,t\ge 0\,. \qquad (2.57)$$

*Similar to the $\alpha$ introduced in (2.32), for any fixed $(s_1,s_2)\in\mathbb{R}_+^2$, $\big(\alpha_t^1(s_1)\alpha_t^2(s_2)\big)_{t\ge 0}$ is an $(\mathbb{F},P)$-martingale.*

*Let $\tau_1 = s$ and $\tau_2 = t$ such that $s<t$, we take the probability measures ${}^sP^t$ and ${}^sQ^t$ on $(\Omega,\mathcal{F}_t)$ as follows*

$$\frac{d\,{}^sP^t}{d\,P} := \frac{\alpha_t(s,t)}{E[\alpha_t(s,t)|\mathcal{F}_s]} = \Big[\mathrm{erf}\Big(\frac{W_s-a}{\sqrt{2(t-s)}}\Big)\Big]^{-1}1_{\{\tau(a)>t\}},$$

*and ${}^sQ^t$ is defined by (2.33) and (2.34) with $\Theta_u(s)\equiv\sigma>0$ constant. For this setting, we find an explicit form for the solution to problem (2.27)-(2.28). Let $\tilde{x}\in[0,1]$ to be given, again we need to determine $\tilde{a}_s\in\mathcal{F}_s$ such that $\tilde{A}(s,t) = \Big\{\dfrac{d\,{}^sP^t}{d\,P} > \tilde{a}_s\,\dfrac{d\,{}^sQ^t}{d\,P}\Big\}$ satisfies the following equation*

$$\begin{aligned}
\tilde{x} &= {}^sQ^t\big(\tilde{A}(s,t)\,|\,\mathcal{F}_s\big)\\
&= {}^sQ^t\big(\{W_t^*-W_s^*<\tilde{b}_s\}\,|\,\mathcal{F}_s\big)\\
&- {}^sQ^t\big(\{W_t^*-W_s^*<\tilde{b}_s\}\cap\{\tau(a)\le t\}\,|\,\mathcal{F}_s\big)\\
&= 1_{\{\tau(a)>s\}}\,{}^sQ^t\big(\{W_t^*-W_s^*<\tilde{b}_s\}\,|\,\mathcal{F}_s\big)\\
&- {}^sQ^t\big(\{W_t^*-W_s^*<\tilde{b}_s\}\cap\{s<\tau(a)\le t\}\,|\,\mathcal{F}_s\big),
\end{aligned} \qquad (2.58)$$

*where* $\tilde{b}_s = \dfrac{-\ln\Big(\tilde{a}_s \,\mathrm{erf}\big(\frac{W_s - a}{\sqrt{2(t-s)}}\big)\Big) + \frac{1}{2}\sigma^2(t-s)}{\sigma}$ *and* $W_u^* = W_u - \sigma u$ *for* $u \geq 0$. *The first term on the right hand side of* (2.58) *is easily calculated as below:*

$$ {}^sQ^t\big(\{W_t^* - W_s^* < \tilde{b}_s\}|\mathcal{F}_s\big) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\tilde{b}_s} \exp\big(\frac{-y^2}{2(t-s)}\big)\, dy. \tag{2.59} $$

*Now let us calculate the second term in* (2.58). *Conditioned on* $\mathcal{F}_s$, *we have*

$$ \begin{aligned} \tau(a)1_{\{\tau(a)>s\}} &= \inf\{u > s \,:\, W_u = a\} \\ &= \inf\{u - s > 0 \,:\, W_u - W_s = a - W_s\} \\ &= \tau(a - W_s)1_{\{\tau(a-W_s)>0\}}. \end{aligned} \tag{2.60} $$

*For the ease of notation, in the following we set* $\tau_s := \tau(a - W_s)$ *and* $\bar{t} := t - s - \tau_s$. *By taking into account* (2.60), *one can write*

$$ \begin{aligned} & {}^sQ^t\Big(\{W_t^* - W_s^* < \tilde{b}_s\} \cap \{s < \tau(a) \leq t\} \,\big|\, \mathcal{F}_s\Big) \\ &= {}^sQ^t\Big(\{W_{t-s}^* < \tilde{b}_s\} \cap \{\tau_s \leq t-s\}\Big)\Big|_{W_s=z} \\ &= {}^sQ^t\Big({}^sQ^t\big(\{W_{\bar{t}+\tau_s}^* - W_{\tau_s}^* < \tilde{b}_s - W_{\tau_s}^*\} \,\big|\, \mathcal{F}_{\tau_s}\big)1_{\{\tau_s \leq t-s\}}\Big)\Big|_{W_s=z} \\ &= \int_0^{t-s} \frac{1}{\sqrt{2\pi(t-s-u)}} \int_{-\infty}^{\tilde{b}_s - a + W_s + \sigma u} \exp\Big(\frac{-y^2}{2(t-s-u)}\Big)\, dy\, {}^sQ^t\big(\tau_s \in du\big)\Big|_{W_s=z}. \end{aligned} \tag{2.61} $$

*We point out that* $W_{\tau_s}^* = a - W_s - \sigma\tau_s$, *and for a given* $W_s = z$, $\tau_s$ *is an* $\mathbb{F}$*-stopping time. Thus* $\{\tau_s \leq t - s\} \in \mathcal{F}_{\tau_s}$ *and* $\big(W_{u+\tau_s} - W_{\tau_s}\big)_{u\geq 0}$ *is a* ${}^sQ^t$*-Brownian motion.*

*To proceed, we provide an explicit formula for the probability density func-*

*tion of* $1_{\{u>0\}}\,{}^sQ^t\big(\tau_s \in du \,|\, \mathcal{F}_s\big)$*. For* $u \le t-s$*, we have*

$$\begin{aligned}
&{}^sQ^t\big(\{\tau_s \le u\} \,|\, \mathcal{F}_s\big) = {}^sQ^t\big(\tau_s \le u\big)\big|_{W_s=z} \\
&= E\big[\exp\big(\sigma W_{t-s} - \frac{1}{2}\sigma^2(t-s)\big)1_{\{\tau_s \le u\}}\big]\big|_{W_s=z} \\
&= E\Big[E\big[\exp\Big(\sigma(W_{\bar{t}+\tau_s} - W_{\tau_s}) + \sigma W_{\tau_s} - \frac{1}{2}\sigma^2(t-s)\Big) \,|\, \mathcal{F}_{\tau_s}\big]1_{\{\tau_s \le u\}}\Big]\Big|_{W_s=z} \\
&= \int_0^u \frac{1}{\sqrt{2\pi(t-s-v)}} \int_{-\infty}^{+\infty} \exp\Big(\sigma w + \sigma(a - W_s) - \frac{1}{2}\sigma^2(t-s)\Big) \\
&\times \exp\Big(\frac{-w^2}{2(t-s-v)}\Big)\, dw\, \frac{|a - W_s|}{\sqrt{2\pi v^3}} \exp\Big(\frac{-(a-W_s)^2}{2v}\Big)\, dv\,\Big|_{W_s=z}.
\end{aligned}$$

*In the last equality, we used the probability density of* $\tau_s$*, see for instance Jeanblanc et al. [18]. By taking derivative with respect to* $u$*, we obtain*

$$\begin{aligned}
&{}^sQ^t\big(\tau_s \in du \,|\, \mathcal{F}_s\big)1_{\{u<t-s\}} \\
&= \frac{1}{\sqrt{2\pi(t-s-u)}} \int_{-\infty}^{+\infty} \exp\Big(\sigma w + \sigma(a - W_s) - \frac{1}{2}\sigma^2(t-s)\Big) \\
&\times \exp\Big(\frac{-w^2}{2(t-s-u)}\Big)\, dw\, \frac{|a - W_s|}{\sqrt{2\pi u^3}} \exp\Big(\frac{-(a-W_s)^2}{2u}\Big)\,\Big|_{W_s=z}
\end{aligned} \tag{2.62}$$

*Finally by combining* (2.58)*,* (2.59)*,* (2.61)*, and* (2.62)*, the* $\mathcal{F}_s$ *measurable random variable* $\tilde{b}_s$ *can be determined.*

*Notice that*

$$\begin{aligned}
\Big\{\frac{d\,{}^sP^t}{d\,P} = \tilde{a}_s\,\frac{d\,{}^sQ^t}{d\,P}\Big\} &= \{W_t^* - W_s^* = \tilde{b}_s\} \cap \{\tau(a) > t\} \\
&\subseteq \{W_t^* - W_s^* = \tilde{b}_s\}.
\end{aligned} \tag{2.63}$$

*On the other hand,* $W_t^* - W_s^*$ *conditioned on* $\mathcal{F}_s$ *is normally distributed. Therefore,* ${}^sQ^t\big(\{W_t^* - W_s^* = \tilde{b}_s\} \,\big|\, \mathcal{F}_s\big) = 0$ *and as an immediate consequence the assumption of Corollary 3.13 is satisfied.*

**Example 2.5.** *Let* $W = (W_1, W_2)$ *to be a two-dimensional standard Brownian* *motion on the probability space* $(\Omega, \mathbb{F} \subset \mathcal{G}, P)$ *with* $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ *the filtration generated by* $W$.

*Following Karoui et al. [26], we introduce the random times* $\tau_1$ *and* $\tau_2$ *such that*

$$\begin{aligned}
&P(\tau_1 > s \,|\, \mathcal{F}_t) = \Phi\Big(\frac{\sigma W_1(t) - s}{\sigma\sqrt{T-t}}\Big), \qquad \textit{for some constant } \delta > 0, \\
&\textit{and} \\
&P(\tau_2 > s \,|\, \mathcal{F}_t) = \frac{1}{\sqrt{1 + 2(T-t)s}} \exp\Big(\frac{-sW_2^2(t)}{1+2(T-t)s}\Big)
\end{aligned} \tag{2.64}$$

*for all* $s, t \geq 0$. *Using the definition of standard normal distribution function* $\Phi$ *and then differentiating w.r.t* $s$ *in* (2.64), *we obtain the* $\mathbb{F}$*-conditional density of* $\tau_1$ *and* $\tau_2$, *respectively, as follows:*

$$\begin{aligned}
&\alpha_t(s) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\Big(\frac{-\big(\sigma W_1(t) - s\big)^2}{2\sigma^2(T-t)}\Big), \\
&\textit{and} \\
&\beta_t(s) = \Big[(T-t)\big(1 + 2(T-t)s\big) + W_2^2(t)\Big] \frac{P(\tau_2 > s \,|\, \mathcal{F}_t)}{\big(1 + 2(T-t)s\big)^2}\,.
\end{aligned} \tag{2.65}$$

*Considering* $\alpha$ *and* $\beta$, *we assume that* $\tau_1$ *and* $\tau_2$ *are independent and their joint* $\mathcal{F}_t$*-conditional density, denoted by* $\gamma_t(s,t)$, *is given by*

$$\gamma_t(s,t) = \alpha_t(s)\beta_t(t), \qquad \textit{for all } s, t \geq 0. \tag{2.66}$$

*Now for* $s < t$ *we take* ${}^{s}P^{t}$ *and* ${}^{s}Q^{t}$ *as below*

$$\frac{d\,{}^{s}P^{t}}{d\,P} := \frac{\alpha_t(s)\beta_t(t)}{\alpha_s(s)\beta_s(t)}, \tag{2.67}$$

*and*

$$\frac{d\,{}^sQ^t}{d\,P} := \exp\Big(\sigma_1\big(W_1(t) - W_s(s)\big) - \frac{1}{2}\sigma_1^2(t-s)\Big) \times \exp\Big(\sigma_2\big(W_2(t) - W_2(s)\big) - \frac{1}{2}\sigma_2^2(t-s)\Big), \tag{2.68}$$

*for some constant $\sigma_1, \sigma_2 > 0$. To solve problem (2.27)-(2.28), for a given $\tilde{x} \in [0,1]$, we investigate $\tilde{A}(s,t) = \Big\{\frac{d\,{}^sP^t}{d\,P} > \tilde{a}_s \frac{d\,{}^sQ^t}{d\,P}\Big\}$ for some $\tilde{a}_s \in \mathcal{F}_s$ to be determined from Corollary 3.13. By (2.65) and (2.68) and some tedious calculations, we can see that*

$$\begin{aligned}\tilde{A}(s,t) = \Big\{&\sigma_1 W_1^*(t) + \sigma_2 W_2^*(t)\\ &+ \frac{\Big(W_1^*(t) + W_1(s) + \sigma_1(t-s) - \frac{s}{\sigma}\Big)^2}{2(T-t)} + \frac{t\Big(W_2^*(t) + W_2(s) + \sigma_2(t-s)\Big)^2}{1+2(T-t)t}\\ &+ \ln\Big(\Big(W_2^*(t) + W_2(s) + \sigma_2(t-s)\Big)^2 + (T-t)\big(1+2(T-t)t\big)\Big) < \tilde{b}_s\Big\},\end{aligned}$$

*where $W_i^*(t) = W_i(t) - W_i(s) - \sigma_i(t-s)$ for $i = 1,2$. Let us define the function $F$ as the following*

$$\begin{aligned}F(x_1,x_2) =&\sigma_1 x_1 + \sigma_2 x_2\\ &+ \frac{\Big(x_1 + W_1(s) + \sigma_1(t-s) - \frac{s}{\sigma}\Big)^2}{2(T-t)} + \frac{t\Big(x_2 + W_2(s) + \sigma_2(t-s)\Big)^2}{1+2(T-t)t}\\ &+ \ln\Big(\Big(x_2 + W_2(s) + \sigma_2(t-s)\Big)^2 + (T-t)\big(1+2(T-t)t\big)\Big).\end{aligned}$$

*Using the above calculation, Corollary 3.13 and the Girsanov theorem, we write*

$$\begin{aligned}\tilde{x} &= {}^sQ^t\big(\tilde{A}(s,t)\,|\,\mathcal{F}_s\big) = {}^sQ^t\Big(\Big\{F\Big(W_1^*(t), W_2^*(t)\Big) < \tilde{b}_s\Big\}\,\Big|\,\mathcal{F}_s\Big)\\ &= \frac{1}{2\pi(t-s)}\iint_{\{F(x_1,x_2)<\tilde{b}_s\}} \exp\Big(\frac{-(x_1^2+x_2^2)}{2(t-s)}\Big)\,dx_1\,dx_2.\end{aligned}$$

*Solution to this equation gives us $\tilde{b}_s$ or equivalently $\tilde{a}_s$.*

## 2.3 Conclusion

The main objective of this chapter is to generalize the classical Neyman-Pearson lemma to the case that a simple null hypothesis and alternative are revealed to the statistician as a surprise. Traditionally, it is supposed that the hypotheses of a statistical test are determined at time $t = 0$. We study a setting that the time, and the hypotheses of the hypothesis test all reveal to us stochastically. This randomness is modelled by using random times in the underlying filtered probability space.

Our results have interesting and meaningful applications in insurance and financial markets in terms of mortality risk and default times. As an example, we can consider a defaultable financial model which after an unpredictable default time we want to maximize the probability of a successful hedge with the available random capital at the time of default. The main point of dealing with this problem is the measurability requirements which we utilized a measurable selection argument to deal with.

# Chapter 3

# Change of probability measures and superhedging in a default-density framework

## 3.1 A regime switching model

We study a financial model exposed to a counterparty risk, this exogenous source of risk results in a jump in the price of the underlying asset. After the default event, the rate of return and volatility switch to a new rate of return and volatility. In fact, this is the model considered by Jiao and Pham [20] to study the problem of optimal investment with counterparty risk.

In our model, we denote the price of the underlying asset by $(S_t)_{t\in[0,T]}$. Depending on the default timing, $S_t$ is described with different stochastic differential equations. In the following, we introduce this family of SDEs:

- *The before-default asset price:*

Let us consider a probability space $(\Omega, \mathcal{G}, P)$ equipped with $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, for

$T < +\infty$, where $\mathbb{F}$ is the filtration generated by a standard Brownian motion $W = (W_t)_{t\in[0,T]}$. The dynamic of before-default asset price $(S_t^{\mathbb{F}})_{t\in[0,T]}$ is given by

$$dS_t^{\mathbb{F}} = S_t^{\mathbb{F}}(\mu_t^{\mathbb{F}}dt + \sigma_t^{\mathbb{F}}dW_t), \qquad S_0^{\mathbb{F}} = S_0 > 0, \quad t \in [0,T], \tag{3.1}$$

where $\mu^{\mathbb{F}}$ and $\sigma^{\mathbb{F}}$ are $\mathbb{F}$-adapted process with $\sigma_t^{\mathbb{F}} > 0$ for all $t \in [0,T]$, and $S_0$ is the initial asset price. To guarantee the existence and uniqueness of the solution to (3.1), we assume that

$$\int_0^T |\frac{\mu_t^{\mathbb{F}}}{\sigma_t^{\mathbb{F}}}|^2 dt + \int_0^T |\sigma_t^{\mathbb{F}}|^2 dt < +\infty, \quad P\text{-a.s.} \tag{3.2}$$

At time $t \in [0,T]$, if the default has not occurred yet then $S_t = S_t^{\mathbb{F}}$.

- *The after-default asset price:*

We represent the default time by a nonnegative and finite random variable $\tau$ on $(\Omega, \mathcal{G}, P)$ such that $P(\tau = 0) = 0$. At the time of default $\tau = \theta$, the asset price jumps to a new value $S_\theta^d(\theta) := S_\theta^{\mathbb{F}}(1-\gamma_\theta)$, where $\gamma$ is an $\mathbb{F}$-adapted process determining the size and direction of this jump. For $t \in (\theta, T]$, process $S_t^d(\theta)$ is governed by the following SDE:

$$\begin{cases} dS_t^d(\theta) = S_t^d(\theta)\big(\mu_t^d(\theta)dt + \sigma_t^d(\theta)dW_t\big) & , \ t \in (\theta, T] \\ \\ S_\theta^d(\theta) = S_\theta^{\mathbb{F}}(1-\gamma_\theta) \end{cases} \tag{3.3}$$

where $(\omega, \theta) \to \mu_t^d(\theta)(\omega)$, $\sigma_t^d(\theta)(\omega)$ are $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable functions for all $t \in [0,T]$ with $\sigma_t^d(\theta) > 0$ for all $\theta \in [0,T]$ and $t \in [\theta, T]$. Similar to $\mu^{\mathbb{F}}$ and $\sigma^{\mathbb{F}}$, we impose the following condition:

$$\int_\theta^T \Big|\frac{\mu_t^d(\theta)}{\sigma_t^d(\theta)}\Big|^2 dt + \int_\theta^T |\sigma_t^d(\theta)|^2 dt < +\infty, \quad P\text{-a.s., for all } \theta \in [0,T]. \tag{3.4}$$

In addition, to make $S^d_\theta(\theta) > 0$ $P$-a.s., for all $\theta \in [0, T]$ we assume that

$$-\infty < \gamma_\theta < 1, \quad \text{and} \quad \int_0^T |\frac{\gamma_t}{\sigma^{\mathbb{F}}_t}|^2 dt < +\infty, \quad P\text{-a.s.}, \tag{3.5}$$

for all $\theta \in [0, T]$.

- *The global asset price:*

At each moment of time $t \in [0, T]$, we aggeregate information from the counterparty risk $\tau$ with the information generated by $S^{\mathbb{F}}$ to include both sources of randomness in our model. In other words, we enlarge the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ with $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]} := \big(\sigma(\tau \wedge t)\big)_{t\in[0,T]}$, the enlarged filtration is denoted by $\mathbb{G} := (\mathcal{G}_t)_{t\in[0,T]}$ where $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$ for all $t \in [0, T]$. Using SDEs (3.1) and (3.3), the $\mathbb{G}$-adapted process $(S_t)_{t\in[0,T]}$ is given by:

$$S_t = S^{\mathbb{F}}_t 1_{\{t<\tau\}} + S^d_t(\tau) 1_{\{t\geq\tau\}}, \quad t \in [0, T]. \tag{3.6}$$

We recall the next proposition without the proof to describe the connection between measurability in $\mathbb{G}$ and $\mathbb{F}$ filtrations.

**Proposition 3.1.** *Let $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ be introduced as above. Then:*

1. *For any $\mathbb{G}$-optional process $(Y_t)_{t\in[0,T]}$, we have a decomposition as:*

$$Y_t = Y^{\mathbb{F}}_t 1_{\{t<\tau\}} + Y^d_t(\tau) 1_{\{t\geq\tau\}}, \quad t \in [0, T],$$

   *where $(Y^{\mathbb{F}}_t)_{t\in[0,T]}$ is an $\mathbb{F}$-optional process, $Y^d_t(\theta)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable for all $\theta \in [0, T]$ and $t \in [\theta, T]$, and $\big(Y^d_t(\theta)\big)_{t\in[\theta,T]}$ is $\mathbb{F}$-optional.*

2. *For any $\mathbb{G}$-predictable process $(Y_t)_{t\in[0,T]}$, we have a decomposition as:*

$$Y_t = Y^{\mathbb{F}}_t 1_{\{t\leq\tau\}} + Y^d_t(\tau) 1_{\{t>\tau\}}, \quad t \in [0, T],$$

*where $(Y_t^{\mathbb{F}})_{t\in[0,T]}$ is an $\mathbb{F}$-predictable process, $Y_t^d(\theta)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable for all $\theta \in [0,T]$ and $t \in [\theta, T]$, and $\big(Y_t^d(\theta)\big)_{t\in[\theta,T]}$ is $\mathbb{F}$-predictable.*

*Proof.* For a generalization of this result to the case of multiple default times consult with Pham [44]. □

- *Density hypothesis:*

We assume that for any $t \in [0,T]$ there exists an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable positive function $(\omega, \theta) \to \alpha_t(\omega, \theta)$ such that

$$P\big(\tau \in d\theta | \mathcal{F}_t\big) = \alpha_t(\theta) d\theta,$$

We can show that for any fixed $\theta \geq 0$ the process $\big(\alpha_t(\theta)\big)_{t\in[0,T]}$ is an $(\mathbb{F}, P)$-martingale. Given $\mathcal{F}_t$, the family $\alpha_t(.)$ is the conditional density of $\tau$ w.r.t the Lebesgue measure. This means that for any bounded Borel function $f$:

$$E\big[f(\tau)|\mathcal{F}_t\big] = \int_0^{+\infty} f(\theta)\alpha_t(\theta)d\theta, \qquad P\text{-a.s.}$$

In particular, we have

$$\int_0^{+\infty} \alpha_t(\theta)d\theta = 1, \qquad P\text{-a.s.}$$

The *survival process* of $\tau$ with respect to $\mathcal{F}_t$ is defined as follows:

$$G_t := P\big(\tau > t|\mathcal{F}_t\big) = \int_t^{+\infty} \alpha_t(\theta)d\theta, \qquad P\text{-a.s.}, \tag{3.7}$$

for any $t \in [0,T]$.

For a comprehensive study of the density hypothesis and it applications in defaultable markets see Karoui et al. [25], Pham [44], and Karoui et al. [27].

- *Density hypothesis versus the intensity hypothesis:*

In comparison to the density hypothesis, a *global default rate* for $\tau$, it is possible to consider a *local default rate* for $\tau$, i.e. a non-negative $\mathbb{F}$-predictable process $(\lambda_t)_{t\in[0,T]}$ such that:

$$P\big(\tau \in (t, t+dt)|\mathcal{F}_t\big) = \lambda_t dt, \quad P\text{-a.s.}, \quad \text{for any } t \in [0,T].$$

The process $(\lambda_t)_{t\in[0,T]}$ is called the $\mathbb{F}$-intensity process of $\tau$ and it is well known that under this assumption $1_{\{\tau\leq t\}} - \int_0^{\tau\wedge t} \lambda_s ds$ follows a $(\mathbb{G}, P)$-martingale. The intensity process can be recovered from the conditional density $\alpha_t(\theta)$. In fact, we can show that for any $t \in [0,T]$

$$\lambda_t = \frac{\alpha_t(t)}{G_t}, \quad P\text{-a.s.}$$

However, the intensity $\lambda$ determines $\alpha_t(\theta)$ only partly, more precisely only for $t \leq \theta$. For more discussion about this subject, see Karoui et al. [25] and Jiao and Pham [20].

The main advantage of the density hypothesis is that fact that it provides more information about the behaviour of the model after the default. The intensity hypothesis fails to describe $\tau$ after the default. In practice, this restriction does not allow us to provide explicit solutions for problems in defaultable models. For instance, the optimization problems studied by Nakano [42] under the intensity hypothesis for which explicit solutions are derived only for the case of zero recovery rate. With a nonzero recovery rate, due to the lack of information from the intensity hypothesis, he could not solve the underlying maximization problem explicitly.

Motivated by Karoui et al. [25], Karoui et al. [27], Jiao and Pham [20], and also the above discussion, we consider the density hypothesis for our financial model in this chapter. In the following, first we provide an explicit representation for the family of probability martingale measures for $(S_t)_{t\in[0,T]}$ with respect to $(\Omega, \mathbb{G} \subseteq \mathcal{G}, P)$. Then we apply our results to find an explicit form for the problem of superhedging in the underlying defaultable market. The main idea is to utilize the density hypothesis to break down the desired representations into filtration $\mathbb{F}$ and the $\mathbb{F}$-Brownian motion $W$. In fact, to get simpler representations we want to avoid using a $\mathbb{G}$-Brownian motion.

**Definition 3.2.** *Let $\pi_t = \pi_t^{\mathbb{F}} 1_{\{t\le\tau\}} + \pi_t^d(\tau)1_{\{t>\tau\}}$, $t \in [0,T]$, be a $\mathbb{G}$-predictable process where $(\pi_t^{\mathbb{F}})_{t\in[0,T]}$ is $\mathbb{F}$-predictable, and for any fixed $\theta \in [0,T]$ process $\big(\pi_t^d(\theta)\big)_{t\in[\theta,T]}$ is $\mathbb{F}$-predictable. In addition, $\pi_t^d(\theta)$ defines a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable functions.*

*Then $\pi$ is called a $\mathbb{G}$-adapted portfolio process if for all $\theta \in [0,T]$:*

$$\int_0^T |\pi_t^{\mathbb{F}}\sigma_t^{\mathbb{F}}|^2 dt + \int_\theta^T |\pi_t^d(\theta)\sigma_t^d(\theta)|^2 dt < +\infty, \;\; P\text{-a.s.}$$

**Definition 3.3.** *A nonnegative $\mathbb{G}$-optional process*

$$(c_t)_{t\in[0,T]} = (c_t^{\mathbb{F}} 1_{\{t<\tau\}} + c_t^d(\tau)1_{\{t\ge\tau\}})_{t\in[0,T]}$$

*such that*

$$\int_0^T c_t^{\mathbb{F}} dt + \int_\theta^T c_t^d(\theta) dt < +\infty, \quad \text{for all } \theta \in [0,T], \;\; P\text{-a.s.}$$

*is called a consumption process. Notice that $(c_t^{\mathbb{F}})_{t\in[0,T]}$ is a nonnegative $\mathbb{F}$-optional process, $c_t^d(\theta)$ is nonnegative and $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable for all $\theta \in [0,T]$ and $t \in [\theta,T]$, and $\big(c_t^d(\theta)\big)_{t\in[\theta,T]}$ is $\mathbb{F}$-optional.*

**Definition 3.4.** *For an initial wealth $x_0 > 0$, a $\mathbb{G}$-adapted portfolio $\pi$ is called a self-financing portfolio process corresponding to the consumption process $c$ if* *the associated wealth process to $(\pi, c)$ is defined as follows:*

$$X_t^{x_0,\pi,c} = X_t^{\mathbb{F}} 1_{\{t<\tau\}} + X_t^d(\tau) 1_{\{t\geq\tau\}}, \quad t \in [0,T]\,. \tag{3.8}$$

*Processes $X^{\mathbb{F}}$ and $X^d(\tau)$ are described by the following SDEs:*

$$dX_t^{\mathbb{F}} = X_t^{\mathbb{F}} \pi_t^{\mathbb{F}} \frac{dS_t^{\mathbb{F}}}{S_t^{\mathbb{F}}} - c_t^{\mathbb{F}} dt\,, \;\; X_0^{\mathbb{F}} = x_0\,, \;\; t \in [0,T]. \tag{3.9}$$

*Similarly, for any fixed $\theta \in [0,T]$, and $t \in (\theta, T]$, we have:*

$$dX_t^d(\theta) = X_t^d(\theta)\pi_t^d(\theta)\frac{dS_t^d(\theta)}{S_t^d(\theta)} - c_t^d(\theta)dt\,, \;\; X_\theta^d(\theta) = X_\theta^{\mathbb{F}}(1 - \pi_\theta^{\mathbb{F}}\gamma_\theta)\,. \tag{3.10}$$

The process $\pi_t^{\mathbb{F}}$ represents the fraction of total wealth invested in $S_t^{\mathbb{F}}$ before the default, and $\pi_t^d(\theta)$ denotes the fraction of the total wealth invested in $S_t^d(\theta)$ after the default at time $\tau = \theta$.

**Definition 3.5.** *For a given initial wealth $x_0 > 0$, a pair of a $\mathbb{G}$ self-financing portfolio process and consumption process $(\pi, c)$ is called $\mathbb{G}$-admissible if the corresponding wealth process satisfies*

$$X_t^{x_0,\pi^{\mathbb{F}},c^{\mathbb{F}}} > 0 \quad \text{and} \quad \pi_t^{\mathbb{F}}\gamma_t < 1\,, \quad P\text{-a.s.} \;\; \text{for all } t \in [0,T],$$

*and for any fixed $\theta \in [0,T]$*

$$X_t^{d,\pi^d,c^d}(\theta) \geq 0, \quad P\text{-a.s.} \;\; \text{for all } t \in (\theta, T].$$

*The set of all $\mathbb{G}$-admissible $(\pi, c)$ for the initial wealth $x_0 > 0$ is denoted by $\mathcal{A}^{\mathbb{G}}(x_0)$.*

The set of all $\mathbb{F}$-admissible pairs $(\pi^{\mathbb{F}}, c^{\mathbb{F}})$ is defined similarly, and is denoted by $\mathcal{A}^{\mathbb{F}}(x_0)$.

Notice that in the case of $c_t \equiv 0$, conditions $x_0 > 0$ and $\pi_\theta^{\mathbb{F}}\gamma_\theta < 1$ along with (3.9) and (3.10) guarantee that

$$X_t^{x_0,\pi} > 0, \quad P\text{-a.s.} \quad \text{for all } t \in [0,T].$$

In other words, the condition $\pi_\theta^{\mathbb{F}}\gamma_\theta < 1$ is equivalent to the positivity requirement of $X_t^d$ in the traditional definition of admissibility of a portfolio.

**Remark 3.6.** *For any $t \in [0,T]$, let $\mu_t = \mu_t^{\mathbb{F}} 1_{\{t<\tau\}} + \mu_t^d(\tau)1_{\{t\geq\tau\}}$ and $\sigma_t = \sigma_t^{\mathbb{F}} 1_{\{t<\tau\}} + \sigma_t^d(\tau)1_{\{t\geq\tau\}}$. Then, from price process representation (3.6), we can write*

$$\begin{aligned} dS_t &= S_{t-}\big(\mu_t dt + \sigma_t dW_t - \gamma_t d(1_{\{\tau\leq t\}})\big) \\ &= \big(S_t^{\mathbb{F}} 1_{\{t\leq\tau\}} + S_t^d(\tau)1_{\{t>\tau\}}\big)\big[\mu_t^{\mathbb{F}} 1_{\{t<\tau\}}dt + \mu_t^d(\tau)1_{\{t\geq\tau\}}dt \\ &+ \sigma_t^{\mathbb{F}} 1_{\{t<\tau\}}dW_t + \sigma_t^d(\tau)1_{\{t\geq\tau\}}dW_t - \gamma_t d(1_{\{\tau\leq t\}})\big]. \end{aligned} \tag{3.11}$$

*This is reduced to:*

$$dS_t = 1_{\{t\leq\tau\}}dS_t^{\mathbb{F}} - \gamma_t S_t^{\mathbb{F}} d(1_{\{\tau\leq t\}}) + 1_{\{t>\tau\}}dS_t^d(\tau), \tag{3.12}$$

*for all $t \in [0,T]$ with $S_0 = S_0^{\mathbb{F}}$.*

*Let $x_0 > 0$ be a constant, and $\pi = \big(\pi^{\mathbb{F}}, \pi^d(\tau)\big) \in \mathcal{A}^{\mathbb{G}}(x_0)$ satisfies Definitions 3.2 and 3.4. In addition, choose $\mathbb{F}$-predictable processes $(\phi_t^{\mathbb{F}})_{t\in[0,T]}$ and $(\phi_t^d(\tau))_{t\in[0,T]}$ such that:*

$$\begin{cases} \phi_t^{\mathbb{F}} S_t^{\mathbb{F}} = \pi_t^{\mathbb{F}} X_t^{\mathbb{F}} & ;\ t \in [0,T] \\ \\ \phi_t^d(\tau)S_t^d(\tau) = \pi_t^d(\tau)X_t^d(\tau) & ;\ t \in [\tau,T] \end{cases} \tag{3.13}$$

*Hence, by above we get*

$$
\begin{aligned}
X_t^{x_0,\pi} &= X_t^{\mathbb{F}} 1_{\{t<\tau\}} + X_t^d(\tau) 1_{\{t\geq\tau\}} \\
&= \big(x_0 + \int_0^t \phi_u^{\mathbb{F}} dS_u^{\mathbb{F}}\big) 1_{\{t<\tau\}} + \big(X_\tau^{\mathbb{F}} - \gamma_\tau \phi_\tau^{\mathbb{F}} S_\tau^{\mathbb{F}} + \int_\tau^t \phi_u^d(\tau) dS_u^d(\tau)\big) 1_{\{t\geq\tau\}} \\
&= x_0 + \int_0^{\tau\wedge t} \phi_u^{\mathbb{F}} dS_u^{\mathbb{F}} - \gamma_\tau \phi_\tau^{\mathbb{F}} S_\tau^{\mathbb{F}} 1_{\{t\geq\tau\}} + \int_\tau^t \phi_u^d(\tau) dS_u^d(\tau). \qquad (3.14)
\end{aligned}
$$

*Combining* (3.12) *and* (3.14), *we have shown that*

$$
X_t^{x_0,\pi} = x_0 + \int_0^t \phi_u^{\mathbb{G}} dS_u, \qquad \textit{for all } t \in [0,T], \qquad (3.15)
$$

*where* $\phi_t^{\mathbb{G}} = \phi_t^{\mathbb{F}} 1_{\{t\leq\tau\}} + \phi_t^d(\tau) 1_{\{t>\tau\}}$ *can be interpreted as the number of shares of the defaultable asset* $S$ *held at time* $t \in [0,T]$. *In other words, Definitions 3.2 and 3.4 for a portfolio and* $\mathbb{G}$*-admissible wealth process is consistent with the classic definition of admissibility using a stochastic integral with respect to* $(S_t)_{t\in[0,T]}$.

## 3.2 Change of probability measures in filtration $\mathbb{G}$

In the next theorem, we provide a representation to fully characterize the Radon-Nikodym density of any change of probability in the setting of Section 3.1. This result can be considered as a version of the Girsanov's theorem for the model described by (3.1) - (3.6).

**Theorem 3.7.** *For an* $\mathbb{F}$*-adapted cadlag process* $(q_t)_{t\in[0,T]}$ *and an* $\mathcal{O}(\mathbb{F}) \otimes$

$\mathcal{B}(\mathbb{R}_+)$*-optional process* $(q_t(\theta))_{t\in[\theta,T]}$*, both strictly positive, define:*

$$\rho_t^{\mathbb{G}} := \frac{q_t}{M_0}1_{\{\tau>t\}} + \frac{q_t(\tau)}{M_0}1_{\{\tau\le t\}}, \qquad t \in [0,T], \tag{3.16}$$

*where* $M_0 > 0$ *is a constant value. Then* $\rho^{\mathbb{G}}$ *is the Radon-Nikodym density of a change of probability measure on* $(\Omega, \mathbb{G} \subseteq \mathcal{G}, P)$ *if and only if all of the following conditions hold:*

1. *For any* $\theta > 0$*, there exists a strictly positive* $(\mathbb{F}, P)$*-martingale* $\big(\beta_t(\theta)\big)_{t\in[\theta,T]}$ *such that*
$$q_t(\theta) = \frac{\beta_t(\theta)}{\alpha_t(\theta)}, \qquad t \in [\theta, T]. \tag{3.17}$$
*We set* $\beta_t(\theta) := E[\beta_\theta(\theta) \,|\, \mathcal{F}_t]$ *for any* $t < \theta$*.*

2. *There exists a positive* $\mathcal{F}_T$*-measurable random variable* $Y_T$ *such that* $q_t$ *satisfies:*
$$q_t G_t = E\big[Y_T G_T + \int_t^T \beta_s(s)ds \,|\, \mathcal{F}_t\big], \quad \text{for any } t \in [0,T], \tag{3.18}$$
*where* $G_t$ *is the survival process defined by* (3.7)*.*

3. *Let* $M_t^\beta := E[Y_T G_T \,|\, \mathcal{F}_t] + \int_0^T \beta_t(s)ds$*. Then* $E\Big[\int_0^T \beta_\theta(\theta)d\theta\Big] < +\infty$*, and* $M_0 = E[M_t^\beta] = q_0$*, for all* $t \in [0,T]$*.*

*Proof.* By part 1, process $\big(q_t(\theta)\alpha_t(\theta)\big)_{t\in[\theta,T]}$ is an $(\mathbb{F}, P)$-martingale. In addition, part 2 implies that $q_t G_t + \int_0^t q_s(s)\alpha_s(s)ds$ is an $(\mathbb{F}, P)$-local martingale. Using Proposition 5.6 of Karoui et al. [25], we can see that $(\rho_t^{\mathbb{G}})_{t\in[0,T]}$ is a positive $(\mathbb{G}, P)$-local martingale and consequently a supermartingale. By the definition of $\rho^{\mathbb{G}}$ we have

$$\begin{aligned} E[\rho_t^{\mathbb{G}}] &= \frac{1}{M_0} E\big[E[q_t 1_{\{\tau>t\}} + q_t(\tau)1_{\{\tau\le t\}} \,|\, \mathcal{F}_t]\big] \\ &= \frac{1}{M_0} E\big[q_t G_t + \int_0^t q_t(s)\alpha_t(s)ds\big] \\ &= \frac{1}{M_0} E\big[q_t G_t + \int_0^t E[q_t(s)\alpha_t(s)|\mathcal{F}_s]ds\big] \\ &= \frac{1}{M_0} E\big[q_t G_t + \int_0^t q_s(s)\alpha_s(s)ds\big] \\ &= \frac{1}{M_0} E\big[Y_T G_T + \int_0^T q_s(s)\alpha_s(s)ds\big]. \end{aligned} \tag{3.19}$$

From (3.18), similar to (3.19) one can see

$$q_0 = E\big[Y_T G_T + \int_0^T q_s(s)\alpha_s(s)ds\big] = E[M_t^\beta], \quad \text{for any } t\in[0,T]. \tag{3.20}$$

Combining equations (3.19) and (3.20), we conclude that $\rho^{\mathbb{G}}$ is a $(\mathbb{G},P)$-supermartingale with constant expectation which proves its martingale property. By (3.20) and $M_0 = E[M_t^\beta]$ from part 3, we also see that $E[\rho_t^{\mathbb{G}}] = 1$. This proves the sufficiency of conditions $1-3$ for the theorem.

On the other hand, let us suppose that $(\rho_t^{\mathbb{G}})_{t\in[0,T]}$ is the Radon-Nidokym density of a change of probaility, i.e., it is a positive $(\mathbb{G},P)$-martingale with $E[\rho_T^{\mathbb{G}}] = 1$. Therefore, by Karoui et al. [25] we have

*(i)* $\beta_t(\theta) = q_t(\theta)\alpha_t(\theta),\ t\in[\theta,T]$, is a positive $(\mathbb{F},P)$-martingale.

*(ii)* $q_t G_t + \int_0^t q_s(s)\alpha_s(s)ds,\ t\in[0,T]$, is a positive $(\mathbb{F},P)$-martingale.

Now part 1 of the theorem is immediate by $(i)$, and we define $\beta_t(\theta)$ for $t<\theta$ as described in part 1. By $(ii)$, one can write

$$E\big[q_T G_T + \int_0^T q_s(s)\alpha_s(s)ds \,\big|\, \mathcal{F}_t\big] = q_t G_t + \int_0^t q_s(s)\alpha_s(s)ds,$$

by taking $Y_T = q_T$, we get part 2. To verify the last part, we combine (3.19) and (3.20) with $E[\rho_T^{\mathbb{G}}] = 1$, and taking into account positivity of $Y_T G_T$, (3.20) implies $E\Big[\int_0^T \beta_\theta(\theta)d\theta\Big] \leq q_0 < +\infty$. □

**Remark 3.8.** *Let us elaborate more on representations (3.16) and (3.17) in Theorem 3.7. The main ideas behind these two forms are the fact that $(\rho_t^{\mathbb{G}})_{t\in[0,T]}$ is a $(\mathbb{G}, P)$-martingale and the computation of $\mathcal{G}_t$-conditional expectations in terms of $\mathcal{F}_t$-conditional expectations.*

*Suppose that $\rho_t^{\mathbb{G}} = \dfrac{q_t}{q_0} 1_{\{\tau>t\}} + \dfrac{q_t(\tau)}{q_0} 1_{\{\tau\leq t\}}$, for $t \in [0, T]$. To determine $q_t$ and $q_t(\tau)$, first we use the $(\mathbb{G}, P)$-martingale property of $(\rho_t^{\mathbb{G}})_{t\in[0,T]}$, and then Theorem 3.1 of Karoui et al. [25]. This gives us:*

$$\begin{aligned}\rho_t^{\mathbb{G}} = E[\rho_T^{\mathbb{G}} \,|\, \mathcal{G}_t] &= \frac{1}{G_t} E\Big[\int_t^{+\infty} \rho_T^{\mathbb{G}}(\theta)\alpha_T(\theta)d\theta \,\Big|\, \mathcal{F}_t\Big] 1_{\{\tau>t\}} \\ &= \frac{1}{\alpha_t(\theta)} E\big[\rho_T^{\mathbb{G}}(\theta)\alpha_T(\theta) \,\big|\, \mathcal{F}_t\big]\Big|_{\theta=\tau} 1_{\{\tau\leqslant t\}},\end{aligned} \tag{3.21}$$

*where*

$$\rho_T^{\mathbb{G}}(\theta) := \begin{cases} \dfrac{q_T}{q_0} & ;\ \theta > T \\ \\ \dfrac{q_T(\theta)}{q_0} & ;\ \theta \leqslant T \end{cases} \tag{3.22}$$

*Hence:*

$$\begin{aligned}\rho_t^{\mathbb{G}} &= \frac{1}{q_0 G_t} E\Big[\int_t^{T} E\big[q_T(\theta)\alpha_T(\theta) \,|\, \mathcal{F}_\theta\big]d\theta + \int_T^{+\infty} q_T\alpha_T(\theta)d\theta \,\Big|\, \mathcal{F}_t\Big] 1_{\{\tau>t\}} \\ &= \frac{1}{q_0\alpha_t(\theta)} E\big[q_T(\theta)\alpha_T(\theta) \,\big|\, \mathcal{F}_t\big]\Big|_{\theta=\tau} 1_{\{\tau\leqslant t\}}.\end{aligned} \tag{3.23}$$

*By setting $\beta_t(\theta) = E\big[q_T(\theta)\alpha_T(\theta) \,\big|\, \mathcal{F}_t\big]$ for $t \in [\theta, T]$ and comparing (3.23) with*

(3.16), *we get:*

$$q_t = \frac{1}{G_t} E\Big[ q_T G_T + \int_t^T \beta_\theta(\theta) d\theta \,\Big|\, \mathcal{F}_t \Big], \qquad \textit{for } t \in [0,T], \tag{3.24}$$

*and*

$$q_t(\theta) = \frac{\beta_t(\theta)}{\alpha_t(\theta)}, \qquad \textit{for } t \in [\theta,T], \tag{3.25}$$

*as desired.*

Using the above theorem, now we determine an explicit representation for those $(\rho_t^{\mathbb{G}})_{t\in[0,T]}$ where process $(\rho_t^{\mathbb{G}} S_t)_{t\in[0,T]}$ is a $(\mathbb{G}, P)$-local martingale, i.e. the family of probability martingale measures for $(S_t)_{t\in[0,T]}$ with respect to $(\Omega, \mathbb{G} \subseteq \mathcal{G}, P)$.

To proceed, let us define

$$Z_t := \exp\Big( -\int_0^t \frac{\mu_u^{\mathbb{F}}}{\sigma_u^{\mathbb{F}}} dW_u - \frac{1}{2}\int_0^t \big(\frac{\mu_u^{\mathbb{F}}}{\sigma_u^{\mathbb{F}}}\big)^2 du \Big), \qquad \text{for any } t \in [0,T], \tag{3.26}$$

and for any fixed $\theta \in [0,T]$ introduce:

$$Z_t^d(\theta) := \exp\Big( -\int_\theta^t \frac{\mu_u^d(\theta)}{\sigma_u^d(\theta)} dW_u - \frac{1}{2}\int_\theta^t \big(\frac{\mu_u^d(\theta)}{\sigma_u^d(\theta)}\big)^2 du \Big), \tag{3.27}$$

for all $t \in [\theta, T]$.

**Definition 3.9.** *Let* $\big\{\tilde\beta_\theta(\theta);\ \theta \in [0,T]\big\}$ *be a family of positive* $\mathcal{F}_\theta \otimes \mathcal{B}(\mathbb{R}_+)$*-measurable random variables such that:*

*(i)* $E\Big[\int_0^T \tilde\beta_\theta(\theta) d\theta\Big] < +\infty.$

*(ii)* $\int_0^T \Big(\frac{\tilde\beta_\theta(\theta)\gamma_\theta}{\sigma_\theta^{\mathbb{F}}}\Big)^2 d\theta < +\infty, \qquad P\text{-a.s.}$

*(iii)* $1+\int_0^T \frac{\tilde{\beta}_\theta(\theta)\gamma_\theta}{Z_\theta \sigma_\theta^{\mathbb{F}}} dW_\theta + \int_0^T \frac{\tilde{\beta}_\theta(\theta)}{Z_\theta}\Big(\frac{\gamma_\theta \mu_\theta^{\mathbb{F}}}{(\sigma_\theta^{\mathbb{F}})^2} - 1\Big) d\theta > 0,\ \ P\text{-a.s.}$

*The family of all* $\{\tilde{\beta}_\theta(\theta);\ \theta \in [0,T]\}$ *satisfying the above conditions is denoted by* $\mathcal{B}$.

Keeping in mind the above definition, the next theorem presents an additive form for the class of probability martingale measures for $S$ on $(\Omega, \mathbb{G}, P)$. We denote the family of all of these probability martingale measures by $\mathcal{Q}$.

**Theorem 3.10.** *Consider positive* $\mathbb{G}$*-adapted process* $(\rho_t^{\mathbb{G}})_{t\in[0,T]}$. *Then* $\frac{dQ}{dP} := \rho_T^{\mathbb{G}}$ *defines a probability martingale measure for* $(S_t)_{t\in[0,T]}$ *with respect to* $(\Omega, \mathbb{G}, P)$ *if and only if there exists a positive* $\mathbb{F}$*-adapted process* $(q_t)_{t\in[0,T]}$, *and a family of positive* $\mathbb{F}$*-adapted processes* $\{(q_t(\theta))_{t\in[\theta,T]};\ \theta\in[0,T]\}$ *such that*

$$\rho_t^{\mathbb{G}} := \frac{q_t}{q_0} 1_{\{\tau>t\}} + \frac{q_t(\tau)}{q_0} 1_{\{\tau\le t\}}, \qquad \text{for any } t\in[0,T]. \tag{3.28}$$

*Moreover, there exists* $\{\tilde{\beta}_\theta(\theta); \theta\in[0,T]\} \in \mathcal{B}$ *such that* $(\frac{q_t}{q_0})_{t\in[0,T]}$ *and* $\big(\frac{q_t(\theta)}{q_0}\big)_{t\in[\theta,T]}$ *satisfy the following conditions:*

1. *For any fixed* $\theta\in[0,T]$, *we have*

$$\frac{q_t(\theta)}{q_0} = \frac{\tilde{\beta}_\theta(\theta) Z_t^d(\theta)}{\alpha_t(\theta)}, \tag{3.29}$$

   *for all* $t\in[\theta,T]$.

2. *Process* $\big(\frac{q_t}{q_0}\big)_{t\in[0,T]}$ *is determined from:*

$$\frac{q_t}{q_0} G_t = 1 + \int_0^t \Big(\frac{\tilde{\beta}_u(u)\gamma_u}{\sigma_u^{\mathbb{F}}} - \frac{q_u G_u \mu_u^{\mathbb{F}}}{q_0 \sigma_u^{\mathbb{F}}}\Big) dW_u - \int_0^t \tilde{\beta}_u(u) du. \tag{3.30}$$

*Equivalently, we have:*

$$\frac{q_t}{q_0} G_t = Z_t \Psi_t, \tag{3.31}$$

*where*

$$\Psi_t := 1 + \int_0^t \frac{\tilde{\beta}_u(u)\gamma_u}{Z_u \sigma_u^{\mathbb{F}}} dW_u + \int_0^t \frac{\tilde{\beta}_u(u)}{Z_u} \Big( \frac{\gamma_u \mu_u^{\mathbb{F}}}{(\sigma_u^{\mathbb{F}})^2} - 1 \Big) du \,. \tag{3.32}$$

*Proof.* Recall that, from Theorem 3.7, $Q$ is a change of probability measure if and only if $\rho^{\mathbb{G}}$ has a representation as (3.28). Furthermore, by definition, $Q$ is a probability martingale measure for $(S_t)_{t\in[0,T]}$ with respect to $(\Omega, \mathbb{G}, P)$ iff $(\rho_t^{\mathbb{G}} S_t)_{t\in[0,T]}$ is a $(\mathbb{G}, P)$-local martingale. From the definition of $\rho^{\mathbb{G}}$ and $S$, we get

$$\rho_t^{\mathbb{G}} S_t = \frac{1}{q_0} \Big[ S_t^{\mathbb{F}} q_t 1_{\{\tau > t\}} + S_t^d(\tau) q_t(\tau) 1_{\{\tau \le t\}} \Big], \qquad t \in [0,T],$$

where we used the fact that $M_0 = q_0$ as it was proven in Theorem 3.7.

By Proposition 5.6 of Karoui et al. [25], $(\mathbb{G}, P)$-local martingale property of $\rho_t^{\mathbb{G}} S_t$ holds true iff:

*(i)* For any $\theta \in [0,T]$, $\big(S_t^d(\theta) q_t(\theta) \alpha_t(\theta)\big)_{t\in[\theta,T]}$ is an $(\mathbb{F}, P)$-local martingale.

*(ii)* Process $(N_t)_{t\in[0,T]} := \Big( S_t^{\mathbb{F}} q_t G_t + \int_0^t S_u^d(u) q_u(u) \alpha_u(u) du \Big)_{t\in[0,T]}$ is an $(\mathbb{F}, P)$-local martingale.

Let $Q$ be a probability martingale measure. Then, as described in Theorem 3.7, for any $\theta \in [0,T]$ there exists a strictly positive $(\mathbb{F}, P)$-martingale $\big(\beta_t(\theta)\big)_{t\in[\theta,T]}$ with $\beta_t(\theta) := E[\beta_\theta(\theta)|\mathcal{F}_t]$ for $t < \theta$. Thus, by martingale representation theorem for Brownian filtrations, there is an $(\mathcal{F}_t)_{t\in[\theta,T]}$ adapted process $(f_t(\theta))_{t\in[\theta,T]}$ such that $\beta_t(\theta) = \beta_\theta(\theta) + \int_\theta^t f_u(\theta) dW_u$ with $\int_\theta^T f_u^2(\theta) du < +\infty$, $P$-a.s. We recall that $\beta_t(.)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable, thus by measurable selection theorem, we can choose $f_t(\theta)$ as a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable functions. In addition, from Theorem 3.7 part 3, we have $\int_0^T \beta_\theta(\theta) d\theta \in L^1(P)$.

By applying the Ito formula, we obtain

$$S_t^d(\theta)\beta_t(\theta) = S_\theta^d(\theta)\beta_\theta(\theta) + \int_\theta^t S_u^d(\theta) f_u(\theta) dW_u + \int_\theta^t \beta_u(\theta) S_u^d(\theta)\sigma_u^d(\theta) dW_u + \int_\theta^t \beta_u(\theta) S_u^d(\theta)\mu_u^d(\theta) du + \int_\theta^t S_u^d(\theta)\sigma_u^d(\theta) f_u(\theta) du. \tag{3.33}$$

To guarantee the martingale property of $S_t^d(\theta)\beta_t(\theta)$, we must have

$$\beta_u(\theta)\mu_u^d(\theta) + \sigma_u^d(\theta) f_u(\theta) = 0, \quad \text{for all } u \in [\theta, T].$$

This implies

$$\frac{d\beta_t(\theta)}{\beta_t(\theta)} = \frac{f_t(\theta)}{\beta_t(\theta)} dW_t = -\frac{\mu_t^d(\theta)}{\sigma_t^d(\theta)} dW_t. \tag{3.34}$$

If for any $\theta > 0$ we set

$$\tilde{\beta}_\theta(\theta) := \frac{\beta_\theta(\theta)}{q_0}, \tag{3.35}$$

then (3.17) and the solution to the above SDE give us (3.29). Condition $(i)$ of Definition 3.9 can be easily seen as a result of (3.20) and (3.35).

To show the second part, by Theorem 3.7, one can see that $q_t G_t + \int_0^t \beta_u(u) du$ is an $(\mathbb{F}, P)$-martingale, thus

$$q_t G_t + \int_0^t \beta_u(u) du = q_0 + \int_0^t h_u dW_u, \quad t \in [0, T], \tag{3.36}$$

for some $\mathbb{F}$-predictable process $(h_t)_{t\in[0,T]}$.

On the other hand, we apply the Ito formula on $S_t^{\mathbb{F}}(q_t G_t)$ to get

$$dN_t = \left[S_t^{\mathbb{F}} h_t + S_t^{\mathbb{F}} \sigma_t^{\mathbb{F}} q_t G_t\right] dW_t + \left[S_t^{\mathbb{F}} \mu_t^{\mathbb{F}} q_t G_t + S_t^{\mathbb{F}} \sigma_t^{\mathbb{F}} h_t - S_t^{\mathbb{F}} \beta_t(t) + S_t^d(t)\beta_t(t)\right] dt. \tag{3.37}$$

Taking into account $(ii)$ and $S_t^d(t) = S_t^{\mathbb{F}}(1-\gamma_t)$, we can write

$$S_t^{\mathbb{F}} \sigma_t^{\mathbb{F}} h_t = \beta_t(t)\gamma_t S_t^{\mathbb{F}} - S_t^{\mathbb{F}} \mu_t^{\mathbb{F}} q_t G_t, \tag{3.38}$$

which implies

$$h_t = \frac{\beta_t(t)\gamma_t}{\sigma_t^{\mathbb{F}}} - q_t G_t \frac{\mu_t^{\mathbb{F}}}{\sigma_t^{\mathbb{F}}}. \tag{3.39}$$

Keeping in mind the integrability condition from part $(ii)$ of Definition 3.9, then equation (3.36) and (3.39) give us

$$d(q_t G_t) = \Big(\frac{\beta_t(t)\gamma_t}{\sigma_t^{\mathbb{F}}} - q_t G_t \frac{\mu_t^{\mathbb{F}}}{\sigma_t^{\mathbb{F}}}\Big) dW_t - \beta_t(t)dt. \tag{3.40}$$

This is a non-homogeneous SDE and we use Melnikov and Shiryaev [39] to find its solution:

$$\frac{q_t}{q_0} G_t = \mathcal{E}\Big(-\int_0^{\cdot} \frac{\mu_u^{\mathbb{F}}}{\sigma_u^{\mathbb{F}}}\, dW_u\Big)_t \Big[1 + \int_0^t \mathcal{E}^{-1}\Big(-\int_0^{\cdot} \frac{\mu_v^{\mathbb{F}}}{\sigma_v^{\mathbb{F}}}\, dW_v\Big)_u d\psi_u\Big], \tag{3.41}$$

where $\mathcal{E}(\cdot)$ denotes the stochastic exponential and the process $\psi_t$ is defined as

$$\psi_t := 1 + \int_0^t \frac{\tilde{\beta}_u(u)\gamma_u}{\sigma_u^{\mathbb{F}}}\, dW_u + \int_0^t \Big(\frac{\tilde{\beta}_u(u)\gamma_u \mu_u^{\mathbb{F}}}{(\sigma_u^{\mathbb{F}})^2} - \tilde{\beta}_u(u)\Big)\, du, \tag{3.42}$$

for all $t \in [0,T]$. Taking into account the definition of $\mathcal{E}(\cdot)$, representation (3.31) is concluded by equations (3.41) and (3.42).

In addition, to satisfy the positivity of process $(q_t)_{t\in[0,T]}$, we need to introduce appropriate constraints on $\tilde{\beta}_\theta(\theta)$. In part 2 of Theorem 3.7, since both $\beta_\theta(\theta)$ and $G_t$ are positive, $Y_T = q_T > 0$ ensures positivity of $q_t$ for all $t \in [0,T]$. Using $q_0 > 0$ and (3.31), $q_T > 0$ if and only if $\Psi_T > 0$, $P$-a.s., i.e. $\tilde{\beta}_\theta(\theta)$ must be chosen such that condition $(iii)$ of Definition 3.9 holds true.

To complete the proof, suppose that $\rho^{\mathbb{G}}$ satisfies part 1 and 2 of the theorem. Using the above arguments, we can see that $\rho^{\mathbb{G}}$ fulfills the criteria of Theorem 3.7 and also $\left(\rho^{\mathbb{G}}_t S_t\right)_{t\in[0,T]}$ is a $(\mathbb{G}, P)$-local martingale. □

**Remark 3.11.** *In fact Theorem 3.10 parametrizes $\mathcal{Q}$ by parameter $\tilde{\beta} \in \mathcal{B}$. As, it is expected, it determines infinitely many probability martingale measures for the incomplete model $S$ on $(\Omega, \mathbb{G}, P)$. For instance, in the case of $\gamma \equiv 0$ for $n \geq 2$ let $\tilde{\beta}(t; n) := \frac{Z_t}{nT}$ for all $t \in [0, T]$. It is a straightforward calculation to show that $\tilde{\beta}(.; n) \in \mathcal{B}$. From the Fubini's theorem and the martingale property of $(Z_t)_{t\in[0,T]}$ with $Z_0 = 1$:*

$$E\Big[\int_0^T \frac{Z_u}{nT} du\Big] = \frac{1}{nT}\int_0^T E\big[Z_u\big] du = \frac{1}{nT}\int_0^T du = \frac{1}{n} < +\infty$$

*This corresponds to condition $(i)$ of Definition 3.9, condition $(ii)$ is trivial for $\gamma \equiv 0$, and to see $(iii)$ in the definition, notice that*

$$\Psi_T = 1 - \int_0^T \frac{Z_t}{nTZ_t} dt = 1 - \frac{1}{n} > 0, \qquad \textit{for all } n \geq 2.$$

*Thus probability measure $Q_n$ defined by $\tilde{\beta}(.; n)$, as in Theorem 3.10, belongs to $\mathcal{Q}$, i.e. $\{Q_n\}_{n\geq 2} \subseteq \mathcal{Q}$.*

We point out that Theorem 3.10 does not use a $\mathbb{G}$-Brownian motion in the decomposition of $\rho^{\mathbb{G}}$. Although we study our model in the enlarged filtration $\mathbb{G}$ but the calculations are reduced to the filtration $\mathbb{F}$. In the next section, this property helps us to determine superhedging trading strategy of a $\mathcal{G}$-contingent claim in terms of $S^{\mathbb{F}}_t$ and $S^d_t(\theta)$ rather than $S_t$.

## 3.3 Superhedging in the defaultable market

In the remaining of this chapter we investigate the superhedging problem in the incomplete defaultable market described in Section 3.1.

Let $H = H^{\mathbb{F}} 1_{\{\tau > T\}} + H^d(\tau) 1_{\{\tau \leq T\}}$ be a nonnegative $\mathcal{G}_T$-measurable contingent claim such that

$$\sup_{Q \in \mathcal{Q}} E^Q[H] < +\infty \,,$$

where $H^{\mathbb{F}}$ is a nonnegative $\mathcal{F}_T$-measurable random variable and $H^d(\tau)$ is a nonnegative $\mathcal{F}_T \otimes \sigma(\tau)$-random variable. We are looking for a minimal initial capital $U_0$ and a $\mathbb{G}$-admissible portfolio process which cover $H$ at the maturity time $T$. Mathematically speaking, this problem is identified by the upper Snell envelope of $H$ with respect to the set $\mathcal{Q}$, i.e.

$$U_t := \operatorname*{ess\,sup}_{Q \in \mathcal{Q}} E^Q \big[ H \big| \mathcal{G}_t \big], \qquad t \in [0, T], \tag{3.43}$$

where $U_0 = \sup_{Q \in \mathcal{Q}} E^Q[H]$ determines the initial cost of superhedging $H$. From the optional decomposition theorem, we know that there exists a $\mathbb{G}$-prdictable process $(\varphi_t)_{t \in [0,T]}$ and a nonnegative $\mathbb{G}$-optional process $(C_t)_{t \in [0,T]}$ with $C_0 = 0$ such that

$$U_t := U_0 + \int_0^t \varphi_s \, dS_s - C_t, \qquad t \in [0, T]. \tag{3.44}$$

See El Karoui and Quenez [12] for more details on pricing in incomplete markets, and consult with Föllmer and Schied [16] for an excellent demonstration of the problem in a discrete time setting.

To obtain the superhedging $\mathbb{G}$-admissible portfolio $\pi = (\pi^{\mathbb{F}}, \pi^d)$, first we find a decomposition for $Y_t$ in terms of $S_t$ and $S_t^d(\tau)$. This result, Theorem 3.14, provides us a tool to characterize $\mathbb{G}$-admissible wealth processes in our model.

The next proposition is a well-known result in complete markets, for reader's convenience, we recall this result from Karatzas and Shreve [23].

**Proposition 3.12.** *Let $(Y_t^{\mathbb{F}})_{t \in [0,T]}$ be a positive $\mathbb{F}$-adapted process. Then*

$(Z_tY_t^{\mathbb{F}})_{t\in[0,T]}$ *is an* $(\mathbb{F},P)$*-local martingale iff*

$$Y_t^{\mathbb{F}} = Y_0^{\mathbb{F}} + \int_0^t \pi_u^{\mathbb{F}} Y_u^{\mathbb{F}} \frac{dS_u^{\mathbb{F}}}{S_u^{\mathbb{F}}}, \quad t \in [0,T], \tag{3.45}$$

*where* $(\pi_t^{\mathbb{F}})_{t\in[0,T]}$ *is an* $\mathbb{F}$*-predictable process such that*

$$\int_0^T (\pi_t^{\mathbb{F}} \sigma_t^{\mathbb{F}})^2 dt < +\infty, \quad P\text{-a.s.} \tag{3.46}$$

*In particular,* $(Y_t^{\mathbb{F}})_{t\in[0,T]}$ *is an* $\mathbb{F}$*-wealth process corresponding to the* $\mathbb{F}$*-admissible trading strategy* $\pi^{\mathbb{F}}$ *and the initial capital* $Y_0^{\mathbb{F}}$*.*

*Proof.* If $(Z_tY_t^{\mathbb{F}})_{t\in[0,T]}$ is a positive $(\mathbb{F},P)$-local martingale then, by martingale representation theorem, there exists an $\mathbb{F}$-predictable process $(\phi_t)_{t\in[0,T]}$ satisfying $\int_0^T \phi_t^2 dt < +\infty$, $P$-a.s., such that

$$Z_tY_t^{\mathbb{F}} = Y_0^{\mathbb{F}} + \int_0^t \phi_u dW_u, \quad \text{for all } t \in [0,T]. \tag{3.47}$$

Define

$$\pi_t^{\mathbb{F}} := (\sigma_t^{\mathbb{F}})^{-1}\Big(\frac{\phi_t}{Z_tY_t^{\mathbb{F}}} + \frac{\mu_t^{\mathbb{F}}}{\sigma_t^{\mathbb{F}}}\Big). \tag{3.48}$$

Then, one can show that

$$\begin{aligned}\int_0^T (\pi_t^{\mathbb{F}}\sigma_t^{\mathbb{F}})^2 dt &\le 2\Big[\int_0^T \frac{\phi_t^2}{(Z_tY_t^{\mathbb{F}})^2} dt + \int_0^T \big(\frac{\mu_t^{\mathbb{F}}}{\sigma_t^{\mathbb{F}}}\big)^2 dt\Big] \\ &\le \frac{2}{\min_{t\in[0,T]}(Z_tY_t^{\mathbb{F}})^2}\int_0^T \phi_t^2 dt + 2\int_0^T \big(\frac{\mu_t^{\mathbb{F}}}{\sigma_t^{\mathbb{F}}}\big)^2 dt < +\infty,\end{aligned} \tag{3.49}$$

since $Z_tY_t^{\mathbb{F}}$ is a continuous positive process, and due to assumption (3.2). Since

$$dZ_t^{-1} = Z_t^{-1}\Big(\frac{\mu_t^{\mathbb{F}}}{\sigma_t^{\mathbb{F}}} dW_t + \big(\frac{\mu_t^{\mathbb{F}}}{\sigma_t^{\mathbb{F}}}\big)^2 dt\Big), \tag{3.50}$$

and $Y_t^{\mathbb{F}} = Z_t^{-1}\big(Z_t Y_t^{\mathbb{F}}\big)$, using equations (3.47) , (3.48), and (3.50) along with a straightforward calculation by Ito formula we can prove (3.45).

Conversely, suppose that (3.45) and (3.46) hold true. Having $\pi_t$, we can define $\phi_t$ using (3.48) which deduces (3.47). In addition, similar to (3.49), we can see that

$$\int_0^T \phi_t^2 dt \leq 2 \max_{t\in[0,T]} (Z_t Y_t^{\mathbb{F}})^2 \Big[ \int_0^T (\sigma_t^{\mathbb{F}} \pi_t^{\mathbb{F}})^2 dt + \int_0^T \big(\frac{\mu_t^{\mathbb{F}}}{\sigma_t^{\mathbb{F}}}\big)^2 dt \Big] < +\infty. \tag{3.51}$$

The last part of the proposition is an immediate result of (3.2) and Definition 3.2. □

As a consequence of Proposition 3.12, we can characterize $\mathbb{G}$-adapted wealth processes in terms of $(\mathbb{F}, P)$-local martingales:

**Corollary 3.13.** *Let $(Y_t)_{t\in[0,T]}$ be a positive $\mathbb{G}$-adapted process. Then $Y_t$ is the wealth process corresponding to a $\mathbb{G}$-admissible portfolio process $\pi = (\pi^{\mathbb{F}}, \pi^d)$ with $c \equiv 0$ iff there exists an $\mathbb{F}$-adapted process $(Y_t^{\mathbb{F}})_{t\in[0,T]}$ and an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-optional process $Y_t^d(.)$, both positive, such that $Y_t = Y_t^{\mathbb{F}} 1_{\{t<\tau\}} + Y_t^d(\tau) 1_{\{t\geq\tau\}}$ and the following conditions hold:*

*(i) $(Z_t Y_t^{\mathbb{F}})_{t\in[0,T]}$ is a positive $(\mathbb{F}, P)$-local martingale.*

*(ii) For any fixed $\theta \in [0, T]$, $\big(Z_t^d(\theta) Y_t^d(\theta)\big)_{t\in[\theta,T]}$ is a positive $(\mathbb{F}, P)$-local martingale.*

*(iii) For all $\theta \in [0, T]$, we have $Y_\theta^d(\theta) = Y_\theta^{\mathbb{F}}(1 - \pi_\theta^{\mathbb{F}} \gamma_\theta)$, $P$-a.s.*

*Proof.* First, let us assume that $Y_t = X_t^{\pi} = X_t^{\pi^{\mathbb{F}}} 1_{\{t\leq\tau\}} + X_t^{\pi^d}(\tau) 1_{\{t>\tau\}}$ for some $\mathbb{G}$-admissible portfolio $\pi = (\pi^{\mathbb{F}}, \pi^d)$. Set $Y_t^{\mathbb{F}} = X_t^{\pi^{\mathbb{F}}}$ and $Y_t^d(\theta) = X_t^{\pi^d}(\theta)$ then part $(i)$ can be easily seen from Proposition 3.12, and part $(ii)$ is verified similar to the proof of this proposition. Part $(iii)$ comes from Definition 3.4 for $X_t^{\pi}$.

Keeping in mind Definition 3.4, one can also show the reverse statement using the arguments of the proof of Proposition 3.12. $\square$

The next theorem can be considered as a counterpart to Proposition 3.12 in the enlarged filtration $\mathbb{G}$. However, in this case, $\mathbb{G}$-local martingale property w.r.t probability measures $Q \in \mathcal{Q}$ does not fully characterize the $\mathbb{G}$-wealth processes. We need to add $(\mathbb{F}, P)$-local martingale property of the before-default process as well:

**Theorem 3.14.** *A positive $\mathbb{G}$-adapted process $Y_t = Y_t^{\mathbb{F}} 1_{\{t<\tau\}} + Y_t^d(\tau) 1_{\{t \geq \tau\}}$ is the wealth process associated to some $\mathbb{G}$-admissible porfolio $\pi = (\pi^{\mathbb{F}}, \pi^d)$ iff $(Z_t Y_t^{\mathbb{F}})_{t\in[0,T]}$ is a positive $(\mathbb{F}, P)$-local martingale, and for any $Q \in \mathcal{Q}$ process $(Y_t)_{t\in[0,T]}$ is a positive $(\mathbb{G}, Q)$-local martingale.*

*Proof.* Taking into account representation (3.28), Proposition (5.6) of Karoui et al. [25] shows that $\rho_t^{\mathbb{G}} Y_t = \dfrac{q_t}{q_0} Y_t^{\mathbb{F}} 1_{\{\tau>t\}} + \dfrac{q_t(\tau)}{q_0} Y_t^d(\tau) 1_{\{\tau \leq t\}}$ is a positive $(\mathbb{G}, P)$-local martingale iff both of the following conditions are satisfied

*(1)* $\left(\dfrac{q_t}{q_0} Y_t^{\mathbb{F}} G_t + \displaystyle\int_0^t \dfrac{q_u(u)}{q_0} Y_u^d(u) \alpha_u(u) du\right)_{t\in[0,T]}$ is a positive $(\mathbb{F}, P)$-local martingale.

*(2)* For any fixed $\theta \in [0,T]$, $\left(\dfrac{q_t(\theta)}{q_0} Y_t^d(\theta) \alpha_t(\theta)\right)_{t\in[\theta,T]}$ is a positive $(\mathbb{F}, P)$-local martingale.

If $Y_t$ is a $\mathbb{G}$-wealth process, condition (2) results from part $(ii)$ of Corollary 3.13 and (3.29). To show $(\mathbb{G}, P)$-local martingale property of $(Y_t)_{t\in[0,T]}$, it is enough to prove condition (1).

From (3.30) and (3.31), $Y_t^{\mathbb{F}} = X_t^{\mathbb{F}}$, and Ito formula we have

$$
\begin{aligned}
&\frac{q_t}{q_0} Y_t^{\mathbb{F}} G_t + \int_0^t \frac{q_u(u)}{q_0} Y_u^d(u)\alpha_u(u)du = Z_t \Psi_t X_t^{\mathbb{F}} + \int_0^t \tilde{\beta}_u(u) Y_u^d(u)du \\
&= X_0^{\mathbb{F}} + \int_0^t Z_u \Psi_u dX_u^{\mathbb{F}} + \int_0^t X_u^{\mathbb{F}} d(Z_u \Psi_u) + \big[Z\Psi, X^{\mathbb{F}}\big]_t + \int_0^t \tilde{\beta}_u(u) Y_u^d(u)du \\
&= X_0^{\mathbb{F}} + \int_0^t Z_u \Psi_u X_u^{\mathbb{F}} \pi_u^{\mathbb{F}} \big(\sigma_u^{\mathbb{F}} dW_u + \mu_u^{\mathbb{F}} du\big) + \int_0^t X_u^{\mathbb{F}} \big(\frac{\tilde{\beta}_u(u)\gamma_u}{\sigma_u^{\mathbb{F}}} - Z_u \Psi_u \frac{\mu_u^{\mathbb{F}}}{\sigma_u^{\mathbb{F}}}\big) dW_u \\
&- \int_0^t X_u^{\mathbb{F}} \tilde{\beta}_u(u)du + \int_0^t X_u^{\mathbb{F}} \pi_u^{\mathbb{F}} \sigma_u^{\mathbb{F}} \big(\frac{\tilde{\beta}_u(u)\gamma_u}{\sigma_u^{\mathbb{F}}} - Z_u \Psi_u \frac{\mu_u^{\mathbb{F}}}{\sigma_u^{\mathbb{F}}}\big) du + \int_0^t \tilde{\beta}_u(u) Y_u^d(u)du \\
&= X_0^{\mathbb{F}} + \int_0^t X_u^{\mathbb{F}} \big(Z_u \Psi_u \pi_u^{\mathbb{F}} \sigma_u^{\mathbb{F}} + \frac{\tilde{\beta}_u(u)\gamma_u}{\sigma_u^{\mathbb{F}}} - Z_u \Psi_u \frac{\mu_u^{\mathbb{F}}}{\sigma_u^{\mathbb{F}}}\big) dW_u \\
&+ \int_0^t \tilde{\beta}_u(u) \Big(Y_u^d(u) - X_u^{\mathbb{F}} \big(1 - \pi_u^{\mathbb{F}} \gamma_u\big)\Big) du
\end{aligned}
\tag{3.52}
$$

By the last equality and part $(iii)$ of Corollary 3.13, condition (1) is now satisfied. The $(\mathbb{F}, P)$-local martingale property of $(Z_t Y_t^{\mathbb{F}})_{t\in[0,T]}$ is given by part $(i)$ of Corollary 3.13.

On the other hand, if $(\rho_t^{\mathbb{G}} Y_t)_{t\in[0,T]}$ is a positive $(\mathbb{G}, P)$-local martingale then conditions (1) and (2) are immediate as discussed above. From (2), it is easy to drive part $(ii)$ of Corollary 3.13. Keeping in mind equation (3.52), condition (1) is fulfilled iff we have part $(iii)$ of Corollary 3.13. Therefore, by combing these with the $(\mathbb{F}, P)$-local martingale property of $(Z_t Y_t^{\mathbb{F}})_{t\in[0,T]}$, Corollary 3.13 finishes the proof. □

Now, we turn back to the superhedging problem (3.43) and finding $Y_t$ and $C_t$ in the optional decomposition (3.44). The next proposition reduces the underlying $Q$-conditional expectations in $\mathbb{G}$ to $P$-conditional expectations in the original filtration $\mathbb{F}$.

**Proposition 3.15.** *Let $H = H^{\mathbb{F}} 1_{\{\tau > T\}} + H^d(\tau) 1_{\{\tau \leq T\}}$ be a nonnegative $\mathcal{G}_T$-*

*measurable contingent claim such that*

$$\sup_{Q\in\mathcal{Q}} E^Q[H] < +\infty\,,$$

*where* $H^{\mathbb{F}}$ *is a nonnegative* $\mathcal{F}_T$*-measurable random variable and* $H^d(\tau)$ *is a nonnegative* $\mathcal{F}_T \otimes \sigma(\tau)$*-random variable. Then for any* $Q \in \mathcal{Q}$ *we obtain:*

$$\begin{aligned} E^Q[H|\mathcal{G}_t] = 1_{\{t<\tau\}} \frac{1}{Z_t\Psi_t} E\Big[Z_T\Psi_T H^{\mathbb{F}} + \int_t^T \tilde{\beta}_\theta(\theta) E\big[Z_T^d(\theta)H^d(\theta)\big|\mathcal{F}_\theta\big]d\theta\Big|\mathcal{F}_t\Big] \\ + 1_{\{t\geq\tau\}} \frac{1}{Z_t^d(\theta)} E\Big[Z_T^d(\theta)H^d(\theta)\Big|\mathcal{F}_t\Big]\Big|_{\theta=\tau} \end{aligned} \tag{3.53}$$

*for all* $t \in [0, T]$.

*Proof.* Using Theorem 3.10, for an arbitrary $Q \in \mathcal{Q}$ we can write $\frac{dQ}{dP} = \rho_T^{\mathbb{G}} = \frac{q_T}{q_0}1_{\{T<\tau\}} + \frac{q_T(\tau)}{q_0}1_{\{T\geq\tau\}}$ with $\frac{q_t}{q_0}$ and $\frac{q_t(\tau)}{q_0}$ as described therein. Hence:

$$\begin{aligned} E^Q[H|\mathcal{G}_t] &= \frac{1}{\rho_t^{\mathbb{G}}} E\Big[\frac{q_T}{q_0}H^{\mathbb{F}}1_{\{T<\tau\}} + \frac{q_T(\tau)}{q_0}H^d(\tau)1_{\{T\geq\tau\}}\Big|\mathcal{G}_t\Big] \\ &= E\Big[\frac{q_T/q_0}{q_t/q_0}H^{\mathbb{F}}1_{\{T<\tau\}}\Big|\mathcal{G}_t\Big] + E\Big[\frac{q_T(\tau)/q_0}{q_t/q_0}H^d(\tau)1_{\{t<\tau\leq T\}}\Big|\mathcal{G}_t\Big] \\ &+ E\Big[\frac{q_T(\tau)/q_0}{q_t(\tau)/q_0}H^d(\tau)1_{\{t\geq\tau\}}\Big|\mathcal{G}_t\Big]. \end{aligned} \tag{3.54}$$

In the following, we simplify these three conditional expectations to reduce them into conditional expectations with respect to the $\mathbb{F}$ filtration. To do so, we utilize (3.31) and Corollary 5.1.1 of Bielecki and Rutkowski [3] to get

$$\begin{aligned} E\Big[\frac{q_T/q_0}{q_t/q_0}H^{\mathbb{F}}1_{\{T<\tau\}}\Big|\mathcal{G}_t\Big] &= 1_{\{t<\tau\}}E\Big[\frac{(q_T/q_0)G_T}{(q_t/q_0)G_t}H^{\mathbb{F}}\Big|\mathcal{F}_t\Big] \\ &= 1_{\{t<\tau\}}E\Big[\frac{Z_T\Psi_T}{Z_t\Psi_t}H^{\mathbb{F}}\Big|\mathcal{F}_t\Big]. \end{aligned} \tag{3.55}$$

To compute the second term in (3.54), we apply Theorem 3.1 of Karoui et al. [25]. By approximating $\frac{q_T(\tau)/q_0}{q_t/q_0} H^d(\tau) 1_{\{\tau \le T\}}$ by $\left(\frac{q_T(\tau)/q_0}{q_t/q_0} H^d(\tau) 1_{\{\tau \le T\}}\right) \wedge n$ and then using monotone convergence theorem, we have

$$
\begin{aligned}
& E\Big[\frac{q_T(\tau)/q_0}{q_t/q_0} H^d(\tau) 1_{\{\tau\le T\}} 1_{\{t<\tau\}} \Big| \mathcal{G}_t\Big] \\
&= \frac{1}{(q_t/q_0)G_t} E\Big[\int_t^T \frac{q_T(\theta)}{q_0} H^d(\theta)\alpha_T(\theta) d\theta \Big| \mathcal{F}_t\Big] 1_{\{t<\tau\}} \\
&= \frac{1}{Z_t \Psi_t} E\Big[\int_t^T \tilde{\beta}_\theta(\theta) E\big[Z_T^d(\theta) H^d(\theta) \big| \mathcal{F}_\theta\big] d\theta \Big| \mathcal{F}_t\Big] 1_{\{t<\tau\}},
\end{aligned}
\tag{3.56}
$$

to obtain the last equality, we used (3.29) and the property of conditional expectation w.r.t $\mathcal{F}_\theta$ and $\mathcal{F}_t$ for $t \le \theta$.

Finally, let us focus on the third term in (3.54). This is done with a similar argument as used in (3.56), i.e. Theorem 3.1 of Karoui et al. [25], approximating by a bounded sequence and monotone convergence theorem:

$$
\begin{aligned}
& E\Big[\frac{q_T(\tau)/q_0}{q_t(\tau)/q_0} H^d(\tau) 1_{\{t\ge\tau\}} \Big| \mathcal{G}_t\Big] \\
&= 1_{\{t\ge\tau\}} E\Big[\frac{(q_T(\theta)/q_0)\alpha_T(\theta)}{(q_t(\theta)/q_0)\alpha_t(\theta)} H^d(\theta) \Big| \mathcal{F}_t\Big]\Big|_{\theta=\tau} \\
&= 1_{\{t\ge\tau\}} E\Big[\frac{Z_T^d(\theta)}{Z_t^d(\theta)} H^d(\theta) \Big| \mathcal{F}_t\Big]\Big|_{\theta=\tau},
\end{aligned}
\tag{3.57}
$$

where again we applied (3.29) in the second equality. □

The following theorem and its proof provide an algorithm how to determine the superhedging trading strategy and the consumption process for an arbitrary $\mathcal{G}$-measurable contingent claim $H$. The strategy consists of two parts:

*(1)* A before-default trading strategy in the original filtration $(\mathcal{F})_{t\in[0,T]}$ with investing in $(S_t^{\mathbb{F}})_{t\in[0,T]}$.

*(2)* An after-default trading strategy in the filtration $(\mathcal{F})_{t\in[\theta,T]}$ with investing

in $\left(S_t^d(\theta)\right)_{t\in[\theta,T]}$, where $\theta = \tau$ is the default occurrence time.

**Theorem 3.16.** *Let $H = H^{\mathbb{F}} 1_{\{T<\tau\}} + H^d(\tau) 1_{\{T\geq\tau\}} \in \mathcal{G}_T$ be as in Proposition 3.15. Then, the superhedging value process of $H$ is given by:*

$$\operatorname*{ess\,sup}_{Q\in\mathcal{Q}} E^Q\left[H\middle|\mathcal{G}_t\right] = 1_{\{t<\tau\}} \operatorname*{ess\,sup}_{\tilde{\beta}\in\mathcal{B}} \frac{1}{Z_t\Psi_t} E\left[Z_T\Psi_T H^{\mathbb{F}} + \int_t^T \tilde{\beta}_\theta(\theta) E\left[Z_T^d(\theta)H^d(\theta)\middle|\mathcal{F}_\theta\right] d\theta \,\middle|\, \mathcal{F}_t\right]$$
$$+ 1_{\{t\geq\tau\}} \frac{1}{Z_t^d(\theta)} E\left[Z_T^d(\theta)H^d(\theta) \,\middle|\, \mathcal{F}_t\right]\Big|_{\theta=\tau} \tag{3.58}$$

*In particular, we have*

$$\sup_{Q\in\mathcal{Q}} E^Q[H] = \sup_{\tilde{\beta}\in\mathcal{B}} E\left[Z_T\Psi_T H^{\mathbb{F}} + \int_0^T \tilde{\beta}_\theta(\theta) E\left[Z_T^d(\theta)H^d(\theta)\middle|\mathcal{F}_\theta\right] d\theta\right]. \tag{3.59}$$

*Proof.* Let us define

$$Y_t^{\mathbb{F}} := \frac{1}{Z_t\Psi_t} E\left[Z_T\Psi_t H^{\mathbb{F}} + \int_t^T Z_\theta\tilde{\Psi}_\theta E\left[Z_T^d(\theta)H^d(\theta)\middle|\mathcal{F}_\theta\right] d\theta \,\middle|\, \mathcal{F}_t\right], \tag{3.60}$$

for any $t \in [0,T]$. In addition, for any fixed $\theta \in [0,T]$, we introduce $Y_t^d(\theta)$ as follows

$$Y_t^d(\theta) := \frac{1}{Z_t^d(\theta)} E\left[Z_T^d(\theta)H^d(\theta) \,\middle|\, \mathcal{F}_t\right], \qquad \text{for all } t \in [\theta, T]. \tag{3.61}$$

Let $Y_t = Y_t^{\mathbb{F}} 1_{\{t<\tau\}} + Y_t^d(\theta) 1_{\{t\geq\tau\}}$, for $t \in [0,T]$. By the definition of $Y_t^d(\theta)$, it is clear that $\left(Z_t^d(\theta)Y_t^d(\theta)\right)_{t\in[\theta,T]}$ is a nonnegative $(\mathbb{F}, P)$-local martingale. Hence there is an $\mathbb{F}$-predictable process $\left(\tilde{\pi}_t^d(\theta)\right)_{t\in[\theta,T]}$ such that

$$Y_t^d(\theta) = X_t^{\tilde{\pi}^d}(\theta) = Y_\theta^d(\theta) + \int_\theta^t X_u^{\tilde{\pi}^d}(\theta)\tilde{\pi}_u^d(\theta) \frac{dS_u^d(\theta)}{S_u^d(\theta)}. \tag{3.62}$$

By applying a measurable selection argument, we can choose the family of $\tilde{\pi}_t^d(\theta)$ such that they are $\mathcal{F}_t \otimes \mathbb{R}_+$-measurable functions for all $t \in [0,T]$.

To obtain (3.58) it is enough to combine (3.43) and (3.53) with (3.62). □

## 3.4 Conclusion

In this chapter, we study a regime-switching model exposed to a counterparty risk where the associated default time satisfies the so-called *density hypothesis*.

Our main goal is to provide closed form representations for the class of probability martingale measures and the superhedging problem for this model. This framework is an interesting case with potential applications in stochastic volatility models, defaultable markets, risk minimization and utility maximization problems. Our explicit solutions facilitate the studying of pricing and optimization problems in defaultable markets which lack concrete examples with explicit representations. Another importance of our results is the fact that our techniques introduce a method to reduce the calculations in the enlarged filtration $\mathbb{G}$ to the Brownian filtration $\mathbb{F}$.

# Chapter 4

# Efficient Hedging for Defaultable Securities and its Application to Equity-Linked Life Insurance Contracts

## 4.1 Model Setting

In this Chapter we consider a financial model consisting of two assets $B$ and $S$, defined by their prices processes $(B_t)_{0\leq t\leq T}$ and $(S_t)_{0\leq t\leq T}$. Let us call this model $(B,S)$-market and assume its price evolution as follows

$$\begin{aligned} dS_t &= S_t\big(m_t dt + \sigma_t dW_t\big), \qquad && S_0 \in (0,\infty) \\ dB_t &= B_t r_t dt, && B_0 = 1 \end{aligned} \tag{4.1}$$

for $t \in [0,T]$. $(r_t)_{0\leq t\leq T}$ is the risk free interest rate of our bank account $B$, volatility and appreciation rate of $S$ as the risky asset are given by $\sigma > 0$ and

$m$ respectively, and $(W_t)_{0\leq t\leq T}$ is a standard Brownian motion on the complete probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T} \subseteq \mathcal{G}, P)$. For the sake of simplicity, we assume $r \equiv 0$.

We postulate that $(B, S)$ is a complete market. In other words, there exists a unique *equivalent martingale measure* $P^*$ defined by

$$\frac{dP^*}{dP} := \rho^*_T$$

such that

$$\rho^*_t := \exp\Big( - \int_0^t \frac{m_s}{\sigma_s} dW_s - \int_0^t \frac{1}{2} (\frac{m_s}{\sigma_s})^2 ds \Big) \tag{4.2}$$

for $t \in [0, T]$. In addition, $(\rho^*_t)_{0\leq t\leq T}$ satisfies the following integrability conditions

*(1)* $\int_0^T (\frac{m_s}{\sigma_s})^2 ds < +\infty, \quad P$-a.s.

*(2)* $E[\rho^*_T] = 1$

The default times are represented by $\tau_i$ for $i = 1, ..., n$. They are some positive $\mathcal{G}$-measurable random variables ($\mathcal{G}$-random times) with $P(\tau_i = 0) = 0$ such that $P(\tau_i > t) > 0$ for all $i = 1, ..., n$ and $t \in [0, T]$. For $J \subseteq \{1, ..., n\}$, by progressively enlargement of the filtrations, define

$$\mathcal{G}^J_t := \mathcal{F}_t \vee \mathcal{H}^J_t \quad \text{and} \quad \mathbb{G}^J := (\mathcal{G}^J_t)_{0\leq t\leq T}$$

where $\mathcal{H}^J_t := \sigma\big(\tau_i \wedge t; i \in J\big)$, for $t \in [0, T]$. If $J = \{1, ..., n\}$, we simply write $\mathcal{G}_t$, $\mathcal{H}_t$, and $\mathbb{G}$.

We make the following assumptions on the default times:

**Assumption 4.1.** *The default times* $\{\tau_i : i = 1, ..., n\}$ *and* $(W_t)_{0\leq t\leq T}$ *are mutually independent.*

**Assumption 4.2.** *$P(\tau_i = \tau_j) = 0$, for all $i, j = 1, ..., n$ and $i \neq j$.*

In fact, Assumption 4.1 with the help of Bielecki and Rutkowski [3] (Lemma 6.1.2) imply that any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale. This guarantees no arbitrage condition in the defaultable $(B, S, \tau)$-market.

There are some crucial processes associated to each random time $\tau_i$ for $i = 1, ..., n$:

- The $\mathbb{F}$-supermatingale

$$G_t^i := P\big(\tau_i > t | \mathcal{F}_t\big) \tag{4.3}$$

  is called the *Azéma supermatingale* or *the survival process of $\tau_i$ with respect to* $\mathbb{F}$. We assume that $G_t^i > 0$ for all $t \in [0, T]$. Due to Assumption 4.1, in our model (4.3) is simplified to $G_t^i = P\big(\tau_i > t\big)$.

- For $i = 1, ..., n$, if there exists a nonnegative $\mathbb{F}$-predictable process $(\mu_t^i)_{0 \leq t \leq T}$ such that

$$G_t^i = \exp\Big( - \int_0^t \mu_s^i ds \Big), \qquad t \in [0, T] \tag{4.4}$$

  then $(\mu_t^i)_{0 \leq t \leq T}$ is called $\mathbb{F}$-intensity of the random time $\tau_i$. This assumption on the default time is well known as *intensity hypothesis*. Since $G_t^i = P\big(\tau_i > t\big)$ by Assumption 4.1, for all $i = 1, ..., n$ the intensity $\mu_t^i$ is only a nonnegative function of the variable $t \in [0, T]$.

- If $(\mu_t^i)_{0 \leq t \leq T}$ exists then

$$M_t^i := 1_{\{\tau_i \leq t\}} - \int_0^{\tau_i \wedge t} \mu_s^i ds, \qquad t \in [0, T]$$

  is a $\mathbb{G}$-martingale.

We can interpret the intensity process as a *local default rate*, i.e.

$$P\Big(\tau_i \in (t, t+dt] | \mathcal{F}_t\Big) = \mu_t^i dt. \tag{4.5}$$

For more details regarding the above processes, see for example Bielecki and Rutkowski [3] or Nikeghbali [43].

Let us define

$$\mathcal{D} := \Big\{ (\kappa_t)_{0\leq t\leq T} : \text{bounded}, \mathbb{G}\text{-predictable and } \kappa_t > -1 \;\; dt \times dP \text{ a.e.} \Big\}.$$

Keeping in mind the above assumptions and notations, we can show that the class of equivalent martingale measures for our defaultable model is given by:

$$\mathcal{Q} := \Big\{ Q^\kappa \mid \kappa := (\kappa_t^1, ..., \kappa_t^n)_{0\leq t\leq T} \in \mathcal{D}^n \Big\} \tag{4.6}$$

where the set $\mathcal{D}^n$ consists of all $n$-tuples of the elements of $\mathcal{D}$, and

$$\frac{dQ^\kappa}{dP} := \rho_T^* \rho_T^\kappa, \qquad \kappa \in \mathcal{D}^n$$

with

$$\rho_t^\kappa = 1 + \sum_{i=1}^n \int_0^t \kappa_s^i \rho_{s-}^{\kappa^i} dM_s^i, \qquad t \in [0, T],$$

for more details, see Bielecki and Rutkowski [3] or Kusuoka [31].

Due to Assumption 4.2, it is not difficult to see that the jumps of $\tau_i$ and $\tau_j$ for $i \neq j$ do not coincide. Thus we can use the definition of stochastic exponential to prove that

$$\rho_t^\kappa = \Big( \prod_{i=1}^n \big(1 + \kappa_{\tau_i}^i 1_{\{\tau_i \leqslant t\}}\big)\Big) \exp\Big( - \sum_{i=1}^n \int_0^{\tau_i \wedge t} \kappa_s^i \mu_s^i ds \Big), \qquad t \in [0, T]. \tag{4.7}$$

Based on the available information ($\mathbb{G}$) to the traders in the defaultable market, we have the following definition for admissible strategies.

**Definition 4.3.**

1. *A* $\mathbb{G}$-trading strategy *is a* $\mathbb{G}$*-predictable process* $\pi := \left(\pi_t^0, \pi_t^1\right)_{t\in[0,T]}$ *such that*
$$\int_0^T |\pi_t^0| dt < \infty, \quad \text{and} \quad \int_0^T \left(\pi_t^1 S_t\right)^2 dt < \infty \quad P\text{-a.s.}$$

2. *At time* $t \in [0,T]$*, the* value process *associated to* $\left(\pi_t^0, \pi_t^1\right)_{t\in[0,T]}$ *is defined by*
$$V_t := \pi_t^0 + \pi_t^1 S_t.$$
*(Since we assume that* $r \equiv 0$*.)*

3. *For a given initial value* $v_0 \geqslant 0$*, the trading strategy* $\pi$ *is called* self-financing *if its corresponding* value process *satisfies*
$$V_t = v_0 + \int_0^t \pi_s^1 dS_s, \quad P\text{-a.s.}$$
*for all* $t \in [0,T]$*.*

4. *A self-financing strategy* $\left(v_0, \pi_t\right)_{t\in[0,T]}$ *is called* $\mathbb{G}$-admissible*, if for its corresponding value process* $(V_t^{v_0,\pi})_{t\in[0,T]}$ *we have*
$$V_t^{v_0,\pi} \geq 0, \quad P\text{-a.s.} \quad \forall t \in [0,T].$$
*The set of all* $\mathbb{G}$*-admissible strategies with initial value* $v_0$ *is denoted by* $\mathcal{A}^{\mathbb{G}}(v_0)$*. In a similar way to above, we can define* $\mathcal{A}^{\mathbb{F}}(v_0)$*, the* $\mathbb{F}$*-admissible strategies with initial value* $v_0$*.*

In this chapter we investigate the efficient hedging problem of the defaultable contingent claims defined as

$$H_\delta := H \prod_{i=1}^{n} \left(1_{\{\tau_i > T\}} + \delta_i 1_{\{\tau_i \le T\}}\right) \tag{4.8}$$

where $H$ is a nonnegative $\mathcal{F}_T$-measurable random variable and $\delta_i \in [0,1]$ is called the *i-recovery rate*, for $i = 1, ..., n$.

To formulate the problem of minimizing the shortfall risk weighted by a loss function $l$ we recall the following definition from Follmer and Leukert [15].

**Definition 4.4.** *A loss function $l$ is an increasing convex function on $[0, +\infty)$ with*

*(i)* $l(0) = 0$

*(ii)* $E[l(H)] < +\infty$

Additionally, we make some differentiability assumptions about $l$.

**Assumption 4.5.** *$l$ has the following properties:*

- $l \in C^1(0, +\infty)$
- $l'$ *is strictly increasing on* $(0, +\infty)$
- $l'(0+) = 0$ *and* $l'(+\infty) = +\infty$

By above, the inverse function of $l'$ exists and is denoted by $I$, $I := (l')^{-1}$.

In this framework, we apply the superhedging techniques to hedge $H_\delta$ in the incomplete market $(B, S, \tau)$. The initial cost of superhedging is defined by

$$U_0 := \inf\left\{u \ge 0 : \; V_T^{u,\pi} \ge H_\delta, \text{ for some } \pi \in \mathcal{A}^{\mathbb{G}}(u)\right\}.$$

Equivalently, we can show that

$$U_0 = \sup_{Q \in \mathcal{Q}} E^Q \big[ H_\delta \big] \tag{4.9}$$

where $\mathcal{Q}$ is the class of all martingale measures for $S$ with respect to $(\Omega, \mathbb{G}, P)$. If we define

$$\bar{X}_t := \operatorname*{ess\,sup}_{Q \in \mathcal{Q}} E^Q \big[ H_\delta \big| \mathcal{G}_t \big] \tag{4.10}$$

for $t \in [0, T]$, then it is known that for some $\bar{\pi} \in \mathcal{A}^{\mathbb{G}}(U_0)$ and an increasing optional process $\bar{C}$ with $\bar{C}_0 = 0$ we have

$$\bar{X}_t = U_0 + \int_0^t \bar{\pi}_s dS_s - \bar{C}_t. \tag{4.11}$$

This decomposition is called the *optional decomposition* of $H_\delta$, see El Karoui and Quenez [12] or Karatzas [22].

Using the structure of $\mathcal{Q}$ we provide a useful representation for $U_0$. Nakano [42] has a similar result for the case $n = 1$ and $\delta \in [0, 1]$. In the following, the expectation $E[\rho_T^* H]$ is denoted by $E^*[H]$.

**Lemma 4.6.** *If $E^*[H] < +\infty$ then $U_0 = E^*[H]$.*

*Proof.* For simplicity of notations, we suppose that $n = 2$, $\delta_1 \neq 0$ and $\delta_2 = 0$, other cases can be treated similarly. In this case, $\{H_\delta > 0\} = \{H > 0\} \cap \{\tau_2 >$

$T\}$ and

$$
\begin{aligned}
U_0 &= \sup_{\kappa\in\mathcal{D}^2} E\big[\rho^*_T\rho^\kappa_T H_\delta\big] \\
&= \sup_{\kappa\in\mathcal{D}^2} \Big\{ E\big[\rho^*_T\rho^\kappa_T H 1_{\{\tau_2>T\}\cap\{\tau_1>T\}}\big] \\
&+ \delta_1 E\big[\rho^*_T\rho^\kappa_T H 1_{\{\tau_2>T\}}(1-1_{\{\tau_1>T\}})\big]\Big\} \\
&= \sup_{\kappa\in\mathcal{D}^2} \Big\{ \delta_1 E\big[\rho^*_T\rho^\kappa_T H 1_{\{\tau_2>T\}}\big] \\
&+ (1-\delta_1) E\big[\rho^*_T\rho^\kappa_T H 1_{\{\tau_2>T\}\cap\{\tau_1>T\}}\big]\Big\}.
\end{aligned} \tag{4.12}
$$

Now, let us consider $\kappa = (\kappa^1, \kappa^2) \in \mathcal{D}^2$ with $\kappa^1 \in \mathcal{D}$ arbitrary but $\kappa^2 \in \mathcal{D}$ constant such that $\kappa^2 \searrow -1$. Then we have

$$
\begin{aligned}
U_0 \geq \lim_{\kappa^2\searrow -1} &\Big\{ \delta_1 E\big[\rho^*_T\rho^{\kappa^1}_T\rho^{\kappa^2}_T H 1_{\{\tau_2>T\}}\big] \\
&+ (1-\delta_1) E\big[\rho^*_T\rho^{\kappa^1}_T\rho^{\kappa^2}_T H 1_{\{\tau_2>T\}\cap\{\tau_1>T\}}\big]\Big\} \\
&= \delta_1 E\Big[\rho^*_T\rho^{\kappa^1}_T \exp\Big(\int_0^T \mu_s^2 ds\Big) H 1_{\{\tau_2>T\}}\Big] \\
&+ (1-\delta_1) E\Big[\rho^*_T\rho^{\kappa^1}_T \exp\Big(\int_0^T \mu_s^2 ds\Big) H 1_{\{\tau_2>T\}\cap\{\tau_1>T\}}\Big].
\end{aligned} \tag{4.13}
$$

We work on each term of the right-hand side of (4.13), separately:

$$
E\big[\rho^*_T\rho^{\kappa^1}_T H 1_{\{\tau_2>T\}}\big] = E\Big[\rho^{\kappa^1}_T E\big[\rho^*_T H 1_{\{\tau_2>T\}} | \mathcal{F}_T \vee \mathcal{H}^2_T\big]\Big]. \tag{4.14}
$$

However, by Kusuoka [31], $\big(\rho^{\kappa^1}_t\big)_{t\in[0,T]}$ and $\Big(E\big[\rho^*_T H 1_{\{\tau_2>T\}} | \mathcal{F}_t \vee \mathcal{H}^2_t\big]\Big)_{t\in[0,T]}$ are two orthogonal $(\mathbb{G}, P)$-local martingales. This implies that their product is a $(\mathbb{G}, P)$-local martingale as well. By considering $(H\wedge m)_{m\geq 1}$ and then using the monotone convergence theorem as $m \to +\infty$, Eq. (4.14) becomes

$$
\begin{aligned}
E\Big[\rho^{\kappa^1}_T E\big[\rho^*_T H 1_{\{\tau_2>T\}} | \mathcal{F}_T \vee \mathcal{H}^2_T\big] \big| \mathcal{G}_0\Big] &= \rho^{\kappa^1}_0 E\big[\rho^*_T H 1_{\{\tau_2>T\}} | \mathcal{F}_0 \vee \mathcal{H}^2_0\big] \\
&= \exp\Big(-\int_0^T \mu_s^2 ds\Big) E^*[H].
\end{aligned} \tag{4.15}
$$

On the other hand, if we first apply (4.15) in (4.13), and then choose $\kappa^1 \in \mathcal{D}$ constant such that $\kappa^1 \searrow -1$ in the second term. Then

$$U_0 \geq \delta_1 E^*[H] + (1-\delta_1)E^*[H] = E^*[H], \tag{4.16}$$

where we used the fact that $\mathcal{F}_T$ and $\{\tau_i\}_{i=1,2}$ are independent and $P\Big(\bigcap_{i=1}^{2}\{\tau_i > T\}\Big) = \exp\Big(-\sum_{i=1}^{2}\int_0^T \mu_s^i ds\Big)$.

To prove the reverse inequality in (4.16), notice that since $\kappa > -1$, we have

$$\begin{aligned} \rho_T^{\kappa} 1_{\{\tau_2 > T\}} &= \rho_T^{\kappa^1} \exp\Big(-\int_0^T \kappa_s^2 \mu_s^2 ds\Big) 1_{\{\tau_2 > T\}} \\ &\leq \rho_T^{\kappa^1} \exp\Big(\int_0^T \mu_s^2 ds\Big) 1_{\{\tau_2 > T\}}, \end{aligned} \tag{4.17}$$

and

$$\rho_T^{\kappa} 1_{\{\tau_2 > T\} \cap \{\tau_1 > T\}} \leq \exp\Big(\sum_{i=1}^{2}\int_0^T \mu_s^i ds\Big) 1_{\{\tau_2 > T\} \cap \{\tau_1 > T\}} \tag{4.18}$$

for any $\kappa = (\kappa^1, \kappa^2) \in \mathcal{D}^2$. By (4.12) and inequalities (4.17) and (4.18), we get

$$\begin{aligned} U_0 &\leq \delta_1 E[\rho_T^* \rho_T^{\kappa^1} H] + (1-\delta_1)E^*[H] \\ &= \delta_1 E^*[H] + (1-\delta_1)E^*[H] = E^*[H]. \end{aligned} \tag{4.19}$$

To get the first equality in (4.19), we need to repeat the arguments applied to $E\big[\rho_T^* \rho_T^{\kappa^1} H 1_{\{\tau_2 > T\}}\big]$ in (4.14) and (4.15) for the case of $E[\rho_T^* \rho_T^{\kappa^1} H]$. □

## 4.2 Formulation of the Problem and Main Results

Clearly, the client is not willing to pay $E^*[H]$ for buying $H_\delta$. S/he can buy $H$ in the default-free market for this price and receive $H$, $P$-a.s., without the risk of default. To offer a competitive price which is reasonable for the client, the premium charged by the company should be less than $E^*[H]$. However, if the premium is less than $E^*[H]$ there is a possibility of shortfall for the company (still there is the possibility of payment $H$). This naturally leads us to the problem of minimizing the *shortfall risk* with an *initial capital* $\tilde{u} < E^*[H]$. In this case, we consider the problem of minimizing the expectation of the shortfall risk weighted by a loss function. More precisely, we want to solve the following problem

$$\min_{\substack{\pi \in \mathcal{A}^{\mathbb{G}}(v) \\ v \leq \tilde{u}}} E\big[l\big((H_\delta - V_T^{v,\pi})^+\big)\big]. \tag{4.20}$$

Due to Proposition 3.1 and Theorem 3.2 of Follmer and Leukert [15], there exists a solution for (4.20). We recall these results in A.

In the next lemma, we show that the solution to the minimization problem (1.1) solves a maximization problem. Then applying this lemma and the results of Cvitanić and Karatzas [8], we can find a closed form expression for the solution, $\tilde{\varphi}$.

**Lemma 4.7.** *Let us define* $\mathcal{R} := \big\{\varphi : \Omega \longrightarrow [0,1] \;\big|\; \varphi \in \mathcal{G}_T\big\}$*, and* $\tilde{\varphi} \in \mathcal{R}$ *to be determined from Theorem 1. Then the random variable* $\tilde{\varphi}$ *is a solution to the following maximization problem*

$$\max_{\varphi \in \mathcal{R}} E^{\bar{P}}[\varphi] \tag{4.21}$$

*subject to the constraint*

$$\sup_{\kappa \in \mathcal{D}^n} E\big[\rho_T^* \rho_T^\kappa \varphi H_\delta\big] \leq \tilde{u}, \tag{4.22}$$

*where*

$$\frac{d\bar{P}}{dP} := \begin{cases} \dfrac{l'\big((1-\tilde{\varphi})H_\delta\big)H_\delta}{E\big[l'\big((1-\tilde{\varphi})H_\delta\big)H_\delta\big]} & ;\ on\ \big\{H_\delta > 0\big\} \cap \big\{\tilde{\varphi} \neq 1\big\} \\ \\ 0 & ;\ otherwise \end{cases} \tag{4.23}$$

*Proof.* Let us define the function $F : L^1(P) \longrightarrow \mathbb{R}$ as $F(\psi) := E\big[l\big((1-\psi)H_\delta\big)\big]$ for $\psi \in L^1(P)$. Then Theorem 1 indicates that $\tilde{\varphi}$ minimizes $F$ over the convex set

$$\mathcal{R}(\tilde{u}) := \big\{\varphi \in \mathcal{R} \ \big|\ \sup_{\kappa \in \mathcal{D}^n} E\big[\rho_T^* \rho_T^\kappa \varphi H_\delta\big] \leq \tilde{u}\big\} \subset L^1(P).$$

As a consequence, above and Theorem 7.4.2 of Luenberger [34] imply that the following inequality holds for the Gateaux derivative of $F$ at $\tilde{\varphi}$ with the increment $\varphi - \tilde{\varphi}$

$$\frac{dF\big(\tilde{\varphi} + t(\varphi - \tilde{\varphi})\big)}{dt}\Big|_{t=0} = DF\big(\tilde{\varphi}; \varphi - \tilde{\varphi}\big) \geqslant 0, \qquad \text{for all } \varphi \in \mathcal{R}(\tilde{u}).$$

Using monotone convergence theorem we have

$$-\,E\big[l'\big((1-\tilde{\varphi})H_\delta\big)(\varphi - \tilde{\varphi})H_\delta\big] \geq 0, \tag{4.24}$$

for all $\varphi \in \mathcal{R}(\tilde{u})$. Equation (4.24) implies the following crucial inequality

$$E\big[l'\big((1-\tilde{\varphi})H_\delta\big)\tilde{\varphi}H_\delta\big] \geq E\big[l'\big((1-\tilde{\varphi})H_\delta\big)\varphi H_\delta\big].$$

This inequality proves the optimality of $\tilde{\varphi}$ to the desired problem. □

**Remark 4.8.** *Lemma 4.7 can also be proved by the method of Karlin (see Karlin [24]) that was used by Follmer and Leukert [15] in the proof of Theo-*

*rem* 5.1*, therein. In fact, both ideas reduce to the same calculation analogous to Eq.* (4.24)*.*

Let $\mathcal{L}$ be the closed convex hull of $\{\rho_T^\kappa\}_{\kappa\in\mathcal{D}^n}$ under $P$-a.s. convergence. It is clear that $\mathcal{L}$ is a convex, bounded set in $L^1(P)$ such that

$$\{\rho_T^\kappa\}_{\kappa\in\mathcal{D}^n} \subseteq \mathcal{L}.$$

Now notice that

$$\begin{aligned} E^{\bar{P}}[\varphi] &= E\big[\varphi\big(\frac{d\bar{P}}{dP} - z\rho_T^* L H_\delta\big)\big] + E\big[z\rho_T^* L\varphi H_\delta\big] \\ &\leqslant E\big[\big(\frac{d\bar{P}}{dP} - z\rho_T^* L H_\delta\big)^+\big] + z\tilde{u}, \end{aligned} \tag{4.25}$$

and

$$E\big[\rho_T^* L\varphi H_\delta\big] \leqslant \tilde{u} \tag{4.26}$$

for all $\varphi \in \mathcal{R}(\tilde{u})$, $L \in \mathcal{L}$ and $z > 0$. To get inequality (4.26), we applied Fatou's lemma and (4.22).

By (4.25) and (4.26), we introduce the dual problem of primal problem (4.21)–(4.22) as follows:

$$V_*(\tilde{u}) := \inf_{\substack{z>0 \\ L\in\mathcal{L}}} \Big\{\tilde{u}z + E\Big[\big(\frac{d\bar{P}}{dP} - z\rho_T^* L H_\delta\big)^+\Big]\Big\}. \tag{4.27}$$

Cvitanić and Karatzas [8] adapted the techniques of nonsmooth convex analysis along with a theorem of Komlós (Komlós [28]) to prove that there exists a solution $(\tilde{z}, \tilde{L}) \in \mathbb{R}^+ \times \mathcal{L}$ to this dual problem. Using inequality (4.25), they showed that $\tilde{\varphi}$ has the following representation

$$\tilde{\varphi} = 1_{\big\{\tilde{z}\rho_T^*\tilde{L}H_\delta < \frac{d\bar{P}}{dP}\big\}} + \tilde{B}1_{\big\{\tilde{z}\rho_T^*\tilde{L}H_\delta = \frac{d\bar{P}}{dP}\big\}}, \qquad P\text{-a.s.} \tag{4.28}$$

where $\tilde{B}$ is a $\mathcal{G}_T$-measurable random variable with values in $[0,1]$. In addition, $\tilde{\varphi}$, $\tilde{L}$, and $\tilde{z}$ satisfy the following conditions

$$E\left[\rho_T^* \tilde{L} \tilde{\varphi} H_\delta\right] = \tilde{u}, \tag{4.29}$$

and if we introduce

$$\tilde{V}(\tilde{z}) := \inf_{L \in \mathcal{L}} E\Big[\big(\frac{d\bar{P}}{dP} - \tilde{z}\rho_T^* L H_\delta\big)^+\Big], \tag{4.30}$$

then $\tilde{V}(\tilde{z}) = E\Big[\big(\frac{d\bar{P}}{dP} - \tilde{z}\rho_T^* \tilde{L} H_\delta\big)^+\Big]$.

In B, for the convenience of reader, we summarize the algorithm of finding $(\tilde{z}, \tilde{L})$. In the next lemma, we provide a more explicit description of $\tilde{L}$.

**Lemma 4.9.** *Consider the dual problem (4.27). For $\tilde{u} < U_0$, let $\tilde{V}(\tilde{z})$, $\tilde{z}$ and $\tilde{L}$ to be defined as above. Then*

1. *$\tilde{V}$ vanishes, more precisely:*

$$\tilde{V}(\tilde{z}) = E\big[\big(\frac{d\bar{P}}{dP} - \tilde{z}\rho_T^* \tilde{L} H_\delta\big)^+\big] = 0, \tag{4.31}$$

*and*

$$E^{\bar{P}}[\tilde{\varphi}] = V_*(\tilde{u}) = \tilde{u}\tilde{z}. \tag{4.32}$$

2. *Moreover, we have:*

$$\tilde{L} \prod_{i=1}^n 1_{\{\tau_i > T\}} = \Big(\prod_{i=1}^n 1_{\{\tau_i > T\}}\Big) \exp\Big(\sum_{i=1}^n \int_0^T \mu_s^i ds\Big). \tag{4.33}$$

*Proof.* *(1)* First, notice that by $l(0) = 0$ we can assume $\tilde{\varphi} = 1$ on the set $\{H_\delta = 0\}$. Now taking into account (4.28), we describe $\tilde{\varphi}$ on $\{H_\delta > 0\}$.

From representation (4.28), we have $\tilde{\varphi} = 1$ on

$$A_1 := \{\tilde{z}\rho_T^* \tilde{L} H_\delta < \frac{d\bar{P}}{dP}\}, \tag{4.34}$$

and $\tilde{\varphi} = \tilde{B}$ on

$$A_2 := \{\tilde{z}\rho_T^* \tilde{L} H_\delta = \frac{d\bar{P}}{dP}\} = \{\tilde{\alpha}\tilde{z}\rho_T^* \tilde{L} = l'\big((1-\tilde{\varphi})H_\delta\big)\}, \tag{4.35}$$

where $\tilde{\alpha} := E\big[l'\big((1-\tilde{\varphi})H_\delta\big)H_\delta\big]$. Furthermore, it is clear that $\tilde{\varphi} = 0$ on

$$A_3 := \big(A_1 \cup A_2\big)^c. \tag{4.36}$$

Let us recall that we defined $\frac{d\bar{P}}{dP} = 0$ on $\tilde{\varphi} = 1$. On the other hand, we know all $\tilde{z}$, $\rho_T^*$, $\tilde{L}$ and $H_\delta$ are nonnegative. This implies that $A_1 = \emptyset$, and consequently $\tilde{V}(\tilde{z}) = 0$. Equation (4.32) is now obvious from (4.31) and (2.1).

*(2)* Without loss of generality, we suppose that $\delta = (\delta_1, ..., \delta_n) \in [0,1)^n$. In addition, let $j \in \{1, ..., n\}$ to be chosen such that $\delta_1, ..., \delta_j \in (0,1)$ and $\delta_i = 0$ for all $i = j+1, ..., n$ (up to a rearrangement of $\tau_i$'s). If $\delta_i = 0$ for all $i = 1, ..., n$ we take $j = 0$. For the case that $\delta \equiv 1$ see Remark 4.12.

Now, we split the proof into two cases:

*(i)* $\delta_i = 0$ for all $i = 1, ..., n$. We already proved that $A_1 = \emptyset$ and $\tilde{\varphi} = 0$ on $A_3$. In this case, we only need to investigate (4.35) on $\{H_\delta > 0\} = \{H > 0\} \cap \big(\bigcap_{i=1}^{n} \{\tau_i > T\}\big)$. Since for all $\kappa \in \mathcal{D}^n$ we have

$\kappa > -1$, then

$$\rho_T^\kappa \prod_{i=1}^n 1_{\{\tau_i > T\}} \le \bar{L} := \Big(\prod_{i=1}^n 1_{\{\tau_i > T\}}\Big) \exp\Big(\sum_{i=1}^n \int_0^T \mu_s^i ds\Big) \\ = \lim_{\substack{\bar{\kappa} \searrow -1 \\ \bar{\kappa} \text{ consatnt}}} \rho_T^{\bar{\kappa}} \prod_{i=1}^n 1_{\{\tau_i > T\}} \in \mathcal{L}. \tag{4.37}$$

In particular, this implies

$$\tilde{L} \prod_{i=1}^n 1_{\{\tau_i > T\}} \le \bar{L}. \tag{4.38}$$

This inequality gives us

$$0 \le E\Big[\Big(\frac{d\bar{P}}{dP} - \tilde{z}\rho_T^* \bar{L} H_\delta\Big)^+ \prod_{i=1}^n 1_{\{\tau_i > T\}}\Big] \le \tilde{V}(\tilde{z}). \tag{4.39}$$

Combining with part (1), $\bar{L}$ is, in fact, a solution to $\tilde{V}(\tilde{z})$.

*(ii)* For some $j \in \{1, ..., n\}$, $\delta_1, ..., \delta_j \in (0, 1)$ and $\delta_i = 0$ for all $i = j+1, ..., n$. In this case, we have

$$\{H_\delta > 0\} = \{H > 0\} \cap \Big(\bigcap_{i=j+1}^n \{\tau_i > T\}\Big).$$

By (4.37) and (4.38), similar to (4.39) we can show that:

$$0 \le E\Big[\Big(\frac{d\bar{P}}{dP} - \tilde{z}\rho_T^* \bar{L} H_\delta\Big)^+ \prod_{i=1}^n 1_{\{\tau_i > T\}}\Big] \\ \le E\Big[\Big(\frac{d\bar{P}}{dP} - \tilde{z}\rho_T^* \tilde{L} H_\delta\Big)^+ \prod_{i=1}^n 1_{\{\tau_i > T\}}\Big] \le \tilde{V}(\tilde{z}). \tag{4.40}$$

Using this and a similar argument as in case $(i)$, (4.33) is proved.

□

**Remark 4.10.** *Let us point out that if the recovery rate* $\delta \neq 0$*,* $\tilde{L}$ *does not necessarily coincide with* $\left(\prod_{i=1}^{n} 1_{\{\tau_i > T\}}\right) \exp\left(\sum_{i=1}^{n} \int_0^T \mu_s^i ds\right)$ *on* $\{H_\delta > 0\}$*. In this case, inequality (4.38) does not hold on* $\left(\bigcap_{i=1}^{j}\{\tau_i > T\}\right)^c \cap \left(\bigcap_{i=j+1}^{n}\{\tau_i > T\}\right) \cap \{H > 0\} \subset \{H_\delta > 0\}$*. For instance, for* $n = 1$ *and* $\delta \neq 0$*, the family of* $\mathcal{G}_T$*-measurable random variables* $\rho_T^\kappa 1_{\{\tau \leq T\}} = (1+\kappa_\tau)\exp\left(-\int_0^\tau \kappa_s \mu_s ds\right) 1_{\{\tau \leq T\}}$*, for* $\kappa > -1$*, does not possess an upper bound. However the modified option* $\tilde{\varphi} H_\delta$*, with* $\tilde{\varphi}$ *given by (4.28), still provides an implicit solution for the efficient hedging problem (4.20).*

*In the case of* $\delta \equiv 0$*, (4.33) fully describes* $\tilde{L}$ *on* $\bigcap_{i=1}^{n}\{\tau_i > T\} \supseteq \{H_\delta > 0\}$*. Considering this discussion, we will find an explicit representation for* $\tilde{\varphi}$ *when the recovery rate* $\delta \equiv 0$*.*

Henceforth, in this chapter, we assume that the recovery rate $\delta_i = 0$ for all $i = 1, ..., n$. In particular, let us define

$$H_0 := H \prod_{i=1}^{n} 1_{\{\tau_i > T\}}, \tag{4.41}$$

where $H$ is a nonnegative $\mathcal{F}_T$-measurable random variable.

**Theorem 4.11.** *Under assumption 4.5 on the loss function* $l$*, the optimal randomized test* $\tilde{\varphi}$ *described in Theorem 1 is given by*

$$\tilde{\varphi} = \begin{cases} 1 - \left(I(\tilde{\lambda}\rho_T^*)/H_0\right) \wedge 1 & ; \{H_0 > 0\} \\ 1 & ; \{H_0 = 0\} \end{cases} \tag{4.42}$$

*where the constant* $\tilde{\lambda}$ *can be determined by the constraint*

$$E^*[H - I(\tilde{\lambda}\rho_T^*) \wedge H] = \tilde{u}. \tag{4.43}$$

*Moreover, $(\tilde{u},\tilde{\pi})$ obtained from the optional decomposition of the modified claim $\tilde{\varphi}H_0$ gives us the the optimal strategy for the efficient hedging problem (4.20).*

*Proof.* From (4.28) and Lemma 4.9 we can see that

$$\tilde{\varphi} = \tilde{B}1_{\left\{\tilde{z}\rho_T^*\tilde{L}H_0=\frac{d\bar{P}}{dP}\right\}},$$

on $\{H_0 > 0\}$ and $\tilde{\varphi} = 1$ on $\{H_0 = 0\}$. Since $\delta \equiv 0$, recall that $\tilde{L} = \bar{L}$ by (4.37) and Remark 4.10. By some straightforward calculations, this becomes

$$\tilde{\varphi} = \begin{cases} 1 - I(\tilde{\lambda}\rho_T^*)/H_0 & ;\big\{\tilde{\varphi} = 1 - I(\tilde{\lambda}\rho_T^*)/H_0\big\} \cap \{H_0 > 0\} \\ 0 & ;\big\{\tilde{\varphi} > 1 - I(\tilde{\lambda}\rho_T^*)/H_0\big\} \\ 1 & ;\{H_0 = 0\} \end{cases} \tag{4.44}$$

where $\tilde{\lambda} := \tilde{\alpha}\tilde{z}\exp\Big(\sum_{i=1}^{n}\int_0^T \mu_s^i ds\Big)$ is a constant. (4.44) is still an implicit form for $\tilde{\varphi}$. To find an explicit representation, we exploit a similar idea to Follmer and Leukert [15]. To do so, for $\lambda > 0$, let us define

$$\varphi_\lambda := \begin{cases} 1 - \big(I(\lambda\rho_T^*)/H_0\big) \wedge 1 & ;\{H_0 > 0\} \\ 1 & ;\{H_0 = 0\} \end{cases} \tag{4.45}$$

Because $\{\tau_i\}_{i=1,\dots,n}$ and $\mathcal{F}_T$ are independent, we get

$$E\big[\rho_T^*\tilde{L}\varphi_\lambda H_0\big] = E^*\big[H - I(\lambda\rho_T^*) \wedge H\big]. \tag{4.46}$$

By dominated convergence theorem, it is easy to see that $E^*\big[H - I(\lambda\rho_T^*) \wedge H\big]$ decreases continuously from $E^*[H]$ to zero as $\lambda$ increases from 0 to $+\infty$. Thus, for $\tilde{u} \in \big(0, E^*[H]\big)$, there exists $\tilde{\lambda} > 0$ such that

$$E\big[\rho_T^* \tilde{L} \varphi_{\tilde{\lambda}} H_0\big] = \tilde{u}. \tag{4.47}$$

Let us consider $\varphi_{\tilde{\lambda}}$ defined by (4.45) and $\tilde{\lambda}$ chosen by (4.47). In the following, we show that $\varphi_{\tilde{\lambda}}$, in fact, satisfies (4.44).

On the set $\big\{\varphi_{\tilde{\lambda}} = 1 - I(\tilde{\lambda}\rho_T^*)/H_0\big\} \cap \{H_0 > 0\}$, it is clear that

$$I(\tilde{\lambda}\rho_T^*)/H_0 = 1 - \varphi_{\tilde{\lambda}} \leq 1.$$

This implies $\varphi_{\tilde{\lambda}} = 1 - \big(I(\tilde{\lambda}\rho_T^*)/H_0\big) \wedge 1 = 1 - I(\tilde{\lambda}\rho_T^*)/H_0$, same as (4.44). Similarly, if $\varphi_{\tilde{\lambda}} > 1 - I(\tilde{\lambda}\rho_T^*)/H_0$ then one can see that

$$\big(I(\tilde{\lambda}\rho_T^*)/H_0\big) \wedge 1 = 1 - \varphi_{\tilde{\lambda}} < I(\tilde{\lambda}\rho_T^*)/H_0, \tag{4.48}$$

where the equality comes from the definition of $\varphi_{\tilde{\lambda}}$, Eq. (4.44). This means $\big(I(\tilde{\lambda}\rho_T^*)/H_0\big) \wedge 1 = 1$, and again by the definition of $\varphi_{\tilde{\lambda}}$, it gives us $\varphi_{\tilde{\lambda}} = 0$. Finally, if we suppose that $\big\{\varphi_{\tilde{\lambda}} < 1 - I(\tilde{\lambda}\rho_T^*)/H_0\big\} \neq \emptyset$ we get the following contradiction

$$\big(I(\tilde{\lambda}\rho_T^*)/H_0\big) \wedge 1 > I(\tilde{\lambda}\rho_T^*)/H_0.$$

The last statement of the theorem is an immediate consequence of Theorem 1 part (2). □

**Remark 4.12.** *$\delta \equiv 1$ implies that $H_\delta = H$ is default free in a complete market. In other words, for all $\tau_i$'s we have $1_{\{\tau_i > T\}} \equiv 1$. By repeating the* *same arguments as above on $\{H > 0\}$ and $\{H = 0\}$ we get*

$$\tilde{\varphi} = \begin{cases} 1 - \big(I(\tilde{\lambda}\rho_T^*)/H\big) \wedge 1 & ; \{H > 0\} \\ 1 & ; \{H = 0\} \end{cases} \tag{4.49}$$

*such that $E^*[\tilde{\varphi}H] = \tilde{u}$. In this case, our result is consistent with Theorem 5.1 of Follmer and Leukert [15].*

**Corollary 4.13.** *For $\delta \equiv 0$ and given $\tilde{u} \in (0, E^*[H])$, the following conclusion holds:*

*Consider $l(x) = \dfrac{x^p}{p}$ for some $p > 1$ and let $\tilde{\varphi}_p$ to be the corresponding $\tilde{\varphi}$ represented in Theorem 4.11. Then there exists $c > 0$ such that*

$$(1 - \tilde{\varphi}_p)H_0 1_{\{H_0>0\}} \longrightarrow (c \wedge H) \prod_{i=1}^{n} 1_{\{\tau_i>T\}} 1_{\{H>0\}} \tag{4.50}$$

*almost sure and also w.r.t $L^1(P^*)$-norm, as $p \longrightarrow +\infty$. The constant $c$ is determined by $E^*[c \wedge H] = E^*[H] - \tilde{u}$.*

*Proof.* Let us consider $\tilde{\lambda}_p$ as the corresponding $\tilde{\lambda}$ in Theorem 4.11. First of all, similar to Follmer and Leukert [15], Proposition 5.3, we can show that for some $c > 0$

$$\lim_{p \to +\infty} \tilde{\lambda}_p^{\frac{1}{p-1}} = c.$$

Due to Theorem 4.11, we have

$$(1 - \tilde{\varphi}_p)H_0 1_{\{H_0>0\}} = \big(\tilde{\lambda}_p^{\frac{1}{p-1}} (\rho_T^*)^{\frac{1}{p-1}} \wedge H\big) \prod_{i=1}^{n} 1_{\{\tau_i>T\}} 1_{\{H>0\}},$$

and in addition by (4.29)

$$\begin{aligned}\tilde{u} &= E^*\left[\tilde{L}\tilde{\varphi}_p H_0 1_{\{H_0>0\}}\right] \\ &= \exp\Big(\sum_{i=1}^n \int_0^T \mu_s^i ds\Big) E^*\Big[\Big(H - \tilde{\lambda}_p^{\frac{1}{p-1}} (\rho_T^*)^{\frac{1}{p-1}} \wedge H\Big) \prod_{i=1}^n 1_{\{\tau_i>T\}}\Big].\end{aligned}$$

Since $\lim_{p\to+\infty} (\rho_T^*)^{\frac{1}{p-1}} = 1$, $P$-a.s., the above equations together with dominated convergence theorem prove the corollary. □

Now, let $ESR(\tilde{u})$ to be the minimum of the expectation of the shortfall risk for the default free contingent claim $H$ and initial capital $\tilde{u} < U_0 = E^*[H]$, defined as (4.20) . Similarly, define $ESR^\tau(\tilde{u})$ as the minimum of the expectation of shortfall risk for $H_0$ and the available initial capital $\tilde{u}$. The next theorem provides a useful relation between $ESR(\tilde{u})$ and $ESR^\tau(\tilde{u})$.

**Theorem 4.14.** *Let $\tilde{u} \in (0, E^*[H])$ to be given, then the following properties hold:*

1. *We have*
$$ESR^\tau(\tilde{u}) = \exp\Big(-\sum_{i=1}^n \int_0^T \mu_s^i ds\Big) ESR(\tilde{u}). \tag{4.51}$$

2. *Suppose that $\hat{\pi} \in \mathcal{A}^{\mathbb{F}}(\tilde{u})$ is the optimal trading strategy that attains $ESR(\tilde{u})$ in the default-free market (4.1). Then the optimal trading strategy associated to $ESR^\tau(\tilde{u})$ is given by*
$$(\tilde{\pi}_t)_{t\in[0,T]} := \Big(\hat{\pi}_t \prod_{i=1}^n 1_{\{\tau_i \geq t\}}\Big)_{t\in[0,T]} \in \mathcal{A}^{\mathbb{G}}(\tilde{u}). \tag{4.52}$$

*Proof.* 1. By the results of Theorem 4.11 and Theorem 1:

$$\begin{aligned} ESR^{\tau}(\tilde{u}) &= E\big[l\big((1-\tilde{\varphi})H_0\big)\big] \\ &= E\Big[l\Big((I(\tilde{\lambda}\rho_T^*) \wedge H)1_{\{H>0\}} \prod_{i=1}^{n} 1_{\{\tau_i>T\}}\Big)\Big] \\ &= P\Big(\bigcap_{i=1}^{n}\{\tau_i > T\}\Big) E\big[l\big(I(\tilde{\lambda}\rho_T^*) \wedge H\big)1_{\{H>0\}}\big]. \end{aligned} \tag{4.53}$$

From (4.43), the constant $\tilde{\lambda}$ can be determined from

$$\tilde{u} = E^*\Big[\big(H - I(\tilde{\lambda}\rho_T^*) \wedge H\big)1_{\{H>0\}}\Big], \tag{4.54}$$

keeping in mind this equation, Remark 4.12 implies

$$ESR(\tilde{u}) = E\big[l\big(I(\tilde{\lambda}\rho_T^*) \wedge H\big)1_{\{H>0\}}\big]. \tag{4.55}$$

Now, comparing (4.55) with (4.53) verifies equation (4.51).

2. Since $\hat{\pi} \in \mathcal{A}^{\mathbb{F}}(\tilde{u})$ is a solution to $ESR(\tilde{u})$, by Theorem 1 and Remark 4.12

$$V_T^{\tilde{u},\hat{\pi}} = H - I(\tilde{\lambda}\rho_T^*) \wedge H, \tag{4.56}$$

where $\tilde{\lambda}$ satisfies (4.54). On the other hand, from Theorem 4.11 we know that the optional decomposition of $\tilde{\varphi}H_0$ gives us the the optimal solution corresponding to $ESR^{\tau}(\tilde{u})$. It is easy to see that

$$\begin{aligned} \tilde{\varphi}H_0 &= \Big(H - I(\tilde{\lambda}\rho_T^*) \wedge H\Big) \prod_{i=1}^{n} 1_{\{\tau_i>T\}} \\ &= V_T^{\tilde{u},\hat{\pi}} \prod_{i=1}^{n} 1_{\{\tau_i>T\}}. \end{aligned}$$

Keeping in mind Assumptions (4.1) and (4.2), we apply the multidimensional Ito formula for $\left(V_t^{\tilde{u},\hat{\pi}} \prod_{i=1}^{n} 1_{\{\tau_i > t\}}\right)_{t\in[0,T]}$. Then the optional decomposition of $\tilde{\varphi} H_0$ is given as follows:

$$\begin{aligned}\tilde{\varphi} H_0 = \tilde{u} + \int_0^T \hat{\pi}_t \prod_{i=1}^{n} 1_{\{\tau_i \geq t\}} dS_t - V_{\tau_1}^{\tilde{u},\hat{\pi}} 1_{\{\tau_1 \leq T\}} \prod_{i=2}^{n} 1_{\{\tau_i > T\}} \\ - \sum_{i=1}^{n-1} \Big( \int_0^{\tau_{i+1}} \hat{\pi}_t \prod_{j=1}^{i} 1_{\{\tau_j > t\}} dS_t \Big) 1_{\{\tau_{i+1} \leq T\}} \prod_{k=i+2}^{n} 1_{\{\tau_k > T\}},\end{aligned}$$

where we set $\prod_{k=i+2}^{n} 1_{\{\tau_k > T\}} \equiv 1$ for $i = n-1$. We used the fact that the continuity of $S$ allows us to write $\int_0^T \hat{\pi}_t \prod_{i=1}^{n} 1_{\{\tau_i > t\}} dS_t = \int_0^T \hat{\pi}_t \prod_{i=1}^{n} 1_{\{\tau_i \geq t\}} dS_t$. Using the decomposition of $\mathbb{G}$-predictable processes in terms of $\tau_i$'s and the $\mathbb{F}$-predictable processes, see Pham [44] Remark 2.1, clearly $\left(\hat{\pi}_t \prod_{i=1}^{n} 1_{\{\tau_i \geq t\}}\right)_{t\in[0,T]}$ is a $\mathbb{G}$-predictable process. In addition, we recall that $\mathbb{F}$-admissibility of $\hat{\pi}$ implies

$$\tilde{u} + \int_0^\theta \hat{\pi}_s dS_s \geq 0, \qquad P\text{-a.s.}$$

for all $\theta \in [0,T]$. Now, since $\left(\bigwedge_{i=1}^n \tau_i\right) \wedge t \in [0,T]$ almost sure we have

$$\tilde{u} + \int_0^t \hat{\pi}_s \prod_{i=1}^{n} 1_{\{\tau_i \geq s\}} dS_s = \tilde{u} + \int_0^{(\bigwedge_{i=1}^n \tau_i)\wedge t} \hat{\pi}_s dS_s \geq 0, \qquad P\text{-a.s.}$$

for all $t \in [0,T]$. This argument proves that, in fact, $\tilde{\pi} \in \mathcal{A}^{\mathbb{G}}(\tilde{u})$.

□

In fact, Theorem 4.14 reduces the efficient hedging problem in the defaultable market to the corresponding problem in the default-free market. The advantage of this result is to avoid the complication of working with the optional decomposition of $\tilde{\varphi} H_0$ in the enlarged filtration $\mathbb{G}$. By equations (4.51) and (4.52), for $\delta \equiv 0$ we only need to find the perfect hedging strategy of

$H\Big(1-\big(I(\tilde\lambda\rho_T^*)/H\big)\wedge 1\Big)\in\mathcal{F}_T$ to solve problem (4.20).

In the next lemma, we investigate smoothness of the minimum of shortfall risk as a function of initial capital.

**Lemma 4.15.** *Let us consider* $ESR^\tau:\big(0,E^*[H]\big)\longrightarrow\big(E[l(H_0)],0\big)$ *as a function of available initial capital* $\tilde u$*. Then* $ESR^\tau\in C^1\Big(\big(0,E^*[H]\big)\Big)$ *and*

$$\frac{dESR^\tau}{du}(\tilde u)=-\tilde z E\big[l'\big((1-\tilde\varphi)H_0\big)H_0\big] \tag{4.57}$$

*for all* $\tilde u\in\big(0,E^*[H]\big)$*.*

*Proof.* Define $U:\Psi\longrightarrow\mathbb{R}$ as

$$U(\psi):=E^*[\tilde L\psi H_0]\ \text{ for }\ \psi\in\Psi,$$

where $\Psi:=\big\{\psi\in L^1(P)\ \big|\ E^*[\tilde L\psi H_0]<+\infty\big\}$. Consider $\tilde\varphi$ defined as (4.42), then by equation (4.29) we get

$$ESR^\tau(\tilde u)=ESR^\tau\big(U(\tilde\varphi)\big). \tag{4.58}$$

To proceed, our idea is to exploit Frechet derivative of $ESR^\tau$, and Gateaux derivative of $U$ and $ESR^\tau\circ u$. By equation (4.51), it is clear that $ESR^\tau\in C^1\big((0,E^*[H])\big)$ iff $ESR\in C^1\big((0,E^*[H])\big)$. We can apply Theorem 7.1 of Follmer and Leukert [15] to see that for $\delta\equiv 1$ (a complete market) $ESR\in C^1\big((0,E^*[H])\big)$. It is also known that $ESR^\tau\in C^1\big((0,E^*[H])\big)$ implies Frechet differentiability of $ESR^\tau$. Moreover, we can compute Gateaux derivative of function $U$ at $\tilde\varphi$ with the increment $\tilde\varphi$ as follows

$$\begin{aligned}DU(\tilde\varphi;\tilde\varphi)&=\frac{dU(\tilde\varphi+t\tilde\varphi)}{dt}\Big|_{t=0}\\&=E^*[\tilde L\tilde\varphi H_0]=\tilde u.\end{aligned} \tag{4.59}$$

By the above arguments, the Frechet derivative of $ESR^{\tau}$ exists and $U$ is Gateaux differentiable. Thus we can apply the chain rule to evaluate the Gateaux derivative of $ESR^{\tau} \circ U$

$$D(ESR^{\tau} \circ U)(\tilde{\varphi}; \tilde{\varphi}) = DESR^{\tau}\big(U(\tilde{\varphi}); DU(\tilde{\varphi}; \tilde{\varphi})\big), \tag{4.60}$$

see for instance Kurdila and Zabarankin [30]. On one hand, we have

$$\begin{aligned} D(ESR^{\tau} \circ U)(\tilde{\varphi}; \tilde{\varphi}) &= \frac{dE\big[l\big((1 - \tilde{\varphi} - t\tilde{\varphi})H_0\big)\big]}{dt}\Big|_{t=0} \\ &= -E\big[l'\big((1 - \tilde{\varphi})H_0\big)\tilde{\varphi}H_0\big] \\ &= -\tilde{\alpha}V_*(\tilde{u}) = -\tilde{\alpha}\tilde{u}\tilde{z}, \end{aligned} \tag{4.61}$$

we used Eq. (4.32) and also recall that $\tilde{\alpha} = E[l'\big((1 - \tilde{\varphi})H_0\big)H_0]$. On the other hand

$$\begin{aligned} DESR^{\tau}\big(U(\tilde{\varphi}); DU(\tilde{\varphi}; \tilde{\varphi})\big) &= DESR^{\tau}\big(\tilde{u}; \tilde{u}\big) \\ &= \frac{dESR^{\tau}(\tilde{u} + t\tilde{u})}{dt}\Big|_{t=0} \\ &= \tilde{u}\frac{dESR^{\tau}}{du}(\tilde{u}). \end{aligned} \tag{4.62}$$

Finally, combining (4.60), (4.61) and (4.62) together, we have

$$\frac{dESR^{\tau}}{du}(\tilde{u}) = -\tilde{\alpha}\tilde{z} < 0 \tag{4.63}$$

for all $\tilde{u} \in \big(0, E^*[H]\big)$.

□

With the help of the above lemma, we can provide more qualitative features of $\tilde{\varphi}$ and $\tilde{z}$ corresponding to maximization problem (4.21) and its dual problem (4.27), for $\delta \equiv 0$.

**Lemma 4.16.** *Let us consider $\tilde{z}$, $\tilde{\lambda}$, $\tilde{\alpha}$, and $\tilde{\varphi}$ (defined above) as functions* *of* $\tilde{u} \in \big(0, E^*[H]\big)$*. Assume that* $\{\tilde{u}_m\}_{m\geqslant 0} \subset \big(0, E^*[H]\big)$ *and* $\tilde{u}_m \longrightarrow \tilde{u}_0$ *as* $m \longrightarrow \infty$*. Then* $\lim\limits_{m\to+\infty} \tilde{\lambda}(\tilde{u}_m) = \tilde{\lambda}(\tilde{u}_0)$*, moreover*

$$\tilde{\varphi}(\tilde{u}_m) \longrightarrow \tilde{\varphi}(\tilde{u}_0) \qquad P\text{-a.s. and w.r.t } L^1(P)\text{-norm} \tag{4.64}$$

*as* $m \longrightarrow \infty$*. In particular, we have* $\lim\limits_{m\to+\infty} \tilde{z}(\tilde{u}_m) = \tilde{z}(\tilde{u}_0)$*.*

*Proof.* By $\tilde{\lambda}(u) = \tilde{\alpha}(u)\tilde{z}(u) \exp\Big(\sum_{i=1}^{n} \int_0^T \mu_s^i ds\Big)$ and $\tilde{\alpha}(u)\tilde{z}(u) = -\dfrac{dESR^\tau}{du}(u) \in C\Big(\big(0, E^*[H]\big)\Big)$, it is clear that

$$\lim_{m\to+\infty} \tilde{\lambda}(\tilde{u}_m) = \tilde{\lambda}(\tilde{u}_0).$$

Using the representation of $\tilde{\varphi}$ in Theorem 4.11, continuity of $I$ and the dominated convergence theorem, we can prove (4.64).

Since $E^*[\tilde{L}\tilde{\varphi}(\tilde{u}_m)H_0\big] = \tilde{u}_m \in \big(0, E^*[H]\big)$ for all $m \geqslant 0$, it is easy to see that $\tilde{\varphi}(\tilde{u}_m) \neq 1$ P-a.s and as a result $\tilde{\alpha}(\tilde{u}_m) \neq 0$ . Therefore, continuity of $\tilde{z}(.)$ on $\big(0, E^*[H]\big)$ can be deduced from the same property for $\tilde{\alpha}(.)$ and $\tilde{\lambda}(.)$. The following inequality, (4.64) and dominated convergence theorem together establish the continuity of $\tilde{\alpha}(.)$:

$$0 \leqslant E\Big[l'\big((1-\tilde{\varphi}(\tilde{u}_m))H_0\big)H_0\Big] \leqslant E\Big[l'\big(I(\tilde{\lambda}(\tilde{u}_m)\rho_T^*)\big)H\Big] = \tilde{\lambda}(\tilde{u}_m)E^*[H].$$

□

To demonstrate our results, we consider the power function $l(x) = \dfrac{x^p}{p}$ for some $p > 0$. In this case, problem (4.20) turns into problem of minimizing the lower partial moments with the random target $H_0$.

**Example 4.1.** *Assume* $n = 1$*,* $\delta \equiv 0$*, and* $H = (S_T - K)^+$ *for some* $K > 0$

*as the strike price of the call option $H$. Hence*

$$H_0 = H1_{\{\tau > T\}}.$$

*Working in the framework of Black-Scholes model with constant parameters $\sigma$ and $m > 0$, we get*

$$\begin{aligned}\frac{dP^*}{dP} := \rho_T^* &= \exp\Big(-\frac{m}{\sigma}W_T - \frac{1}{2}(\frac{m}{\sigma})^2 T\Big) \\ &= S_0^{\frac{m}{\sigma^2}} \exp\Big(\frac{m^2}{2\sigma^2}T - \frac{1}{2}mT\Big) S_T^{-\frac{m}{\sigma^2}}\end{aligned} \tag{4.65}$$

*for $t \in [0, T]$. Clearly, by Girsanov's theorem, $(W_t^*)_{0 \leq t \leq T} := \big(W_t + \frac{m}{\sigma}t\big)_{0 \leq t \leq T}$ is an $(\mathbb{F}, P^*)$ standard Brownian motion.*

*Now, by our results, problem (4.20) can be solved in two ways:*

1. *Directly, using our result for defaultable markets (i.e., Theorem 4.11). In this case, we need to find the optional decomposition of $\tilde{\varphi} H_0$ in the enlarged filtration $\mathbb{G}$. However, this method demands some tedious calculations and finally gives us a complicated hedging strategy.*

   *Suppose $\tilde{u} < E^*[(S_T - K)^+]$, then Theorem 4.11 implies*

$$\tilde{\varphi}_p H_0 = \begin{cases} H - (\tilde{\lambda}\rho_T^*)^{\frac{1}{p-1}} \wedge H & ; \{H > 0\} \cap \{\tau > T\} \\ 0 & ; \{H = 0\} \cup \{\tau \leq T\} \end{cases}$$

   *By Follmer and Leukert [15], Theorem 3.2, we know that $(\tilde{u}, \tilde{\pi})$ obtained from the optional decomposition of the modified claim $\tilde{\varphi}_p H_0$ solves the efficient hedging problem (4.20). Similar to (4.10) and (4.11), for $t \in [0, T]$ define*

$$\tilde{X}_t := ess\sup_{\kappa \in \mathcal{D}} E^{Q^\kappa}\big[\tilde{\varphi}_p H_0 \big| \mathcal{G}_t\big].$$

*For $\kappa \in \mathcal{D}$ and $t \in [0,T]$, first we simplify the underlying conditional expectation:*

$$\begin{aligned} E^{Q^\kappa}\big[\tilde{\varphi}_p H_0 \big| \mathcal{G}_t\big] &= \frac{1}{\rho_t^\kappa \rho_t^*} \exp\Big(-\int_0^t \kappa_s \mu_s ds\Big) \\ &\times E\big[\rho_T^* \exp\Big(-\int_t^T \kappa_s \mu_s ds\Big)\big(H - (\tilde{\lambda}\rho_T^*)^{\frac{1}{p-1}} \wedge H\big) 1_{\{\tau > T\}} \big| \mathcal{G}_t\big]. \end{aligned}$$

*Let $\tilde{H} := H - (\tilde{\lambda}\rho_T^*)^{\frac{1}{p-1}} \wedge H$. Then Corollary* 5.1.1 *of Bielecki and Rutkowski [3] and some calculations give us*

$$\begin{aligned} &E\big[\rho_T^* \exp\Big(-\int_t^T \kappa_s \mu_s ds\Big) \tilde{H} 1_{\{\tau > T\}} \big| \mathcal{G}_t\big] = \\ &1_{\{\tau > t\}} E\big[\rho_T^* \tilde{H} \exp\Big(\int_0^t \mu_s ds - \int_t^T \kappa_s \mu_s ds\Big) 1_{\{\tau > T\}} \big| \mathcal{F}_t\big]. \end{aligned}$$

*Therefore, by above*

$$\begin{aligned} &E^{Q^\kappa}\big[\tilde{\varphi}_p H_0 \big| \mathcal{G}_t\big] = \\ &\frac{1}{\rho_t^*} 1_{\{\tau > t\}} E\big[\rho_T^* \tilde{H} \exp\Big(\int_0^t \mu_s ds - \int_t^T \kappa_s \mu_s ds\Big) 1_{\{\tau > T\}} \big| \mathcal{F}_t\big]. \end{aligned} \tag{4.66}$$

*Now, if $\kappa$ is constant in* (4.66) *and $\kappa \searrow -1$ then by Fatou's lemma and the definition of $\tilde{X}_t$ we get*

$$\begin{aligned} \tilde{X}_t &\geqslant \frac{1}{\rho_t^*} 1_{\{\tau > t\}} E\big[\rho_T^* \tilde{H} \exp\Big(\int_0^T \mu_s ds\Big) 1_{\{\tau > T\}} \big| \mathcal{F}_t\big] \\ &= \frac{1}{\rho_t^*} 1_{\{\tau > t\}} \exp\Big(\int_0^T \mu_s ds\Big) E\big[\rho_T^* \tilde{H} E\big[1_{\{\tau > T\}} \big| \mathcal{F}_T\big] \big| \mathcal{F}_t\big] \\ &= 1_{\{\tau > t\}} E^*\big[\tilde{H} \big| \mathcal{F}_t\big]. \end{aligned} \tag{4.67}$$

*On the other hand, due to* (4.66) *and $\kappa > -1 \quad ds \times dP$-a.e. it can be* *seen that*

$$\begin{aligned}\tilde{X}_t &\leqslant \frac{1}{\rho_t^*} 1_{\{\tau>t\}} E\big[\rho_T^* \tilde{H} \exp\Big(\int_0^T \mu_s ds\Big) 1_{\{\tau>T\}}\big|\mathcal{F}_t\big] \\ &= 1_{\{\tau>t\}} E^*\big[\tilde{H}\big|\mathcal{F}_t\big].\end{aligned} \tag{4.68}$$

*The inequalities* (4.67) *and* (4.68) *show that*

$$\begin{aligned}\tilde{X}_t &= 1_{\{\tau>t\}} E^*\big[\tilde{H}\big|\mathcal{F}_t\big] \\ &= E^*\big[\tilde{H}\big|\mathcal{F}_t\big] - 1_{\{\tau\leqslant t\}} E^*\big[\tilde{H}\big|\mathcal{F}_t\big]\end{aligned} \tag{4.69}$$

*for* $t \in [0, T]$.

*By martingale representation theorem for Brownian filtrations, we have*

$$E^*\big[\tilde{H}\big|\mathcal{F}_t\big] = E^*[\tilde{H}] + \int_0^t \pi'_u dS_u$$

*for some* $\mathbb{F}$*-predictable process* $\pi'$.

*Applying Ito formula on the second term of* (4.69) *(in the second equality) and using the above representation along with the continuity of the process* $S$*, we get*

$$\begin{aligned}\tilde{X}_t &= \tilde{u} + \int_0^t \pi'_u 1_{\{\tau\geq u\}} dS_u \\ &\quad - \Big(\tilde{u} + \int_0^\tau \pi'_u dS_u\Big) N_t\end{aligned} \tag{4.70}$$

*where* $N_t := 1_{\{\tau\leqslant t\}}$. *Furthermore, notice that similar to* (4.54)

$$\tilde{u} = E^*\big[H - (\tilde{\lambda}\rho_T^*)^{\frac{1}{p-1}} \wedge H\big].$$

*We can interpret the optional decomposition* (4.70) *as follows: Starting with* $\tilde{u}$ *as the initial capital, if we hold* $\pi'_t 1_{\{\tau\geq t\}}$ *number of shares of the* *stock at time* $t \in [0,T]$*, and withdraw the amount* $\left(\tilde{u} + \int_0^\tau \pi'_u dS_u)\right) N_t$ *then we can guarantee to generate* $\tilde{\varphi}_p H_0$ *at time* $t = T$*.*

2. *In contrast to above, we can apply Theorem 4.14 and Remark 4.12. In other words, instead of solving the efficient hedging problem in the defaultable market, we first solve our problem in the complete market. Then, we apply (4.51) and (4.52) to determine the minimum of shortfall risk and the optimal strategy in the defaultable market.*

*Keeping in mind the second approach, let us fix* $r \equiv 0$*,* $m = 0.02$*,* $\sigma = 0.2$*,* $S_0 = 1$*,* $K = 0.8$*, and* $T = 15$ *(years), thus* $U_0 = E^*[(S_T - K)^+] = 0.3819$*. In addition, assume* $\mu \equiv 0.01$ *which implies* $P(\tau > T) = 0.8607$ *(a probability of* $P(\tau \leqslant T) = 0.1393$ *default before the maturity time* $T = 15$*).*

*Consider* $l(x) = \dfrac{x^2}{2}$ *and the available initial capital* $\tilde{u} = 0.17 < U_0$ *to hedge* $H_0$*. Applying Remark 4.12, we have*

$$ESR(\tilde{u}) = 0.0971.$$

*In the next section, for an analogous claim* $H = (S_T - K)^+ + K$*, we provide the details how to compute* $ESR(\tilde{u}) = \frac{1}{p} E\big[\big((1 - \tilde{\varphi}(\tilde{u}))H\big)^p\big]$ *and the associated optimal trading strategy* $\hat{\pi}$*.*

*By Theorem 4.14 and above, starting with* $\tilde{u} = 0.17$*, the minimum of the expectation of shortfall risk weighted by* $l$ *for* $H_0$ *becomes*

$$ESR^\tau(\tilde{u}) = P(\tau > T) ESR(\tilde{u}) = 0.8607 \times 0.0971 = 0.0836.$$

*For some fixed values of initial capital, Table 4.1 presents the associated minimum shortfall risk versus* $\tilde{u}$*. For a given* $\tilde{u}$*, since* $H_0 \leq H$*, as it is expected* $ESR^\tau(\tilde{u})$ *is less than* $ESR(\tilde{u})$*.*

**Table 4.1:** $ESR^{\tau}(\tilde{u})$ vs. $ESR(\tilde{u})$ for a defaultable call option.

| Initial capital | $ESR^{\tau}(\tilde{u})$ | $ESR(\tilde{u})$ |
|---|---|---|
| $\tilde{u}$ | $\delta = 0$ | $\delta = 1$ |
| \$0.32 | 0.0051 | 0.0059 |
| \$0.27 | 0.0183 | 0.0213 |
| \$0.22 | 0.0429 | 0.0498 |
| \$0.17 | 0.0836 | 0.0971 |

## 4.3 Application to Equity-Linked Life Insurance Contracts

In this section, we want to study equity-linked life insurance contracts in the framework of Section 4.1. Although there are different types of equity-linked life insurance contracts, we concentrate on the contracts called "pure endowment". Mathematically speaking, a pure endowment equity-linked insurance is defined as

$$H 1_{\{T(x)>T\}} \tag{4.71}$$

where $H$ is a nonnegative $\mathcal{F}_T$-measurable random variable and $T(x)$ is a positive random variable defined on the probability space $(\Omega, \mathcal{G}, P)$. In fact, $H$ is a future payment at time $t = T$ which its size depends on the evolution of the risky asset $S$ during the contract period $[0, T]$, and $T(x)$ represents the remaining lifetime (or the future lifetime) of a client who is currently at age $x$. The quantity

$${}_T p_x := P\big(T(x) > T\big) \tag{4.72}$$

is called the *survival probability of the client*. Using "Life Tables" (see for instance Bowers et al. [4]) we can find ${}_T p_x$ of each client for our pricing and hedging purposes. Clearly, ${}_T p_x$ depends on some factors such as age, race, sex, etc. We do not touch a mortality modeling in this chapter, while an appropriate stochastic mortality modeling can bring reasonable advantages in such pricings (see, Melnikov and Romaniuk [37]). For a pure endowment contract (4.71), if the insured is still alive at maturity of the contract the payment is $H$ otherwise zero.

Similarly, we can define a defaultable (pure endowment) equity-linked life insurance contract with recovery rate $\delta$ as a contract with the following payoff function

$$H_\delta\big(\tau, T(x)\big) := \big(H1_{\{\tau>T\}} + \delta H1_{\{\tau\leqslant T\}}\big)1_{\{T(x)>T\}} \tag{4.73}$$

where $\tau$ is a default time for insurance company. Therefore, to receive the payment $H$ the client must be alive at time $T$ and also the insurance company should not default up to this time. In the following, to provide explicit solutions (by applying Theorem 4.11), we let $\delta \equiv 0$. In this case, (4.73) is denoted by $H_0\big(\tau, T(x)\big)$.

**Assumption 4.17.** *We postulate that $S$, $T(x)$, and $\tau$ are mutually independent.*

The three elements of our model, $S$, $T(x)$ and $\tau$, generate two types of risks. There is an uncertainty associated to the asset price and the default time. This risk depends on the behaviour of the financial market, and it is known as *financial/credit risk* from financial literature. Another source of risk is the so-called *mortality risk* from insurance terminology, and it is the risk caused by the mortality time of the client, $T(x)$, which is independent of the financial market. There are different approaches to hedge and price the contingent claim $H_0\big(\tau, T(x)\big)$, we focus on superhedging approach (El Karoui and Quenez [12]) and Brennan-Schwartz approach (Brennan and Schwartz [5]) to deal with these two sources of risk (respectively) in this chapter. See

Moller [41] for a survey on different financial and insurance principles to hedge equity-linked life insurance contracts.

### 4.3.1 Brennan-Schwartz Approach

By Brennan-Schwartz approach, size of the life insurance contracts is considered to be large enough to use the strong law of large numbers. In other words, if the insurer sells the insurance contract (4.73) to $N$ clients then we have

$$\sum_{i=1}^{N} 1_{\{T_i(x)>T\}} \approx N\,{}_T p_x. \tag{4.74}$$

This means that by applying strong law of large numbers, the mortality risk is managed (diversified) by the size of the contracts. Hence, hedging $H_0\big(\tau, T(x)\big)$ reduces to hedging the modified claim ${}_T p_x H 1_{\{\tau>T\}}$.

Keeping in mind that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ and $\mathcal{H}_t = \sigma(\tau \wedge t)$ for $t \in [0, T]$. To hedge the credit risk associated to $H_0\big(\tau, T(x)\big)$, we apply superhedging techniques for $H1_{\{\tau>T\}}$ in the incomplete market $(B, S, \tau)$ equipped with the filtration $\mathbb{G} = (\mathcal{G}_t)_{0\leqslant t\leqslant T}$. In fact, for a single contract $H_0\big(\tau, T(x)\big)$ the insurance company should superhedge ${}_T p_x$ short positions of $H1_{\{\tau>T\}}$ in the defaultable market.

As a particular case, we study *equity-linked life insurance contracts with constant guarantee* $K$, i.e.,

$$H = \max(S_T, K) = (S_T - K)^+ + K$$

in (4.73). Using Brennan-Schwartz argument and Lemma 4.6, we consider the following amount as the premium of insurance contract (4.73)

$$\begin{aligned}\tilde{u} := {}_T p_x U_0 = {}_T p_x E^*\big[\max(S_T, K)\big] \\ = {}_T p_x E^*\big[(S_T - K)^+\big] + {}_T p_x K\end{aligned} \tag{4.75}$$

obviously, $\tilde{u} < U_0 = E^*\big[\max(S_T, K)\big]$.

Same as Example 4.1, we take $l(x) = \dfrac{x^p}{p}$ for some $p > 1$. Starting with the premium $\tilde{u} = {}_T p_x U_0$ as the initial capital, we want to solve the efficient hedging problem (4.20) for the insurance contract $H_0\big(\tau, T(x)\big)$. Let $\hat{\varphi}_p$ to be the optimal solution corresponding to the initial capital $\tilde{u}$ and the problem introduced in Remark 4.12. Then Theorem 4.11 and 4.14 show that the perfect hedging of the modified claim $\hat{\varphi}_p.\max(S_T, K)$ solves our efficient hedging problem. We follow a similar argument to Follmer and Leukert [15] to find the explicit solution. By Remark 4.12:

$$\hat{\varphi}_p.\max(S_T, K) = \max(S_T, K) - \Big(c^{\frac{1}{p-1}} S_T{}^{\frac{-\beta}{p-1}}\Big) \wedge \max(S_T, K), \tag{4.76}$$

where $\beta := \dfrac{m}{\sigma^2}$, and constant $c$ comes from the constants involving Theorem 4.11 and Eq. (4.65). By $\tilde{u} = E^*\big[\hat{\varphi}_p.\max(S_T, K)\big]$, depending on the value of $\tilde{u}$, the decreasing convex function $c^{\frac{1}{p-1}} s^{\frac{-\beta}{p-1}}$ intersects with $\max(s, K)$ at $s = K_1 < K$ or $s = K_2 \geqslant K$. More precisely, we have:

*(i)* If $\tilde{u} > E^*\Big[\big(S_T - K(\frac{S_T}{K})^{\frac{\beta}{1-p}}\big)1_{\{S_T \geqslant K\}}\Big]$ then (4.76) becomes

$$= K1_{\{K_1 \leqslant S_T < K\}} + S_T 1_{\{S_T \geqslant K\}} - K(\frac{S_T}{K_1})^{\frac{\beta}{1-p}} 1_{\{S_T \geqslant K_1\}}. \tag{4.77}$$

*(ii)* If $\tilde{u} \leqslant E^*\big[\big(S_T - K(\frac{S_T}{K})^{\frac{\beta}{1-p}}\big)1_{\{S_T \geqslant K\}}\big]$ then, in this case, (4.76) is equal to

$$= S_T 1_{\{S_T \geqslant K_2\}} - K_2(\frac{S_T}{K_2})^{\frac{\beta}{1-p}} 1_{\{S_T \geqslant K_2\}}. \tag{4.78}$$

Now, applying Theorems 1 and 4.11, we provide an analytic expression for the minimum value of shortfall risk. Additionally, the optimal strategy is derived by using replication principle in complete markets. We only provide the details for the first case, (4.77), the corresponding results for the second case can be obtained by some straightforward modifications. Let us define

$$\begin{aligned} V_t &:= E^*\Big[\hat{\varphi}_p.\max\Big(S_t \exp\big[\sigma(W_T^* - W_t^*) - \frac{1}{2}\sigma^2(T-t)\big], K\Big)\Big|\mathcal{F}_t\Big] \\ &= F_p(t, S_t) \end{aligned} \tag{4.79}$$

for $t \in [0, T]$. In the case of (4.77), the Markov property and log-normal distribution of $S_t$ imply that

$$\begin{aligned} F_p(t,s) = {} & K\Phi\Big(d_-(t,s,K_1)\Big) - K\Phi\Big(d_-(t,s,K)\Big) \\ & + s\Phi\Big(d_+(t,s,K)\Big) \\ & - K(\frac{s}{K_1})^{\frac{\beta}{1-p}} \exp\Big[\frac{m(T-t)}{2(p-1)}(\frac{\beta}{p-1}+1)\Big] \\ & \times \Phi\Big(d_-(t,s,K_1) + \frac{m\sqrt{T-t}}{\sigma(1-p)}\Big), \end{aligned} \tag{4.80}$$

where $\Phi$ is the standard normal distribution function and

$$d_\pm(t,s,K) = \frac{Lns - LnK}{\sigma\sqrt{T-t}} \pm \frac{1}{2}\sigma\sqrt{T-t}.$$

The constant $K_1$, and a priori $c$, can be determined from

$$\tilde{u} = E^*\big[\hat{\varphi}_p.\max(S_T, K)\big] = F_p(0, S_0).$$

After finding $K_1$, by Theorem 4.14 the minimum shortfall risk can be calculated as follows

$$\begin{aligned} ESR^{\tau}(\tilde{u}) = P(\tau > T)ESR(\tilde{u}) = \frac{1}{p}P(\tau > T)E\Big[\Big(\big(1-\hat{\varphi}_p(\tilde{u})\big)H\Big)^p\Big] \\ = \frac{K^p}{p}P(\tau > T)\Big\{1 - \Phi\Big(d_-(0,S_0,K_1) + \frac{m\sqrt{T}}{\sigma}\Big) \\ + (\frac{S_0}{K_1})^{\frac{p\beta}{1-p}} \exp\Big[\frac{p\beta T}{p-1}\Big(\frac{1}{2}\sigma^2(\frac{p\beta}{p-1}+1) - m\Big)\Big] \\ \times \Phi\Big(d_-(0,S_0,K_1) + \frac{m\sqrt{T}}{\sigma} + \frac{mp\sqrt{T}}{\sigma(1-p)}\Big)\Big\}. \end{aligned} \tag{4.81}$$

Moreover, the optimal strategy corresponding to $ESR(\tilde{u})$ is given by

$$\begin{aligned} \hat{\pi}_t &= \frac{\partial}{\partial s}F_p(t,s)|_{s=S_t} \\ &= \Big\{\frac{K}{s\sigma\sqrt{2\pi(T-t)}}\Big[\exp\Big(-\frac{d_-^2(t,s,K_1)}{2}\Big) \\ &- \exp\Big(-\frac{d_-^2(t,s,K)}{2}\Big) + \frac{s}{K}\exp\Big(-\frac{d_+^2(t,s,K)}{2}\Big)\Big] \\ &+ \Phi\Big(d_+(t,s,K)\Big) \\ &- \frac{K}{s}(\frac{s}{K_1})^{\frac{\beta}{1-p}} \exp\Big[\frac{m(T-t)}{2(1-p)}(\frac{\beta}{p-1}+1)\Big] \\ &\times \Big[\frac{\beta}{1-p}\Phi\Big(d_-(t,s,K_1) + \frac{m\sqrt{T-t}}{\sigma(1-p)}\Big) \\ &+ \frac{1}{\sigma\sqrt{2\pi(T-t)}}\exp\Big(-\frac{\big(d_-(t,s,K_1) + \frac{m\sqrt{T-t}}{\sigma(1-p)}\big)^2}{2}\Big)\Big]\Big\}\Big|_{s=S_t}. \end{aligned} \tag{4.82}$$

Therefore, by (4.52) and (4.82), the trading strategy $(\hat{\pi}_t 1_{\{\tau \geq t\}})_{t\in[0,T]}$ is a solution to $ESR^{\tau}(\tilde{u})$.

### 4.3.2 Superhedging Approach

Alternative to above discussion (Brennan-Schwartz method), it is possible to treat both $\tau$ and $T(x)$ as independent default times. In this case, we work in the filtration enlarged by both $\tau$ and $T(x)$, i.e.,

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$$

where $\mathcal{H}_t = \sigma(\tau \wedge t) \vee \sigma(T(x) \wedge t)$ for $t \in [0, T]$. We know that there exists a nonnegative function $\hat{\mu}$ such that for all $t \geqslant 0$

$$_tp_x := P\big(T(x) > t\big) = \exp\Big(-\int_0^t \hat{\mu}(x+s)ds\Big). \tag{4.83}$$

$\hat{\mu}$ is known as the *force of mortality* or the *hazard rate function*, see for example Bowers et al. [4].

Then considering $\tau_1 = \tau$, $\tau_2 = T(x)$, and $\delta_1 = \delta_2 = 0$ in (4.8), we can use Theorem 4.14, Eq. (4.51), to get

$$ESR^{\tau, T(x)}(\tilde{u}) = {_Tp_x} P(\tau > T) ESR(\tilde{u}). \tag{4.84}$$

Notice that $ESR(\tilde{u})$ can be computed as presented in Subsection 4.3.1. Eq. (4.84) means that if we add the information regarding the survival of clients at each $t \in [0, T]$ to the filtration $\mathbb{G}$, then the minimum of shortfall risk is reduced by the ratio of $_Tp_x$.

**Example 4.2.** *Let us consider* $l(X) = \dfrac{X^2}{2}$ *and the parameters of our model same as Example 4.1. In this example, we suppose that the client is at age* $x = 30$*. For* $T = 15$ *and* $x = 30$*, using the life table in Bowers et al. [4], the client will survive to the maturity time of the contract with the probability of* $_Tp_x = 0.949$*. Moreover, it is easy to see that* $U_0 = E^*\big[\max(S_T, K)\big] = 1.182$*.*

*Taking the Brennan-Schwartz approach into consideration and starting with the premium* $\tilde{u} = {}_T p_x U_0 = 1.122$ *as the initial capital, we can employ* (4.81) *to see that*

$$ESR^{\tau}(\tilde{u}) = 0.0014$$

*for* $\delta \equiv 0$*, and* $ESR(\tilde{u}) = 0.0016$ *(*$\delta = 1$*).*

*In the case of superhedging approach, with the same* $\tilde{u} = 1.122$*, we apply* (4.84) *this time to obtain*

$$ESR^{\tau,T(x)}(\tilde{u}) = 0.0013$$

*for* $\delta_1 = \delta_2 = 0$*, and* $ESR^{T(x)}(\tilde{u}) = 0.0015$ *where* $\delta_1 = 1$ *and* $\delta_2 = 0$*.*

*In Table 4.2, we compare Brennan-Schwartz and superhedging methods for some given values of* $\tilde{u}$*. Similar to Table 4.1, from the insurer's point of view,* $ESR^{\tau}(.)$ *is still a decreasing function of the initial capital, and for a fixed* $\tilde{u}$ *it decreases with a higher possibility of the default event.*

*Obviously, using superhedging approach generates smaller values for* $ESR^{\tau}(\tilde{u})$*, and this is consistent with our intuition. In the case of Brennan-Schwartz method, we eliminate the mortality risk of the clients by the constant number* ${}_T p_x$*, but in the superhedging method, more accurate information is available. In the latter case, by the enlargement of the filtrations and adding the new source of randomness to our model, we can provide a better approximation of the risk.*

## 4.4 Conclusion

In the framework of a defaultable Black-scholes model subject to a capital constraint, this chapter studies the problem of minimizing the expectation of

**Table 4.2:** A comparison of the minimum shortfall risk for equity-linked life insurance contracts with guarantee: Brennan-Schwartz approach vs. superhedging approach.

| Initial capital | Brennan-Schwartz approach | | Superhedging approach | |
|---|---|---|---|---|
| | $ESR^{\tau}(\tilde{u})$ | $ESR(\tilde{u})$ | $ESR^{\tau,T(x)}(\tilde{u})$ | $ESR^{T(x)}(\tilde{u})$ |
| $\tilde{u}$ | $\delta = 0$ | $\delta = 1$ | $\delta_1 = \delta_2 = 0$ | $\delta_1 = 1, \delta_2 = 0$ |
| \$1.122 | 0.0014 | 0.0016 | 0.0013 | 0.0015 |
| \$1.066 | 0.0050 | 0.0058 | 0.0048 | 0.0055 |
| \$1.010 | 0.0110 | 0.0128 | 0.0105 | 0.0121 |
| \$0.954 | 0.0193 | 0.0225 | 0.0184 | 0.0213 |

shortfall risk weighted by a loss function. The underlying defaultable contingent claim, with nonzero recovery rates, is exposed to multiple independent default times satisfying the intensity hypothesis. We convert the considered dynamic optimization problem with respect to time into a max-min problem for testing a composite hypothesis against a simple alternative. The latter problem is solved by the techniques of non-smooth convex duality studied by Cvitanić and Karatzas [8]. In the case of the zero recovery rates, we provide an explicit solution for the optimal solution to the desired efficient hedging problem. Moreover, it has been proved that the efficient hedging problem in the defaultable market (the enlarged filtration) can be reduced to a similar problem in the reference default-free market. The results are demonstrated by their application to equity-linked life insurance contracts with guaranteed minimum maturity benefit.

We decide to analyze further measures to quantify and reduce risk in defaultable models. In particular, VaR, CVaR and CaR minimization problems (see, e.g., Rockafellar and Uryasev [45], Rockafellar and Uryasev [46] and Dmitrasinovic-Vidovic et al. [10]) in models with dependent defaults, models subject to default times satisfying the density hypothesis, and the case of

American contingent claims are some of our future research plans.

# Chapter 5

# Bermudan Options and Connections to Equity-Linked Life Insurance Contracts

## 5.1 An investment bridge between mortality protection and equity benefits

In Chapter 4 we studied the guaranteed minimum maturity benefit (GMMB) equity-linked life insurance which guarantees the policyholder the maximum between a predetermined amount (the guarantee) and an underlying stock index at the maturity time. If the guarantee matures in-the-money then the insurer is liable for the shortfall, otherwise the policyholder receives the stock index and the insurer's liability is zero.

The equity participation of the contract exposes the insurer to the market risk in terms of a European call/put option. By traditional actuarial approach, the mortality risk of the contract is usually managed by the client's survival probability over the term of the contract. In contrast to this approch, we used the concept of random times and the enlargement of the filtrations to determine a hedging strategy which takes into account both financial risk and mortality risk dynamically.

In this chapter, we investigate another type of investment guarantee that resembles an American option. Using this connection and enlargement of filtration techniques, from the insurer's point of view, we study the problem of maximizing probability of a successful hedge under a capital constraint.

**Definition 5.1.** *The guaranteed minimum death benefit (GMDB) is a type of equity-linked life insurance contract with two main characteristic features, investment opportunity and protection guarantee. Upon the insured's death during the term of the contract; if the underlying asset price rises then the insured enjoys the benefits of the equity investment, and in the case of a downside risk investment the insurer guarantees a minimum payment to protect the insured against the market risk.*

We consider contracts designed with a *separate account* format. This means that the insurer manages the fund available (from the premium) in the account by investing in the underlying equity, but the actual owner of the account is still the insured. Let $F_t$ be the market value of the separate account and $S_t$ be the price of the underlying equity investment at time $t$. Then

$$F_t = F_0 \frac{S_t}{S_0}(1-m)^t, \tag{5.1}$$

for $t = 0, 1, 2, ..., T$, where $T$ is the maturity time and $m$ denotes the *management charge rate* deducted from the account at the end of each month.

If the insured dies in the time interval $(t-1, t]$, for $t = 1, 2, ..., T$, the policyholder receives $\text{Max}(K, F_t)$. Mathematically speaking, a GMDB contract with guarantee $K > 0$ and the maturity time $T$ has the following payoff:

$$\sum_{t=1}^{T} e^{-rt}\,\mathrm{Max}(K, F_t) 1_{\{t-1<T(x)\leq t\}}, \tag{5.2}$$

where $r$ is the risk free interest rate, and $T(x)$ represents the life time of a client who is currently at age $x$. For a comprehensive study of the investment guarantees and their valuations approaches consult with Hardy [17], Aase and Persson [1] and Ekern and Persson [11].

We suppose that the dynamic of the underlying asset price $(S_t)_{0\leq t\leq T}$ is governed by the following Black-Scholes model:

$$\begin{cases} dS_t = S_t\big(\mu_t dt + \sigma_t dW_t\big) & ; S_0 > 0 \\ dB_t = B_t r_t dt & ; B_0 = 1 \end{cases} \tag{5.3}$$

for $t \in [0, T]$. $(W_t)_{0\leq t\leq T}$ is a standard Brownian motion on the complete probability space $\big(\Omega, \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T} \subseteq \mathcal{G}, P\big)$, $(\mu_t)_{0\leq t\leq T}$ and the positive process $(\sigma_t)_{0\leq t\leq T}$ are $\mathbb{F}$-adapted processes representing the appreciation rate and the volatility of $S$ respectively. The nonnegative deterministic process $(r_t)_{0\leq t\leq T}$ denotes the risk free interest rate.

On the probability space $\big(\Omega, \mathcal{G}, P\big)$, the *equivalent martingale measure* $P^*$ for (5.3) is defined as:

$$\frac{dP^*}{dP} := \rho^*_T$$

where

$$\rho^*_t := \exp\Big( - \int_0^t \frac{\mu_s - r_s}{\sigma_s} dW_s - \int_0^t \frac{1}{2}(\frac{\mu_s - r_s}{\sigma_s})^2 ds\Big) \tag{5.4}$$

for $t \in [0, T]$.

To satisfy the no-arbitrage condition and make the above model complete, we impose the following integrability conditions:

*(1)* $\int_0^T (\frac{\mu_s - r_s}{\sigma_s})^2 ds < +\infty, \quad P\text{-a.s.}$

*(2)* $E[\rho_T^*] = 1$

**Assumption 5.2.** *For the sake of simplicity, we assume that* $r_t \equiv 0$ *with* $\mu_t \equiv \mu \in \mathbb{R}$ *and* $\sigma_t \equiv \sigma > 0$ *constant. In addition, we take the management rate* $m \equiv 0$ *and* $F_0 = S_0$ *in* (5.1)*.*

In this setting, the GMDB contract (5.2) is simplified to:

$$H(D) := \sum_{t=1}^{T} \text{Max}(K, S_t) 1_{\{t-1<T(x)\leq t\}} \tag{5.5}$$

In general, the mortality risk and the financial risk are not correlated. Hence it is natural to assume that:

**Assumption 5.3.** $T(x)$ *is a* $\mathcal{G}$*-measurable random variable which is independent of the risky asset* $S$*.*

## 5.2 GMDB contract and Bermudan option

Minimum guarantee equity-linked life insurance contracts are traditionally priced and hedged by a combination of actuarial methods and modern techniques of mathematical finance. By the law of large numbers, the mortality risk of the client is replaced by its expected value; and the financial risk associated to the underlying equity is managed by the methods of Black and Scholes. See Brennan and Schwartz [5] and Hardy [17] for more details on pricing and hedging principles of these types of contracts. In the setting of Section 5.1, by the aforementioned method, instead of (5.5) the following modified version of $H(D)$ is analyzed:

$$\tilde{H} := \sum_{t=1}^{T} P\big(t-1 < T(x) \leq t\big)\,\mathrm{Max}(K, S_t)\,. \tag{5.6}$$

Let $P^*$ be the unique probability martingale measure of the model (5.3). Then initial price of the modified payoff $\tilde{H}$ is given by

$$\tilde{H}(0) := \sum_{t=1}^{T} P\big(t-1 < T(x) \leq t\big) E^*\big[\,\mathrm{Max}(K, S_t)\big], \tag{5.7}$$

where $E^*$ denotes the expectation with respect to the probability measure $P^*$.

In this chapter, rather than using the law of large numbers, through constructing a new filtration we consider $H(D)$ as an American option with only finitely many permitted exercise dates $\{1, 2, 3, ...., T\}$ in an incomplete market.

To do so, by progressively enlargement of filtrations, let us define

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t, \qquad \text{for all } t \in [0, T], \tag{5.8}$$

where $\mathcal{H}_t := \sigma\big(T(x) \leq t\big)$ is the $\sigma$-field generated by $T(x)$ up to time $t \in [0, T]$. We denote the enlarged filtration $(\mathcal{G}_t)_{t\in[0,T]}$ by $\mathbb{G} := \mathbb{F} \vee \mathbb{H}$.

The financial model (5.3) equipped with the probability space $(\Omega, \mathcal{G}, P)$ and the new filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ is an incomplete market because, for instance, $1_{\{T(x)>T\}}$ is not attainable in this model. Assumption 5.3 combined with Lemma 6.1.2 of Bielecki and Rutkowski [3] imply that any $(\mathbb{F}, P)$-martingale remains a $(\mathbb{G}, P)$-martingale. This guarantees the no arbitrage condition in the new model.

From Bowers et al. [4], we know that there exists a nonnegative function $\hat{\mu}$, known as the *force of mortality* or the *hazard rate function*, such that for all $t \geqslant 0$

$$ {}_tp_x := P\big(T(x) > t\big) = \exp\Big(-\int_0^t \hat{\mu}(x+s)ds\Big). \tag{5.9} $$

We do not touch a mortality modelling in this chapter, while an appropriate stochastic mortality modelling can bring reasonable advantages in such pricing methods (see, Melnikov and Romaniuk [37]).

Let us introduce

$$ \mathcal{D} := \Big\{(\kappa_t)_{0\le t\le T} : \text{bounded}, \mathbb{G}\text{-predictable and } \kappa_t > -1 \ \ dt \times dP \text{ a.e.}\Big\}. $$

For any $\kappa \in \mathcal{D}$, define

$$ \rho_t^\kappa = 1 + \int_0^t \kappa_s \rho_{s-}^\kappa dM_s, \qquad t \in [0,T], $$

with $M_t = 1_{\{T(x)\le t\}} - \displaystyle\int_0^{T(x)\wedge t} \hat{\mu}(x+s)ds$ which is a $(\mathbb{G}, P)$-martingale.

Using the definition of stochastic exponential, the unique solution to the above SDE is given by

$$ \rho_t^\kappa = \big(1 + \kappa_{T(x)} 1_{\{T(x)\leqslant t\}}\big)\exp\Big(-\int_0^{T(x)\wedge t} \kappa_s \hat{\mu}(x+s)ds\Big) \tag{5.10} $$

Keeping in mind the above notations, by Bielecki and Rutkowski [3], Kusuoka [31], or Nakano [42], we can provide an explicit representation for the Radon-Nikodým density of the probability martingale measures of $(S_t)_{t\in[0,T]}$ on $\big(\Omega, \mathbb{G} = (\mathcal{G}_t)_{0\le t\le T} \subseteq \mathcal{G}, P\big)$ as follows:

$$ \mathcal{Q} := \Big\{Q^\kappa \,\Big|\, \frac{dQ^\kappa}{dP} = \rho_T^* \rho_T^\kappa \text{ for some } \kappa \in \mathcal{D}\Big\} \tag{5.11} $$

To see the GMDB life insurance $H(D)$ defined in (5.5) as a Bermudan option, we recall the next definition from Schweizer [48]:

**Definition 5.4.** *A* Bermudan option *is a particular type of American option which can be exercised only at a predetermined* region of permitted exercise dates $R \subseteq [0, T]$. *The* payoff process *is a nonnegative adapted RCLL process denoted by $U = (U_t)_{t\in[0,T]}$ such that $U_t = 0$ for $t \notin R$. The option is exercised by choosing a $\mathbb{G}$-stopping time $\tau$ with values in $R$.*

In this chapter, we consider $R = \{1, 2, 3, ...., T\}$, a suitable finite subset of $[0, T]$. The underlying Bermudan option with the payoff process

$$U = (U_t)_{t\in[0,T]} = \big(\mathrm{Max}(K, S_t)\big)_{t\in[0,T]}$$

and the region of permitted exercise dates $R$ is represented by the pair $(U, R)$.

**Remark 5.5.** *The exercise date of the GMDB contract $H(D)$ is not exactly $T(x)$. In fact, it is the smallest integer greater than or equal to $T(x)$. Let us denote this positive discrete time $\mathcal{G}_T$-random variable by $\tilde{\tau}$, we can use the* ceiling function *to represent $\tilde{\tau}$, i.e.*

$$\tilde{\tau} := ]T(x)[ \tag{5.12}$$

To view $H(D)$ as a Bermudan option, first we need to prove that the random exercise time $\tilde{\tau}$ is a $\mathbb{G}$-stopping time.

**Lemma 5.6.** *Let $\tilde{\tau}$ be defined as in (5.12), then $\tilde{\tau}$ is a $\mathbb{G}$-stopping time.*

*Proof.* For any $t \in [0, T]$, we have

$$\{\tilde{\tau} \le t\} = \{\tilde{\tau} \le [t]\} = \{T(x) \le [t]\} \in \mathcal{G}_{[t]} \subseteq \mathcal{G}_t,$$

where $[t]$ is the *floor function* of $t$, the largest integer less than or equal to $s$. $\square$

We consider the superhedging approach to price and hedge the Bermudan option $(U, R) = \Big( \big( \mathrm{Max}(K, S_t) \big)_{t \in [0,T]}, R \Big)$. By using optional decomposition of supermartingales, Kramkov [29] investigated this approach for the general case of American options . Schweizer [48] utilized a backward argument to find an analytic formula for the superhedging value process of a Bermudan option. In the next section, the Schweizer's techniques are adapted to price $H(D)$ on the probability space $\big(\Omega, \mathbb{G} = (\mathcal{G}_t)_{0 \le t \le T} \subseteq \mathcal{G}, P\big)$.

The superhedging value process of $(U, R)$ at time $t \in [0, T]$ is defined as follows:

$$X_t := \operatorname*{ess\,sup}_{\substack{\kappa \in \mathcal{D} \\ \tau \in \mathcal{S}_{t,T}(R)}} E^{Q^\kappa} \big[ U_\tau | \mathcal{G}_t \big], \tag{5.13}$$

where $\mathcal{S}_{t,T}(R)$ is the set of all $\mathbb{G}$-stopping times with values in $R \cap [t, T]$, and $E^{Q^\kappa}[\cdot]$ denotes expectation w.r.t probability $Q^\kappa$.

First, Schweizer [48] shows that $X_{t_i}$ at $t_i \in R$ is equal to the $\mathcal{Q}$-uniform snell envelope of all the finite possible payoffs $U_{t_j}$ for $j = i, i+1, ..., n$, i.e. he drops the stopping times $\tau \in \mathcal{S}_{t_i,T}(R)$ from the calculation of $X_{t_i}$. Then $X_t$ between two possible exercise dates $t_i$ and $t_{i+1}$ is determined by the price of a European option initiated at time $t_i$, maturity time $t_{i+1}$ and the payoff $X_{t_{i+1}}$ at time $t_{i+1}$.

For the reader's convenience, we summarize the Schweizer's method in Appendix D.

## 5.3 A stochastic game between the death time and financial decisions

Let $v_0 > 0$ be a given initial capital, and $(\pi_t)_{t\in[0,T]}$ a $\mathbb{G}$-predictable $S$-integrable process. Then the self-financing value process $\left(V_t^{v_0,\pi}\right)_{t\in[0,T]}$ corresponding to $(v_0, \pi)$ is defined as:

$$V_t^{v_0,\pi} := v_0 + \int_0^t \pi_s dS_s, \qquad P\text{-a.s., for all } t \in [0,T].$$

If $V_t^{v_0,\pi} \geq 0 \quad P$-a.s. for any $t \in [0,T]$, then the self-financing strategy $(v_0, \pi)$ is called $\mathbb{G}$-admissible. The set of all $\mathbb{G}$-admissible trading strategies with the initial capital $v_0$ is denoted by $\mathcal{A}^{\mathbb{G}}(v_0)$.

Assume that the available initial capital to superhedge the Bermudan option $(U,R)$ is $\tilde{v}_0 > 0$ which is subject to the constraint:

$$\tilde{v}_0 < \sup_{\substack{\kappa\in\mathcal{D} \\ \tau\in\mathcal{S}_{0,T}(R)}} E^{Q^\kappa}\left[U_\tau\right]. \tag{5.14}$$

Since $\tilde{v}_0$ is strictly less than the initial cost of superhedging $(U,R)$, for any choice of $\pi \in \mathcal{A}^{\mathbb{G}}(\tilde{v}_0)$, always there is a possibility of shortfall risk, i.e.

$$\forall \pi \in \mathcal{A}^{\mathbb{G}}(\tilde{v}_0) \ \exists j \in \{1,...,n\} \ \text{ s.t. } \ P\Big(\operatorname{Max}(K, S_{t_j}) > V_{t_j}^{\tilde{v}_0,\pi}\Big) > 0. \tag{5.15}$$

In this chapter, we are looking for an optimal trading strategy $\tilde{\pi} \in \mathcal{A}^{\mathbb{G}}(\tilde{v}_0)$ such that it minimizes the *worst possible scenario* of a shortfall risk as described in (5.15). Equivalently, we formulate the *quantile hedging* problem for the Bermudan option $(U,R)$ with the available initial capital $\tilde{v}_0$ as follows:

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}}(\tilde{v}_0)} \left( \inf_{\tau\in\mathcal{S}_{0,T}(R)} P\big(V_\tau^{\tilde{v}_0,\pi} \geq U_\tau\big)\right) \tag{5.16}$$

This max-min problem is a stochastic game between the death time of the insured and the insurer's trading strategy to hedge $(U, R)$. From the insurer's point of view, by problem (5.16) we want to maximize the worst probability of a successful hedge over all permitted exercise dates in $R$.

As the first step to deal with problem (5.16), we exploit the Schweizer's method to determine the superhedging value process $X_t$ introduced in (5.13). The next proposition gives us a tool to compute the underlying $\mathcal{G}_t$-conditional expectations in the definition of $B_i$'s, in (4.2), in terms of $\mathcal{F}_t$-conditional expectations.

**Proposition 5.7.** *Let $H$ be an $\mathcal{F}_T$-measurable random variable. Then, for any $\kappa \in \mathcal{D}$ and $t \in [0, T]$, we have*

$$E^{Q^\kappa}\big[H\big|\mathcal{G}_t\big] = E^*\big[H\big|\mathcal{F}_t\big]. \tag{5.17}$$

*Proof.* Using Bayes formula, one can write

$$E^{Q^\kappa}\big[H\big|\mathcal{G}_t\big] = \frac{1}{\rho_t^\kappa \rho_t^*} E\big[\rho_T^\kappa \rho_T^* H\big|\mathcal{F}_t\big], \qquad \text{for any } \kappa \in \mathcal{D}. \tag{5.18}$$

$\big(E\big[\rho_T^* H\big|\mathcal{F}_u\big]\big)_{0\le u\le T}$ is an $(\mathbb{F}, P)$-martingale which, due to Assumption 5.3, follows a $(\mathbb{G}, P)$-martingale too. In addition, $(\rho_u^\kappa)_{0\le u\le T}$ is a $(\mathbb{G}, P)$-martingale orthogonal to $\big(E\big[\rho_T^* H\big|\mathcal{F}_u\big]\big)_{0\le u\le T}$, since their quadratic covariation is equal to zero. This implies their product $\big(\rho_u^\kappa E\big[\rho_T^* H\big|\mathcal{F}_u\big]\big)_{0\le u\le T}$ is a $(\mathbb{G}, P)$-local martingale. By passing through $(H \wedge m)_{m\ge 1}$ and using the monotone convergence theorem, we get

$$E\big[\rho_T^\kappa E\big[\rho_T^* H\big|\mathcal{F}_T\big]\big|\mathcal{G}_t\big] = \rho_t^\kappa E\big[\rho_T^* H\big|\mathcal{F}_t\big]. \tag{5.19}$$

Hence

$$\frac{1}{\rho_t^\kappa \rho_t^*} E\big[\rho_T^\kappa \rho_T^* H \big| \mathcal{G}_t\big] = \frac{1}{\rho_t^*} E\big[\rho_T^* H \big| \mathcal{F}_t\big] , \tag{5.20}$$

from this equation, we can easily derive (5.17). □

Keeping in mind Appendix D, in the following theorem we now compute process $(X_t)_{t\in[0,T]}$:

**Proposition 5.8.** *Let $(X_t)_{t\in[0,T]}$ be the superhedging value process of the Bermudan option $(U,R) = \Big(\big(\operatorname{Max}(K,S_t)\big)_{t\in[0,T]}, R\Big)$. Then we have*

$$X_t = S_t + E^*\big[(K - S_T)^+ \big| \mathcal{F}_t\big], \qquad \textit{for all } t \in [0,T]. \tag{5.21}$$

*In particular, the initial cost of superhedging is given by*

$$X_0 = S_0 + E^*\big[(K - S_T)^+\big]. \tag{5.22}$$

*Proof.* By the definition of $B_n$ in (4.1):

$$B_n = U_{t_n} = \operatorname{Max}(K, S_{t_n}).$$

From this and (4.2), we obtain

$$\begin{aligned} B_{n-1} &= \operatorname{Max}\Big(U_{t_{n-1}}, \operatorname*{ess\,sup}_{\kappa\in\mathcal{D}} E^{Q^\kappa}\big[B_n | \mathcal{G}_{t_{n-1}}\big]\Big) \\ &= \operatorname{Max}\Big(\operatorname{Max}(K, S_{t_{n-1}}), E^*\big[\operatorname{Max}(K, S_{t_n}) \big| \mathcal{F}_{t_{n-1}}\big]\Big), \end{aligned} \tag{5.23}$$

where to get the second equality we have used Proposition 5.7. On the other hand, it is easy to see that

$$E^*\big[\operatorname{Max}(K, S_{t_n}) \big| \mathcal{F}_{t_{n-1}}\big] \geq K \tag{5.24}$$

and

$$E^*\big[\operatorname{Max}(K, S_{t_n})\big|\mathcal{F}_{t_{n-1}}\big] \geq E^*\big[S_{t_n}\big|\mathcal{F}_{t_{n-1}}\big] = S_{t_{n-1}} \tag{5.25}$$

Hence (5.23) becomes

$$\begin{aligned} B_{n-1} &= E^*\big[\operatorname{Max}(K, S_{t_n})\big|\mathcal{F}_{t_{n-1}}\big] \\ &= E^*\big[S_{t_n} + (K - S_{t_n})^+\big|\mathcal{F}_{t_{n-1}}\big] \\ &= S_{t_{n-1}} + E^*\big[(K - S_{t_n})^+\big|\mathcal{F}_{t_{n-1}}\big]\,. \end{aligned} \tag{5.26}$$

By induction, we can show that for all $i = 0, 1, ..., n-1$

$$\begin{aligned} B_i &= \operatorname{Max}\Big(U_{t_i}, \operatorname*{ess\,sup}_{\kappa\in\mathcal{D}} E^{Q^\kappa}\big[B_{i+1}|\mathcal{G}_{t_i}\big]\Big) \\ &= \operatorname{Max}\Big(\operatorname{Max}(K, S_{t_i}), \operatorname*{ess\,sup}_{\kappa\in\mathcal{D}} E^{Q^\kappa}\big[E^*\big[\operatorname{Max}(K, S_{t_n})\big|\mathcal{F}_{t_{i+1}}\big]\big|\mathcal{G}_{t_i}\big]\Big) \\ &= \operatorname{Max}\Big(\operatorname{Max}(K, S_{t_i}), E^*\big[\operatorname{Max}(K, S_{t_n})\big|\mathcal{F}_{t_i}\big]\Big) \\ &= E^*\big[S_{t_n} + (K - S_{t_n})^+\big|\mathcal{F}_{t_i}\big] \\ &= S_{t_i} + E^*\big[(K - S_{t_n})^+\big|\mathcal{F}_{t_i}\big] \end{aligned} \tag{5.27}$$

In this case, we applied Proposition 5.7 on $H = E^*\big[\operatorname{Max}(K, S_{t_n})\big|\mathcal{F}_{t_{i+1}}\big]$ with the fact that $\mathcal{F}_{t_i} \subseteq \mathcal{F}_{t_{i+1}}$. In particular, as a side product of (5.27), we can see that $(B_i)_{i=0,1,...,n}$ is an $(\mathbb{F}, P^*)$-martingale.

Having $B_i$'s determined, by Appendix D and Proposition 5.7 we now calculate $X_t$ for $t \in (t_i, t_{i+1}]$ for each $i = 0, 1, 2, ..., n-1$ as follows:

$$\begin{aligned} X_t &= \operatorname*{ess\,sup}_{\kappa\in\mathcal{D}} E^{Q^\kappa}\Big[S_{t_{i+1}} + E^*\big[(K - S_{t_n})^+\big|\mathcal{F}_{t_{i+1}}\big]\Big|\mathcal{G}_t\Big] \\ &= E^*\Big[S_{t_{i+1}} + E^*\big[(K - S_{t_n})^+\big|\mathcal{F}_{t_{i+1}}\big]\Big|\mathcal{F}_t\Big] \\ &= S_t + E^*\big[(K - S_{t_n})^+\big|\mathcal{F}_t\big] \end{aligned} \tag{5.28}$$

Since $\left(S_t + E^*\left[(K - S_{t_n})^+ \middle| \mathcal{F}_t\right]\right)_{t\in[0,T]}$ is a continuous process and consumption process $C_t^i \equiv 0$ on each subinterval $(t_i, t_{i+1}]$, it is clear that the overall consumption process $(C_t)_{t\in[0,T]}$ defined by (4.6) is zero.

Combining above with equations (4.5)-(4.7) completes the proof. □

Let us come back to the main problem of this chapter, the quantile hedging problem (5.16). Aguilar [2] also studied this problem for a general American option but he only established an upper bound for this max-min problem.

In the setting of this chapter, we solve problem (5.16) for its precise optimal value. In addition, explicit form solutions will be provided for the maximal probability and the optimal hedge which achieves this value.

**Theorem 5.9.** *For the quantile hedging problem* (5.16)*, we have*

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}}(\tilde{v}_0)} \left( \inf_{\tau\in\mathcal{S}_{0,T}(R)} P\left(V_\tau^{\tilde{v}_0,\pi} \geq U_\tau\right)\right) = \sup_{\pi\in\mathcal{A}^{\mathbb{G}}(\tilde{v}_0)} P\left(V_T^{\tilde{v}_0,\pi} \geq U_T\right) \tag{5.29}$$

*Proof.* For an arbitrary $\pi \in \mathcal{A}^{\mathbb{G}}(\tilde{v}_0)$, let $A_T := \{V_T^{\tilde{v}_0,\pi} \geq U_T\} \in \mathcal{G}_T$. Then

$$E^{Q^\kappa}\left[V_T^{\tilde{v}_0,\pi} 1_{A_T} \middle| \mathcal{G}_{t_i}\right] \geq E^{Q^\kappa}\left[U_T 1_{A_T} \middle| \mathcal{G}_{t_i}\right], \tag{5.30}$$

for all $\kappa \in \mathcal{D}$ and $i = 0, 1, 2, ..., n$. Since $\left(V_t^{\tilde{v}_0,\pi}\right)_{t\in[0,T]}$ is a $(\mathbb{G}, Q^\kappa)$-supermartingale, we get

$$V_{t_i}^{\tilde{v}_0,\pi} \geq E^{Q^\kappa}\left[V_T^{\tilde{v}_0,\pi} \middle| \mathcal{G}_{t_i}\right] \geq E^{Q^\kappa}\left[V_T^{\tilde{v}_0,\pi} 1_{A_T} \middle| \mathcal{G}_{t_i}\right]. \tag{5.31}$$

On the other hand, Proposition 5.7 implies

$$\begin{aligned} E^{Q^\kappa}\left[U_T 1_{A_T} \middle| \mathcal{G}_{t_i}\right] + E^{Q^\kappa}\left[U_T 1_{A_T^c} \middle| \mathcal{G}_{t_i}\right] &= E^{Q^\kappa}\left[U_T \middle| \mathcal{G}_{t_i}\right] \\ = E^*\left[U_T \middle| \mathcal{F}_{t_i}\right] = E^*\left[B_n \middle| \mathcal{F}_{t_i}\right] &= B_i \geq U_{t_i}, \end{aligned} \tag{5.32}$$

where we used the martingale property of $(B_i)_{i=0,1,\ldots,n}$ from the proof of Proposition 5.8, and also the fact that by the definition $B_i$ dominates $U_i$. Combing (5.30) - (5.32), we can see that

$$V_{t_i}^{\tilde{v}_0,\pi} \geq U_{t_i} - E^{Q^\kappa}\big[U_T 1_{A_T^c} \big| \mathcal{G}_{t_i}\big]. \tag{5.33}$$

Multiplying both sides of the above inequality by $1_{A_T}$ and then applying Lemma 1 to $E^{Q^\kappa}\big[U_T 1_{A_T^c}\big|\mathcal{G}_{t_i}\big]1_{A_T}$, we have shown that

$$V_{t_i}^{\tilde{v}_0,\pi} 1_{A_T} \geq U_{t_i} 1_{A_T}. \tag{5.34}$$

Therefore, on $A_T = \{V_T^{\tilde{v}_0,\pi} \geq U_T\} \in \mathcal{G}_T$, one can write

$$V_{t_i}^{\tilde{v}_0,\pi} 1_{\{\tau=t_i\}} \geq U_{t_i} 1_{\{\tau=t_i\}}, \qquad \text{for any } \tau \in \mathcal{S}_{0,T}(R). \tag{5.35}$$

This leads to

$$V_\tau^{\tilde{v}_0,\pi} = \sum_{i=0}^{n} V_{t_i}^{\tilde{v}_0,\pi} 1_{\{\tau=t_i\}} \geq \sum_{i=0}^{n} U_{t_i} 1_{\{\tau=t_i\}} = U_\tau\,. \tag{5.36}$$

Hence we obtain

$$\big\{V_T^{\tilde{v}_0,\pi} \geq U_T\big\} \subseteq \big\{V_\tau^{\tilde{v}_0,\pi} \geq U_\tau\big\}, \qquad \text{for any } \tau \in \mathcal{S}_{0,T}(R), \tag{5.37}$$

which this concludes that

$$\inf_{\tau \in \mathcal{S}_{0,T}(R)} P\big(V_\tau^{\tilde{v}_0,\pi} \geq U_\tau\big) = P\big(V_T^{\tilde{v}_0,\pi} \geq U_T\big) \tag{5.38}$$

and the proof is complete. □

Theorem 5.9, in fact, reduces quantile hedging problem of the Bermudan option $(U, R)$ to the corresponding problem for the European option $U_T = \text{Max}(K, S_T)$ at the maturity time $T$. We recall that GMDB contract is managed in a separate account format. This means the underlying asset $(S_t)_{t\in[0,T]}$ is reserved and available at the maturity to be paid to the insurer, and the actual liability of the insurer to fulfill his financial obligation is the put option $(K - S_T)^+$ not $\text{Max}(K, S_T) = S_T + (K - S_T)^+$.

By Theorem 5.9 and the above discussion, problem 5.16 simplifies to

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}}(\tilde{v}_0)} P\big(V_T^{\tilde{v}_0,\pi} \geq (K - S_T)^+\big)\,. \tag{5.39}$$

Notice that the above maximization problem runs over trading strategies $\pi \in \mathcal{A}^{\mathbb{G}}(\tilde{v}_0)$. We will exploit a Neyman-Pearson lemma argument from Follmer and Leukert [14], Assumption 5.3 and the decomposition of $\mathbb{G}$-adapted processes in terms of $\mathbb{F}$-adapted processes to replace $\mathcal{A}^{\mathbb{G}}(\tilde{v}_0)$ with $\mathcal{A}^{\mathbb{F}}(\tilde{v}_0)$.

**Theorem 5.10.** *Let $Y$ be a nonnegative $\mathcal{F}_T$-measurable random variable and $\tilde{v}_0 > 0$. Then we have*

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}}(\tilde{v}_0)} P\big(V_T^{\tilde{v}_0,\pi} \geq Y\big) = \sup_{\pi\in\mathcal{A}^{\mathbb{F}}(\tilde{v}_0)} P\big(V_T^{\tilde{v}_0,\pi} \geq Y\big)\,. \tag{5.40}$$

*Proof.* Let $\hat{A} \in \mathcal{G}_T$ be a solution to the problem

$$\underset{A\in\mathcal{G}_T}{\text{Max}}\, P(A) \tag{5.41}$$

subject to the constraint

$$\sup_{\kappa\in\mathcal{D}} E^{Q^\kappa}[Y 1_A] \leq \tilde{v}_0\,. \tag{5.42}$$

Then the trading strategy $\hat{\pi} \in \mathcal{A}^{\mathbb{G}}(\tilde{v}_0)$ obtained from the optional decomposition of

$$\operatorname*{ess\,sup}_{\kappa\in\mathcal{D}} E^{Q^\kappa}\left[Y1_{\hat{A}}\,\middle|\,\mathcal{G}_t\right]$$

solves the problem

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}}(\tilde{v}_0)} P\Big(V_T^{\tilde{v}_0,\pi}\geq Y\Big)\,. \tag{5.43}$$

Moreover, the maximal probability of success is given by:

$$P\Big(V_T^{\tilde{v}_0,\hat{\pi}}\geq Y\Big)=P(\hat{A})\,. \tag{5.44}$$

Similarly, suppose $\tilde{A}\in\mathcal{F}_T$ is a solution to the following problem:

$$\operatorname*{Max}_{A\in\mathcal{F}_T} P(A) \tag{5.45}$$

subject to the constraint

$$E^*[Y1_A]\leq\tilde{v}_0\,. \tag{5.46}$$

Then $\tilde{\pi}\in\mathcal{A}^{\mathbb{F}}(\tilde{v}_0)$ the perfect hedge of the modified claim $Y1_{\tilde{A}}$ solves the problem

$$\sup_{\pi\in\mathcal{A}^{\mathbb{F}}(\tilde{v}_0)} P\Big(V_T^{\tilde{v}_0,\pi}\geq Y\Big), \tag{5.47}$$

and the maximum probability is equal to:

$$P\Big(V_T^{\tilde{v}_0,\tilde{\pi}}\geq Y\Big)=P(\tilde{A})\,. \tag{5.48}$$

Since $\mathcal{A}^{\mathbb{F}}(\tilde{v}_0)\subseteq\mathcal{A}^{\mathbb{G}}(\tilde{v}_0)$, it is easy to see that

$$P(\tilde{A})\leq P(\hat{A})\,. \tag{5.49}$$

To finish the proof, we establish the reverse inequality. By Pham [44], there exists an $\mathbb{F}$-predictable process $(\hat{\pi}_t^{\mathbb{F}})_{t\in[0,T]}$ and a family of $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}_+)$-measurable functions

$$\left\{\left(\hat{\pi}_t^d(\theta)\right)_{\theta\leq t\leq T}\ :\ \text{for } \theta\in[0,T]\right\}$$

where for any fixed $\theta \in [0,T]$ the process $\left(\hat{\pi}_t^d(\theta)\right)_{\theta\leq t\leq T}$ is $\mathbb{F}$-predictable. Moreover, the $\mathbb{G}$-predictable process $(\hat{\pi}_t)_{t\in[0,T]} \in \mathcal{A}^{\mathbb{G}}(\tilde{v}_0)$ admits the following decomposition:

$$\hat{\pi}_t = \hat{\pi}_t^{\mathbb{F}} 1_{\{t\leq T(x)\}} + \hat{\pi}_t^d(\theta) 1_{\{t>T(x)\}}, \tag{5.50}$$

for any fixed $\theta \in [0,T]$ and $t \in [\theta, T]$.

This, in particular, implies

$$V_T^{\tilde{v}_0,\hat{\pi}} = V_T^{\tilde{v}_0,\hat{\pi}^{\mathbb{F}}} 1_{\{T<T(x)\}} + V_T^{\tilde{v}_0,\hat{\pi}^d(\theta)} 1_{\{T\geq\theta\}}\Big|_{\theta=T(x)}. \tag{5.51}$$

By (5.47) and (5.48), we get

$$P\big(V_T^{\tilde{v}_0,\hat{\pi}^{\mathbb{F}}} \geq Y\big) \leq P(\tilde{A}). \tag{5.52}$$

With a similar argument, for any $\theta \in [0,T]$

$$P\big(V_T^{\tilde{v}_0,\hat{\pi}^{\mathbb{F}}(\theta)} \geq Y\big) \leq P(\tilde{A}). \tag{5.53}$$

Using the results of Dellacherie and Meyer [9] and Coculescu and Nikeghbali [6] for computing expectation involving random times, we write

$$\begin{gathered}
P\big(V_T^{\tilde{v}_0,\hat{\pi}^{\mathbb{F}}(\theta)} \geq Y\big) P\big(T(x)=\theta\big) \leq P(\tilde{A}) P\big(T(x)=\theta\big) \\
\int_0^T P\big(V_T^{\tilde{v}_0,\hat{\pi}^{\mathbb{F}}(\theta)} \geq Y\big) P\big(T(x)=\theta\big) d\theta \leq \int_0^T P(\tilde{A}) P\big(T(x)=\theta\big) d\theta \\
E\Big[\int_0^T 1_{\left\{V_T^{\tilde{v}_0,\hat{\pi}^{\mathbb{F}}(\theta)}\geq Y\right\}} P\big(T(x)=\theta\big) d\theta\Big] \leq P(\tilde{A}) \int_0^T P\big(T(x)=\theta\big) d\theta
\end{gathered}$$

$$P\Big(\big\{V_T^{\tilde{v}_0,\hat{\pi}^{\mathbb{F}}(T(x))} \geq Y\big\} \cap \big\{T(x) \leq T\big\}\Big) \leq P(\tilde{A})P\big(T(x) \leq T\big) \tag{5.54}$$

By multiplying both sides of (5.52) by $P\big(T(x) > T\big)$ and then combining with (5.54) and (5.44), we can see that

$$\begin{aligned} P(\hat{A}) &= P\big(V_T^{\tilde{v}_0,\hat{\pi}} \geq Y\big) \\ &= P\Big(\big\{V_T^{\tilde{v}_0,\hat{\pi}^{\mathbb{F}}} \geq Y\big\} \cap \big\{T(x) > T\big\}\Big) + P\Big(\big\{V_T^{\tilde{v}_0,\hat{\pi}^{\mathbb{F}}(T(x))} \geq Y\big\} \cap \big\{T(x) \leq T\big\}\Big) \\ &\leq P(\tilde{A}) \end{aligned}$$

Therefore the reverse inequality of (5.49) is proved, and this means $\tilde{A} = \big\{V_T^{\tilde{v}_0,\tilde{\pi}} \geq Y\big\}$ solves problem (5.41)-(5.42). As an immediate consequence, $\tilde{\pi} \in \mathcal{A}^{\mathbb{F}}(\tilde{v}_0)$ is a solution to problem (5.39) and the proof is finished.

□

Finally, we utilize the techniques of Follmer and Leukert [14] to provide an explicit form solution:

**Theorem 5.11.** *For a given initial capital $\tilde{v}_0 > 0$ consider the optimization problem (5.39). Define the European option $\tilde{H}$ as follows*

$$\tilde{H} := (K - S_T)^+ - (\tilde{K} - S_T)^+ - (K - \tilde{K})1_{\{S_T \leq K\}},$$

*where $\tilde{K} \in (0, K)$ is a constant subject to the constraint*

$$E^*[\tilde{H}] = \tilde{v}_0\,.$$

*Then the perfect hedge $(\tilde{\pi}_t)_{t\in[0,T]} \in \mathcal{A}^{\mathbb{F}}(\tilde{v}_0)$ for $\tilde{H}$ solves problem (5.39), and the maximal probability of success is given by:*

$$P\big(V_T^{\tilde{v}_0,\tilde{\pi}} \geq (K - S_T)^+\big) = 1 - \Phi\Big(\frac{\ln \tilde{K} - \ln S_0}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\Big),$$

*where $\Phi$ is the standard normal distribution function.*

*Proof.* Keeping in mind Theorem 5.10, define

$$\tilde{A} := \Big\{ \frac{dP}{dP^*} > const.\,(K - S_T)^+ \Big\}.$$

By Follmer and Leukert [14], the replicating strategy $(\tilde{\pi}_t)_{t \in [0,T]} \in \mathcal{A}^{\mathbb{F}}(\tilde{v}_0)$ for the modified claim $(K - S_T)^+ 1_{\tilde{A}}$ is an optimal solution to the problem

$$\sup_{\pi \in \mathcal{A}^{\mathbb{F}}(\tilde{v}_0)} P\big(V_T^{\tilde{v}_0,\pi} \geq (K - S_T)^+\big) \tag{5.55}$$

with the maximal probability of success

$$P\big(V_T^{\tilde{v}_0,\tilde{\pi}} \geq (K - S_T)^+\big) = P(\tilde{A})\,.$$

By the definition of $P^*$ from (5.4) and the unique solution to SDE (5.3), we can rewrite $\tilde{A}$ as follows

$$\tilde{A} = \Big\{ S_T^{\frac{\mu}{\sigma^2}} > \tilde{\lambda}(K - S_T)^+ \Big\}, \tag{5.56}$$

where $\tilde{\lambda}$ is a positive constant that can be determined from

$$E^*\big[(K - S_T)^+ 1_{\tilde{A}}\big] = \tilde{v}_0\,. \tag{5.57}$$

Regardless of $\frac{\mu}{\sigma^2} \leq 1$ or $\frac{\mu}{\sigma^2} > 1$, the increasing function $s^{\frac{\mu}{\sigma^2}}$ intersects with $\tilde{\lambda}(K - s)^+$ at exactly one point $\tilde{K} \in (0, K)$. Hence, from (5.56), we obtain

$$\begin{aligned} \tilde{A} = \big\{S_T > \tilde{K}\big\} &= \Big\{ S_0 \exp\big(\sigma W_T + (\mu - \frac{1}{2}\sigma^2)T\big) > \tilde{K} \Big\} \\ &= \Big\{ S_0 \exp\big(\sigma W_T^* - \frac{1}{2}\sigma^2 T\big) > \tilde{K} \Big\} = \big\{ W_T^* > \bar{K} \big\}, \end{aligned} \tag{5.58}$$

where $\bar{K} > 0$ is again a constant to be determined, and

$$\big(W_t^*\big)_{t\in[0,T]} := \big(W_t + \frac{\mu}{\sigma}t\big)_{t\in[0,T]},$$

by Girsanov's theorem, is a standard $(\mathbb{F}, P^*)$-Brownian motion.

To exploit the Black-Scholes formula, we represent the modified claim $(K - S_T)^+ 1_{\tilde{A}}$ as follows

$$(K - S_T)^+ 1_{\tilde{A}} = (K - S_T)^+ - (\tilde{K} - S_T)^+ - (K - \tilde{K}) 1_{\{S_T \le K\}}$$

By this and (5.57), constant $\tilde{K}$ can be computed from

$$\begin{aligned}\tilde{v}_0 &= E^*\big[(K - S_T)^+ 1_{\tilde{A}}\big] = K\Phi(-d_-(K)) - S_0\Phi(-d_+(K)) \\ &- \tilde{K}\Phi(-d_-(\tilde{K})) + S_0\Phi(-d_+(\tilde{K})) - (K - \tilde{K})\Phi(-d_-(K)) \\ &= S_0\Phi(-d_+(\tilde{K})) - \tilde{K}\Phi(-d_-(\tilde{K})) - S_0\Phi(-d_+(K)) + \tilde{K}\Phi(-d_-(K)),\end{aligned} \tag{5.59}$$

where for any $z > 0$

$$d_\pm(z) := \frac{\ln S_0 - \ln z}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}.$$

After finding $\tilde{K}$ from (5.57) and (5.58), the maximal probability of success is given by

$$\begin{aligned}P(\tilde{A}) &= P\Big(S_0 \exp\big(\sigma W_T + (\mu - \frac{1}{2}\sigma^2)T\big) > \tilde{K}\Big) \\ &= 1 - \Phi\big(-d_-(\tilde{K})\big).\end{aligned} \tag{5.60}$$

□

## 5.4 Conclusion

Considering a Black-Scholes model, we define a Guaranteed minimum Death Benefit (GMDB) life insurance contract in this market. The main aim of this chapter is to solve the quantile hedging problem (5.16) under an initial capital constraint. To do so, we progressively enlarge the filtration generated by the underlying asset with the filtration generated by the survival process of the insured. Then the GMDB contract is considered as a Bermudan option on the probability space equipped with the enlarged filtration. Independency assumption between the mortality risk and the financial risk combined with the minimum guarantee structure of the payment simplifies the superhedging method into a perfect hedge in the original complete market. Moreover, the max-min problem corresponding to the quantile hedging problem of the GMDB contract in the enlarged filtration is converted into a straightforward quantile hedging problem for a European put option in the filtration generated by the Black-Sholes model.

Our approach has the potential to be generalized to the case of any American style contingent claim exercised at random times independent of the underlying risky asset. The results of this chapter show how to deal with conditional expectations and value processes in the enlarged filtration, these results can be applied in a similar framework.

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# Appendix A

# An Equivalent Static Optimization Problem

The following theorem by Follmer and Leukert [15] guarantees a solution for the minimization problem (4.20).

**Theorem 1.** *Let* $\mathcal{R} = \big\{\varphi : \Omega \longrightarrow [0,1] \;\big|\; \varphi \in \mathcal{G}_T\big\}$*, defined as in Lemma 4.7, then*

1. *There exists* $\tilde{\varphi} \in \mathcal{R}$ *that solves the problem*

$$\min_{\varphi \in \mathcal{R}} E\big[l\big((1-\varphi)H_\delta\big)\big] \tag{1.1}$$

*under the constraint*

$$\sup_{\kappa \in \mathcal{D}^n} E\big[\rho_T^* \rho_T^\kappa \varphi H_\delta\big] \leq \tilde{u}.$$

*Since* $l' > 0$ *then any two solutions are equal* $P$*-a.s. on* $\{H_\delta > 0\}$*. Moreover we can assume that* $\tilde{\varphi} = 1$ *on* $\{H_\delta = 0\}$*.*

2. *The hedging strategy* $(\tilde{u}, \tilde{\pi})$ *of the modified claim* $\tilde{\varphi} H_\delta$ *(obtained from the optional decomposition theorem) solves minimization problem* (4.20)*.*

# Appendix B

# The Algorithm of Solving the Dual Problem

By Cvitanić and Karatzas [8], the solution to the dual problem (4.27), $(\tilde{z}, \tilde{L})$, is determined by the following method:

*(1)* For any $z \geqslant 0$, there exists $\tilde{L}(z) \in \mathcal{L}$ that solves

$$\tilde{V}(z) := \inf_{L \in \mathcal{L}} E\Big[\big(\frac{d\bar{P}}{dP} - z\rho_T^* L H_\delta\big)^+\Big]$$

*(2)* For any given $\tilde{u} \in (0, E^*[H])$, let $V_*(\tilde{u})$ to be defined as (4.27) then we have

$$V_*(\tilde{u}) = \inf_{z>0} \big\{\tilde{u}z + \tilde{V}(z)\big\}.$$

In addition, there exists $\tilde{z}(\tilde{u}) > 0$ that attains this infimum.

*(3)* For any given $\tilde{u} \in \big(0, E^*[H]\big)$, there exists $(\tilde{z}, \tilde{L}) := \Big(\tilde{z}(\tilde{u}), \tilde{L}\big(\tilde{z}(\tilde{u})\big)\Big) \in \mathbb{R}^+ \times \mathcal{L}$ such that

$$V_*(\tilde{u}) = \tilde{u}\tilde{z} + E\Big[\big(\frac{d\bar{P}}{dP} - \tilde{z}\rho_T^* \tilde{L} H_\delta\big)^+\Big]. \tag{2.1}$$

*(4)* By above, $(\tilde{z}, \tilde{L})$ is given by

$$\tilde{z} := \arg\min_{z>0} \big[\tilde{u}z + \tilde{V}(z)\big]$$

and

$$\tilde{L} := \tilde{L}(\tilde{z}).$$

After finding $(\tilde{z}, \tilde{L})$ by the above algorithm, the optimal randomized test $\tilde{\varphi}$ is given by

$$\tilde{\varphi} = 1_{\left\{\tilde{z}\rho_T^*\tilde{L}H_\delta < \frac{d\bar{P}}{dP}\right\}} + \tilde{B}1_{\left\{\tilde{z}\rho_T^*\tilde{L}H_\delta = \frac{d\bar{P}}{dP}\right\}}, \qquad P\text{-a.s.}$$

where $\tilde{B}$ is a $\mathcal{G}_T$-random variable with values in $[0,1]$, and $\tilde{\varphi}$ satisfies the constraint

$$E\left[\rho_T^*\tilde{L}\tilde{\varphi}H_\delta\right] = \tilde{u}.$$

Moreover, we have

$$E^{\bar{P}}\left[\tilde{\varphi}\right] = V_*(\tilde{u}). \tag{2.2}$$

# Appendix C

# Measurable Selection

Suppose that $(\Omega, \mathcal{F}, P)$ is a complete probability space and $(M, \mathcal{B}(M), d)$ is a complete separable metric space with the Borel $\sigma$-field $\mathcal{B}(M)$ and the metric $d$.

**Theorem 1 (Aumann's Measurable Selection Theorem).** *Let $\Gamma : \Omega \longrightarrow 2^M$ be a nonempty set-valued $\mathcal{F} \otimes \mathcal{B}(M)$-measurable function. In other words, for all $\omega \in \Omega$, $\Gamma(\omega)$ is a nonempty subset of $M$ and*

$$Graph(\Gamma) := \big\{(\omega, m) \,:\, m \in \Gamma(\omega)\big\} \in \mathcal{F} \otimes \mathcal{B}(M).$$

*Then there exists a measurable function $f : \Omega \longrightarrow M$ such that*

$$f(\omega) \in \Gamma(\omega), \qquad \text{for all } \omega \in \Omega.$$

*The function $f$ is known as a measurable selection for the set-valued function $\Gamma$.*

See, for instance, Wagner [49] and Kabanov and Pergamenshchikov [21] for more details on this topic.

In the proof of Theorem 2.1, we have applied the above theorem for the set-valued function

$$\Gamma : \Omega \times \mathbb{R}_+ \longrightarrow 2^{\mathbb{R}_+}$$

with $\Gamma(\omega, \tilde{x}) = \tilde{a}_s(w, \tilde{x})$ defined as in equation (2.8).

# Appendix D

# Bermudan Option Hedging

In the following, we recall the superhedging strategy of a Bermudan option from Schweizer [48]:

Assume $(U, R)$ be a Bermudan option with $R = \{t_1, t_2, ...., t_n\} \subseteq [0, T]$ for some $n \in \mathbb{N}$ and $0 =: t_0 < t_1 < t_2 < ... < t_n = T$ such that

$$X_0 = \sup_{\substack{\kappa \in \mathcal{D} \\ i=1,2,...,n}} E^{Q^\kappa}\left[U_{t_i}\right] < +\infty.$$

Using a backward induction argument, let us define process $(B_i)_{i=0,1,2,...,n}$ as follows

$$B_n := U_{t_n} \tag{4.1}$$

and

$$B_i := \text{Max}\left(U_{t_i}, \operatorname*{ess\,sup}_{\kappa \in \mathcal{D}} E^{Q^\kappa}\left[B_{t_{i+1}} | \mathcal{G}_{t_i}\right]\right) \tag{4.2}$$

for $i = 0, 1, 2, ..., n-1$.

Then for the superhedging value process $(X_t)_{t \in [0,T]}$ introduced in (5.13) we have

$$X_{t_i} = B_i \qquad P\text{-a.s., for all } i = 0, 1, 2, ..., n. \tag{4.3}$$

Moreover, process $(X_t)_{t \in [0,T]}$ has an RCLL version on each subinterval $(t_i, t_{i+1})$ such that

$$X_t = X_{t_i+} + \int_{t_i}^{t} \pi_s^{(i)} dS_s - C_t^i \qquad \text{for } t \in (t_i, t_{i+1}], \tag{4.4}$$

for some $S$-integrable $\mathbb{R}$-valued $\mathbb{G}$-predictable process $(\pi_t^{(i)})_{t \in (t_i, t_{i+1}]}$ and a non-negative increasing $\mathbb{G}$-optional process $(C_t^{(i)})_{t \in (t_i, t_{i+1}]}$ with $C_{t_i}^{(i)} \equiv 0$. We set $X_{0+} := X_0$.

By attaching all the above $n$ trading strategies $\pi^{(i)}$ and the consumption processes $C^i$, for all $t \in [0, T]$ we now define:

$$\pi_t := \sum_{i=0}^{n-1} \pi_t^{(i)} 1_{]\!]t_i, t_{i+1}]\!]} \tag{4.5}$$

and

$$C_t := \sum_{\substack{t_i \le t \\ i=0,1,\dots,n-1}} C^i_{t_i} + \sum_{i=0}^{n-1} C^i_t 1_{\rrbracket t_i, t_{i+1} \llbracket} + \sum_{\substack{t_i < t \\ i=0,1,\dots,n-1}} (X_{t_i} - X_{t_i+}). \tag{4.6}$$

Combining (4.5) and (4.6), we obtain

$$X_t = X_0 + \int_0^t \pi_s dS_s - C_t \qquad \text{for } t \in [0,T]. \tag{4.7}$$

We used the next lemma in the proof of Theorem 5.9.

**Lemma 1.** *Let $s \le t$ and $A \in \mathcal{G}_t$. Then for any nonnegative $\mathcal{G}_t$-measurable random variable $Y$, we have*

$$E\big[Y 1_{A^c} \big| \mathcal{G}_s\big] 1_A = 0, \qquad P\text{-}a.s. \tag{4.8}$$

*Proof.* We prove this lemma by contradiction. Let

$$B := \left\{\omega \in A \,:\, E\big[Y 1_{A^c} \big| \mathcal{G}_s\big] \neq 0\right\}.$$

Then, by $B \in \mathcal{G}_s$, we get

$$0 < \int_B E\big[Y 1_{A^c} \big| \mathcal{G}_s\big] dP = \int_B Y 1_{A^c} dP = \int_{B \cap A^c} Y dP \tag{4.9}$$

On the other hand, $B \subseteq A$ implies $\int_{B \cap A^c} Y dP = 0$. Therefore $B$ must be $P$ almost sure empty and (4.8) is satisfied. □