Topics in Convex Geometric Analysis and Discrete Tomography

by

Ning Zhang

A thesis submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

MATHEMATICS

Department of Mathematical and Statistical Sciences

University of Alberta

Abstract

In this thesis, some topics in convex geometric analysis and discrete tomography are studied. Firstly, let K be a convex body in the n-dimensional Euclidean space. Is K uniquely determined by its sections? There are classical results that explain what happens in the case of sections passing through the origin. However, much less is known about sections that do not contain the origin. Here, several problems of this type and the corresponding uniqueness results are established. We also establish a discrete analogue of the Aleksandrov theorem for the areas and the surface areas of projections. Finally, we find the best constant for the Grünbaum's inequality for projections, which generalizes both Grünbaum's inequality, and an old inequality of Minkowski and Radon.

Preface

Chapter 3 of this thesis has been accepted for publication as V. Yaskin and N. Zhang, "Non-central sections of convex bodies", Israel J. Math., DOI:10.1007/s11856-017-1532-9. I was responsible for mathematical proof as well as the manuscript composition. Prof. V. Yaskin was the supervisory author and was involved in concept formation, mathematical proof, and manuscript composition.

Chapter 4 of this thesis is an original work by N. Zhang and is based on the preprint N. Zhang, "On bodies with congruent sections by cones or non-central planes".

Chapter 5 of this thesis is an original work by N. Zhang and has been published as N. Zhang, "An analogue of the Aleksandrov projection theorem for convex lattice polygons", Proc. Amer. Math. Soc., 145 (2017), 2305–2310.

Chapter 6 of this thesis has been published as D. Ryabogin, V. Yaskin, and N. Zhang, "Unique determination of convex lattice sets", Discrete Comput. Geom. (3) 57 (2017), 582–589. I was responsible for mathematical proof as well as the manuscript composition. Prof. D. Ryabogin was involved in concept formation and manuscript edits. Prof. V. Yaskin was the supervisory author and was involved in concept formation, mathematical proof, and manuscript composition.

Chapter 7 of this thesis has been published as M. Stephen and N. Zhang, "Grünbaum's inequality for projections", J. Funct. Anal. (6) 272 (2017), 2628–2640. I was responsible for mathematical proof as well as the manuscript composition. M. Stephen was involved in concept formation, mathematical proof, and manuscript composition.

Acknowledgements

I would like to express my deepest appreciation to my supervisor, Prof. Vladyslav Yaskin, for his enthusiastic encouragement, patient guidance and useful critiques during my studies. Fortunately, his original thought, creative insight, and rich knowledge have inspired me throughout this thesis. I particularly express my gratitude to him for bringing me to the subjects of convex geometry and geometric functional analysis, in particular, introducing to me the geometric tomography and promoting the progress of my thesis.

Besides my advisor, I would like to thank the rest of my thesis committee members for their insightful comments. I would also like to thank Prof. Alexander Litvak and Prof. Nicole Tomczak-Jaegermann for their words of encouragement and support.

My sincere thanks also go to Prof. Jie Xiao, Prof. Dmitry Ryabogin, and Prof. Deping Ye, who provided me an opportunity to work with them, and who gave hard questions which incented me to widen my research from various perspectives.

I thank my fellow workmates, Matthew Stephen and Han Hong, for the stimulating discussions, for the sleepless nights we were working together, and for all the fun we have had in the last four years.

Last but not the least, I would like to thank my parents for supporting me spiritually throughout writing this thesis and my life in general.

Table of Contents

List of Figures

Chapter 1

Introduction

The contents of this thesis have two main directions pertaining to the results obtained in [29], [33], [35], [38], and [37]. One is geometric tomography dealing with the unique determination of convex bodies or discrete convex lattice sets from the size of their sections or projections. Another one is about the Grünbaum inequality.

1.1 Geometric tomography

The area of mathematics dealing with properties of objects (e.g. convex bodies or star bodies) based on the size of sections, projections, etc, is known as geometric tomography. It gives a mathematical basis for imaging by sections or projections, through the use of penetrating waves. This method can be used to reconstruct a three-dimensional object from its two-dimensional images and is applicable in archaeology, astrophysics, atmospheric science, biology, geophysics, materials science, oceanography, plasma physics, quantum information theory, and radiology.

Below is Minkowski-Funk's section theorem (coming from Minkowski's and Funk's works on projections and central sections (cf. [11, 22, 31]). Here and below, set $n \geq 2$.

Theorem 1.1.1. *Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$. Assume that $\mathrm{vol}_{n-1}(K\cap\xi^\perp) = \mathrm{vol}_{n-1}(L\cap\xi^\perp)$ for every $\xi \in S^{n-1}$, where $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi\rangle = 0\}$. Then $K = L$.*

The original approach to this theorem is to use spherical harmonics. A Fourier transform proof can be found in Koldobsky's book [22]. This theorem is false without the symmetry condition. For non-symmetric bodies, Falconer [8] and Gardner [11] proved the following result independently.

Theorem 1.1.2. *Let K and L be convex bodies in $\mathbb{R}^n$ containing two distinct points p and q in their interior. If*

$$\mathrm{vol}_{n-1}((K-p)\cap\xi^\perp) = \mathrm{vol}_{n-1}((L-p)\cap\xi^\perp)$$

and

$$\mathrm{vol}_{n-1}((K-q)\cap\xi^\perp) = \mathrm{vol}_{n-1}((L-q)\cap\xi^\perp)$$

for every $\xi \in S^{n-1}$, then $K = L$.

Recently, a lot of attention has been attracted to the following problem about non-central sections. It was posed by Barker and Larman in [3], though a similar question on the sphere was considered earlier by Santaló [30].

Problem 1.1.3. *Let K and L be convex bodies in $\mathbb{R}^n$ that contain a Euclidean ball B in their interiors. If $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ for every hyperplane H that supports B, does it follow that $K = L$?*

This problem is still open even in $\mathbb{R}^2$. Some particular cases are known to be true. In particular, a body K in $\mathbb{R}^2$ all of whose intersections with lines supporting a disk B have the same length, must itself be a disk; see [3]. The problem also has a positive answer in the class of convex polytopes in $\mathbb{R}^n$; see [34]. Barker and Larman also suggested a more general version of Problem 1.1.3.

Problem 1.1.4. *Let K and L be convex bodies in $\mathbb{R}^n$ that contain a convex body D in their interiors. If $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ for every hyperplane H that supports D, does it follow that $K = L$?*

In Chapter 3, we study the following question in the spirit of Gardner-Falconer's result.

Problem 1.1.5. *Let K and L be convex bodies in $\mathbb{R}^n$ that contain two convex bodies D_1 and D_2 in their interiors. If $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ for every hyperplane H that supports either D_1 or D_2, does it follow that $K = L$?*

We prove that Problem 1.1.5 has a positive answer in $\mathbb{R}^2$ under some mild assumptions on D_1 and D_2. We also study the following closely related problem.

Problem 1.1.6. *Let K and L be convex bodies in $\mathbb{R}^n$ and let D be a convex body in the interior of $K \cap L$. If $\mathrm{vol}_n(K \cap H^+) = \mathrm{vol}_n(L \cap H^+)$ for every hyperplane H supporting D, does it follow that $K = L$? Here, H^+ is the half-space bounded by the hyperplane H that does not intersect the interior of D.*

In $\mathbb{R}^n$ $(n \geq 3)$, we give a positive answer to a certain modification of this problem. In $\mathbb{R}^2$, we obtain some partial results. If D is a disk in $\mathbb{R}^2$ and $K \subset \mathbb{R}^2$ is a convex body such that $\mathrm{vol}_2(K \cap H^+) = \mathrm{const}$ for all H supporting D, then K is also a disk. We also solve a modification of Problem 1.1.6 by adding another body inside $K \cap L$ as in Problem 1.1.5. In higher dimensions, we established the following results.

Theorem 1.1.7. *Let K and L be convex bodies in $\mathbb{R}^n$ containing two distinct points p and q in their interiors. If for every half-space E whose boundary contains either p or q one has $\mathrm{vol}_n(K \cap E) = \mathrm{vol}_n(L \cap E)$, then $K = L$.*

Theorem 1.1.8. *Let K and L be convex bodies in $\mathbb{R}^n$ (where n is even) and let D be a cube in the interior of $K \cap L$. If $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ for any hyperplane passing through a vertex of D and an interior point of D, then $K = L$.*

Groemer showed in [16] that convex bodies are uniquely determined by the areas of "half-sections".

Theorem 1.1.9. *Let K and L be star bodies in $\mathbb{R}^n$. For $\xi \in S^{n-1}$ and $v \in \xi^\perp \cap S^{n-1}$ define $H(\xi, v) = \{x \in \mathbb{R}^n : x \in \xi^\perp$ and $\langle x, v\rangle \geq 0\}$. If*

$$\mathrm{vol}_{n-1}(K \cap H(\xi, v)) = \mathrm{vol}_{n-1}(L \cap H(\xi, v))$$

for every $\xi \in S^{n-1}$ and $v \in \xi^\perp \cap S^{n-1}$, then $K = L$.

We obtain a version of this result for non-central half-sections.

Theorem 1.1.10. *Let K and L be convex bodies in $\mathbb{R}^n$, $n \geq 3$ that contain a ball D in their interiors. For a fixed point $p = D \cap H$ and $v \in S^{n-1}$ set $v_p^+ = \{x \in \mathbb{R}^n : \langle x - p, v\rangle \geq 0\}$. If $\mathrm{vol}_{n-1}(K \cap H \cap v_p^+) = \mathrm{vol}_{n-1}(L \cap H \cap v_p^+)$ for every H supporting D and every unit vector $v \in H - p$, then $K = L$.*

Our next result gives a solution Problem 1.1.4 for bodies of revolution when the body D is in some special position.

Theorem 1.1.11. *Let K and L be convex bodies of revolution in $\mathbb{R}^n$ with the same axis of revolution. Let D be a convex body in the interior of both K and L such that D does not intersect the axis of revolution. If $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ for every hyperplane H supporting D, then $K = L$.*

There are also many questions in geometric tomography concerning bodies with congruent sections or projections. The following problem is contained in Gardner's book [11, Page 289].

Problem 1.1.12. *Suppose that $2 \leq k \leq n-1$ and that K and L are star bodies in $\mathbb{R}^n$ such that the section $K \cap H$ is congruent to $L \cap H$ for all $H \in G(n,k)$. Is K a translate of $\pm L$?*

Here, $K \cap H$ being congruent to $L \cap H$ means that there exists an orthogonal transformation φ in H such that $\varphi(K \cap H)$ is a translate of $L \cap H$. The answer is affirmative when considering translates only (cf. [11, Theorem 7.1.1]). For the case of rotations only, in [28] Ryabogin gave an affirmative answer when $k = 2$. Some partial results were obtained by Alfonseca, Cordier, and Ryabogin in [1]. In general, this problem is still open. Below we study a version of this problem. For $t \in (0,1)$, we define

$$C_t(\xi) := \{x \in \mathbb{R}^n : \langle x, \xi \rangle = t|x|\}$$

to be a cone in the direction of ξ.

Problem 1.1.13. *Let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin in their interiors and $t \in (0,1)$. Assume that for every $\xi \in S^{n-1}$ there is a rigid motion ϕ_ξ such that $K \cap C_t(\xi) = \phi_\xi(L \cap C_t(\xi))$. Does it follow that $K = L$?*

In Chapter 4, we give an affirmative answer to this problem under the assumption that the bodies have C^2 boundaries.

Theorem 1.1.14. *Let $K, L \subset \mathbb{R}^3$ be C^2 convex bodies containing the origin in their interiors and $t \in (0,1)$. Assume that for every $\xi \in S^{n-1}$ there is a rotation ϕ_ξ preserving ξ such that $K \cap C_t(\xi) = \phi_\xi(L \cap C_t(\xi))$. Then $K = L$.*

1.2 Discrete tomography

One of subareas of geometric tomography is discrete tomography which is concerned with the problem of reconstructing finite subsets of the integer lattice from their sections or projections (see [20]). The following is known as Shepp's problem.

Problem 1.2.1. *Let K be a finite subset of $\mathbb{Z}^n$. Is K determined by all its discrete point X-rays?*

Here, the *discrete point X-ray* of K at p in the direction $u \in S^n$ is defined by

$$X_p K(u) = |K \cap (L[O, u] + p)|,$$

where $|\cdot|$ is the cardinality of the corresponding finite set, and $L[O, u]$ is the line passing through O in the direction u. The answer is negative even in dimension 2. Therefore, it is natural to ask whether it is true for discrete point X-rays at two distinct points. Dulio, Gardner, and Peri [7] gave a positive answer to this question under some conditions. They studied convex lattice sets in $\mathbb{Z}^n$ (i.e. those finite subsets $K \subset \mathbb{Z}^n$, for which $K = \operatorname{conv}(K) \cap \mathbb{Z}^n$, where $\operatorname{conv}(K)$ is the convex hull of K) and established

Theorem 1.2.2. *Let K_1 and K_2 be convex lattice sets in $\mathbb{Z}^2$. If all the discrete point X-rays of K_1 and K_2 at two distinct points $p_1, p_2 \in \mathbb{Z}^2$ coincide, $L[p_1, p_2] \cap K_i = \emptyset$, for $i = 1, 2$, and $\operatorname{conv}(K_1), \operatorname{conv}(K_2)$ either both meet $[p_1, p_2]$ or both meet $L[p_1, p_2] \backslash [p_1, p_2]$, then $K_1 = K_2$. Here $L[p_1, p_2]$ denote the line passing through p_1 and p_2 and $[p_1, p_2]$ is the segment connecting p_1 and p_2.*

Later, Gardner, Gronchi, and Zong [12] proposed a discrete analogue of the Aleksandrov theorem.

Problem 1.2.3. *Let $K, L \subset \mathbb{Z}^n$ be origin-symmetric convex lattice sets. If $|K|\xi^\perp| = |L|\xi^\perp|$ for every $\xi \in \mathbb{Z}^n$, is it true that $K = L$?*

Here, $|K|\xi^\perp|$ is the cardinality of the projection of K onto the hyperplane $\xi^\perp$. They gave a negative answer in dimension 2. For higher dimensions, this problem is still open. Since the answer is negative in dimension 2, can we impose additional conditions to make the answer affirmative? Another question is whether the counterexample from [12] is the only counterexample in $\mathbb{Z}^2$. Zhou [39] showed that the example given in [12] is the only counterexample when $|K| \leq 17$.

In Chapter 5, we give a positive answer to Problem 1.2.3 in $\mathbb{Z}^2$ under an additional hypothesis.

Theorem 1.2.4. *Let K and L be origin-symmetric convex lattice polygons in $\mathbb{R}^2$. If $|(K \cap \mathbb{Z}^2)|\xi^\perp| = |(L \cap \mathbb{Z}^2)|\xi^\perp|$ and $|(2K \cap \mathbb{Z}^2)|\xi^\perp| = |(2L \cap \mathbb{Z}^2)|\xi^\perp|$ for all $\xi \in S^1$, then $K = L$.*

In Chapter 6, we study a modification of Problem 1.2.3.

Problem 1.2.5. *Let $K, L \subset \mathbb{Z}^n$ be the origin-symmetric convex lattice sets. If $|\partial(K|\xi^\perp)| = |\partial(L|\xi^\perp)|$ for every $\xi \in \mathbb{Z}^n$, is it true that $K = L$?*

Here, the perimeter (or surface area) of the projection of K onto $\xi^\perp$, for $\xi \in S^{n-1}$, denoted by $|\partial(K|\xi^\perp)|$, is the number of points on the boundary of the convex hull of $K|\xi^\perp$. We solve Problem 1.2.5 affirmatively when $n = 3$. In higher dimensions, a positive answer is obtained in the class of convex lattice sets whose convex hulls are zonotopes (i.e. finite vector sums of line segments).

Theorem 1.2.6. *Let K and L be origin-symmetric convex lattice sets in $\mathbb{Z}^n$ with $\operatorname{conv}(K)$ and $\operatorname{conv}(L)$ being zonotopes for $n \geq 3$. If $|\partial(K|\xi^\perp)| = |\partial(L|\xi^\perp)|$ for any $\xi \in \mathbb{Z}^n$, then $K = L$.*

1.3 Geometric inequalities

A well-known result in asymptotic geometry is the Grünbaum inequality giving a lower bound for the volume of halves of a convex body split by an affine hyperplane passing through the centroid. Let K be a convex body in $\mathbb{R}^n$. The centroid of K is the point

$$g(K) := \mathrm{vol}_n(K)^{-1} \int_K x\, dx \in K.$$

Grünbaum's inequality states that if $g(K) = 0$ then

$$\mathrm{vol}_n(K \cap \xi^+) \geq \left(\frac{n}{n+1}\right)^n \mathrm{vol}_n(K) \qquad \forall\, \xi \in S^{n-1},$$

where ξ^+ denotes the half-space $\{x \in \mathbb{R}^n : \langle x, \xi \rangle \geq 0\}$. Here, the equality holds for the direction ξ if and only if K is a cone of the form $\mathrm{conv}\,\{y_1 + L, y_2\}$ with $g(K) = 0$, L is an $(n-1)$-dimensional convex body in $\xi^\perp$, and $y_1, y_2 \in \mathbb{R}^n$ are points with $\langle y_1, \xi \rangle < 0 < \langle y_2, \xi \rangle$. Mityagin [25] obtained the same result using a different method. Of a similar nature is the following result of Minkowski and Radon (see Pages 57–58 of [5] and Section 6.1 of [19]). If $g(K) = 0$, then

$$h_K(\xi) \geq \left(\frac{1}{n+1}\right) \Big(h_K(-\xi) + h_K(\xi)\Big) \quad \forall\, \xi \in S^{n-1}, \tag{1.1}$$

where $h_K(x) := \max_{y \in K} \langle x, y \rangle$ is the support function of K. Here, the equality holds for ξ if and only if K is a cone of the form $\mathrm{conv}\,\{y_1, y_2 + L\}$ with $g(K) = 0$ and y_1, y_2, L are as above. An equivalent form of the previous result is $-K \subset nK$, which can be written as

$$\frac{\mathrm{vol}_1(K \cap E \cap \xi^+)}{\mathrm{vol}_1(K \cap E)} \geq \left(\frac{1}{n+1}\right) \quad \forall\, E \in G(n,1), \quad \forall\, \xi \in S^{n-1} \cap E;$$

where $G(n,k)$ denotes the Grassmanian of k-dimensional subspaces of $\mathbb{R}^n$.

Recently, in [9] Fradelizi, Meyer, and Yaskin posed and studied an analogue of the

Grünbaum inequality for sections. Note that one cannot apply Grünbaum's result to this problem since the centroid of a section is generally different from the centroid of the body.

Problem 1.3.1. *For a convex body K in $\mathbb{R}^n$ with its centroid at the origin, is there a constant $c = c(n,k) > 0$ such that*

$$\mathrm{vol}_k(K \cap E \cap \xi^+) \geq c\, \mathrm{vol}_k(K \cap E) \quad \forall\ E \in G(n,k), \quad \forall\ \xi \in S^{n-1} \cap E?$$

In [9] it was shown that

$$\mathrm{vol}_k(K \cap E \cap \xi^+) \geq \frac{c_0}{(k+1)^2} \left(1 + \frac{k+1}{n-k}\right)^{-(n-k-2)} \mathrm{vol}_k(K \cap E) \tag{1.2}$$

for some absolute constant $c_0 > 0$. However, the best bound for the latter question is still unknown.

In Chapter 7, partially motivated by Fradelizi-Meyer-Yaskin's work (the Grünbaum inequality for sections), a similar problem is considered:

Problem 1.3.2. *For a convex body K in $\mathbb{R}^n$ with its centroid at the origin, is there a constant $c = c(n,k) > 0$ such that*

$$\mathrm{vol}_k\Big((K|E) \cap \xi^+\Big) \geq c\, \mathrm{vol}_k\Big(K|E\Big) \quad \forall\ E \in G(n,k), \quad \forall\ \xi \in S^{n-1} \cap E?$$

Here, $K|E$ is the orthogonal projection of K onto E.

We completely solve this problem and obtain an optimal constant together with equality conditions.

Theorem 1.3.3. *Let K be a convex body in $\mathbb{R}^n$ with its centroid at the origin, and* *let* $k \in \mathbb{Z}$ *be such that* $1 \leq k \leq n$*. Then*

$$\frac{\mathrm{vol}_k\Big((K|E) \cap \xi^+\Big)}{\mathrm{vol}_k\Big(K|E\Big)} \geq \left(\frac{k}{n+1}\right)^k \quad \forall\ E \in G(n,k), \quad \forall\ \xi \in S^{n-1} \cap E;$$

there is an equality for some E *and* ξ *if and only if* $K = \mathrm{conv}\{y_1 + L_1, y_2 + L_2\}$ *where*

$$\begin{cases} L_1 \subset \xi^{\perp} \textit{ and } L_1|(E \cap \xi^{\perp}) \textit{ are } (k-1)\textit{-dimensional convex bodies}; \\ L_2 \subset E^{\perp} \textit{ is an } (n-k)\textit{-dimensional convex body}; \\ \langle y_1, \xi \rangle < 0 < \langle y_2, \xi \rangle; \\ g(K) = 0. \end{cases}$$

Chapter 2

Definitions and preliminaries

In this chapter we collect some basic concepts and definitions that we use in the thesis. For further facts in convex geometry and geometric tomography the reader is referred to the books by Gardner [11] and Schneider [31].

A set in $\mathbb{R}^n$ is called *convex* if it contains the closed line segment joining any two of its points. A convex set is a *convex body* if it is compact and has non-empty interior. A convex body is *strictly convex* if its boundary contains no line segments.

We say that the set K is *origin-symmetric* if $K = -K$, where $tK := \{tx \in \mathbb{R}^n : x \in K\},\ t \in \mathbb{R}$.

For an integer $1 \leq k \leq n$, let $\mathrm{vol}_k(\cdot)$ denote k-dimensional Hausdorff measure on $\mathbb{R}^n$.

A hyperplane H *supports* a set E at a point x if $x \in E \cap H$ and E is contained in one of the two closed half-spaces bounded by H. We say H is a *supporting hyperplane* of E if H supports E at some point.

The *support function* of K is defined by

$$h_K(x) = \max\{\langle x, y\rangle : y \in K\},$$

for $x \in \mathbb{R}^n$. If h_K is of class C^k on $\mathbb{R}^n \backslash \{O\}$, we will simply say that K has a C^k

support function. For a convex body $K \subset \mathbb{R}^2$ it is often convenient to write h_K as a function of the polar angle θ. So, abusing notation, we will use $h_K(\theta)$ to denote $h_K((\cos\theta, \sin\theta))$. If H is the supporting line to $K \subset \mathbb{R}^2$ with the outer normal vector $(\cos\theta, \sin\theta)$, and K has a C^1 support function, then K has a unique point of contact with H, and $|h_K'(\theta)|$ is the distance from this point to the foot of the perpendicular from the origin O to H; see [11, p. 24].

The *width function* of K in the direction u is

$$w_K(u) := h_K(u) + h_K(-u).$$

In $\mathbb{R}^2$, $w_K(u^\perp)$ means the width in the direction perpendicular to u.

Recall that the *centroid* of K is the point

$$g(K) := \mathrm{vol}_n(K)^{-1} \int_K x\,dx \in K.$$

The *convex hull* of a set A, denoted by $\mathrm{conv}\,A$, is the smallest convex set containing A.

A set K in $\mathbb{R}^n$ is called a *convex polytope* if it is a convex hull of finitely many points.

A *convex lattice polytope* is a polytope all of whose vertices are in $\mathbb{Z}^n$.

We say A is a *convex lattice set* if $(\mathrm{conv}\,A) \cap \mathbb{Z}^n = A$.

A compact set L is called a *star body* if the origin O is an interior point of L, every line through O meets L in a line segment, and its *Minkowski functional* defined by

$$\|x\|_L = \min\{a \geq 0 : x \in aL\}$$

is a continuous function on $\mathbb{R}^n$.

The *radial function* of L is given by $\rho_L(x) = \|x\|_L^{-1}$, for $x \in \mathbb{R}^n \backslash \{O\}$. If $x \in S^{n-1}$, then $\rho_L(x)$ is just the radius of L in the direction of x.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$. Functions from this space are called test functions. For a function $\psi \in \mathcal{S}(\mathbb{R}^n)$, its *Fourier transform* is defined by

$$\hat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(x) e^{-i\langle x,\xi\rangle}\, dx,\ \ \xi \in \mathbb{R}^n.$$

By $\mathcal{S}'(\mathbb{R}^n)$ we denote the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. Elements of this space are referred to as distributions. By $\langle f, \psi\rangle$ we denote the action of the distribution f on the test function ψ. Note that $\hat{\psi}$ is also a test function, which allows to introduce the following definition. We say that the distribution $\hat{f}$ is *the Fourier transform of the distribution* f, if

$$\langle \hat{f}, \psi\rangle = \langle f, \hat{\psi}\rangle,$$

for every test function ψ.

If f is an even homogeneous of degree $-n+1$ continuous function on $\mathbb{R}^n \backslash \{O\}$, then f can be thought of as a distribution that acts on test functions by integration. Its Fourier transform is a continuous function on $\mathbb{R}^n \backslash \{O\}$, homogeneous of degree -1, whose restriction to S^{n-1} is given by

$$\hat{f}(\xi) = \pi \int_{S^{n-1}\cap\xi^{\perp}} f(\theta)\, d\theta, \quad \xi \in S^{n-1}.$$

The reader is referred to the book [22] for applications of Fourier transforms to the study of convex bodies.

Chapter 3

Non-central sections of convex bodies

3.1 Introduction

This chapter is based on [35] and is concerned with the problem of Barker and Larman (Problem 1.1.3). First of all, it is interesting to see what happens if the hypotheses of Problems 1.1.4 and 1.1.6 hold for two distinct bodies D_1 and D_2 simultaneously (i.e. if we double the amount of information). We show that in this case the answer in $\mathbb{R}^2$ is affirmative under some mild assumptions on D_1 and D_2.

We also discuss some higher-dimensional analogues. In particular, Groemer [16] considered half-planes of the form $H(u, w) = \{x \in \mathbb{R}^n : x \in u^\perp, \langle x, w \rangle \geq 0\}$, where $u \in S^{n-1}$ and $w \in S^{n-1} \cap u^\perp$. We proved that the equality $\mathrm{vol}_{n-1}(K \cap H(u, w)) = \mathrm{vol}_{n-1}(L \cap H(u, w))$ for all such half-planes implies that $K = L$. We give a version of this result for half-planes that do not pass through the origin. Some other types of sections are also discussed.

3.2 Main results: 2-dimensional cases.

We will start with the following definition. We say that convex bodies D_1 and D_2 in $\mathbb{R}^2$ are *admissible* if they have C^2 support functions, $D_1 \cup D_2$ is not convex, and there are only two lines that support both D_1 and D_2 and do not separate D_1 and D_2. The last condition is satisfied, when, for example, the bodies D_1 and D_2 are disjoint, or they touch each other, or they overlap, but their boundaries have only two common points.

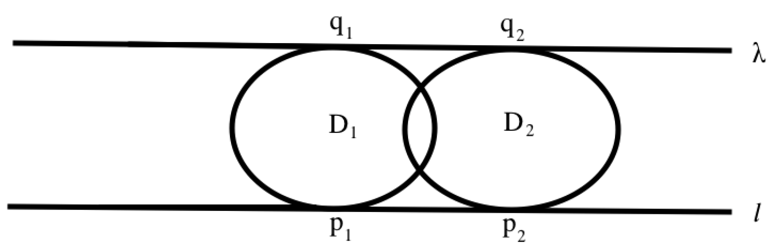

Figure 3.1: Two supporting lines are parallel.

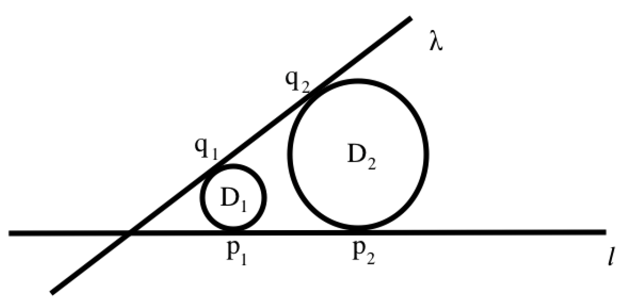

Figure 3.2: Two supporting lines are intersecting.

Figures 3.1 and 3.2 show two examples of admissible convex bodies. For simplicity, the reader could just think of two disks (not necessarily of the same radius) such that none of them is contained in the other.

We will now prove the following two results.

Theorem 3.2.1. *Let K and L be convex bodies in $\mathbb{R}^2$ and let D_1 and D_2 be two admissible convex bodies in the interior of $K \cap L$. If the chords $K \cap H$ and $L \cap H$ have equal length for all H supporting either D_1 or D_2, then $K = L$.*

If H is a supporting line to a body $D \subset \mathbb{R}^2$, we will denote by H^+ the half-plane bounded by H and disjoint from the interior of D.

Theorem 3.2.2. *Let K and L be convex bodies in $\mathbb{R}^2$ and let D_1 and D_2 be two admissible convex bodies in the interior of $K \cap L$. If $\mathrm{vol}_2(K \cap H^+) = \mathrm{vol}_2(L \cap H^+)$ for every H supporting D_1 or D_2, then $K = L$.*

We will obtain these theorems as particular cases of a more general statement, Theorem 3.2.4 below. First, we will need the following lemma.

Lemma 3.2.3. *Let $D \subset \mathbb{R}^2$ be a convex body with a C^2 support function. Let $Q \in \partial D$ and l be the supporting line to D at Q. Suppose the origin O is located on the line perpendicular to l and passing through Q, and $O \neq Q$. Consider a polar coordinate system centered at O with the polar axis $\overrightarrow{OQ}$. Then, for θ small enough, we have*

$$h_D'(\theta)\sin\theta + h_D(0) - h_D(\theta)\cos\theta \approx \theta \approx \sin^2\theta, \tag{3.1}$$

where $f \approx g$ means there exist two constants C_1, C_2, such that, $C_1 g \leq f \leq C_2 g$; here, C_1, C_2 are only dependent on D.

Proof. Since Q is both the point where l supports D and the foot of the perpendicular from O to l, it follows that $h_D'(0) = 0$. Thus,

$$h_D(\theta) = h_D(0) + \frac{h_D''(0)}{2}\theta^2 + o(\theta^2).$$

Therefore, for θ small enough, we have

$$
\begin{aligned}
& h_D(0) - h_D(\theta)\cos\theta \\
&= h_D(0) - \left(h_D(0) + \frac{h_D''(0)}{2}\theta^2 + o(\theta^2)\right)\left(1 - \frac{1}{2}\theta^2 + o(\theta^2)\right) \\
&= \frac{h_D(0) - h_D''(0)}{2}\theta^2 + o(\theta^2) \\
&\approx \sin^2\theta,
\end{aligned}
$$

and $h_D'(\theta) = h_D''(0)\theta + o(\theta) \approx \theta$, thus $h_D'(\theta)\sin\theta \approx \sin^2\theta$. $\square$

Now let K be a convex body in $\mathbb{R}^n$, and D be a strictly convex body in the interior of K. Let H be a supporting plane to D with outer unit normal vector ξ, and $p = D \cap H$ be the corresponding point of contact. If $u \in S^{n-1} \cap \xi^\perp$, we denote by $\rho_{K,D}(u,\xi) = \rho_{K,p}(u)$ the radial function of $K \cap H$ with respect to p.

Theorem 3.2.4. *Let K and L be convex bodies in $\mathbb{R}^2$ and let D_1 and D_2 be two admissible convex bodies in the interior of $K \cap L$. Assume that for some $i > 0$ one of the following two conditions holds:*

(I) $\rho^i_{K,D_j}(u,\xi) + \rho^i_{K,D_j}(-u,\xi) = \rho^i_{L,D_j}(u,\xi) + \rho^i_{L,D_j}(-u,\xi)$, *for* $j = 1, 2$,

(II) $\partial K \cap \partial L \neq \emptyset$ *and* $\rho^i_{K,D_j}(u,\xi) - \rho^i_{K,D_j}(-u,\xi) = \rho^i_{L,D_j}(u,\xi) - \rho^i_{L,D_j}(-u,\xi)$, *for* $j = 1, 2$,

for all $\xi, u \in S^1$ such that $u \perp \xi$.
Then $K = L$.

Proof. We will present the proof of the theorem only using condition (I). The other case is similar and we will just make a brief comment on how the proof should be adjusted.

For the reader's convenience let us first outline the idea of the proof. The proof consists of four steps. In Step 1 we fix a common supporting line to D_1 and D_2 that has a certain property. Denoting this line by l, in Step 2 we show that $K \cap l = L \cap l$. In Step 3 we prove that the boundaries of K and L coincide in some neighborhood of the line l. This allows to conclude in Step 4 that the boundaries of K and L coincide everywhere.

Step 1. Since there are two common supporting lines to D_1 and D_2 (that do not separate D_1 and D_2), we will denote them by l and λ, and let $p_1 = D_1 \cap l$, $q_1 = D_1 \cap \lambda$, $p_2 = D_2 \cap l$, $q_2 = D_2 \cap \lambda$; see Figures 1 and 2. We claim that at least one of the (possibly degenerate) segments $[p_1, p_2]$ or $[q_1, q_2]$ is not entirely contained in $D_1 \cup D_2$. We will prove this claim in a slightly more general setting, i.e. without the assumption that D_1 and D_2 are strictly convex. In that case, instead of single points of contact we may have intervals, and $[p_1, p_2]$ or $[q_1, q_2]$ will just stand for the convex hulls of the corresponding support sets. To prove the claim, we will argue by contradiction. Assume that $[p_1, p_2]$ and $[q_1, q_2]$ are contained in $D_1 \cup D_2$. Then there are points $p \in [p_1, p_2]$ and $q \in [q_1, q_2]$ that both belong to $D_1 \cap D_2$. We can assume that the origin is an interior point of the interval $[p, q]$. Since there are only two common supporting lines to D_1 and D_2, we have exactly two directions u_1 and u_2, such that $h_{D_1}(u_1) = h_{D_2}(u_1)$ and $h_{D_1}(u_2) = h_{D_2}(u_2)$. These directions divide the circle S^1 into two open arcs U_1 and U_2, satisfying $h_{D_1}(u) > h_{D_2}(u)$ for all $u \in U_1$, and $h_{D_1}(u) < h_{D_2}(u)$ for all $u \in U_2$. Thus the line $l(p, q)$ through the points p and q cuts each of the bodies D_1 and D_2 into two convex parts: $D_1 = D_{11} \cup D_{12}$ and $D_2 = D_{21} \cup D_{22}$, such that $D_{11} \supset D_{21}$ and $D_{12} \subset D_{22}$. In other words, $D_1 \cup D_2 = D_{11} \cup D_{22}$, where D_{11} and D_{22} are separated by $l(p, q)$. Now, if we take two points $X, Y \in D_1 \cup D_2$, then we have two cases: either they lie on one side of $l(p, q)$, or on different sides. In the first case, either $X, Y \in D_{11}$, or $X, Y \in D_{22}$, which means that $[X, Y] \subset D_1 \cup D_2$. In the second case, the segment $[X, Y]$ intersects $[p, q]$ since p and q belong to supporting lines, and thus one part of $[X, Y]$ lies in D_{11}, and the other in D_{22}, which again implies that $[X, Y] \subset D_1 \cup D_2$, meaning that $D_1 \cup D_2$ is convex. Contradiction. Thus, we have proved that at least one of the segments $[p_1, p_2]$ or $[q_1, q_2]$ is not entirely contained in $D_1 \cup D_2$. We will assume it is the segment $[p_1, p_2]$ and will fix the corresponding supporting line l.

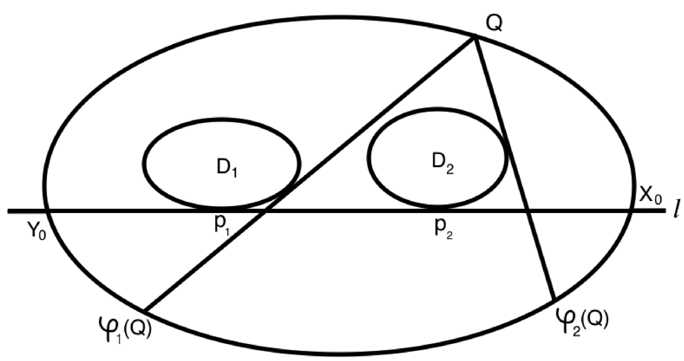

Figure 3.3: Mapping through supporting lines.

Step 2. Here we will show that $\partial K \cap l = \partial L \cap l$. To this end, we define two mappings φ_1 and φ_2 (see Figure 3.3). We will start with φ_1; the other is similar. Let Q be a point outside of D_1. There are two unique supporting lines to D_1 passing through Q. Choose the one that lies on the left of the body D_1, when viewing from the point Q. Let T be the point of contact of the chosen supporting line and the body D_1. On this line we take a point $\varphi_1(Q)$, such that T is inside the segment $[Q, \varphi_1(Q)]$ and

$$|QT|^i + |\varphi_1(Q)T|^i = \rho^i_{K,D_1}(u, \xi) + \rho^i_{K,D_1}(-u, \xi),$$

where u is a unit vector parallel to $\overrightarrow{TQ}$ and ξ is the outward unit normal vector to D_1 at T (which is perpendicular to u). The definition for φ_2 is similar; one only needs to replace D_1 by D_2. Note that the domains of φ_1 and φ_2 include the symmetric difference $K \triangle L$. An important observation is that if Q is on the boundary of K (resp. L), then $\varphi_1(Q)$, $\varphi_1^{-1}(Q)$, $\varphi_2(Q)$, and $\varphi_2^{-1}(Q)$ are also on the boundary of K (resp. L).

Note that there exists at least one point $Q \in \partial K \cap \partial L$. Otherwise, one of ∂K or ∂L would be strictly contained inside the other, thus violating condition (1) of the proposition. The line l divides the plane into two closed half-planes l^+ and l^-, where l^+ is the one that contains D_1 and D_2. If $Q \in l^+$, then applying φ_1 finitely many times, we will get a point in l^- (since φ_1 cannot miss the whole half-plane), which is also a common point of the boundaries of K and L. Thus from now on we will assume that $Q \in l^-$. If $Q \in l$, then the proof of Step 2 is finished. If Q is strictly below l, we will apply the following procedure.

Without loss of generality, we can assume that, if the line λ intersects l, then the point of intersection lies to the left of the point p_1, as in Figure 3.2. Let us also denote by X_0 and Y_0 the points of intersection of the boundary of K with the line l, as in Figure 3.3. Let $Q_0 = \varphi_2^{-1}(Q)$. The line $l(Q, Q_0)$ through Q and Q_0 is tangent to D_2 and therefore cannot have common points with D_1 (otherwise rolling this line along the boundary of D_2 we would find a third common supporting line to both D_1 and D_2). Now consider $\varphi_1(Q_0)$ and the line $l(\varphi_1(Q_0), Q_0)$ through $\varphi_1(Q_0)$ and Q_0. Note that $\varphi_1(Q_0)$ is below l. Since $l(Q, Q_0)$ and $l(\varphi_1(Q_0), Q_0)$ are different, the points Q and $\varphi_1(Q_0)$ are also different. Moreover, we have $\angle(\overrightarrow{\varphi_1(Q_0)Q_0}, \overrightarrow{p_1X_0}) < \angle(\overrightarrow{QQ_0}, \overrightarrow{p_1X_0})$. Repeating this procedure, we construct $Q_1 = \varphi_2^{-1}(\varphi_1(Q_0))$ and observe that $\angle(\overrightarrow{\varphi_1(Q_0)Q_1}, \overrightarrow{p_1X_0}) < \angle(\overrightarrow{\varphi_1(Q_0)Q_0}, \overrightarrow{p_1X_0})$, as in Figure 3.4.

Continuing in this manner, we obtain a sequence of points $\{Q_j\}_{j=0}^{\infty}$ and a corresponding sequence of angles $\{\theta_j\}_{j=0}^{\infty}$, defined by $Q_{j+1} = \varphi_2^{-1}(\varphi_1(Q_j))$ and $\theta_j = \angle(\overrightarrow{\varphi_1(Q_j)Q_j}, \overrightarrow{p_1X_0})$. We note that $Q_j \in l^+ \cap \partial K \cap \partial L$, and $\theta_j > \theta_{j+1}$, for all j. Thus, the sequence $\{\theta_j\}$ is strictly decreasing and positive, and therefore convergent.

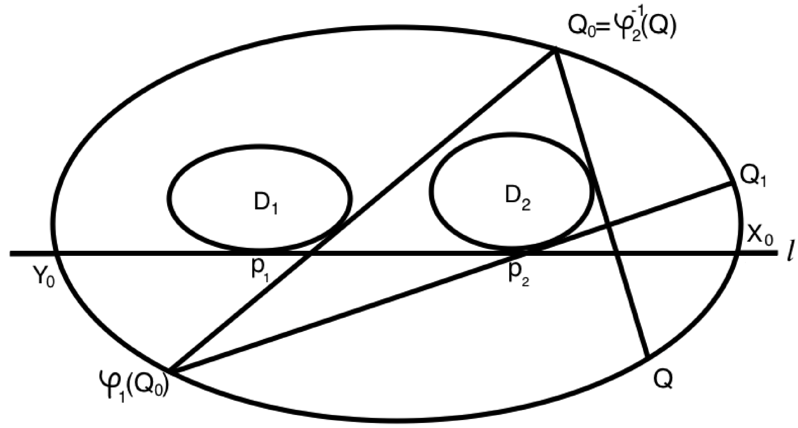

Figure 3.4: Sequence of points.

To reach a contradiction, let us assume that the limit is not zero. Then there is a point $\tilde{Q} = \lim_{j\to\infty} Q_j$ that lies above the line l and satisfies $\varphi_1(\tilde{Q}) = \varphi_2(\tilde{Q})$. Thus, we have a third line that supports both D_1 and D_2. Contradiction. Hence, $\lim_{j\to\infty} \theta_j = 0$, and we conclude that $\partial K \cap l = \partial L \cap l = \{X_0, Y_0\}$.

Step 3. We will prove that ∂K and ∂L coincide in some one-sided neighborhood of the point X_0. Since

$$\frac{|Y_0p_1||X_0p_2|}{|X_0p_1||Y_0p_2|} < 1,$$

we can choose positive numbers a, b, c, d such that

$$0 < a < |X_0p_1|, \ |Y_0p_1| < b, \ 0 < c < |Y_0p_2|, \ |X_0p_2| < d, \text{ and } \frac{bd}{ac} < 1.$$

By the continuity of the boundaries of K, L, D_1, and D_2, there exist neighborhoods,

$\mathcal{N}(X_0)$, $\mathcal{N}(Y_0)$, of X_0 and Y_0 respectively, such that

$$\begin{cases} |XT_1| > a \text{ and } |XT_2| < d, & \text{if } X \in \mathcal{N}(X_0), \\ \\ |YT_3| > c \text{ and } |YT_4| < b, & \text{if } Y \in \mathcal{N}(Y_0), \end{cases} \tag{3.2}$$

where T_1 is the point of intersection of l and the line through X supporting D_1 (if X is itself on the line l, then we let $T_1 = p_1$). Similarly, T_2 is the point of intersection of l and the line through X supporting D_2 (again, if X is on the line l, then we let $T_2 = p_2$). Here and below, by the supporting lines we mean those that are closest to l. There is no ambiguity, since X is sufficiently close to l. (The points T_3 and T_4 are defined similarly, if we replace X by Y).

Next we claim that there are points of $\partial K \cap \partial L$ in the set $\mathcal{N}(X_0) \cap l^+$. Indeed, if in Step 2 there was a point $Q \in \partial K \cap \partial L$ strictly below the line l, then the points from the corresponding sequence $\{Q_i\}$ all lie in $\partial K \cap \partial L \cap \mathcal{N}(X_0) \cap l^+$ for i large enough. If in Step 2 the point Q was on the line l, then we can take $\varphi_1(\varphi_2^{-1}(X_0))$, which will be strictly below l, and repeat the same procedure.

Our goal is to show that ∂K and ∂L coincide in $\mathcal{N}(X_0) \cap l^+$. Taking a smaller neighborhood $\mathcal{N}(X_0)$ if needed, we can assume that $\varphi_1(\mathcal{N}(X_0) \cap l^+) \subset \mathcal{N}(Y_0)$. Discarding finitely many terms of the sequence $\{Q_j\}$, we can also assume that $Q_j \in \mathcal{N}(X_0) \cap l^+$ for all $j \geq 0$. Now consider the segments of the boundaries of ∂K and ∂L between the points Q_0 and Q_1. If they coincide, then we are done, since the boundaries of ∂K and ∂L would have to coincide between Q_j and Q_{j+1} for all j. So, we will next assume that ∂K and ∂L are not identically the same between Q_0 and Q_1. Let E_0 be the component of $K \triangle L$ with endpoints Q_0 and Q_1, i.e. E_0 is the subset of $(K \triangle L) \cap l^+$ located between the lines $l(Q_0, \varphi_1(Q_0))$ and $l(Q_1, \varphi_1(Q_0))$. We will define a sequence of sets $\{E_j\}_{j=0}^{\infty}$, where $E_{j+1} = \varphi_2^{-1}(\varphi_1(E_j))$. Each E_j is a component of $K \triangle L$ with endpoints Q_j and Q_{j+1}.

Now consider a Cartesian coordinate system with l being the x-axis, and the y-axis perpendicular to l. We will be using ideas similar to those in [11, Section 5.2]. For a measurable set E define

$$\nu_i(E) = \iint_E |y|^{i-2}\, dx\, dy. \tag{3.3}$$

Note that $\nu_i(E)$ is invariant under shifts parallel to the x-axis. This allows us to associate with each D_1 and D_2 their own Cartesian systems. In both systems l is the x-axis, but in the coordinate system associated with D_1 the origin is at p_1, while in the system associated with D_2 the origin is at p_2.

Our goal is to estimate $\nu_i(E_j)$. Fix the Cartesian system associated with D_1, with p_1 being the origin. For a point $(x, y) \in \mathcal{N}(X_0) \cup \mathcal{N}(Y_0)$ we will introduce new coordinates (r, θ) as follows. Let $\theta = \angle(l_{\theta,1}, l)$, where $l_{\theta,1}$ is the line passing through (x, y) and supporting D_1. Define r to be the signed distance between (x, y) and the foot of the perpendicular from the point $(0, 1)$ to the line $l_{\theta,1}$. (The word "signed" means that $r > 0$ in the neighborhood of X_0 and $r < 0$ in the neighborhood of Y_0). Let $h_{D_1}(\theta)$ be the support function of D_1 measured from the point $(0, 1)$ in the direction of $(\sin\theta, -\cos\theta)$. Using that

$$(x, y) = h_{D_1}(0) \cdot (0, 1) + r(\cos\theta, \sin\theta) + h_{D_1}(\theta) \cdot (\sin\theta, -\cos\theta),$$

we will write the integral (3.3) in the (r, θ)-coordinates associated with D_1. Since the Jacobian is $|r - h'_{D_1}(\theta)|$, and $r = h'_{D_1}(\theta)$ corresponds to the point of contact of $l_{\theta,1}$ and D_1, we get

$$\nu_i(E_j) = \iint_{E_j} |y|^{i-2}\, dx\, dy$$

$$= \int_{\theta_{j+1}}^{\theta_j} \left| \int_{\rho_{K,D_1}(u,\xi)-h'_{D_1}(\theta)}^{\rho_{L,D_1}(u,\xi)-h'_{D_1}(\theta)} |r\sin\theta + h_{D_1}(0) - h_{D_1}(\theta)\cos\theta|^{i-2}|r - h'_{D_1}(\theta)|\, dr \right| d\theta$$
$$= \int_{\theta_{j+1}}^{\theta_j} \left| \int_{\rho_{K,D_1}(u,\xi)}^{\rho_{L,D_1}(u,\xi)} |r\sin\theta + h'_{D_1}(\theta)\sin\theta + h_{D_1}(0) - h_{D_1}(\theta)\cos\theta|^{i-2} r\, dr \right| d\theta,$$

where $u = (\cos\theta, \sin\theta)$, and $\xi = (\sin\theta, -\cos\theta)$. Here the absolute value of the integral with respect to r is needed, since we do not know which of ρ_K or ρ_L is greater.

For small θ, Lemma 3.2.3 yields that

$$h'_{D_1}(\theta)\sin\theta + h_{D_1}(0) - h_{D_1}(\theta)\cos\theta \approx \sin^2\theta.$$

Since E_j is inside $\mathcal{N}(X_0)$, there exists a constant $C > 0$ such that

$$(1 - C\sin\theta) r\sin\theta \leq r\sin\theta + h'_{D_1}(\theta)\sin\theta + h_{D_1}(0) - h_{D_1}(\theta)\cos\theta \leq (1 + C\sin\theta) r\sin\theta,$$

where we assume that θ is small enough so that $1 - C\sin\theta > 0$.

If $i \geq 2$, for small $\theta > 0$ we have

$$\left(\frac{1 - C\sin\theta}{1 + C\sin\theta}\right)^{i-2} (r\sin\theta)^{i-2} \leq (1 - C\sin\theta)^{i-2} (r\sin\theta)^{i-2}$$
$$\leq |r\sin\theta + h'_{D_1}(\theta)\sin\theta + h_{D_1}(0) - h_{D_1}(\theta)\cos\theta|^{i-2}$$
$$\leq (1 + C\sin\theta)^{i-2} (r\sin\theta)^{i-2} \leq \left(\frac{1 + C\sin\theta}{1 - C\sin\theta}\right)^{i-2} (r\sin\theta)^{i-2}.$$

On the other hand, for $i < 2$,

$$\left(\frac{1 + C\sin\theta}{1 - C\sin\theta}\right)^{i-2} (r\sin\theta)^{i-2} \leq (1 + C\sin\theta)^{i-2} (r\sin\theta)^{i-2}$$
$$\leq |r\sin\theta + h'_{D_1}(\theta)\sin\theta + h_{D_1}(0) - h_{D_1}(\theta)\cos\theta|^{i-2}$$
$$\leq (1 - C\sin\theta)^{i-2} (r\sin\theta)^{i-2} \leq \left(\frac{1 - C\sin\theta}{1 + C\sin\theta}\right)^{i-2} (r\sin\theta)^{i-2}.$$

Thus, for both $i \geq 2$ and $i < 2$, we have

$$\begin{aligned}\frac{1}{i}\int_{\theta_{j+1}}^{\theta_j}\left(\frac{1-C\sin\theta}{1+C\sin\theta}\right)^{|i-2|}(\sin\theta)^{i-2}\left|\rho^i_{K,D_1}(u,\xi)-\rho^i_{L,D_1}(u,\xi)\right|\,d\theta \leq \nu_i(E_j)\\ \leq \frac{1}{i}\int_{\theta_{j+1}}^{\theta_j}\left(\frac{1+C\sin\theta}{1-C\sin\theta}\right)^{|i-2|}(\sin\theta)^{i-2}\left|\rho^i_{K,D_1}(u,\xi)-\rho^i_{L,D_1}(u,\xi)\right|\,d\theta.\end{aligned} \tag{3.4}$$

Now apply the same estimates to $\nu_i(\varphi_1(E_j))$. Since $\varphi_1(E_j) \subset \mathcal{N}(Y_0)$, and assuming that the constant C chosen above works for both $\mathcal{N}(X_0)$ and $\mathcal{N}(Y_0)$, we get

$$\begin{aligned}&\nu_i(\varphi_1(E_j))\\ \geq&\frac{1}{i}\int_{\theta_{j+1}}^{\theta_j}\left(\frac{1-C\sin\theta}{1+C\sin\theta}\right)^{|i-2|}(\sin\theta)^{i-2}\left|\rho^i_{K,D_1}(-u,\xi)-\rho^i_{L,D_1}(-u,\xi)\right|\,d\theta\\ =&\frac{1}{i}\int_{\theta_{j+1}}^{\theta_j}\left(\frac{1-C\sin\theta}{1+C\sin\theta}\right)^{|i-2|}(\sin\theta)^{i-2}\left|\rho^i_{K,D_1}(u,\xi)-\rho^i_{L,D_1}(u,\xi)\right|\,d\theta\\ =&\frac{1}{i}\int_{\theta_{j+1}}^{\theta_j}\left(\frac{1-C\sin\theta}{1+C\sin\theta}\right)^{2|i-2|}\left(\frac{1+C\sin\theta}{1-C\sin\theta}\right)^{|i-2|}(\sin\theta)^{i-2}\left|\rho^i_{K,D_1}(u,\xi)-\rho^i_{L,D_1}(u,\xi)\right|\,d\theta\\ \geq&\left(\frac{1-C\sin\theta_j}{1+C\sin\theta_j}\right)^{2|i-2|}\nu_i(E_j),\end{aligned}$$

since $\dfrac{1-C\sin\theta}{1+C\sin\theta}$ is decreasing.

Define another sequence of angles $\eta_j = \angle(\overrightarrow{\varphi_1(Q_j)Q_{j+1}}, \overrightarrow{p_1X_0})$. Then calculations similar to those above give

$$\nu_i(E_{j+1}) \geq \left(\frac{1-C\sin\eta_j}{1+C\sin\eta_j}\right)^{2|i-2|}\nu_i(\varphi_1(E_j)).$$

Thus,

$$\nu_i(E_{j+1}) \geq \left(\frac{1-C\sin\eta_j}{1+C\sin\eta_j}\right)^{2|i-2|}\left(\frac{1-C\sin\theta_j}{1+C\sin\theta_j}\right)^{2|i-2|}\nu_i(E_j). \tag{3.5}$$

Observe that (3.2) implies, for all j,

$$\frac{\sin\theta_{j+1}}{\sin\theta_j} = \frac{\sin\theta_{j+1}}{\sin\eta_j}\frac{\sin\eta_j}{\sin\theta_j} \leq \frac{db}{ac} < 1,$$

and, similarly,

$$\frac{\sin\eta_{j+1}}{\sin\eta_j} \leq \frac{db}{ac}.$$

Set $\sigma = \dfrac{db}{ac}$, where $\sigma \in (0,1)$. Then $\sin\theta_j \leq \sigma^j \sin\theta_0 \leq \sigma^j$ and $\sin\eta_j \leq \sigma^j \sin\eta_0 \leq \sigma^j$. For sufficiently small $x > 0$, we have the following inequalities: $1 + x \leq e^x$ and $1 - x \geq e^{-2x}$. Let $N > 0$ be large enough so that $x = C\sigma^j$ satisfies the latter two inequalities for all $j \geq N$. Then for all $j \geq N$, we have

$$\nu_i(E_{j+1}) \geq \left(\frac{1 - C\sigma^j}{1 + C\sigma^j}\right)^{4|i-2|} \nu_i(E_j) \geq \left(\frac{e^{-2C\sigma^j}}{e^{C\sigma^j}}\right)^{4|i-2|} \nu_i(E_j) = e^{-12C|i-2|\sigma^j}\nu_i(E_j).$$

Using the latter estimate inductively, we get

$$\begin{aligned}\nu_i(E_{j+1}) &\geq \prod_{m=N}^{j} e^{-12C|i-2|\sigma^m}\nu_i(E_N)\\ &= \exp\left\{-12C|i-2|\sum_{m=N}^{j}\sigma^m\right\}\nu_i(E_N)\\ &\geq \gamma\nu_i(E_N),\end{aligned}$$

where

$$\gamma = \exp\left\{-12C|i-2|\sum_{m=N}^{\infty}\sigma^m\right\} > 0.$$

Since all E_j are disjoint, and since $\nu_i(E_N) \geq \tilde{C}\nu_i(E_0) > 0$, for some constant $\tilde{C}$ (by virtue of (3.5)), we conclude that

$$\nu_i\left(\bigcup_{j=N+1}^{\infty} E_j\right) = \sum_{j=N+1}^{\infty}\nu_i(E_j) \geq \gamma\sum_{j=N+1}^{\infty}\nu_i(E_N) = \infty.$$

Since $l \cap (K \triangle L) = \{X_0, Y_0\}$, there exists a triangle T with one vertex at X_0 satisfying $T \cap l = X_0$ and $\cup_{j=N+1}^{\infty} E_j \subset T$, implying

$$\nu_i(T) \geq \nu_i \left(\bigcup_{j=N+1}^{\infty} E_j \right) = \infty.$$

However, by [11, Lemma 5.2.4], any triangle of the form $T = \{(x,y) : a|x - x_0| \leq y \leq b\}$, for $a, b > 0$, has finite ν_i-measure. We get a contradiction. Thus, $\partial K = \partial L$ in $\mathcal{N}(X_0) \cap l^+$.

Step 4. To finish the proof, we take any point $A \in \partial K$. Applying φ_1 to A finitely many times, we can get a point A' in $l^- \cap \partial K$. As in Step 2, produce a sequence of points $A_{j+1} = \varphi_2^{-1}(\varphi_1(A_j))$ with $A_0 = \varphi_2^{-1}(A')$. As we have seen above, there is a number M large enough such that $A_M \in \mathcal{N}(X_0) \cap l^+$. Applying the conclusion of Step 3, we get $A_M \in \partial K \cap \partial L$. Tracing the sequence $\{A_i\}$ backwards, we conclude that $A \in \partial K \cap \partial L$. Therefore, $K = L$.

We now briefly comment on how to proceed if we use condition (II) of the theorem. Note that here we require that there is a point $Q \in \partial K \cap \partial L$. We define φ_1 and φ_2 in a similar way as above, with the only difference that

$$|QT|^i - |\varphi_j(Q)T|^i = \rho^i_{K,D_j}(u,\xi) - \rho^i_{K,D_j}(-u,\xi),$$

for $j = 1, 2$. The rest of the proof goes without any changes.

□

Remark 3.2.5. The C^2-smoothness assumption for the support functions of the bodies D_1 and D_2 can be relaxed. As we saw above, we only need the C^2 condition in some neighborhoods of the points p_1 and p_2 correspondingly. Moreover, D_1 or D_2 can also be polygons. In the latter case, ρ_{K,D_j} is not well defined for finitely many supporting lines, but this is not an issue. Step 1 of the proof does not need any changes, since it was proved for bodies that are not necessarily strictly convex. In Step 2, we consider small one-sided neighborhoods of X_0 and Y_0, where ρ_{K,D_j} is well-defined. As for Step 3, the proof will be similar to [11, Section 5.2], since all supporting lines to a polygon D_j passing through points $X \in \mathcal{N}(X_0) \cap l^+$ will contain the same vertex of D_j. Thus, as in [11], the measure ν_i would be invariant under φ_j. So, whenever we speak about admissible bodies, one can consider a larger class of admissible bodies by including the bodies described in this remark.

Theorem 3.2.1 (with admissible bodies as in the above remark) is now a consequence of Theorem 3.2.4 (use part (I) with $i = 1$). The following is an immediate corollary of Theorem 3.2.1.

Corollary 3.2.6. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^2$ and let D be a convex body in the interior of $K \cap L$, such that D and $-D$ are admissible bodies. If the chords $K \cap H$ and $L \cap H$ have equal length for all H supporting D, then $K = L$. In particular, D can be a disk not centered at the origin.*

Using the same ideas, one can prove the following.

Corollary 3.2.7. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^2$ and let D be a convex body outside of $K \cup L$ (either a polygon or a body with a C^2 support function). If the chords $K \cap H$ and $L \cap H$ have equal length for all H supporting D, then $K = L$.*

We will now prove Theorem 3.2.2 using the class of admissible bodies described in Remark 3.2.5.

Proof. First we will prove the following claim. Let K an L be convex bodies in $\mathbb{R}^2$, D be a convex body in the interior of $K \cap L$, where D is either a body with C^2 support function or a polygon. If $\mathrm{vol}_2(K \cap H^+) = \mathrm{vol}_2(L \cap H^+)$ for every H supporting D, then

$$\rho^2_{K,D}(u,\xi) - \rho^2_{K,D}(-u,\xi) = \rho^2_{L,D}(u,\xi) - \rho^2_{L,D}(-u,\xi),$$

for every $\xi \in S^1$ and $u \in S^1 \cap \xi^\perp$, whenever well-defined. (Note that in the case when D is a polygon, the radial functions above are not well-defined for finitely many directions ξ that are orthogonal to the edges of D).

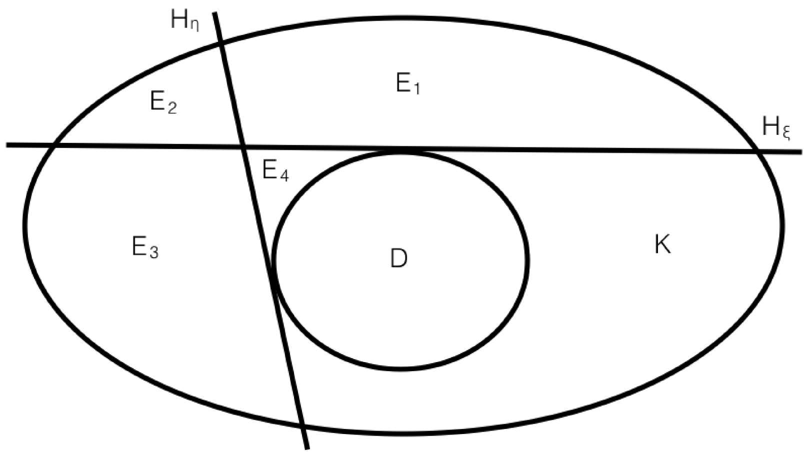

Figure 3.5: Proof of Corollary 3.2.7.

We will treat simultaneously both the case of smooth bodies and polygons. To prove the claim, let ξ be any unit vector (and ξ is not orthogonal to an edge of D, if D is a polygon). Let H_ξ be the supporting line orthogonal to ξ. Let $\zeta \in S^1 \cap \xi^\perp$. For a small angle $\phi > 0$ let $\eta = \cos\phi\,\xi + \sin\phi\,\zeta$, and denote by H_η the supporting line orthogonal to η. Define the following sets: $E_1 = H^+_\xi \backslash H^+_\eta$, $E_2 = H^+_\xi \cap H^+_\eta$, $E_3 = H^+_\eta \backslash H^+_\xi$, and E_4 is the curvilinear triangle enclosed by H_ξ, H_η, and the boundary of D; see Figure 5. Note that when η and ξ are close enough, we have $E_4 \subset K \cap L$, and E_4 is empty if D is a polygon. By the assumption of the theorem,

$$\mathrm{vol}_2((E_1 \cup E_2) \cap K) - \mathrm{vol}_2((E_3 \cup E_2) \cap K) = \mathrm{vol}_2((E_1 \cup E_2) \cap L) - \mathrm{vol}_2((E_3 \cup E_2) \cap L),$$

implying

$$\mathrm{vol}_2((E_1 \cup E_4) \cap K) - \mathrm{vol}_2((E_3 \cup E_4) \cap K) = \mathrm{vol}_2((E_1 \cup E_4) \cap L) - \mathrm{vol}_2((E_3 \cup E_4) \cap L). \tag{3.6}$$

Now we will consider the following coordinate system (r, θ) associated with D. For a point (x, y) outside of D, we let $(x, y) = h_D(\theta)\,(\cos\theta\,\xi + \sin\theta\,\zeta) + r(\sin\theta\,\xi - \cos\theta\,\zeta)$, where $h_D(\theta)$ is the support function of D in the direction of $v = \cos\theta\,\xi + \sin\theta\,\zeta$. Setting $w = \sin\theta\,\xi - \cos\theta\,\zeta$, and observing that the Jacobian is $|r + h'_D(\theta)|$, we get

$$\begin{aligned}
&\int_0^{\phi}\int_{h'_D(\theta)}^{\rho_{K,D}(w,v)+h'_D(\theta)} |r + h'_D(\theta)|\,dr\,d\theta - \int_0^{\phi}\int_{h'_D(\theta)}^{\rho_{K,D}(-w,v)+h'_D(\theta)} |r + h'_D(\theta)|\,dr\,d\theta \\
&= \int_0^{\phi}\int_{h'_D(\theta)}^{\rho_{L,D}(w,v)+h'_D(\theta)} |r + h'_D(\theta)|\,dr\,d\theta - \int_0^{\phi}\int_{h'_D(\theta)}^{\rho_{L,D}(-w,v)+h'_D(\theta)} |r + h'_D(\theta)|\,dr\,d\theta,
\end{aligned}$$

which after a variable change becomes

$$\begin{aligned}
&\int_0^{\phi}\int_0^{\rho_{K,D}(w,v)} r\,dr\,d\theta - \int_0^{\phi}\int_0^{\rho_{K,D}(-w,v)} r\,dr\,d\theta \\
&= \int_0^{\phi}\int_0^{\rho_{L,D}(w,v)} r\,dr\,d\theta - \int_0^{\phi}\int_0^{\rho_{L,D}(-w,v)} r\,dr\,d\theta.
\end{aligned}$$

Differentiating both sides with respect to ϕ, and setting $\phi = 0$, we get

$$\rho^2_{K,D}(u, \xi) - \rho^2_{K,D}(-u, \xi) = \rho^2_{L,D}(u, \xi) - \rho^2_{L,D}(-u, \xi),$$

as claimed.

To finish the proof of the theorem, note that $\partial K \cap \partial L \cap l^- \neq \emptyset$, where l is the common supporting line to D_1 and D_2 as in Theorem 3.2.4; otherwise we would have $\mathrm{vol}_2(K \cap l^-) < \mathrm{vol}_2(L \cap l^-)$ or $\mathrm{vol}_2(K \cap l^-) > \mathrm{vol}_2(L \cap l^-)$, which contradicts the hypotheses.

Now the conclusion follows from Theorem 3.2.4. $\square$

Corollary 3.2.8. *Let K be a convex body in $\mathbb{R}^2$ and let D be a disk in the interior of K. If $\mathrm{vol}_2(K \cap H^+) =$ const for every H supporting D, then K is also a disk.*

Proof. From the proof of Theorem 3.2.2 we see that the condition $\mathrm{vol}_2(K\cap H^+) = C$ for every line H supporting D implies $\rho^2_{K,D}(u,\xi) - \rho^2_{K,D}(-u,\xi) = 0$ for all $\xi \in S^1$ and $u \in S^1 \cap \xi^\perp$. Without loss of generality, let D be a disk of radius 1. Consider the mapping φ defined as follows. Let Q be a point outside of D. There are two unique supporting lines to D passing through Q. Choose the one that lies on the right of the disk D when viewing from the point Q. Let T be the point of contact of the chosen supporting line and the disk D. On this line we take a point $\varphi(Q)$, such that T is the midpoint of the segment $[Q, \varphi(Q)]$.

For a point $Q \in \partial K$ introduce the coordinates (θ, r) so that

$$Q = (\cos\theta, \sin\theta) + r(\sin\theta, -\cos\theta).$$

Then,

$$\varphi(Q) = (\theta + 2\arctan r, r).$$

Applying φ to $\varphi(Q)$ and iterating this procedure, we get a set

$$E = \{((\theta + 2n\arctan r) \bmod 2\pi), r) : \ n \in \mathbb{N}\} \subset \partial K.$$

Note that all points in this set are at the same distance from the origin. If $\arctan r$ is an irrational multiple of π, E is a dense subset of ∂K, implying that K is a disk. If $\arctan r$ is a rational multiple of π, we will argue by contradiction. Assume K is not a disk. By the continuity of ∂K, there exists a point on the boundary of K with coordinates (θ', r'), such that, $\arctan r'$ is an irrational multiple of π. Contradiction.

□

Remark 3.2.9. *Corollary 3.2.8 was independently obtained by Kurusa and Ódor [23]. It also appears that Theorem 3.2.1 was probably known to Barker and Larman. Here, we get it as a simple consequence of Theorem 3.2.4.*

3.3 Main results: Higher dimensional cases.

Theorem 3.3.1. *Let K and L be convex bodies in $\mathbb{R}^n$ (where n is even) and let D be a cube in the interior of $K \cap L$. If $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ for any hyperplane passing through a vertex of D and an interior point of D, then $K = L$.*

For $\epsilon > 0$ and $\xi \in S^{n-1}$, denote by

$$U_\epsilon(\xi) = \{\eta \in S^{n-1} : \langle \eta, \xi \rangle > \sqrt{1-\epsilon^2}\}$$

the spherical cap centered at ξ, and by

$$E_\epsilon(\xi) = \{\eta \in S^{n-1} : |\langle \eta, \xi \rangle| < \epsilon\}$$

the neighborhood of the equator $S^{n-1} \cap \xi^\perp$.

Lemma 3.3.2. *Let K and L be convex bodies in $\mathbb{R}^n$ (where n is even) containing the origin in their interiors. Let $\xi \in S^{n-1}$ and $\epsilon > 0$. If $\mathrm{vol}_{n-1}(K \cap u^\perp) = \mathrm{vol}_{n-1}(L \cap u^\perp)$ for every $u \in E_\epsilon(\xi)$, then $\rho_K^{n-1}(\eta) + \rho_K^{n-1}(-\eta) = \rho_L^{n-1}(\eta) + \rho_L^{n-1}(-\eta)$ for every $\eta \in U_\epsilon(\xi)$.*

Proof. For every even function $\psi \in C^\infty(S^{n-1})$ with support in $U_\epsilon(\xi) \cup U_\epsilon(-\xi)$, we have

$$\int_{S^{n-1}} (\|x\|_K^{-n+1} + \|-x\|_K^{-n+1}) \psi(x)\, dx$$

$$= (2\pi)^{-n} \int_{S^{n-1}} (\|x\|_K^{-n+1} + \|-x\|_K^{-n+1})^\wedge(u) (\psi(x/|x|)|x|^{-1})^\wedge(u)\, du,$$

where we used Parseval's formula on the sphere; see [22, Section 3.4].

Since $(\|x\|_K^{-n+1} + \|-x\|_K^{-n+1})^\wedge(u) = 2\pi(n-1)\mathrm{vol}_{n-1}(K \cap u^\perp)$ by [22, Lemma 3.7], the assumption of the lemma yields

$$(\|x\|_K^{-n+1} + \|-x\|_K^{-n+1})^\wedge(u) = (\|x\|_L^{-n+1} + \|-x\|_L^{-n+1})^\wedge(u)$$

for every $u \in E_\epsilon(\xi)$. On the other hand, by formula (3.6) from [14] or [26, Lemma 5.1], we see that $(\psi(x/|x|)|x|^{-1})^\wedge\Big|_{S^{n-1}}$ is supported in $E_\epsilon(\xi)$.

Therefore,

$$\begin{aligned}
&\int_{S^{n-1}} (\|x\|_K^{-n+1} + \|-x\|_K^{-n+1}) \psi(x)\, dx \\
&= (2\pi)^{-n} \int_{S^{n-1}} (\|x\|_L^{-n+1} + \|-x\|_L^{-n+1})^\wedge(u) (\psi(x/|x|)|x|^{-1})^\wedge(u)\, du \\
&= \int_{S^{n-1}} (\|x\|_L^{-n+1} + \|-x\|_L^{-n+1}) \psi(x)\, dx.
\end{aligned}$$

Since this true for any $\psi \in C^\infty(S^{n-1})$ with support in $U_\epsilon(\xi) \cup U_\epsilon(-\xi)$, the conclusion follows. $\square$

Definition 3.3.3. *Let D be a convex polytope and v_k one of its vertices. Define $C_D(v_k)$ to be the double cone centered at v_k with the property that every point in $C_D(v_k)$ lies on a line through v_k that has non-empty intersection with $D \setminus \{v_k\}$.*

Note that when D is a cube, $\cup_k C_D(v_k) = \mathbb{R}^n$.

Remark 3.3.4. *For simplicity, we stated Theorem 3.3.1 only in the case when D is a cube, but, in fact, it remains valid for a larger class of polytopes. In particular, any centrally symmetric polytope D satisfying the following condition will work:*

$\cup_k C_D(v_k) = \mathbb{R}^n$. *Indeed, this condition does not works for all the centrally symmetric polytopes; for example, consider polytope in* $\mathbb{R}^3$ *with vertices* $(1,1,0)$, $(-1,1,0)$, $(-1,-1,0)$, $(1,-1,0)$, $(2,0,1)$, *and* $(-2,0,-1)$, *then* $(0,0,\lambda) \notin \cup_k C_D(v_k)$ *for sufficient large* λ.

Proof of Theorem 3.3.1. We will prove the theorem for the class of polytopes described in Remark 3.3.4. Assume that D is such a polytope and its center of symmetry is at the origin O.

By Lemma 3.3.2, if v_i is a vertex of D, then

$$\rho^{n-1}_{K,v_i}(\xi) + \rho^{n-1}_{K,v_i}(-\xi) = \rho^{n-1}_{L,v_i}(\xi) + \rho^{n-1}_{L,v_i}(-\xi),$$

for every $\xi \in S^{n-1} \cap (C_D(v_i) - v_i)$. Here, if p is a point in the interior of L, and $L - p$ is a star body, then we will use $\rho_{L,p}$ to denote ρ_{L-p}.

For a point $Q \in C_D(v_i)$ define a mapping φ_i as follows. Let $\varphi_i(Q)$ be the point on the line through Q and v_i, such that v_i lies between Q and $\varphi_i(Q)$, and

$$|Qv_i|^{n-1} + |\varphi_i(Q)v_i|^{n-1} = \rho^{n-1}_{K,v_i}(\xi) + \rho^{n-1}_{K,v_i}(-\xi) = \rho^{n-1}_{L,v_i}(\xi) + \rho^{n-1}_{L,v_i}(-\xi),$$

where ξ is the unit vector in the direction of $\overrightarrow{v_iQ}$. Note that the domain of φ_i is not the entire set $C_D(v_i)$, but it will be enough that φ_i is defined in some neighborhood of $(K \triangle L) \cap C_D(v_i)$.

Note that $\partial K \cap \partial L \neq \emptyset$. Otherwise one of the bodies K or L would be strictly contained inside the other body, thus violating the condition $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ from the statement of the theorem. Consider a point $Q \in \partial K \cap \partial L$. There exists a vertex v_i of D, such that $Q \in C_D(v_i)$. Since D is origin-symmetric, there is a vertex $v_j = -v_i$. Our first goal is to show that $l(v_i, v_j) \cap \partial K = l(v_i, v_j) \cap \partial L$, where $l(v_i, v_j)$ is the line through v_i and v_j. If Q belongs to this line, we are done. If not, we will argue as follows.

Since $Q \in C_D(v_i) \cap \partial K \cap \partial L$, then $\varphi_i(Q)$ is also in $C_D(v_i) \cap \partial K \cap \partial L$. Let $\{F_m\}$ be the collection of the facets of D that contain the vertex v_i, and let $\{n_m\}$ be collection of the corresponding outward unit normal vectors. Note that the condition $Q \in C_D(v_i)$ means that either $\langle \overrightarrow{v_i Q}, n_m \rangle \geq 0$ for all m, or $\langle \overrightarrow{v_i Q}, n_m \rangle \leq 0$ for all m. Without loss of generality we can assume that $\langle \overrightarrow{v_i Q}, n_m \rangle \geq 0$ for all m (otherwise, take $\varphi_i(Q)$ instead of Q).

We claim that $Q \in C_D(v_i) \cap C_D(v_j)$. Indeed, the outward unit normal vectors to the facets that contain v_j are $\{-n_m\}$. Thus,

$$\langle \overrightarrow{v_j Q}, n_m \rangle = \langle \overrightarrow{v_i Q}, n_m \rangle + \langle \overrightarrow{v_j v_i}, n_m \rangle = \langle \overrightarrow{v_i Q}, n_m \rangle + 2\langle \overrightarrow{O v_i}, n_m \rangle \geq 0.$$

Next we claim that $\varphi_j(Q) \in C_D(v_i) \cap C_D(v_j)$. It is clear that $\varphi_j(Q) \in C_D(v_j)$. Thus, it is enough to show that $\langle \overrightarrow{v_i \varphi_j(Q)}, n_m \rangle \leq 0$ for all m. We have

$$\overrightarrow{v_i \varphi_j(Q)} = \overrightarrow{OQ} + \overrightarrow{Q \varphi_j(Q)} - \overrightarrow{O v_i} = \overrightarrow{OQ} + \alpha \overrightarrow{Q v_j} - \overrightarrow{O v_i},$$

where $\alpha = \dfrac{|Q\varphi_j(Q)|}{|Qv_j|} > 1$. So,

$$\overrightarrow{v_i \varphi_j(Q)} = \overrightarrow{OQ} + \alpha \overrightarrow{O v_j} - \alpha \overrightarrow{OQ} - \overrightarrow{O v_i} = (1-\alpha)\overrightarrow{OQ} - (1+\alpha)\overrightarrow{O v_i} = (1-\alpha)\overrightarrow{v_i Q} - 2\alpha \overrightarrow{O v_i}.$$

Thus, for every m,

$$\langle \overrightarrow{v_i \varphi_j(Q)}, n_m \rangle = (1-\alpha)\langle \overrightarrow{v_i Q}, n_m \rangle - 2\alpha \langle \overrightarrow{O v_i}, n_m \rangle \leq 0.$$

In a similar fashion one can show that $\varphi_i(\varphi_j(Q)) \in C_D(v_i) \cap C_D(v_j)$. Thus we can produce a sequence of points $\{Q_k\}_{k=0}^{\infty}$, where $Q_0 = Q$ and $Q_k = \varphi_i(\varphi_j(Q_{k-1}))$, and such that $Q_k \in C_D(v_i) \cap C_D(v_j) \cap \partial K \cap \partial L$ for all $k \geq 0$. Moreover, all these points belong to the 2-dimensional plane spanned by the points Q, v_i, and v_j. As in Proposition 3.2.4 we have the corresponding sequence of angles $\theta_k = \angle(\overrightarrow{v_i Q_k}, \overrightarrow{v_i v_j})$, with $\theta_k < \theta_{k-1}$. One can see that $\lim_{k\to\infty} \theta_k = 0$. Since $Q_k \in \partial K \cap \partial L$ for all k, we have proved that $l(v_i, v_j) \cap \partial K = l(v_i, v_j) \cap \partial L$.

Denote the points of intersection of the latter line with the boundaries of K and L by X_0 and Y_0, and consider any 2-dimensional plane H through X_0 and Y_0. Using [8, Lemma 7], we see that there are neighborhoods $\mathcal{N}(X_0)$ and $\mathcal{N}(Y_0)$ of X_0 and Y_0 correspondingly, such that

$$H \cap \mathcal{N}(X_0) \cap \partial K = H \cap \mathcal{N}(X_0) \cap \partial L, \text{ and } H \cap \mathcal{N}(Y_0) \cap \partial K = H \cap \mathcal{N}(Y_0) \cap \partial L.$$

If P is a point in $C_D(v_i) \cap H$ that does not belong to $\mathcal{N}(X_0)$ or $\mathcal{N}(Y_0)$, then we apply φ_j and φ_i to produce a sequence of points P_k, which after finitely many steps will belong to $\mathcal{N}(X_0)$ or $\mathcal{N}(Y_0)$. Thus, $P_N \in \partial K \cap \partial L$ for some large N. Applying inverse maps φ_i^{-1} and φ_j^{-1}, we conclude that $P \in \partial K \cap \partial L$. Thus, we have shown that

$$H \cap C_D(v_i) \cap \partial K = H \cap C_D(v_i) \cap \partial L.$$

Since this is true for every H, we have $C_D(v_i) \cap \partial K = C_D(v_i) \cap \partial L$.

Now consider any other vertex of D, say v_m, that is connected to v_i by an edge. One can see that

$$C_D(v_i) \cap C_D(v_m) \cap \partial K \cap \partial L \neq \emptyset.$$

Repeating the same process as above, we get

$$C_D(v_m) \cap \partial K = C_D(v_m) \cap \partial L.$$

Since we can do this for every vertex, it follows that $C_D(v_k) \cap \partial K = C_D(v_k) \cap \partial L$ for every k, and thus $K = L$. $\square$

Remark 3.3.5. How to prove this in odd dimensions? Is there a different condition that guarantees a positive answer in odd dimensions? It is still an open question. If we replace the equality of sections by the equality of derivatives of the parallel section functions, then, for example, in $\mathbb{R}^3$ first derivatives are not enough; cf. [21, Remark 1].

The next theorem is an analogue of Groemer's result for half-sections. The difference is that we look at half-sections that do not pass through the origin. We will adopt the following notation. For a point $p \in \mathbb{R}^n$ and a vector $v \in S^{n-1}$, define $v_p^\perp = \{x \in \mathbb{R}^n : \langle x - p, v \rangle = 0\}$ and $v_p^+ = \{x \in \mathbb{R}^n : \langle x - p, v \rangle \geq 0\}$.

Theorem 3.3.6. *Let K and L be convex bodies in $\mathbb{R}^n$, $n \geq 3$, that contain a strictly convex body D in their interiors. Assume that*

$$\mathrm{vol}_{n-1}(K \cap H \cap v_p^+) = \mathrm{vol}_{n-1}(L \cap H \cap v_p^+),$$

for every hyperplane H supporting D and every unit vector $v \in H - p$, where $p = D \cap H$. Then $K = L$.

Proof. Let us fix a supporting plane H and consider the equality

$$\mathrm{vol}_{n-1}(K \cap H \cap v_p^+) = \mathrm{vol}_{n-1}(L \cap H \cap v_p^+),$$

for every unit vector $v \in H - p$. Then [16] implies that

$$\rho_{K,p}^{n-1}(u) - \rho_{K,p}^{n-1}(-u) = \rho_{L,p}^{n-1}(u) - \rho_{L,p}^{n-1}(-u),$$

for every vector $u \in S^{n-1} \cap (H - p)$, where $p = D \cap H$.

Now observe that $\partial K \cap \partial L \neq \emptyset$; otherwise the condition $\mathrm{vol}_{n-1}(K \cap H \cap v_p^+) = \mathrm{vol}_{n-1}(L \cap H \cap v_p^+)$ would be violated. Moreover, if $Q \in \partial K \cap \partial L$, then by [3, Lemma 3] there exists a neighborhood $\mathcal{N}(Q)$ of Q, such that $\mathcal{N}(Q) \cap \partial K \subset \partial K \cap \partial L$. Hence, $\partial K \cap \partial L$ is open in ∂K. On the other hand, by the continuity of the boundaries of K and L, $\partial K \cap \partial L$ is closed in ∂K. Therefore,

$$\partial K \cap \partial L = \partial K = \partial L.$$

$\square$

Corollary 3.3.7. *Let K be a convex body in $\mathbb{R}^n$, $n \geq 3$, that contains a ball D of radius t in its interior. If*

$$\mathrm{vol}_{n-1}(K \cap \{\xi^\perp + t\xi\} \cap v^+) = \mathit{const},$$

for every $\xi \in S^{n-1}$ and every vector $v \in S^{n-1} \cap \xi^\perp$, then K is a Euclidean ball.

In the next theorem we will consider a different type of half-sections.

Theorem 3.3.8. *Let K and L be convex bodies in $\mathbb{R}^n$, $n \geq 3$, that contain a ball D in their interiors. Assume that*

$$\mathrm{vol}_{n-1}(K \cap H^+ \cap v^\perp) = \mathrm{vol}_{n-1}(L \cap H^+ \cap v^\perp)$$

for every hyperplane H supporting D and every unit vector $v \in H - p$, where $p = D \cap H$. Then $K = L$.

Proof. Let us fix a unit vector v, and consider $\xi, \zeta \in S^{n-1} \cap v^\perp$ such that $\xi \perp \zeta$. For a small ϕ let $\eta = \cos\phi\, \xi + \sin\phi\, \zeta$. Without loss of generality we will assume that D has radius 1 and is centered at the origin. Consider the affine hyperplanes $H_\xi = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 1\}$ and $H_\eta = \{x \in \mathbb{R}^n : \langle x, \eta \rangle = 1\}$. Let the $(n-3)$-dimensional subspace W be the orthogonal compliment of span$\{\xi, \zeta\}$ in $v^\perp$. Consider the orthogonal projection of the convex body $K \cap v^\perp$ onto the 2-dimensional subspace spanned by ξ and ζ. The picture is identical to Figure 5, with E_1, E_2, E_3, and E_4 defined similarly. If $n = 3$, we repeat the argument from the proof of Theorem 3.2.2. If $n \geq 4$, we will use the following modification of this argument.

Let $\bar{E}_i = E_i \times W$, for $i = 1, 2, 3, 4$. Then the equality

$$\mathrm{vol}_{n-1}(K \cap v^\perp \cap H_\xi^+) - \mathrm{vol}_{n-1}(K \cap v^\perp \cap H_\eta^+) = \mathrm{vol}_{n-1}(L \cap v^\perp \cap H_\xi^+) - \mathrm{vol}_{n-1}(L \cap v^\perp \cap H_\eta^+)$$

implies

$$\begin{aligned} &\mathrm{vol}_{n-1}(K \cap v^\perp \cap (\bar{E}_1 \cup \bar{E}_4)) - \mathrm{vol}_{n-1}(K \cap v^\perp \cap (\bar{E}_3 \cup \bar{E}_4)) \\ &\qquad = \mathrm{vol}_{n-1}(L \cap v^\perp \cap (\bar{E}_1 \cup \bar{E}_4)) - \mathrm{vol}_{n-1}(L \cap v^\perp \cap (\bar{E}_3 \cup \bar{E}_4)). \end{aligned} \tag{3.7}$$

For $x \in \mathrm{span}\{\xi, \zeta\}$, consider the following parallel section function:

$$A_{K \cap v^\perp, W}(x) = \mathrm{vol}_{n-3}(K \cap v^\perp \cap \{W + x\}).$$

Then equation (3.7) and the Fubini theorem imply

$$\begin{aligned} &\int_{E_1 \cup E_4} A_{K \cap v^\perp, W}(x)dx - \int_{E_3 \cup E_4} A_{K \cap v^\perp, W}(x)dx \\ &= \int_{E_1 \cup E_4} A_{L \cap v^\perp, W}(x)dx - \int_{E_3 \cup E_4} A_{L \cap v^\perp, W}(x)dx. \end{aligned}$$

Now we will pass to new coordinates (r, θ) as in the proof of Theorem 3.2.2, by letting

$x(r,\theta) = \cos\theta\,\xi + \sin\theta\,\zeta + r(\sin\theta\,\xi - \cos\theta\,\zeta)$. Then

$$\int_0^\phi \int_0^\infty |r| A_{K\cap v^\perp,W}(x(r,\theta))drd\theta - \int_0^\phi \int_{-\infty}^0 |r| A_{K\cap v^\perp,W}(x(r,\theta))drd\theta$$
$$= \int_0^\phi \int_0^\infty |r| A_{L\cap v^\perp,W}(x(r,\theta))drd\theta - \int_0^\phi \int_{-\infty}^0 |r| A_{L\cap v^\perp,W}(x(r,\theta))drd\theta.$$

Differentiating with respect to ϕ and letting $\phi = 0$, we get

$$\int_{-\infty}^\infty r A_{K\cap v^\perp,W}(x(r,0))dr = \int_{-\infty}^\infty r A_{L\cap v^\perp,W}(x(r,0))dr. \tag{3.8}$$

Note that

$$\begin{aligned} A_{K\cap v^\perp,W}(x(r,0)) &= A_{K\cap v^\perp,W}(\xi - r\zeta) \\ &= A_{(K-\xi)\cap v^\perp,W}(-r\zeta) \\ &= \int_{x\in\xi^\perp\cap v^\perp\cap\{\langle x,\zeta\rangle=-r\}} \chi(\|x\|_{K-\xi})dx. \end{aligned}$$

Therefore, (3.8) and the Fubini theorem give

$$\int_{\xi^\perp\cap v^\perp} \langle x,\zeta\rangle \chi(\|x\|_{K-\xi})dx = \int_{\xi^\perp\cap v^\perp} \langle x,\zeta\rangle \chi(\|x\|_{L-\xi})dx.$$

Passing to polar coordinates in $\xi^\perp \cap v^\perp$, we get

$$\int_{S^{n-1}\cap\xi^\perp\cap v^\perp} \langle w,\zeta\rangle \|w\|_{K-\xi}^{-n+1}dw = \int_{S^{n-1}\cap\xi^\perp\cap v^\perp} \langle w,\zeta\rangle \|w\|_{L-\xi}^{-n+1}dw.$$

Observe, that this is true for any $\zeta \in \xi^\perp \cap v^\perp$. Furthermore, for any vector $\vartheta \in \xi^\perp$ there is a vector $\zeta \in \xi^\perp \cap v^\perp$ and a number β such that $\vartheta = \zeta + \beta v$. Therefore, for every $\vartheta \in \xi^\perp$ we have

$$\int_{S^{n-1}\cap\xi^\perp\cap v^\perp} \langle w, \vartheta\rangle \|w\|_{K-\xi}^{-n+1} dw = \int_{S^{n-1}\cap\xi^\perp\cap v^\perp} \langle w, \vartheta\rangle \|w\|_{L-\xi}^{-n+1} dw.$$

Fixing ξ and ϑ, and looking at all $v \in S^{n-1}\cap\xi^\perp$, we can consider the latter equality as the equality of the spherical Radon transforms on $S^{n-1}\cap\xi^\perp$. Since the spherical Radon transform only allows to reconstruct even parts, we get

$$\langle w, \vartheta\rangle \|w\|_{K-\xi}^{-n+1} + \langle -w, \vartheta\rangle \|-w\|_{K-\xi}^{-n+1} = \langle w, \vartheta\rangle \|w\|_{L-\xi}^{-n+1} + \langle -w, \vartheta\rangle \|-w\|_{L-\xi}^{-n+1},$$

for all $w, \vartheta \in S^{n-1}\cap\xi^\perp$. That is,

$$\|w\|_{K-\xi}^{-n+1} - \|-w\|_{K-\xi}^{-n+1} = \|w\|_{L-\xi}^{-n+1} - \|-w\|_{L-\xi}^{-n+1}, \text{ for all } w \in S^{n-1}\cap\xi^\perp.$$

We finish the proof as in Theorem 3.3.6. □

Below we will prove an analogue of the result of Falconer [8] and Gardner [11] for halfspaces. We will need the following lemma.

Lemma 3.3.9. *Suppose $i > 0$. Let K and L be convex bodies in $\mathbb{R}^n$, p_1 and p_2 be distinct points in the interior of $K \cap L$. If for all $\xi \in S^{n-1}$,*

$$\rho^i_{K,p_j}(\xi) - \rho^i_{K,p_j}(-\xi) = \rho^i_{L,p_j}(\xi) - \rho^i_{L,p_j}(-\xi), \text{ for } j = 1, 2, \tag{3.9}$$

and $\partial K \cap \partial L \neq \emptyset$, then $K = L$.

Proof. Let l be the line passing through p_1 and p_2. Our first goal is to prove that $\partial K \cap l = \partial L \cap l$. Let $Q_0 \in \partial K \cap \partial L$. If $Q_0 \in l$, we are done. Otherwise, we define two maps φ_1, φ_2 as follows. If Q is a point distinct from p_1, then $\varphi_1(Q)$ is defined to be the point on the line passing through Q and p_1, such that p_1 lies between Q and $\varphi_1(Q)$ and

$$|Qp_1|^i - |p_1\varphi_1(Q)|^i = \rho^i_{K,p_1}(\xi) - \rho^i_{K,p_1}(-\xi),$$

where $\xi = \dfrac{\overrightarrow{p_1Q}}{|p_1Q|}$.
Note that the domain of φ_1 contains the set $K\triangle L$. The map φ_2 is defined similarly with p_1 replaced by p_2.
For the chosen point $Q_0 \in \partial K \cap \partial L$ consider the 2-dimensional plane H passing through Q_0, p_1, and p_2. Construct a sequence of points $\{Q_j\} \subset \partial K \cap \partial L \cap H$, satisfying $Q_{j+1} = \varphi_2^{-1}(\varphi_1(Q_j))$, and a sequence of angles $\{\theta_j\} = \{\angle(\overrightarrow{Q_j\varphi_1(Q_j)}, l)\}$. One can see that $\lim_{j\to\infty}\theta_j = 0$, and therefore the limit

$$X_0 = \lim_{j\to\infty} Q_j$$

is a point on $l \cap \partial K \cap \partial L$. The claim that $\partial K \cap l = \partial L \cap l$ is now proved.
Let V be any 2-dimensional affine subspace of $\mathbb{R}^n$ that contains the line l. Consider the bodies $K \cap V$ and $L \cap V$ in V. The line l cuts both these bodies in two parts, $K \cap V = K_1 \cup K_2$ and $L \cap V = L_1 \cup L_2$, so that K_1 and L_1 are on the same side of l. Since $K \cap l = L \cap l$, the following star bodies are well-defined: $\tilde{K} = K_1 \cup L_2$ and $\tilde{L} = K_2 \cup L_1$. Condition (3.9) now implies

$$\rho^i_{\tilde{K},p_j}(\xi) + \rho^i_{\tilde{K},p_j}(-\xi) = \rho^i_{\tilde{L},p_j}(\xi) + \rho^i_{\tilde{L},p_j}(-\xi), \text{ for } j = 1, 2.$$

Now we can use [11, Theorem 6.2.3] to show that $\tilde{K} = \tilde{L}$, implying that $K \cap V = L \cap V$. Since V was an arbitrary affine subspace containing l, it follows that $K = L$. $\square$

Remark 3.3.10. *A version of this lemma for a smaller set of values of i (but without the assumption $\partial K \cap \partial L \neq \emptyset$) was proved by Koldobsky and Shane, [21, Lemma*

6]. They also showed (see [21, Remark 1]) that one can take two balls that satisfy condition (3.9) with $i = 1$, but whose boundaries do not intersect. .

With the help of Lemma 3.3.9 we obtain the following result.

Theorem 3.3.11. *Let K and L be convex bodies in $\mathbb{R}^n$ containing two distinct points p_1 and p_2 in their interiors. If for every $v \in S^{n-1}$, we have*

$$\mathrm{vol}_n(K \cap v_{p_j}^+) = \mathrm{vol}_n(L \cap v_{p_j}^+) \text{ for } j = 1, 2,$$

then $K = L$.

Proof. By [16], we have $\rho_{K,p_j}^n(\xi) - \rho_{K,p_j}^n(-\xi) = \rho_{L,p_j}^n(\xi) - \rho_{L,p_j}^n(-\xi)$, for $j = 1, 2$, and every $\xi \in S^{n-1}$. Also observe that $\partial K \cap \partial L \neq \emptyset$. Otherwise one of K or L would be strictly contained inside the other, which would contradict the hypothesis of the theorem. Now the result follows from Lemma 3.3.9. $\square$

Note that Problem 1.1.3 is open even in the case of bodies of revolution when the center of the ball lies on the axis of revolution. However, if we consider a ball that does not intersect the axis of revolution, then the problem has a positive answer.

Theorem 3.3.12. *Let K and L be convex bodies of revolution in $\mathbb{R}^n$ with the same axis of revolution. Let D be a convex body in the interior of both K and L such that D does not intersect the axis of revolution. If for every hyperplane H supporting D we have*

$$\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H),$$

then $K = L$.

Proof. Consider the two supporting hyperplanes of D that are perpendicular to the axis of revolution. Let p and q be the points where these hyperplanes intersect the axis of revolution.

Note that every plane passing through p (or q) can be rotated around the axis of revolution until it touches the body D. Due to the rotational symmetry of the bodies K and L we obtain that

$$\mathrm{vol}_{n-1}(K \cap (p + \xi^{\perp})) = \mathrm{vol}_{n-1}(L \cap (p + \xi^{\perp})),$$

and

$$\mathrm{vol}_{n-1}(K \cap (q + \xi^{\perp})) = \mathrm{vol}_{n-1}(L \cap (q + \xi^{\perp})),$$

for every $\xi \in S^{n-1}$.

The conclusion now follows from the corresponding result of Falconer [8] and Gardner [11], described in the introduction. $\square$

Chapter 4

On bodies with congruent sections

4.1 Introduction and main result

This chapter deals with Problem 1.1.13 motivated by Problem 1.1.12; see also [11, Page 289]. Let us recall the statement of the problem.

Problem 4.1.1. *Let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin in their interiors and $t \in (0,1)$. Assume that for every $\xi \in S^{n-1}$ there is a rigid motion ϕ_ξ such that $K \cap C_t(\xi) = \phi_\xi(L \cap C_t(\xi))$. Does it follow that $K = L$?*

Here, for $t \in (0,1)$, we define

$$C_t(\xi) := \{x \in \mathbb{R}^n : \langle x, \xi \rangle = t|x|\}$$

to be a cone in the direction of ξ. For some special values of t, Problem 4.1.1 has an affirmative answer (see Sacco [30] for details); but in general it is still open.

4.1.1 Main Result

We solve Problem 4.1.1 in $\mathbb{R}^3$ in the class of C^2 convex bodies, i.e. convex bodies with C^2 boundaries.

Theorem 4.1.2. *Let $f, g \in C^2(S^2)$ and $t \in (0,1)$. Assume that for every $\xi \in S^2$ there is a rotation ϕ_ξ around ξ such that*

$$f(\phi_\xi(\theta)) = g(\theta)$$

for all $\theta \in S^2 \cap (\xi^\perp + t\xi)$. Then $f = g$.

As a corollary of Theorem 4.1.2, we get a positive answer to a version of Problem 4.1.1.

Corollary 4.1.3. *Let $K, L \subset \mathbb{R}^3$ be C^2 convex bodies containing the origin in their interiors and $t \in (0,1)$. Assume that for every $\xi \in S^{n-1}$ there is a rotation ϕ_ξ around ξ such that $K \cap C_t(\xi) = \phi_\xi(L \cap C_t(\xi))$. Then $K = L$.*

4.2 Proof of the main result

For a unit vector $\xi \in S^2$, we define an open ball on S^2 with centre at ξ to be

$$B_\epsilon(\xi) := \{\theta \in S^2 : \|\theta - \xi\| < \epsilon\},$$

where $\|\cdot\|$ is the Euclidean distance. We also define $\phi_\xi = \phi_{\xi,\alpha} \in \mathrm{SO}(3)$ to be the rotation around ξ by an angle α in the anticlockwise direction. Namely, for any $\theta \in S^2$,

$$\phi_{\xi,\alpha}(\theta) = \theta\cos(\alpha\pi) + (\xi \times \theta)\sin(\alpha\pi) + \xi\langle\xi, \theta\rangle(1 - \cos(\alpha\pi)),$$

where $\xi \times \theta$, $\langle\xi, \theta\rangle$ are usual vector and scalar products in $\mathbb{R}^3$.

Definition 4.2.1. *Let f, g, t be as in Theorem 4.1.2. Define the following three sets in S^2,*

$$\Xi_0 = \{\xi \in S^2 : f(\theta) = g(\theta),\ \forall \theta \in S^2 \cap (\xi^\perp + t\xi)\};$$
$$\Xi_n = \{\xi \in S^2 : f(\phi_{\xi, \frac{2}{n}}(\theta)) = f(\theta),\ \forall \theta \in S^2 \cap (\xi^\perp + t\xi)\}, \quad n = 2, 3, \dots$$
$$\Xi_{\text{con}} = S^2 \backslash (\Xi_0 \cup (\cup_{n=2}^\infty \Xi_n)).$$

Lemma 4.2.2. *Ξ_n are closed, for all $n = 0, 2, 3, \cdots$.*

Proof. First, given two non-parallel directions $\xi_1, \xi_2 \in S^2$, we define a map $\psi_{\xi_1,\xi_2} : \xi_1^\perp + t\xi_1 \to \xi_2^\perp + t\xi_2$ as follows:
Consider the great circle passing through ξ_1, ξ_2, which intersects $\xi_1^\perp + t\xi_1$ and $\xi_2^\perp + t\xi_2$ at θ_{11}, θ_{12} and θ_{21}, θ_{22} respectively. θ_{11}, θ_{12} are chosen in such a way that the triple $\theta_{11}, \theta_{12}, \xi_1 \times \xi_2$ has a positive orientation. The same is assumed to hold for θ_{21}, θ_{22}. For any point $\theta \in S^2 \cap (\xi_1^\perp + t\xi_1)$, there exists $\phi_{\xi_1,\alpha} \in \mathrm{SO}(3)$, such that $\theta = \phi_{\xi_1,\alpha}(\theta_{11})$. We define $\psi_{\xi_1,\xi_2}(\theta) := \phi_{\xi_2,\alpha}(\theta_{21})$. If $\xi_1 = \xi_2$, we define $\psi_{\xi_1,\xi_2}(\theta) = \theta$. Note that for any $\theta \in S^2 \cap (\xi_1^\perp + t\xi_1)$,

$$\begin{aligned}
&\|\psi_{\xi_1,\xi_2}(\theta) - \theta\| = \|\phi_{\xi_2,\alpha}(\theta_{21}) - \phi_{\xi_1,\alpha}(\theta_{11})\| \\
=&\|\theta_{21}\cos(\alpha\pi) + (\xi_2 \times \theta_{21})\sin(\alpha\pi) + \xi_2\langle \xi_2, \theta_{21}\rangle(1 - \cos(\alpha\pi)) \\
&- \theta_{11}\cos(\alpha\pi) - (\xi_1 \times \theta_{11})\sin(\alpha\pi) - \xi_1\langle \xi_1, \theta_{11}\rangle(1 - \cos(\alpha\pi))\| \\
=&\|\theta_{21}\cos(\alpha\pi) + t\xi_2(1 - \cos(\alpha\pi)) - \theta_{11}\cos(\alpha\pi) - t\xi_1(1 - \cos(\alpha\pi))\| \\
\le&\|\theta_{21}\cos(\alpha\pi) - \theta_{11}\cos(\alpha\pi)\| + \|t\xi_2(1 - \cos(\alpha\pi)) - t\xi_1(1 - \cos(\alpha\pi))\| \\
\le&\|\theta_{21} - \theta_{11}\| + 2\|\xi_1 - \xi_2\| \\
=&3\|\xi_1 - \xi_2\|
\end{aligned}$$

and

$$\phi_{\xi_2,\beta}(\psi_{\xi_1,\xi_2}(\theta)) = \psi_{\xi_1,\xi_2}(\phi_{\xi_1,\beta}(\theta)), \quad \text{for any } \beta.$$

Given a sequence $\xi_i \in \Xi_0$ with $\lim_{i\to\infty} \xi_i = \xi$, for any $\theta \in S^2 \cap (\xi^\perp + t\xi)$, we have

$$\begin{aligned}
&|f(\theta) - g(\theta)| \\
\leq &|f(\theta) - f(\psi_{\xi,\xi_i}(\theta))| + |f(\psi_{\xi,\xi_i}(\theta)) - g(\psi_{\xi,\xi_i}(\theta))| + |g(\psi_{\xi,\xi_i}(\theta)) - g(\theta)| \\
= &|f(\theta) - f(\psi_{\xi,\xi_i}(\theta))| + |g(\psi_{\xi,\xi_i}(\theta)) - g(\theta)|.
\end{aligned}$$

As $\xi_i \to \xi$, $\psi_{\xi,\xi_i}(\theta) \to \theta$; hence, by continuity of f and g,

$$|f(\theta) - g(\theta)| = 0, \quad \forall \theta \in S^2 \cap (\xi^\perp + t\xi),$$

which implies $\xi \in \Xi_0$.

Similarly, given a sequence $\xi_i \in \Xi_n$ with $\lim_{i\to\infty} \xi_i = \xi$, for any $\theta \in S^2 \cap (\xi^\perp + t\xi)$, we have

$$\begin{aligned}
&|f(\phi_{\xi,\frac{2}{n}}(\theta)) - f(\theta)| \\
\leq &|f(\phi_{\xi,\frac{2}{n}}(\theta)) - f(\phi_{\xi_i,\frac{2}{n}}(\psi_{\xi,\xi_i}(\theta)))| + |f(\phi_{\xi_i,\frac{2}{n}}(\psi_{\xi,\xi_i}(\theta))) - f(\psi_{\xi,\xi_i}(\theta))| \\
&+ |f(\psi_{\xi,\xi_i}(\theta)) - f(\theta)| \\
= &|f(\phi_{\xi,\frac{2}{n}}(\theta)) - f(\psi_{\xi,\xi_i}(\phi_{\xi,\frac{2}{n}}(\theta)))| + |f(\psi_{\xi,\xi_i}(\theta)) - f(\theta)|.
\end{aligned}$$

As $\xi_i \to \xi$, $\psi_{\xi,\xi_i}(\theta) \to \theta$; hence, by continuity of f,

$$|f(\phi_{\xi,\frac{2}{n}}(\theta)) - f(\theta)| = 0, \quad \forall \theta \in S^2 \cap (\xi^\perp + t\xi),$$

which implies $\xi \in \Xi_n$. $\square$

Lemma 4.2.3. *Suppose that for some $\xi \in S^2$ there exists $\alpha \in \mathbb{Q}$ such that $f(\phi_{\xi,\alpha}(\theta)) = f(\theta)$, $\forall \theta \in S^2 \cap (\xi^\perp + t\xi)$. Then there exists $n \geq 2$, such that $\xi \in \Xi_n$.*

Proof. Let us write $\alpha = \frac{p}{q}$, where p and q are coprime integers. It is sufficient to show $\frac{2}{n} = m\frac{p}{q} + 2l$ for some $m, n, l \in \mathbb{Z}$. Indeed, this would imply that

$$f(\phi_{\xi,\frac{2}{n}}(\theta)) = f(\phi_{\xi,m\frac{p}{q}+2l}(\theta)) = f(\phi_{\xi,m\frac{p}{q}}(\theta)) = f(\theta)$$

But, since p, q are coprime, there exist $k, r \in \mathbb{Z}$, such that $pk + qr = 1$. If we set $n = q$, then

$$\frac{2}{n} = \frac{2(pk+qr)}{q} = 2k\frac{p}{q} + 2r.$$

□

Lemma 4.2.4. *Let f, g, t be as in Theorem 4.1.2. Then Ξ_{con} is open and $\lambda(\xi)$ is a continunous function on Ξ_{con}, where*

$$\lambda(\xi) := \{\alpha \in [0,2) : f(\phi_{\xi,\alpha}(\theta)) = g(\theta), \forall \theta \in S^2 \cap (\xi^\perp + t\xi)\}.$$

Proof. If $\xi \in \Xi_{\text{con}}$, $\lambda(\xi)$ is a single-valued function; otherwise, if $\alpha, \beta \in \lambda(\xi)$ with $\alpha \neq \beta$,

$$f(\phi_{\xi,\alpha}(\theta)) = g(\theta) = f(\phi_{\xi,\beta}(\theta)),\ \forall \theta \in S^2 \cap (\xi^\perp + t\xi),$$

implying

$$f(\phi_{\xi,\alpha-\beta}(\theta)) = f(\theta),\ \forall \theta \in S^2 \cap (\xi^\perp + t\xi).$$

If $\alpha - \beta$ is irrational, then $f(\theta) \equiv C \equiv g(\theta)$, $\forall \theta \in S^2 \cap (\xi^\perp + t\xi)$, which means $\xi \in \Xi_0$, contradiction. If $\alpha - \beta$ is rational, then by Lemma 4.2.3, $\xi \in \Xi_n$, contradiction.

Now assume that Ξ_{con} is not open. There exists $\xi \in \Xi_{\text{con}}$, such that, for any $i \in \mathbb{N}$, there exists $\xi_i \in B_{\frac{1}{i}}(\xi)$, such that $\xi_i \in \Xi_{n_i}$, for some n_i. If there are infinitely many ξ_i

that belong to Ξ_0, then $0 \in \lambda(\xi)$, that is $\xi \in \Xi_0$, contradiction. If there are infinitely many ξ_i, for which $n_i \neq 0$, then $\lambda(\xi_i)$ is a multivalued function. Thus there exists $\alpha_i \in \lambda(\xi_i)$, such that $|\alpha_i - \lambda(\xi)| > \varepsilon$, for some $\varepsilon > 0$. By compactness of $[0, 2]$, there exists a subsequence ξ_{i_k}, such that $\lim_{k\to\infty} \alpha_{i_k} = \alpha$, where $|\alpha - \lambda(\xi)| \geq \varepsilon$.
Then for any $\theta \in S^2 \cap (\xi^\perp + t\xi)$,

$$\begin{aligned}
&|f(\phi_{\xi,\lambda(\xi)}(\theta)) - f(\phi_{\xi,\alpha}(\theta))| \\
\leq &|f(\phi_{\xi,\lambda(\xi)}(\theta)) - g(\theta)| + |g(\theta) - g(\psi_{\xi,\xi_{i_k}}(\theta))| \\
&+ |g(\psi_{\xi,\xi_{i_k}}(\theta)) - f(\phi_{\xi_{i_k},\alpha_{i_k}}(\psi_{\xi,\xi_{i_k}}(\theta)))| + |f(\phi_{\xi_{i_k},\alpha_{i_k}}(\psi_{\xi,\xi_{i_k}}(\theta))) - f(\phi_{\xi,\alpha_{i_k}}(\theta))| \\
= &|g(\theta) - g(\psi_{\xi,\xi_{i_k}}(\theta))| + |f(\psi_{\xi,\xi_{i_k}}(\phi_{\xi,\alpha_{i_k}}(\theta)) - f(\phi_{\xi,\alpha}(\theta))|.
\end{aligned}$$

As $k \to \infty$, we have $\psi_{\xi,\xi_{i_k}}(\theta) \to \theta$ and $\psi_{\xi,\xi_{i_k}}(\phi_{\xi,\alpha_{i_k}}(\theta)) \to \phi_{\xi,\alpha}(\theta)$; hence, by continuity of f and g,

$$|f(\phi_{\xi,\lambda(\xi)}(\theta)) - f(\phi_{\xi,\alpha}(\theta))| = 0, \ \forall \theta \in S^2 \cap (\xi^\perp + t\xi),$$

implying

$$|f(\phi_{\xi,\alpha}(\theta)) - g(\theta)| = 0, \ \forall \theta \in S^2 \cap (\xi^\perp + t\xi),$$

contradiction.
For the continuity, since $\lambda(\xi)$ is a single-valued function when $\xi \in \Xi_{\text{con}}$, consider a sequence $\{\xi_i\}_{i=1}^\infty \in \Xi_{\text{con}}$, such that $\Xi_{\text{con}} \ni \xi = \lim_{i\to\infty} \xi_i$. By compactness of $[0, 2]$, there exists a subsequence $\{\xi_{i_k}\}_{k=1}^\infty$, such that $\alpha = \lim_{k\to\infty} \lambda(\xi_{i_k})$. Then for any $\theta \in S^2 \cap (\xi^\perp + t\xi)$,

$$\begin{aligned}
&|f(\phi_{\xi,\alpha}(\theta)) - g(\theta)| \\
\leq &|f(\phi_{\xi,\alpha}(\theta)) - f(\phi_{\xi_{i_k},\lambda(\xi_{i_k})}(\psi_{\xi,\xi_{i_k}}(\theta)))| + |f(\phi_{\xi_{i_k},\lambda(\xi_{i_k})}(\psi_{\xi,\xi_{i_k}}(\theta))) - g(\psi_{\xi,\xi_{i_k}}(\theta))| \\
&+ |g(\psi_{\xi,\xi_{i_k}}(\theta)) - g(\theta)|
\end{aligned}$$

$$=|f(\phi_{\xi,\alpha}(\theta)) - f(\phi_{\xi_{i_k},\lambda(\xi_{i_k})}(\psi_{\xi,\xi_{i_k}}(\theta)))| + |g(\psi_{\xi,\xi_{i_k}}(\theta)) - g(\theta)|.$$

As $k \to \infty$, we have $\phi_{\xi_{i_k},\lambda(\xi_{i_k})}(\psi_{\xi,\xi_{i_k}}(\theta)) \to \phi_{\xi,\alpha}(\theta)$ and $\psi_{\xi,\xi_{i_k}}(\theta) \to \theta$; hence by continuity of f and g,

$$|f(\phi_{\xi,\alpha}(\theta)) - g(\theta)| = 0, \ \forall\theta \in S^2 \cap (\xi^\perp + t\xi),$$

that is, $\lambda(\xi) = \alpha$. If $\{\lambda(\xi_i)\}_{i=1}^\infty$ has another subsequence with a different limit $\beta \neq \alpha$, then $\{\alpha, \beta\} \subset \lambda(\xi)$, contradicting to the fact that $\xi \in \Xi_{\text{con}}$. □

Lemma 4.2.5. *Let f, g, t be as in Theorem 4.1.2. Then either $\{\theta \in S^2 : f(\theta) = g(\theta)\} = S^2$ or the set*

$$\{\theta \in S^2 : f(\theta) \neq g(\theta)\} \cap [\{\theta \in S^2 : \nabla_{S^2} f(\theta) \neq 0\} \cup \{\theta \in S^2 : \nabla_{S^2} g(\theta) \neq 0\}]$$

is not empty.

Here, ∇_{S^2} is the spherical gradient, that is, for a function f on S^2,

$$(\nabla_{S^2} f)(x/|x|) = \nabla(f(x/|x|)), \quad x \in \mathbb{R}^3/\{0\}$$

where $f(x/|x|)$ is the 0-degree homogeneous extension of the function f to $\mathbb{R}^3/\{0\}$ and ∇ is the gradient in the ambient Euclidean space.

Proof. Assume

$$\{\theta \in S^2 : f(\theta) \neq g(\theta)\} \cap [\{\theta \in S^2 : \nabla_{S^2} f(\theta) \neq 0\} \cup \{\theta \in S^2 : \nabla_{S^2} g(\theta) \neq 0\}] = \emptyset.$$

Then

$$\{\theta \in S^2 : f(\theta) \neq g(\theta)\} \subset \{\theta \in S^2 : \nabla_{S^2} f(\theta) = 0\} \cap \{\theta \in S^2 : \nabla_{S^2} g(\theta) = 0\}.$$

Since $f, g \in C^2(S^2)$, the set

$$\Upsilon_0 := \{\theta \in S^2 : \nabla_{S^2} f(\theta) = 0\} \cap \{\theta \in S^2 : \nabla_{S^2} g(\theta) = 0\}$$

is closed and f and g are constant in any connected subset of Υ_0.

Note that the set $\{\theta \in S^2 : f(\theta) \neq g(\theta)\}$ is not the whole sphere; otherwise without loss of generality let $f(\theta) < g(\theta)$. Then

$$\begin{aligned} \int_{S^2 \cap (\xi^\perp + t\xi)} g(\theta)\, d\theta &= \int_{S^2 \cap (\xi^\perp + t\xi)} f(\phi_\xi(\theta))\, d\theta \\ = \int_{S^2 \cap (\xi^\perp + t\xi)} f(\theta)\, d\theta &< \int_{S^2 \cap (\xi^\perp + t\xi)} g(\theta)\, d\theta. \end{aligned}$$

Assume there exists $x \in \{\theta \in S^2 : f(\theta) \neq g(\theta)\}$. Choose the largest connected open neighbourhood $\mathcal{N}_x$ of x in $\{\theta \in S^2 : f(\theta) \neq g(\theta)\}$, $\forall \theta \in S^2$. Then the closure of $\mathcal{N}_x$ is in Υ_0 and the boundary of $\mathcal{N}_x$ is a subset of $\{\theta \in S^2 : f(\theta) = g(\theta)\}$, which implies $C_1 = f = g = C_2$ in the closure of $\mathcal{N}_x$; contradiction. Hence, $\{\theta \in S^2 : f(\theta) \neq g(\theta)\} = \emptyset$.

$\square$

Lemma 4.2.6. *Let f, g, t be as in Theorem 4.1.2 and $\xi \in \Xi_{\text{con}}$. For any point $\theta \in S^2 \cap (\xi^\perp + t\xi)$ consider the curve*

$$\Lambda(\theta) := \cup_{\zeta \in (\eta^\perp + t\eta) \cap B_\epsilon(\xi)} \phi_{\zeta, \lambda(\zeta)}(\eta), \quad \text{where } \eta = \phi_{\xi, -\lambda(\xi)}(\theta),$$

passing through θ. *If* $\lambda(\xi) \neq 1$, *then*

$$\Lambda(\theta) \cap S^2 \cap \operatorname{int}((\xi^\perp + t\xi)_+) \neq \emptyset.$$

Here, $(\xi^\perp + t\xi)_+ := \{x \in \mathbb{R}^3 : \langle x, \xi\rangle \geq t\}$ *and* $\operatorname{int}((\xi^\perp + t\xi)_+) := \{x \in \mathbb{R}^3 : \langle x, \xi\rangle > t\}$

Proof. Without loss of generality, we can assume that $0 < \lambda(\xi) < 1$. The other case is similar. Since $\xi \in \Xi_{\text{con}}$ and $0 < \lambda(\xi) < 1$, by Lemma 4.2.4 there exists $0 < \iota < 1/2$ and a ball $B_\epsilon(\xi) \subset \Xi_{\text{con}}$ such that $\iota \leq \lambda(\zeta) \leq 1 - \iota$ for any $\zeta \in B_\epsilon(\xi)$.

Now take any $\theta \in S^2 \cap (\xi^\perp + t\xi)$ and define $\eta = \phi_{\xi,-\lambda(\xi)}(\theta)$. We set $\zeta = \phi_{\eta,\alpha}(\xi)$ for some small $\alpha > 0$ and $\omega = \phi_{\zeta,\lambda(\zeta)}(\eta)$. Then we have

$$\begin{aligned}
&\zeta \times \eta = \phi_{\eta,\alpha}(\xi) \times \eta \\
=&(\xi\cos(\alpha\pi) + (\eta \times \xi)\sin(\alpha\pi) + \eta\langle\eta,\xi\rangle(1 - \cos(\alpha\pi))) \times \eta \\
=&\xi \times \eta\cos(\alpha\pi) + (\xi\langle\eta,\eta\rangle - \eta\langle\xi,\eta\rangle)\sin(\alpha\pi) \\
=&\xi \times \eta\cos(\alpha\pi) + (\xi - t\eta)\sin(\alpha\pi)
\end{aligned}$$

and

$$\begin{aligned}
&\langle\xi,\zeta\rangle = \langle\xi,\phi_{\eta,\alpha}(\xi)\rangle \\
=&\langle\xi,\xi\cos(\alpha\pi) + (\eta \times \xi)\sin(\alpha\pi) + \eta\langle\eta,\xi\rangle(1 - \cos(\alpha\pi))\rangle \\
=&\cos(\alpha\pi) + t^2(1 - \cos(\alpha\pi)).
\end{aligned} \tag{4.1}$$

Therefore,

$$\begin{aligned}
&\langle\xi,\omega\rangle - t = \langle\xi,\phi_{\zeta,\lambda(\zeta)}(\eta)\rangle - t \\
=&\langle\xi,\eta\cos(\lambda(\zeta)\pi) + (\zeta \times \eta)\sin(\lambda(\zeta)\pi) + t\zeta(1 - \cos(\lambda(\zeta)\pi))\rangle - t
\end{aligned}$$

$$
\begin{aligned}
&= t\cos(\lambda(\zeta)\pi) + \langle \xi, \xi \times \eta \cos(\alpha\pi) + (\xi - t\eta)\sin(\alpha\pi)\rangle \sin(\lambda(\zeta)\pi) \\
&\quad + t(1-\cos(\lambda(\zeta)\pi))(\cos(\alpha\pi) + t^2(1-\cos(\alpha\pi))) - t \\
&= t\cos(\lambda(\zeta)\pi) + (1-t^2)\sin(\alpha\pi)\sin(\lambda(\zeta)\pi) \\
&\quad + t(1-\cos(\lambda(\zeta)\pi))(\cos(\alpha\pi) + t^2(1-\cos(\alpha\pi))) - t \\
&= (1-t^2)\sin(\alpha\pi)\sin(\lambda(\zeta)\pi) + t(1-\cos(\lambda(\zeta)\pi))(t^2-1)(1-\cos(\alpha\pi)) \\
&= (1-t^2)(\sin(\alpha\pi)\sin(\lambda(\zeta)\pi) - t(1-\cos(\lambda(\zeta)\pi))(1-\cos(\alpha\pi))) \\
&\geq (1-t^2)(\sin(\alpha\pi)\sin(\iota\pi) - t(1-\cos((1-\iota)\pi))(1-\cos(\alpha\pi))) \\
&> 0 \quad \text{for sufficiently small } \alpha.
\end{aligned}
$$

To show

$$\sin(\alpha\pi)\sin(\iota\pi) - t(1-\cos((1-\iota)\pi))(1-\cos(\alpha\pi)) > 0$$

for sufficiently small $\alpha > 0$, we used that for $a, b > 0$, and $x > 0$ sufficiently small,

$$
\begin{aligned}
f'(x) &= (a\sin x - b(1-\cos x))' \\
&= a\cos x - b\sin x > 0
\end{aligned}
$$

and $f(0) = 0$.

Hence, $\omega \in \Lambda(\theta) \cap S^2 \cap \operatorname{int}((\xi^\perp + t\xi)_+)$. □

Proof of Theorem 4.1.2. Assume the set $\{\theta \in S^2 : f(\theta) \neq g(\theta)\}$ is not empty. By Lemma 4.2.5, we have

$$\{\theta \in S^2 : f(\theta) \neq g(\theta)\} \cap [\{\theta \in S^2 : \nabla_{S^2} f(\theta) \neq 0\} \cup \{\theta \in S^2 : \nabla_{S^2} g(\theta) \neq 0\}] \neq \emptyset.$$

Without loss of generality, we can choose

$$x \in \{\theta \in S^2 : f(\theta) \neq g(\theta)\} \cap \{\theta \in S^2 : \nabla_{S^2} f(\theta) \neq 0\},$$

and therefore, there exists an open ball

$$B_\epsilon(x) \subset \{\theta \in S^2 : f(\theta) \neq g(\theta)\} \cap \{\theta \in S^2 : \nabla_{S^2} f(\theta) \neq 0\}.$$

By the inverse function theorem, the collection of local level sets of f, $\mathcal{L}(f) := \{\Theta_\tau\}_{a<\tau<b}$, is a collection of disjoint C^2 curves, where $\Theta_\tau := \{\theta \in S^2 : f(\theta) = \tau\} \cap B_\epsilon(x)$.

For curves $\{\Theta_\tau\} \subset S^2$, consider their geodesic curvature $k_g(\cdot)$. If for every $\eta \in \Theta_\tau$ and $\Theta_\tau \in \mathcal{L}(f)$, we have $k_g(\eta) = 0$, then each Θ_τ belongs to some great circle. Choose one of these great circles. It divides S^2 into two hemispheres. Fix one of these hemispheres and denote it by S^2_+. Consider all circles of the form $S^2 \cap (\xi^\perp + t\xi)$ that are tangent to the curves Θ_τ and $\xi \in S^2_+$. Denote by Σ the set of these ξ.

Now consider the case when for some $\tau \in (a, b)$, there exists a $\theta \in \Theta_\tau$, such that $k_g(\theta) \neq 0$. Then by C^2 smoothness of f, there exists a smaller neighbourhood of x, which we will again denote by $B_\epsilon(x)$, and a collection of level sets, which we will again denote by $\{\Theta_\tau\}_{a<\tau<b}$, such that $k_g(\eta) \neq 0$ for any $\eta \in \Theta_\tau$ and $a < \tau < b$. For each point $\eta \in \Theta_\tau$, consider the great circle which is tangent to Θ_τ at η. Then $\{\Theta_\tau\}_{a<\tau<b}$ lie on one side of their tangent great circle. For each τ and each $\eta \in \Theta_\tau$ consider a circle $S^2 \cap (\xi^\perp + t\xi)$ that is tangent to Θ_τ at η and lies on the other side with respect to the tangent great circle. Let Σ be the set of such directions ξ.

Note that for each Θ_τ, these ξ form a parallel set of Θ_τ on S^2, i.e. the envelope of a family of circles on S^2 with centres at Θ_τ and of radius t. Hence the set Σ is a union of such curves and thus contains non-empty interior. We claim $\Xi_{\text{con}} \cap \operatorname{int}(\Sigma) \neq \emptyset$;

otherwise, $\mathrm{int}\,(\Sigma) \subset \Xi_0 \cup (\cup_{n=2}^{\infty}\Xi_n)$, but $\mathrm{int}\,(\Sigma) \cap \Xi_0 = \emptyset$, since $B_\epsilon(x) \subset \{\theta \in S^2 : f(\theta) \neq g(\theta)\}$. Hence, $\mathrm{int}\,(\Sigma) \subset \cup_{n=2}^{\infty}\Xi_n$, which implies that $(\cup_{n=2}^{\infty}\Xi_n)\cap\mathrm{int}\,(\Sigma)$ contains non-empty interior. By the Baire category theorem and Lemma 4.2.2, there exists some $k \in \mathbb{N}$, such that $\Xi_k \cap \mathrm{int}\,(\Sigma)$ contains non-empty interior.

Now assume $\xi \in \mathrm{int}\,(\Xi_k \cap \Sigma)$, there exists $B_\delta(\xi) \subset \mathrm{int}\,(\Xi_k \cap \Sigma)$. For any $\theta \in \xi^\perp + t\xi$, we have

$$f(\eta) = f(\theta), \quad \forall \eta \in \Lambda_\xi(\theta) := \cup_{\zeta \in (\theta^\perp + t\theta)\cap B_\delta(\xi)} \phi_{\zeta,\frac{2}{k}}(\theta)$$

and

$$f(\omega) = f(\theta), \quad \forall \omega \in \Delta_\xi(\theta) := \cup_{\eta\in\Lambda_\xi(\theta)} \cup_{\vartheta \in (\eta^\perp + t\eta)\cap B_\delta(\xi)} \phi_{\vartheta,-\frac{2}{k}}(\eta).$$

Let us show that $\Delta_\xi(\theta)$ has non-empty interior. Note that for any $\eta \in \Lambda_\xi(\theta)$, by Equation 4.1 we have

$$\langle \theta, \eta \rangle = \cos(2\pi/k) + t^2(1 - \cos(2\pi/k)) =: \varsigma(t),$$

where $-1 < \varsigma(t) < 1$. If $\varsigma(t) = 0$, then $\Lambda_\xi(\theta) \subset S^2 \cap \theta^\perp$. Fix $\eta \in \Lambda_\xi(\theta)$, then for each

$$\omega \in \cup_{\vartheta \in (\eta^\perp + t\eta)\cap B_\delta(\xi)} \phi_{\vartheta,-\frac{2}{k}}(\eta),$$

by Equation 4.1 we have

$$\langle \omega, \eta \rangle = \varsigma(t) = 0,$$

which means $\cup_{\vartheta \in (\eta^\perp + t\eta)\cap B_\delta(\xi)} \phi_{\vartheta,-\frac{2}{k}}(\eta)$ is a curve passing through θ and contained in $S^2 \cap \eta^\perp$. Since $\Lambda_\xi(\theta)$ is a continuous curve, by changing η we see that $\Delta_\xi(\theta)$ has the shape of a sand dial, which we will refer to as $\bowtie$ shape.

If $0 < \varsigma(t) < 1$, then $\Lambda_\xi(\theta) \subset S^2 \cap (\theta^\perp + \varsigma(t)\theta)$. Fix $\eta \in \Lambda_\xi(\theta)$, then $\cup_{\vartheta \in (\eta^\perp + t\eta)\cap B_\delta(\xi)} \phi_{\vartheta,-\frac{2}{k}}(\eta)$ gives a curve passing through θ and contained in $S^2 \cap (\eta^\perp + \varsigma(t)\eta)$. Observe that for different $\eta \in \Lambda_\xi(\theta)$, we have different curves $\cup_{\vartheta\in(\eta^\perp+t\eta)\cap B_\delta(\xi)}\phi_{\vartheta,-\frac{2}{k}}(\eta)$ with the only common point θ. Since these curves change continuously, the set $\Delta_\xi(\theta)$ again has a $\bowtie$ shape.

If $-1 < \varsigma(t) < 0$, use the same argument to show that $\Delta_\xi(\theta)$ has a $\bowtie$ shape. Therefore, $\Delta_\xi(\theta)$ is a set with non-empty interior on S^2.

Now to reach a contradiction, assume that f is not constant on $S^2 \cap (\xi^\perp + t\xi)$. Then f takes on infinitely many values and so there are infinitely many disjoint sets $\Delta_\xi(\theta)$ with $m(\Delta_\xi(\theta)) = \nu > 0$, where ν is a number independent of $\theta \in S^2 \cap (\xi^\perp + t\xi)$, which is impossible. Here m is the Hausdorff measure on S^2. On the other hand, if f is a constant on $S^2 \cap (\xi^\perp + t\xi)$, then $\xi \in \Xi_0$, which contradicts to $\mathrm{int}\,(\Sigma) \cap \Xi_0 = \emptyset$. Thus, we have proved $\Xi_{\text{con}} \cap \mathrm{int}\,(\Sigma) \neq \emptyset$.

Now assume that for every $\xi \in \Xi_{\text{con}} \cap \mathrm{int}\,(\Sigma)$, we have $\lambda(\xi) = 1$. Then there exists $B_\delta(\xi) \subset \Xi_{\text{con}} \cap \mathrm{int}\,(\Sigma)$ such that $\lambda(\zeta) = 1$ for any $\zeta \in B_\delta(\xi)$. For any $\theta \in \xi^\perp + t\xi$, we have

$$g(\eta) = f(\theta), \quad \forall \eta \in \Lambda_\xi(\theta) := \cup_{\zeta\in(\theta^\perp+t\theta)\cap B_\delta(\xi)}\phi_{\zeta,1}(\theta)$$

and

$$f(\omega) = g(\eta) = f(\theta), \quad \forall \omega \in \Delta_\xi(\theta) := \cup_{\eta\in\Lambda_\xi(\theta)} \cup_{\vartheta\in(\eta^\perp+t\eta)\cap B_\delta(\xi)} \phi_{\vartheta,1}(\eta).$$

Following the same argument as above, we have that $\Delta_\xi(\theta)$ is a set with non-empty interior on S^2. Therefore, f is a constant on $\xi^\perp + t\xi$; otherwise, if f takes on infinitely many values, then there are infinitely many disjoint sets $\Delta_\xi(\theta)$, where $m(\Delta_\xi(\theta)) = \nu > 0$; contradiction. But if f is a constant on $\xi^\perp + t\xi$, then $\xi \in \Xi_0$, which contradicts to $\xi \in \Xi_{\text{con}}$.

Finally assume that there exists $\xi \in \Xi_{\text{con}} \cap \mathrm{int}\,(\Sigma)$ such that $\lambda(\xi) \neq 1$. Then by Lemma 4.2.4 there exists a neighbourhood $B_\epsilon(\xi) \subset \Xi_{\text{con}} \cap \mathrm{int}\,(\Sigma)$ and $\theta \in \Theta_\tau \in \mathcal{L}(f)$

for some τ, such that $S^2 \cap (\xi^\perp + t\xi)_+ \cap \Theta_\tau = \theta$. On the other hand, by Lemma 4.2.6

$$\mathit{\Lambda}(\theta) = \cup_{\zeta \in (\eta^\perp + t\eta) \cap B_\epsilon(\xi)} \phi_{\zeta,\lambda(\zeta)}(\eta), \quad \text{where } \eta = \phi_{\xi,-\lambda(\xi)}(\theta),$$

gives a curve such that $\mathit{\Lambda}(\theta) \cap S^2 \cap \operatorname{int}((\xi^\perp + t\xi)_+) \neq \emptyset$ and $f(\omega) = f(\theta)$ for any $\omega \in \mathit{\Lambda}(\theta)$. Thus, $\mathit{\Lambda}(\theta) \cup \mathit{\Theta}_\tau$ must be a level set of f at value τ but it is not a 1-manifold; contradiction.

Therefore, $\{\theta \in S^2 : f(\theta) \neq g(\theta)\} = \emptyset$. $\square$

Chapter 5

An analogue of the Aleksandrov projection theorem for convex lattice polygons

5.1 Introduction

This chapter studies the discrete version of the Aleksandrov projection theorem. Since the convex hull of a convex lattice set is a convex lattice polytope, i.e. a polytope all of whose vertices are in $\mathbb{Z}^n$, it would be convenient to restate Problem 1.2.3 as follows. Let $K, L \subset \mathbb{R}^n$ be origin-symmetric convex lattice polytopes. If $|(K \cap \mathbb{Z}^n)|u^\perp| = |(L \cap \mathbb{Z}^n)|u^\perp|$ for every $u \in \mathbb{Z}^n$, is it true that $K = L$?

In [12], the authors gave a negative answer to Problem 1.2.3 in $\mathbb{Z}^2$. However, it is not known whether there are other counterexamples. Zhou [39] and Xiong [36] showed that these counterexamples are unique in some special classes. For higher dimensions, this problem is still open. Since the answer is negative in dimension 2, Gardner, Gronchi, and Zong asked if it is possible to impose reasonable additional conditions to make the answer affirmative. In this chapter, we obtain a positive answer to Problem 1.2.3 in $\mathbb{Z}^2$ under an additional hypothesis.

If K is a convex lattice polytope, then we define the difference set of the underlying convex lattice set by

$$D_1K := DK \cap \mathbb{Z}^n = \{u \in \mathbb{Z}^n : \exists x_1, x_2 \in K \cap \mathbb{Z}^n, u \parallel x_1x_2\},$$

where $\parallel$ means parallel.

For a directed line segment parallel to $u \in \mathbb{Z}^n$ with the initial point $(p_1, \dots, p_n) \in \mathbb{Z}^n$ and the end point $(q_1, \dots, q_n) \in \mathbb{Z}^n$, let

$$\hat{u} := (\frac{q_1 - p_1}{d}, \dots, \frac{q_n - p_n}{d})$$

denote the primitive vector in the direction u, where $d = \gcd(q_1 - p_1, \dots, q_n - p_n)$.

We will need the well-known Pick theorem (see [2, p. 90] or [4, p. 38]). Let $K \subset \mathbb{R}^2$ be a convex lattice polygon. Then

$$\mathrm{vol}_2(K) = |K \cap \mathbb{Z}^2| - \frac{1}{2}|\partial K \cap \mathbb{Z}^2| - 1,$$

where ∂K is the boundary of K.

We are now ready to state our main result.

Theorem 5.1.1. *Let $K, L \subset \mathbb{R}^2$ be origin-symmetric convex lattice polygons. If*

$$|(K \cap \mathbb{Z}^2)|u^{\perp}| = |(L \cap \mathbb{Z}^2)|u^{\perp}|$$

and

$$|(2K \cap \mathbb{Z}^2)|u^{\perp}| = |(2L \cap \mathbb{Z}^2)|u^{\perp}|$$

for all $u \in \mathbb{Z}^2$, *then* $K = L$.

Remark 5.1.2. It will be clear from the proof that we do not need projections in all directions, only in directions parallel to the edges of K and L, and one more direction $\xi \in \mathbb{Z}^2 \backslash (D_1 K \cup D_1 L)$.

5.2 Proof of Theorem 5.1.1

Theorem 5.2.1. *Let K be an origin-symmetric convex lattice polygon in $\mathbb{R}^2$ with edges $\{e_i\}_{i=1}^{2n}$, where e_i and e_{n+i} are symmetric with respect to the origin. Then*

$$|(K \cap \mathbb{Z}^2)|e_i^{\perp}| = |\hat{e}_i| w_K(e_i^{\perp}) + 1, \ \textit{for } 1 \le i \le n,$$

where $|\hat{e}_i|$ is the length of the primitive vector parallel to e_i.

We will first prove the theorem in a simple case.

Lemma 5.2.2. *Let $K \subset \mathbb{R}^2$ be a parallelogram with edges $\{e_i\}_{1 \le i \le 4}$, where $e_1 \parallel e_3$ and $e_2 \parallel e_4$. Then*

$$|(K \cap \mathbb{Z}^2)|e_i^{\perp}| = |\hat{e}_i| w_K(e_i^{\perp}) + 1, \ \textit{for } i = 1, 2.$$

Proof. Consider the point lattice Λ generated by $\hat{e}_1$ and $\hat{e}_2$ and the quotient map $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\Lambda$. Set $l(e_1)$ to be the line passing through the origin and parallel to e_1. If $x \in K \cap \Lambda$, then

$$|(x + l(e_1)) \cap (K \cap \mathbb{Z}^2)| = |e_1 \cap \mathbb{Z}^2|.$$

If $x \in (K \cap \mathbb{Z}^2) \backslash \Lambda$, then $\pi((x + l(e_1)) \cap (K \cap \mathbb{Z}^2))$ contains only one point; otherwise, $x \in \Lambda$. Thus,

$$|(x + l(e_1)) \cap (K \cap \mathbb{Z}^2)| = |e_1 \cap \mathbb{Z}^2| - 1.$$

One can see that,

$$|(K \cap \Lambda)|e_1^{\perp}| = |e_2 \cap \mathbb{Z}^2|. \tag{5.1}$$

Furthermore when projecting $(K \cap \mathbb{Z}^2)\backslash\Lambda$ onto $e_1^{\perp}$, each point in the projection has $|e_1 \cap \mathbb{Z}^2| - 1$ preimages. Thus,

$$|((K \cap \mathbb{Z}^2)\backslash\Lambda)|e_1{}^{\perp}| = \frac{|(K \cap \mathbb{Z}^2)| - |e_1 \cap \mathbb{Z}^2||e_2 \cap \mathbb{Z}^2|}{|e_1 \cap \mathbb{Z}^2| - 1}; \tag{5.2}$$

hence, by (5.1) and (5.2),

$$\begin{aligned}
|(K \cap \mathbb{Z}^2)|e_1^{\perp}| &= \frac{|(K \cap \mathbb{Z}^2)| - |e_1 \cap \mathbb{Z}^2||e_2 \cap \mathbb{Z}^2|}{|e_1 \cap \mathbb{Z}^2| - 1} + |e_2 \cap \mathbb{Z}^2| \\
&= \frac{|(K \cap \mathbb{Z}^2)| - |e_2 \cap \mathbb{Z}^2|}{|e_1 \cap \mathbb{Z}^2| - 1} \\
&= \frac{\mathrm{vol}_2(K) + |e_1 \cap \mathbb{Z}^2| - 1}{|e_1 \cap \mathbb{Z}^2| - 1} \text{ (by Pick's theorem)} \\
&= \frac{|\hat{e}_1|(|e_1 \cap \mathbb{Z}^2| - 1)w_K(e_1^{\perp}) + |e_1 \cap \mathbb{Z}^2| - 1}{|e_1 \cap \mathbb{Z}^2| - 1} \\
&= |\hat{e}_1|w_K(e_1^{\perp}) + 1.
\end{aligned}$$

□

Proof of Theorem 5.2.1. Without loss of generality, we only need to compute $|(K \cap \mathbb{Z}^2)|e_1^{\perp}|$. Create a convex lattice set whose convex hull is a parallelogram with edges e_1 and e_{n+1}, denoted by P. Note that, for any $x \in (K \cap \mathbb{Z}^2)\backslash P$, $x + l(e_1) \cap P \neq \emptyset$. Thus, there exists $m \in \mathbb{Z}$, such that $x \in P + me_1$, which implies $x - me_1 \in P \cap \mathbb{Z}^2$. Therefore, by Lemma 5.2.2

$$|(K \cap \mathbb{Z}^2)|e_1^{\perp}| = |(P \cap \mathbb{Z}^2)|e_1^{\perp}| = |\hat{e}_1|w_K(e_1^{\perp}) + 1.$$

□

Theorem 5.2.1 implies that if the directions of the edges of K are the same as those of L, then there is a uniqueness in Problem 1.2.3.

Lemma 5.2.3. *Let K be an origin-symmetric convex lattice polygon in $\mathbb{Z}^2$. Let $u \in D_1K$. If $2(|(K \cap \mathbb{Z}^2)|u^\perp| - 1) = |(2K \cap \mathbb{Z}^2)|u^\perp| - 1$, then*

$$|(K \cap \mathbb{Z}^2)|u^\perp| = |\hat{u}|w_K(u^\perp) + 1.$$

Proof. Let $u \in D_1K$. If u is parallel to one of the edges of K, then, by Theorem 5.2.1, $|(K \cap \mathbb{Z}^2)|u^\perp| = |\hat{u}|w_K(u^\perp) + 1$; if not, consider the pair of points $(x_1, x_2) \in \{(x, y) \in K \times K : xy \parallel \hat{u}\}$ such that

$$d(O, \overline{x_1x_2}) = \max_{\{(x,y) \in K \times K : xy \parallel \hat{u}\}} d(O, \overline{xy}).$$

Here, we denoted by O, the origin and $d(O, A) = \inf_{x \in A} \|x - O\|_2$, the distance between O and a set A. The set $\{(x, y) \in K \times K : xy \parallel \hat{u}\}$ is not empty, since $u \in D_1K$.
Thus, the lines passing through x_1, x_2 and $-x_1, -x_2$ divide $\mathbb{R}^2$ into three parts E_1, E_2, and E_3, where $O \in E_2$, E_1, E_3 are reflections of each other with respect to O and they overlap on their boundaries.
Note that, $E_2 \cap K \cap \mathbb{Z}^2$ is a convex lattice set and $x_1, x_2, -x_1, -x_2$ lie on two parallel edges of $E_2 \cap K$. (Here, $E_2 \cap K$ can be a segment.) Then, by Theorem 5.2.1, we have

$$|(E_2 \cap K \cap \mathbb{Z}^2)|u^\perp| = |\hat{u}|w_{E_2 \cap K}(u^\perp) + 1.$$

Set $|(E_1 \cap K \cap \mathbb{Z}^2)|u^\perp| = |(E_3 \cap K \cap \mathbb{Z}^2)|u^\perp| = m$. We have

$$|(K \cap \mathbb{Z}^2)|u^\perp| = 2m + |\hat{u}|w_{E_2 \cap K}(u^\perp) - 1. \tag{5.3}$$

On the other hand, $|(2E_2 \cap 2K \cap \mathbb{Z}^2)|u^\perp| = 2|\hat{u}|w_{E_2\cap K}(u^\perp) + 1$. Moreover, a line l parallel to u divides $2E_1 \cap 2K$ into two parts of equal width in the direction perpendicular to u, denoted by $E_{11} \cap 2K$ and $E_{12} \cap 2K$, where $d(O, E_{11}) > d(O, E_{12})$ and they overlap on their boundaries.

Note that there exists a pair of points $y_1, y_2 \in l \cap 2K \cap \mathbb{Z}^2$. To see this, pick a point z from $E_1 \cap K \cap \mathbb{Z}^2$ such that $w_{[-z,z]}(u^\perp) = w_K(u^\perp)$, where $[-z, z]$ is the segment connecting $-z$ and z. Then $2z, 2x_1, 2x_2 \in 2K \cap \mathbb{Z}^2$ implies $y_1 = z + x_1, y_2 = z + x_2 \in 2K \cap l$.

Now we obtain $E_{11} \cap 2K \supset E_1 \cap K + z$. To see this, assume $x_1, x_2 \in \{x \in \mathbb{R}^2 : \langle x, v_1\rangle = a_1\}$, then $\langle z, v_1\rangle = a_1 - w_{E_1\cap K}(v_1)$ and $\langle u, v_1\rangle = 0$. Thus for any $x \in E_1 \cap K$, $\langle x + z, v_1\rangle \le 2a_1 - w_{E_1\cap K}(v_1)$, implying $x + z \in E_{11} \cap 2K$. Therefore, we have

$$|(E_{11} \cap 2K \cap \mathbb{Z}^2)|u^\perp| \ge |(E_1 \cap K \cap \mathbb{Z}^2)|u^\perp| = m.$$

And since $E_{12} \cap 2K$ contains a parallelogram Q with vertices $2x_1, x_1 + x_2, y_1, y_2$,

$$|(E_{12} \cap 2K \cap \mathbb{Z}^2)|u^\perp| = |\hat{u}|w_Q(u^\perp) + 1 = |\hat{u}|w_{E_1\cap K}(u^\perp) + 1.$$

Hence,

$$|(2E_1 \cap 2K \cap \mathbb{Z}^2)|u^\perp| \ge m + |\hat{u}|w_{E_1\cap K}(u^\perp).$$

Therefore, since E_1, E_3 are reflections of each other with respect to O, we have

$$\begin{aligned}|(2K \cap \mathbb{Z}^2)|u^\perp| &= 2|(2E_1 \cap 2K \cap \mathbb{Z}^2)|u^\perp| + |(2E_2 \cap 2K \cap \mathbb{Z}^2)|u^\perp| - 2\\ &\ge 2(m + |\hat{u}|w_{E_1\cap K}(u^\perp)) + 2|\hat{u}|w_{E_2\cap K}(u^\perp) - 1.\end{aligned}$$

Then, by the assumption and (5.3),

$$
\begin{aligned}
&2(2m + |\hat{u}|w_{E_2\cap K}(u^{\perp}) - 2) = 2(|(K \cap \mathbb{Z}^2)|u^{\perp}| - 1) \\
&= |(2K \cap \mathbb{Z}^2)|u^{\perp}| - 1 \geq 2(m + |\hat{u}|w_{E_1\cap K}(u^{\perp})) + 2|\hat{u}|w_{E_2\cap K}(u^{\perp}) - 1,
\end{aligned}
$$

which implies

$$m \geq |\hat{u}|w_{E_1\cap K}(u^{\perp}) + 1.$$

On the other hand, $m \leq |\hat{u}|w_{E_1\cap K}(u^{\perp}) + 1$, by constructing a large parallelogram containing $E_1 \cap K$, that has two edges parallel to u and whose width perpendicular to u is $w_{E_1\cap K}(u^{\perp})$; thus,

$$m = |\hat{u}|w_{E_1\cap K}(u^{\perp}) + 1. \tag{5.4}$$

Finally, by (5.3) and (5.4),

$$
\begin{aligned}
|(K \cap \mathbb{Z}^2)|u^{\perp}| &= 2m + |\hat{u}|w_{E_2\cap K}(u^{\perp}) - 1 \\
&= |\hat{u}|(2w_{E_1\cap K}(u^{\perp}) + w_{E_2\cap K}(u^{\perp})) + 1 \\
&= |\hat{u}|(w_{E_1\cap K}(u^{\perp}) + w_{E_2\cap K}(u^{\perp}) + w_{E_3\cap K}(u^{\perp})) + 1 \\
&= |\hat{u}|w_K(u^{\perp}) + 1
\end{aligned}
$$

□

If we define E_K to be the collection of all directions parallel to the edges of K, one can easily prove the following Lemma.

Lemma 5.2.4. *Let K and L be origin-symmetric convex polygons in $\mathbb{R}^2$. If $w_K(u^{\perp}) = w_L(u^{\perp})$ for all $u \in E_K \cup E_L$, then $K = L$.*

Proof of Theorem 5.1.1. Here, we use the weaker condition mentioned in Remark 5.1.2. Note that, $|(K\cap\mathbb{Z}^2)|u^{\perp}| < |K\cap\mathbb{Z}^2|$, if $u \in D_1K$; but $|(K\cap\mathbb{Z}^2)|u^{\perp}| = |K\cap\mathbb{Z}^2|$, if $u \in \mathbb{Z}^2 \backslash D_1 K$. For any $u \in E_K$, we have $u \in D_1 L$; indeed, if this is not the case, then,

$$|(L \cap \mathbb{Z}^2)|u^\perp| = |L \cap \mathbb{Z}^2| = |(L \cap \mathbb{Z}^2)|\xi^\perp| = |(K \cap \mathbb{Z}^2)|\xi^\perp| = |K \cap \mathbb{Z}^2| > |(K \cap \mathbb{Z}^2)|u^\perp|$$

for some $\xi \in \mathbb{Z}^2 \backslash (D_1 K \cup D_1 L)$. Then by Lemma 5.2.2, we have

$$|(K \cap \mathbb{Z}^2)|u^\perp| = |\hat{u}| w_K(u^\perp) + 1 \quad \text{and} \quad |(2K \cap \mathbb{Z}^2)|u^\perp| = 2|\hat{u}| w_K(u^\perp) + 1.$$

By the assumption,

$$\begin{aligned} &|(2L \cap \mathbb{Z}^2)|u^\perp| - 1 = |(2K \cap \mathbb{Z}^2)|u^\perp| - 1 = 2|\hat{u}| w_K(u^\perp) \\ &= 2(|(K \cap \mathbb{Z}^2)|u^\perp| - 1) = 2(|(L \cap \mathbb{Z}^2)|u^\perp| - 1). \end{aligned}$$

Applying Lemma 5.2.3,

$$|(L \cap \mathbb{Z}^2)|u^\perp| = |\hat{u}| w_L(u^\perp) + 1 = |(K \cap \mathbb{Z}^2)|u^\perp| = |\hat{u}| w_K(u^\perp) + 1. \tag{5.5}$$

Therefore, by (5.5),

$$w_L(u^\perp) = w_K(u^\perp),$$

for any $u \in E_K$. Similarly, we can show $w_L(u^\perp) = w_K(u^\perp)$, for any $u \in E_L$. Then the conclusion follows from Lemma 5.2.4. $\square$

Chapter 6

Unique determination of convex lattice sets

6.1 Introduction and main results

Note that the Aleksandrov projection theorem is also true for other intrinsic volumes of projections. For example, if the projections of two origin-symmetric convex bodies onto all hyperplanes have equal surface areas, then the bodies coincide, [18, p. 115]. In this chapter we suggest to study an analogue of this result in discrete settings. Let K be a convex lattice set in $\mathbb{Z}^n$ and $u \in \mathbb{Z}^n$. By the discrete surface area $|\partial(K|u^\perp)|$ of the projection of K onto $u^\perp$ we will understand the number of points in $K|u^\perp$ that lie on the boundary of the convex hull of $K|u^\perp$, i.e.

$$|\partial(K|u^\perp)| = |(K|u^\perp) \cap \partial(\mathrm{conv}(K)|u^\perp)|.$$

When $K \subset \mathbb{Z}^3$, we will use the term "discrete perimeter".

We say that a finite set K in $\mathbb{R}^n$ is full-dimensional if $\mathrm{conv}(K)$ has non-empty interior. In questions below, we will only consider full-dimensional convex lattice sets.

Problem 6.1.1. *Let K and L be origin-symmetric full-dimensional convex lattice sets in $\mathbb{Z}^n$. If for every $u \in \mathbb{Z}^n$ we have*

$$|\partial(K|u^{\perp})| = |\partial(L|u^{\perp})|,$$

is then necessarily $K = L$?

Below we give a positive answer to this problem in $\mathbb{Z}^3$.

Theorem 6.1.2. *Let K and L be origin-symmetric full-dimensional convex lattice sets in $\mathbb{Z}^3$. If the discrete perimeters of $K|u^{\perp}$ and $L|u^{\perp}$ are equal for all $u \in \mathbb{Z}^3$, then $K = L$.*

As one can see, if we drop the assumption that the sets are full-dimensional, then Problem 6.1.1 in $\mathbb{Z}^3$ has a negative answer, since it reduces to Problem 1.2.3 in $\mathbb{Z}^2$. We also solve Problem 6.1.1 in $\mathbb{Z}^n$, $n \geq 4$, in the class of convex lattice sets whose convex hulls are zonotopes. Recall that a zonotope is a finite Minkowski sum of closed line segments, [18, p. 146].

Theorem 6.1.3. *Let K and L be origin-symmetric full-dimensional convex lattice sets in $\mathbb{Z}^n$, $n \geq 4$, such that* $\operatorname{conv}(K)$ *and* $\operatorname{conv}(L)$ *are zonotopes. If*

$$|\partial(K|u^{\perp})| = |\partial(L|u^{\perp})|$$

for all $u \in \mathbb{Z}^n$, then $K = L$.

Let us briefly mention some facts and concepts that are used in this chapter.

Let $u_1, \ldots, u_m$ be linearly independent vectors in $\mathbb{Z}^n$, with $m \leq n$. The set

$$\Lambda = \left\{ \sum_{i=1}^{m} a_i u_i : a_i \in \mathbb{Z}, \text{ for } 1 \leq i \leq m \right\}$$

is called a sublattice of $\mathbb{Z}^n$ of rank m. The vectors $u_1, \ldots, u_m$ form a basis of Λ.

The set

$$\Pi = \left\{ \sum_{i=1}^{m} b_i u_i : 0 \leq b_i < 1, \text{ for } 1 \leq i \leq m \right\}$$

is called the fundamental parallelepiped of the basis $u_1, \ldots, u_m$. The m-dimensional volume of the fundamental parallelepiped does not depend on the choice of the basis of Λ; it is called the determinant of Λ and denoted $|\Lambda|$. For these and other related results, the reader is referred to the books by Barvinok [2] and Gruber [17].

We will also need the Minkowski uniqueness theorem saying that a convex polytope in $\mathbb{R}^n$ is uniquely determined (up to translation) by the areas of its facets and the normal vectors to the facets; see [31, p. 397].

6.2 Proofs of the main results

Proof of Theorem 1.3. The idea is to show that for every facet F_K of $\text{conv}(K)$, there is a facet F_L of $\text{conv}(L)$ that is parallel to F_K (and vice versa), and

$$|\partial F_K \cap \mathbb{Z}^3| - 2|F_K \cap \mathbb{Z}^3| = |\partial F_L \cap \mathbb{Z}^3| - 2|F_L \cap \mathbb{Z}^3|. \tag{6.1}$$

Using (6.1) and Pick's theorem, we will conclude that for every pair of parallel facets, $\text{vol}_2(F_K) = \text{vol}_2(F_L)$, and will use the Minkowski uniqueness theorem to finish the proof. Below we provide the details.

First we claim that for every facet F_K of $\text{conv}(K)$, there is a facet F_L of $\text{conv}(L)$ that is parallel to F_K, and vice versa. Indeed, assume that there exists a facet F_K such that no facet of $\text{conv}(L)$ is parallel to F_K. Note that $\{\theta \in S^2 : \theta = |u|^{-1}u, \text{ where } u \in \mathbb{Z}^n \setminus \{0\}\}$ is a dense subset of S^2. One can see that in the statement of the theorem we can take vectors from the sphere S^2. Choose a direction $\xi \in S^2$ that is parallel to F_K (and the opposite facet, since K is origin-symmetric) and not parallel to any other facets of either $\text{conv}(K)$ or $\text{conv}(L)$. Then, the boundary of $\text{conv}(K|\xi^{\perp})$ consists of the edges e and $-e$ that are the projections of F_K and $-F_K$, as well as other edges that are the projections of some edges of $\text{conv}(K)$. The boundary of $\text{conv}(L|\xi^{\perp})$ solely consists of the projections of some edges of $\text{conv}(L)$.

Furthermore, we can assume that $|K|\xi^{\perp}| = |K|$ and $|L|\xi^{\perp}| = |L|$, since there are only finitely many directions that do not satisfy these equalities. For ϕ small enough, consider the vectors $\zeta = \cos\phi\,\xi + \sin\phi\,\eta$ and $\theta = \cos\phi\,\xi - \sin\phi\,\eta$, where η is the unit outward normal vector to F_K. Note that the number of points in K that are projected to $K|\xi^{\perp}$, $K|\zeta^{\perp}$, and $K|\theta^{\perp}$, and that do not come from the facets F_K and $-F_K$, is the same. On the other hand, at least one of the points of F_K belongs to the interior of either $\text{conv}(K|\zeta^{\perp})$ or $\text{conv}(K|\theta^{\perp})$. Thus at least one of the two inequalities holds:

$$|\partial(K|\zeta^{\perp})| < |\partial(K|\xi^{\perp})| \text{ or } |\partial(K|\theta^{\perp})| < |\partial(K|\xi^{\perp})|.$$

However,

$$|\partial(L|\zeta^{\perp})| = |\partial(L|\xi^{\perp})| \text{ and } |\partial(L|\theta^{\perp})| = |\partial(L|\xi^{\perp})|.$$

We get a contradiction. Thus, every facet of $\text{conv}(K)$ is parallel to a facet of $\text{conv}(L)$ and vice versa.

To prove (6.1), we will use the following formula:

$$|\partial(K|\zeta^{\perp})| + |\partial(K|\theta^{\perp})| - 2|\partial(K|\xi^{\perp})| = 2|\partial F_K \cap \mathbb{Z}^3| - 4|F_K \cap \mathbb{Z}^3| + 4. \tag{6.2}$$

Let us explain the validity of this equality. First of all, observe that the left-hand side only sees the points that are projected from F_K and $-F_K$. (The contribution of the rest of the boundary of K is annihilated, since the number of points in K that are projected to $\mathrm{conv}(K|\xi^\perp)$, $\mathrm{conv}(K|\zeta^\perp)$, and $\mathrm{conv}(K|\theta^\perp)$, and that do not come from the facets F_K and $-F_K$, is the same). Next we see that $\partial(K|\zeta^\perp)$ gets points from one side of $\partial F_K \cap \mathbb{Z}^3$ (and its reflection about the origin), and $\partial(K|\theta^\perp)$ gets points from the other side of $\partial F_K \cap \mathbb{Z}^3$ (and its reflection about the origin). There are two points on each $F_K \cap \mathbb{Z}^3$ and $-F_K \cap \mathbb{Z}^3$ that are projected into both $\mathrm{conv}(K|\zeta^\perp)$ and $\mathrm{conv}(K|\theta^\perp)$, which yields the constant term equal to 4 in (6.2). Since all points from F_K and $-F_K$ are projected into different points in $\partial(K|\xi^\perp)$, the latter set has exactly $2|F_K \cap \mathbb{Z}^3|$ points coming from those facets. Formula (6.2) follows.

Now equality (6.2) together with the assumption of the theorem yields (6.1) for every pair of parallel facets of $\mathrm{conv}(K)$ and $\mathrm{conv}(L)$.

Let H be the 2-dimensional subspace that is parallel to the facets F_K and F_L. Then, $\Lambda = H \cap \mathbb{Z}^3$ is a lattice of rank 2; see e.g. [32, Chap. I, §2]. Let $|\Lambda|$ be the determinant of the lattice Λ. By Pick's theorem and equality (6.1),

$$\begin{aligned} \mathrm{vol}_2(F_K) &= |\Lambda|(|F_K \cap \mathbb{Z}^3| - \frac{1}{2}|\partial F_K \cap \mathbb{Z}^3| - 1) \\ &= |\Lambda|(|F_L \cap \mathbb{Z}^3| - \frac{1}{2}|\partial F_L \cap \mathbb{Z}^3| - 1) \\ &= \mathrm{vol}_2(F_L). \end{aligned}$$

Thus we have proved that for each facet F_K in $\mathrm{conv}\,(K)$, there is a facet F_L in $\mathrm{conv}\,(L)$ (and vice versa), such that F_K and F_L are parallel and $\mathrm{vol}_2(F_K) = \mathrm{vol}_2(F_L)$. Minkowski's uniqueness theorem then implies that $\mathrm{conv}\,(K) = \mathrm{conv}\,(L)$, or equivalently, $K = L$.

Before we present the proof of Theorem 1.4, let us introduce the following notation. If P is a convex body in $\mathbb{R}^n$, we define the upper boundary $\mathcal{U}_\xi(P)$ of P in the direction $\xi \in S^{n-1}$ to be

$$\mathcal{U}_\xi(P) := \{x \in P : x + \epsilon\xi \notin P,\ \forall \epsilon > 0\},$$

and the lower boundary $\mathcal{L}_\xi(P)$ of P in the direction ξ to be

$$\mathcal{L}_\xi(P) := \{x \in P : x - \epsilon\xi \notin P,\ \forall \epsilon > 0\}.$$

If P is a polytope, then $\mathcal{U}_\xi(P)$ is the union of the facets F_i of P whose outer normal vectors n_i satisfy the inequality $\langle n_i, \xi\rangle > 0$. Similarly, $\mathcal{L}_\xi(P)$ is the union of the facets F_i of P whose outer normal vectors n_i satisfy the inequality $\langle n_i, \xi\rangle < 0$.

We will need the following lemma that will be used as an analogue of Pick's Theorem.

Lemma 6.2.1. *Let Z be a zonotope with vertices in the lattice $\Lambda \subset \mathbb{R}^n$. Let $\xi \in S^{n-1}$ be a direction that is not parallel to any of the facets of K. Then*

$$\mathrm{vol}_n(Z) = |\Lambda|(|Z \cap \Lambda| - |\mathcal{U}_\xi(Z) \cap \Lambda|).$$

This formula is discussed in [6, Section 2.3.2], but, for the sake of completeness, we outline a sketch of our proof below.

First of all, without loss of generality, we can assume that $\Lambda = \mathbb{Z}^n$. Next we proceed by induction on the number of summands of Z. The base case is when Z is the sum of n segments. If Z is a box with facets parallel to the coordinate planes, the formula is obvious. Furthemore, it is not hard to show that it is true for all parallelotopes. The inductive step is as follows. Assume that the formula is true for zonotopes that are the sum of N segments. If Z is the sum of $N + 1$ segments, it can be written as the sum of a segment and a zonotope with N summands. If the latter is full-dimensional, its facets are zonotopes with at most $N - 1$ summands. (If it is not full-dimensional, write it as sum of a segment and a zonotope with $N - 1$ summands). Thus Z can be written as the union of zonotopes (with disjoint interiors) that are sums of no more than N segments. Next use the induction hypothesis, each of the component zonotopes contributes its volume, but also its (from direction ξ) 'invisible' lattice points (and this does not lead to double counts).

We will now present a solution of Problem 6.1.1 in the class of zonotopes in $\mathbb{Z}^n$.

Proof of Theorem 1.4. The proof is similar to that of Theorem 6.1.2, but some additional considerations will be needed. Again, we can assume that the hypothesis of the theorem is true for all directions u from S^{n-1}. Let us denote $Z_K = \operatorname{conv}(K)$ and $Z_L = \operatorname{conv}(L)$. As above, one can show that for every facet F_K of Z_K, there is a facet F_L of Z_L that is parallel to F_K, and vice versa.

Let $\xi \in S^{n-1}$ be a vector that is parallel to a facet F_K (and the opposite facet $-F_K$) of Z_K, but not parallel to any other facet of Z_K. Furthermore, we can assume that $|K|\xi^\perp| = |K|$ and $|L|\xi^\perp| = |L|$, since there are only finitely many directions that do not satisfy these equalities.

Observe that

$$|\partial(K|\xi^\perp)| = 2|(F_K \cap \mathbb{Z}^n)|\xi^\perp| + R(\xi) = 2|F_K \cap \mathbb{Z}^n| + R(\xi), \tag{6.3}$$

where $R(\xi)$ counts those points on the boundary of $(Z_K \cap \mathbb{Z}^n)|\xi^\perp$ that did not come from the facets F_K or $-F_K$.

Let η be the unit outward normal vector to F_K. For $\phi > 0$ small enough consider the vector $\zeta = \cos\phi\,\xi + \sin\phi\,\eta$. We claim that

$$|\partial(K|\zeta^\perp)| = 2|\mathcal{L}_\xi(F_K) \cap \mathbb{Z}^n| + R(\zeta), \tag{6.4}$$

where $R(\zeta)$ counts those points on the boundary of $K|\zeta^\perp$ that did not come from the facets F_K or $-F_K$.

Assume for the moment that the claim is proved. Then, by the hypothesis of the theorem, we have

$$|\partial(K|\xi^{\perp})| - |\partial(K|\zeta^{\perp})| = |\partial(L|\xi^{\perp})| - |\partial(L|\zeta^{\perp})|.$$

Substituting formulas (6.3) and (6.4), and using the fact that $R(\xi) = R(\zeta)$, and

$$|\mathcal{U}_\xi(F_K) \cap \mathbb{Z}^n| = |\mathcal{L}_\xi(F_K) \cap \mathbb{Z}^n|,$$

we get

$$|F_K \cap \mathbb{Z}^n| - |\mathcal{U}_\xi(F_K) \cap \mathbb{Z}^n| = |F_L \cap \mathbb{Z}^n| - |\mathcal{U}_\xi(F_L) \cap \mathbb{Z}^n|.$$

Since any facet of a lattice zonotope is a lattice zonotope, we can apply Lemma 6.2.1 to the facets F_K, F_L and (a shift of) the sublattice $\Lambda = H \cap \mathbb{Z}^n$, where H is the subspace parallel to F_K and F_L. Thus, we obtain that $\mathrm{vol}_{n-1}(F_K) = \mathrm{vol}_{n-1}(F_L)$, and we can use Minkowski's uniqueness theorem to conclude that $\mathrm{conv}(K) = \mathrm{conv}(L)$, or equivalently, $K = L$.

It remains to prove (6.4). We will use the boundary structure of Z_K and its projection $Z_K|\xi^{\perp}$. One can see that

$$\partial Z_K = F_K \cup (-F_K) \cup \mathcal{U}_\xi(Z_K) \cup \mathcal{L}_\xi(Z_K)$$

and

$$Z_K|\xi^{\perp} = \mathcal{U}_\xi(Z_K)|\xi^{\perp} = \mathcal{L}_\xi(Z_K)|\xi^{\perp}.$$

Note that

$$\partial(Z_K|\xi^{\perp}) = \partial(\mathcal{U}_\xi(Z_K))|\xi^{\perp} = \partial(\mathcal{L}_\xi(Z_K))|\xi^{\perp},$$

and for $\zeta = \cos\phi\,\xi + \sin\phi\,\eta$ we have

$$\partial(Z_K|\zeta^\perp) = (\mathcal{U}_\zeta(Z_K) \cap \mathcal{L}_\zeta(Z_K))|\zeta^\perp. \tag{6.5}$$

We see that if $x \in \partial(Z_K|\xi^\perp)$ and $x \notin (F_K \cup (-F_K))|\xi^\perp$, then $x \in (\mathcal{U}_\xi(Z_K) \cap \mathcal{L}_\xi(Z_K))|\xi^\perp$. Therefore, $R(\xi)$ counts the number of lattice points on $(\mathcal{U}_\xi(Z_K) \cap \mathcal{L}_\xi(Z_K)) \setminus (F_K \cup (-F_K))$. Note that the latter number does not change if we replace ξ by another vector ζ that is close enough. In particular,

$$R(\zeta) = \left|\Big((\mathcal{U}_\xi(Z_K) \cap \mathcal{L}_\xi(Z_K)) \setminus (F_K \cup (-F_K))\Big) \cap \mathbb{Z}^n\right|. \tag{6.6}$$

Now observe that

$$\mathcal{U}_\zeta(Z_K) = \mathcal{U}_\xi(Z_K) \cup F_K \qquad \text{and} \qquad \mathcal{L}_\zeta(Z_K) = \mathcal{L}_\xi(Z_K) \cup (-F_K).$$

Hence,

$$\mathcal{U}_\zeta(Z_K) \cap \mathcal{L}_\zeta(Z_K) = (\mathcal{U}_\xi(Z_K) \cap \mathcal{L}_\xi(Z_K)) \cup (\mathcal{U}_\xi(Z_K) \cap (-F_K)) \cup (\mathcal{L}_\xi(Z_K) \cap F_K),$$

that is

$$\mathcal{U}_\zeta(Z_K) \cap \mathcal{L}_\zeta(Z_K) = (\mathcal{U}_\xi(Z_K) \cap \mathcal{L}_\xi(Z_K)) \cup \mathcal{U}_\xi(-F_K) \cup \mathcal{L}_\xi(F_K).$$

In view of the latter formula, and (6.5), (6.6), we get

$$\begin{aligned}
&|\partial(K|\zeta^\perp)| - R(\zeta) \\
=&|\mathcal{U}_\zeta(Z_K) \cap \mathcal{L}_\zeta(Z_K) \cap \mathbb{Z}^n| - \left|\Big((\mathcal{U}_\xi(Z_K) \cap \mathcal{L}_\xi(Z_K)) \setminus (F_K \cup (-F_K))\Big) \cap \mathbb{Z}^n\right| \\
=&|((\mathcal{U}_\xi(Z_K) \cap \mathcal{L}_\xi(Z_K)) \cup \mathcal{U}_\xi(-F_K) \cup \mathcal{L}_\xi(F_K)) \cap \mathbb{Z}^n| \\
&- \left|\Big((\mathcal{U}_\xi(Z_K) \cap \mathcal{L}_\xi(Z_K)) \setminus (F_K \cup (-F_K))\Big) \cap \mathbb{Z}^n\right|
\end{aligned}$$

$$=|\mathcal{U}_\xi(-F_K) \cap \mathbb{Z}^n| + |\mathcal{L}_\xi(F_K) \cap \mathbb{Z}^n| = 2|\mathcal{L}_\xi(F_K) \cap \mathbb{Z}^n|.$$

Thus, formula (6.4) is proved. This finishes the proof of the theorem.

Chapter 7

Grünbaum's inequality for projections

7.1 Introduction

This chapter extends Grünbaum's result [18], bounding the volume of the halves of a convex body which is split by a hyperplane passing through the centroid.

Recall some notation used in this chapter. Let K be a convex body in $\mathbb{R}^n$; that is, a convex and compact set with non-empty interior. For integers $1 \leq k \leq n$, let $\mathrm{vol}_k(\cdot)$ denote k-dimensional Hausdorff measure on $\mathbb{R}^n$. The centroid of K is the point

$$g(K) := \mathrm{vol}_n(K)^{-1} \int_K x\, dx \in K.$$

For $\xi \in S^{n-1}$, let ξ^+ denote the half-space $\{x \in \mathbb{R}^n : \langle x, \xi \rangle \geq 0\}$.

7.2 Auxiliary Lemmas

We associate with a convex body $K \subset \mathbb{R}^n$, $z \in \mathrm{int}(K)$, and $\xi \in S^{n-1}$ the unique cone

$$G = G(K, z, \xi) = \mathrm{conv}\,\{a\xi + B, b\xi\}$$

in $\mathbb{R}^n$ for which

- $B \subset \xi^{\perp}$ is an $(n-1)$-dimensional Euclidean ball centred at the origin;
- $a, b \in \mathbb{R}$ and $a < b$;
- $\mathrm{vol}_{n-1}((K - z) \cap \xi^{\perp}) = \mathrm{vol}_{n-1}((G - z) \cap \xi^{\perp})$;
- $\mathrm{vol}_{n-1}((K - z) \cap \xi^{+}) = \mathrm{vol}_{n-1}((G - z) \cap \xi^{+})$;
- $\mathrm{vol}_k(K) = \mathrm{vol}_k(G)$.

We summarize some simple properties of G in the following lemma.

Lemma 7.2.1. *Let K be a convex body in $\mathbb{R}^n$, $z \in int(K)$, and $\xi \in S^{n-1}$. Let $G = G(K, z, \xi)$ be the previously defined cone. Then*

$$h_G(-\xi) \le h_K(-\xi) \quad \textit{and} \quad h_K(\xi) \le h_G(\xi).$$

Furthermore,

$$\mathrm{vol}_n(\{x \in K : \langle x, \xi\rangle \ge t\}) \le \mathrm{vol}_n(\{x \in G : \langle x, \xi\rangle \ge t\}) \quad \forall\ t \in \mathbb{R}; \tag{7.1}$$

if there is equality for all $t \in \mathbb{R}$, then $K = conv\,\{y_1 + L, y_2\}$ where

$$\begin{cases} L \subset \xi^{\perp} \textit{ is an } (n-1)\textit{-dimensional convex body;} \\ \langle y_1, -\xi\rangle = h_G(-\xi) \quad \textit{and} \quad \langle y_2, \xi\rangle = h_G(\xi). \end{cases}$$

Proof. Assume without loss of generality that z is the origin. Let $\widetilde{K}$ be the Schwarz symmetral of K with respect to the direction ξ (see e.g. [11]). That is, $\widetilde{K}$ is the convex body in $\mathbb{R}^n$ for which $(\widetilde{K} - t\xi) \cap \xi^\perp$ is an $(n-1)$-dimensional Euclidean ball centred at the origin in $\xi^\perp$ with

$$\mathrm{vol}_{n-1}((\widetilde{K} - t\xi) \cap \xi^\perp) = \mathrm{vol}_{n-1}((K - t\xi) \cap \xi^\perp) \quad \forall\, t \in [-h_K(-\xi), h_K(\xi)].$$

It is easy to see that

$$h_{\widetilde{K}}(\pm\xi) = h_K(\pm\xi), \qquad G = G(K, 0, \xi) = G(\widetilde{K}, 0, \xi),$$

and

$$\mathrm{vol}_n(\{x \in \widetilde{K} : \langle x, \xi\rangle \geq t\}) = \mathrm{vol}_n(\{x \in K : \langle x, \xi\rangle \geq t\}) \quad \forall\, t \in \mathbb{R}.$$

Suppose $h_{\widetilde{K}}(\xi) > h_G(\xi)$. We then have

$$G \cap \xi^+ = \mathrm{conv}\{G \cap \xi^\perp, h_G(\xi)\xi\} \subsetneq \mathrm{conv}\{G \cap \xi^\perp, h_{\widetilde{K}}(\xi)\xi\} \subset \widetilde{K} \cap \xi^+,$$

which implies $\mathrm{vol}_n(G \cap \xi^+) < \mathrm{vol}_n(\widetilde{K} \cap \xi^+)$. This is a contradiction, so $h_{\widetilde{K}}(\xi) \leq h_G(\xi)$. Now, there is a $t_0 \in (0, h_K(\xi)]$ for which

$$\{x \in G : 0 \leq \langle x, \xi\rangle \leq t_0\} \subset \{x \in \widetilde{K} : 0 \leq \langle x, \xi\rangle \leq t_0\} \tag{7.2}$$

and

$$\{x \in \widetilde{K} : t_0 < \langle x, \xi\rangle \leq h_{\widetilde{K}}(\xi)\} \subset \{x \in G : t_0 < \langle x, \xi\rangle \leq h_G(\xi)\}; \tag{7.3}$$

otherwise, we will get a contradiction of the convexity of $\widetilde{K}$, or find that $\mathrm{vol}_n(\widetilde{K}\cap\xi^+) < \mathrm{vol}_n(G\cap\xi^+)$. The convexity of $\widetilde{K}$, the containment (7.2), and $\widetilde{K}\cap\xi^\perp = G\cap\xi^\perp$ together imply

$$\widetilde{K} \cap \left\{t\xi + \xi^\perp\right\} \subset G \cap \left\{t\xi + \xi^\perp\right\} \quad \forall\, t \in [-h_G(\xi), 0]. \tag{7.4}$$

Suppose $h_G(-\xi) > h_{\widetilde{K}}(-\xi)$. With (7.4), we then get

$$\{x \in \widetilde{K} :\, -h_{\widetilde{K}}(-\xi) \le \langle x, \xi\rangle \le 0\} \subsetneq \{x \in G :\, -h_G(-\xi) \le \langle x, \xi\rangle \le 0\}$$

and

$$\mathrm{vol}_n(\widetilde{K}) - \mathrm{vol}_n(\widetilde{K} \cap \xi^+) < \mathrm{vol}_n(G) - \mathrm{vol}_n(G \cap \xi^+),$$

which is again a contradiction. So $h_G(-\xi) \le h_{\widetilde{K}}(-\xi)$. Finally, we see that inequality (7.1) follows from the facts $\mathrm{vol}_n(\widetilde{K}) = \mathrm{vol}_n(G)$ and $\mathrm{vol}_n(\widetilde{K} \cap \xi^+) = \mathrm{vol}_n(G \cap \xi^+)$ combined with (7.2), (7.3), and (7.4).

If there is equality in inequality (7.1) for all $t \in \mathbb{R}$, then there will be equality in (7.2), (7.3), and (7.4). This shows $\widetilde{K} = G$. Because its Schwarz symmetral is a cone, K itself must be the cone given in the lemma statement. □

Note 1. *The concave functions in this chapter are always assumed to be continuous on their supports. Of course, the concavity of a function guarantees its continuity on the interior of its support in general.*

Lemma 7.2.2. *Let K be a convex body in $\mathbb{R}^n$, $\xi \in S^{n-1}$, and $p > 0$. Let $\psi : K \to \mathbb{R}^+$ be a concave function, not identically zero. Put $G = G(K, g(\psi^p, K), \xi)$. There is a* *unique function* $\Psi : G \to \mathbb{R}^+$ *for which*

$$\begin{cases} \Psi \equiv f(\langle\,\cdot\,,\xi\rangle) \text{ for some non-decreasing } f : [-h_G(-\xi), h_G(\xi)] \to \mathbb{R}^+; \\ \mathrm{vol}_n(\{x \in K :\ \psi(x) \geq \tau\}) = \mathrm{vol}_n(\{x \in G :\ \Psi(x) \geq \tau\}) \quad \forall\, \tau \in \mathbb{R}. \end{cases}$$

This Ψ *is concave. Furthermore,*

$$\langle g(\psi^p, K), \xi\rangle \leq \langle g(\Psi^p, G), \xi\rangle;$$

if there is equality, then

$$\begin{cases} K \text{ is the cone from the equality case of Lemma 7.2.1}; \\ \psi(x) = f(\langle x, \xi\rangle) \quad \forall\, x \in K. \end{cases}$$

Proof. Put

$$m := \min_{x\in K} \psi(x), \qquad M := \max_{x\in K} \psi(x).$$

Define functions $w : [m, M] \to [-h_G(-\xi), h_G(\xi)]$ and $W : [m, M] \to \mathbb{R}^+$ by

$$W(\tau) := \mathrm{vol}_n(\{x \in K : \psi(x) \geq \tau\}) = \mathrm{vol}_n(\{x \in G : \langle x, \xi\rangle \geq w(\tau)\}) \tag{7.5}$$

for all $\tau \in [m, M]$. Note that $|K| = |G|$ ensures w is well-defined.

The function $W^{\frac{1}{n}}$ is concave and strictly decreasing. As ψ is concave, we have

$$\begin{aligned} &\lambda\{x \in K :\ \psi(x) \geq \tau_1\} + (1-\lambda)\{x \in K :\ \psi(x) \geq \tau_2\} \\ &\subset \{x \in K :\ \psi(x) \geq \lambda\tau_1 + (1-\lambda)\tau_2\} \end{aligned}$$

for all $\lambda \in [0,1]$ and $\tau_1, \tau_2 \in [m, M]$. Applying the Brunn-Minkowski inequality to these level sets shows $W^{\frac{1}{n}}$ is concave. The connectedness of K and the continuity of ψ guarantee $W^{\frac{1}{n}}$ is strictly decreasing.

The function w is convex and strictly increasing. Let $H > 0$ denote the height of the cone G, and let $V > 0$ denote the $(n-1)$-dimensional volume of its base. The set

$$\{x \in G : \langle x, \xi \rangle \geq w(\tau)\}$$

is a cone homothetic to G, with height $h_G(\xi) - w(\tau)$ and a base of some $(n-1)$-dimensional volume $v > 0$. It is necessary that

$$\frac{v}{V} = \left(\frac{h_G(\xi) - w(\tau)}{H}\right)^{n-1} \quad \text{and} \quad \frac{v(h_G(\xi) - w(\tau))}{n} = W(\tau),$$

so

$$w(\tau) = h_G(\xi) - \left(\frac{nH^{n-1}}{V}\right)^{\frac{1}{n}} W^{\frac{1}{n}}(\tau).$$

As $W^{\frac{1}{n}}$ is concave and strictly decreasing, w is convex and strictly increasing. It is then necessary that w has an inverse $w^{-1} : [-h_G(-\xi), \delta] \to [m, M]$ which is concave and strictly increasing, where $\delta := \max w \leq h_G(\xi)$.

Define $f : [-h_G(-\xi), h_G(\xi)] \to \mathbb{R}^+$ by

$$f(t) := w^{-1}(t) \quad \forall\, t \in [-h_G(-\xi), \delta], \quad \text{and} \quad f(t) := M \quad \forall\, t \in [\delta, h_G(\xi)].$$

By construction, f is non-decreasing with

$$\mathrm{vol}_n(\{x \in K :\ \psi(x) \geq \tau\}) = \mathrm{vol}_n(\{x \in G :\ f(\langle x, \xi \rangle) \geq \tau\}) \quad \forall\, \tau \in \mathbb{R}.$$

The uniqueness of f is easy to verify. As w^{-1} is concave and increasing, f is concave. Although the upper level sets for $\Psi := f(\langle \cdot , \xi\rangle)$ have the same volume as the corresponding sets for ψ, they are "pushed" further in the direction ξ. More precisely, by equation (7.5) and Lemma 7.2.1,

$$\begin{aligned}
&\mathrm{vol}_n\Big(\{x \in K : \psi(x) \geq \tau\} \cap \{x \in K : \langle x, \xi\rangle \geq t\}\Big) \\
\leq &\min\Big\{\mathrm{vol}_n(\{x \in K : \psi(x) \geq \tau\}), \mathrm{vol}_n(\{x \in K : \langle x, \xi\rangle \geq t\})\Big\} \\
\leq &\min\Big\{\mathrm{vol}_n(\{x \in G : \Psi(x) \geq \tau\}), \mathrm{vol}_n(\{x \in G : \langle x, \xi\rangle \geq t\})\Big\} \\
= &\mathrm{vol}_n\Big(\{x \in G : \Psi(x) \geq \tau\} \cap \{x \in G : \langle x, \xi\rangle \geq t\}\Big)
\end{aligned} \tag{7.6}$$

for all $\tau, t \in \mathbb{R}$. We have

$$\int_K \psi^p \, dx = p \int_0^\infty \tau^{p-1} W(\tau) \, d\tau = \int_G \Psi^p \, dx$$

using the "layer cake representation" for the L_p-norm of a function (e.g. Theorem 1.13 of [24]). The obvious generalization of Theorem 1.13 to products of functions, and inequality (7.6), give

$$\begin{aligned}
&\int_K \langle x, \xi\rangle \psi^p \, dx \\
= &p \int_0^\infty \int_0^\infty \tau^{p-1} \mathrm{vol}_n(\{x \in K : \psi(x) \geq \tau\} \cap \{x \in K : \langle x, \xi\rangle \geq t\}) \, dt \, d\tau \\
\leq &p \int_0^\infty \int_0^\infty \tau^{p-1} \mathrm{vol}_n(\{x \in G : \Psi(x) \geq \tau\} \cap \{x \in G : \langle x, \xi\rangle \geq t\}) \, dt \, d\tau \\
= &\int_G \langle x, \xi\rangle \Psi^p \, dx,
\end{aligned} \tag{7.7}$$

where we now assume without loss of generality that $h_K(-\xi) = 0$.

Observe that equality in (7.7) implies equality in (7.6) for all $\tau, t \in \mathbb{R}$. Choosing

$\tau = m$ gives

$$\mathrm{vol}_n\Big(\{x \in K : \langle x, \xi\rangle \geq t\}\Big) = \mathrm{vol}_n\Big(\{x \in G : \langle x, \xi\rangle \geq t\}\Big) \quad \forall\, t \in \mathbb{R}, \tag{7.8}$$

so K is the cone from the equality case of Lemma 7.2.1. We need to show that

$$\psi(x) = f(\langle x, \xi\rangle) \quad \forall\, x \in K;$$

this is obvious when $m = M$, so assume $m < M$. Now, choosing $t = w(\tau)$ for $\tau \in [m, M]$ gives

$$\begin{aligned}&\mathrm{vol}_n\Big(\{x \in K : \psi(x) \geq \tau\} \cap \{x \in K : \langle x, \xi\rangle \geq w(\tau)\}\Big)\\ =&\mathrm{vol}_n\Big(\{x \in G : \langle x, \xi\rangle \geq w(\tau)\}\Big),\end{aligned} \tag{7.9}$$

because

$$\{x \in G : \Psi(x) \geq \tau\} = \{x \in G : \langle x, \xi\rangle \geq w(\tau)\}.$$

Equalities (7.5), (7.8), and (7.9) show, respectively, that the sets

$$A_\tau := \{x \in K : \psi(x) \geq \tau\}, \qquad \begin{aligned}B_\tau &:= \{x \in K : \langle x, \xi\rangle \geq w(\tau)\}\\ &= \{x \in K : f(\langle x, \xi\rangle) \geq \tau\},\end{aligned}$$

and $A_\tau \cap B_\tau$ each have the same volume as

$$C_\tau := \{x \in G : \langle x, \xi\rangle \geq w(\tau)\}$$

for $\tau \in [m, M]$. Therefore, A_τ and B_τ must coincide up to a set of measure zero. We also have

$$A_\tau = \overline{\{x \in K : \psi(x) > \tau\}} = \overline{\mathrm{int}(A_\tau)} \qquad \text{and} \qquad B_\tau = \overline{\mathrm{int}(B_\tau)} \tag{7.10}$$

for all $\tau \in [m, M)$, because ψ is continuous and concave, and B_τ is always an n-dimensional cone for $\tau < M$. If $A_\tau \neq B_\tau$ for a given $\tau \in [m, M)$, then (7.10) contradicts the fact that A_τ and B_τ only differ by a set of measure zero. It then follows that

$$A_M = \bigcap_{m \leq \tau < M} A_\tau = \bigcap_{m \leq \tau < M} B_\tau = B_M.$$

Because the upper level sets for ψ coincide exactly with those for $f(\langle\,\cdot\,,\xi\rangle)$, we must have $\psi \equiv f(\langle\,\cdot\,,\xi\rangle)$. □

Remark 7.2.3. *An inspection of Lemma 7.2.2 and its proof shows there is also a unique function $\widetilde{\Psi} : K \to \mathbb{R}^+$ whose upper level sets have the same volume as those for ψ, and which has the form $\widetilde{\Psi} \equiv \widetilde{f}(\langle\,\cdot\,,\xi\rangle)$ for some non-decreasing $\widetilde{f} : [-h_K(-\xi), h_K(\xi)] \to \mathbb{R}^+$. However, it is interesting to note that this $\widetilde{\Psi}$ is not concave in general. For a specific example, take $K = conv\{(0,0),\,(1,0),\,(1,1)\} \subset \mathbb{R}^2$, $\xi = (1,0) \in S^1$, and*

$$\psi(x) := 1 - \langle x, \xi\rangle \qquad \forall\; x \in K.$$

One will find that

$$\widetilde{\Psi}(x) = 1 - \sqrt{1 - \langle x, \xi\rangle^2} \qquad \forall\; x \in K,$$

which is in fact convex.

Lemma 7.2.4. *Let K be a convex body in $\mathbb{R}^n$, $\xi \in S^{n-1}$, and $p > 0$. Consider functions $\phi, \Phi : K \to \mathbb{R}^+$ defined by*

$$\phi(x) := h(\langle x, \xi\rangle) \qquad \text{and} \qquad \Phi(x) := \langle x, \xi\rangle + h_K(-\xi),$$

for some concave function $h : [-h_K(-\xi), h_K(\xi)] \to \mathbb{R}^+$, not identically zero. Then

$$\langle g(\phi^p, K), \xi\rangle \leq \langle g(\Phi^p, K), \xi\rangle;$$

if there is equality, $\phi \equiv \tau \cdot \Phi$ for some $\tau > 0$.

Proof. As Φ is not identically zero, there is a unique $\tau > 0$ so that

$$\int_K \phi^p \, dx = \tau^p \int_K \Phi^p \, dx = \int_K (\tau \cdot \Phi)^p \, dx.$$

Assuming without loss of generality that $h_K(-\xi) = 0$ and $b := h_K(\xi) > 0$,

$$\begin{aligned} &\int_0^b h^p(t)\mathrm{vol}_{n-1}(\{x \in K : \langle x, \xi\rangle = t\}) \, dt \\ = &\int_K \phi^p \, dx = \int_K (\tau \cdot \Phi)^p \, dx = \int_0^b (\tau \cdot t)^p \mathrm{vol}_{n-1}(\{x \in K : \langle x, \xi\rangle = t\}) \, dt. \end{aligned} \tag{7.11}$$

There exists $t_0 \in (0, b)$ such that $h(t_0) = \tau \cdot t_0$; otherwise, equation (7.11) is contradicted. Because h is concave with $h(0) \geq 0$,

$$h(t) \geq \tau \cdot t \quad \forall\, t \in [0, t_0], \quad h(t) \leq \tau \cdot t \quad \forall\, t \in [t_0, b].$$

We then have

$$\int_0^b (t - t_0)\Big(h^p(t) - (\tau \cdot t)^p\Big)\mathrm{vol}_{n-1}(\{x \in K : \langle x, \xi\rangle = t\}) \, dt \leq 0,$$

with $h(t) \equiv \tau \cdot t$ when there is equality. That is,

$$\begin{aligned}&\int_0^b t\, h^p(t)\, \mathrm{vol}_{n-1}(\{x \in K : \langle x, \xi\rangle = t\})\, dt \\ \leq &\int_0^b t\, (\tau \cdot t)^p\, \mathrm{vol}_{n-1}(\{x \in K : \langle x, \xi\rangle = t\})\, dt,\end{aligned}$$

or rather

$$\frac{\int_K \langle x, \xi\rangle \phi^p\, dx}{\int_K \phi^p\, dx} \leq \frac{\int_K \langle x, \xi\rangle (\tau \cdot \Phi)^p\, dx}{\int_K (\tau \cdot \Phi)^p\, dx} = \frac{\int_K \langle x, \xi\rangle \Phi^p\, dx}{\int_K \Phi^p\, dx},$$

with $\phi \equiv \tau \cdot \Phi$ when there is equality. $\square$

Remark 7.2.5. *If we alter the statement of Lemma 7.2.4 so that $\phi : K \to \mathbb{R}^+$ is a concave function without necessarily having the particular form $\phi \equiv h(\langle \cdot\,, \xi\rangle)$, then it is possible that*

$$\langle g(\phi^p, K), \xi\rangle > \langle g(\Phi^p, K), \xi\rangle.$$

For example, consider the closed curves

$$\begin{aligned}\mathcal{C}_1 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 \in [-1, 1],\ x_2 = 1 - \sqrt{1 - x_1^2},\ x_3 = 0\},\\ \mathcal{C}_2 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 \in [-1, 1],\ x_2 = 1,\ x_3 = \sqrt{1 - x_1^2}\},\end{aligned}$$

which are arcs on a sphere in $\mathbb{R}^3$ of radius one and centred at $(0, 1, 0)$. Let $E, H \in G(3, 2)$ denote the x_1, x_2 - plane and the x_2, x_3 - plane, respectively. Then $K :=$ conv$\{\mathcal{C}_1\}$ is half of a Euclidean disk in E, $L :=$ conv$\{\mathcal{C}_1, \mathcal{C}_2\}$ is a convex body in $\mathbb{R}^3$, and $K = L|E$. For $(x_1, x_2, x_3) \in K$, define

$$\phi(x_1, x_2, x_3) := \mathrm{vol}_1(L \cap \{(x_1, x_2, x_3) + E^{\perp}\}) \quad \textit{and} \quad \Phi(x_1, x_2, x_3) := x_2.$$

By the Brunn-Minkowski inequality, ϕ is concave. It can be shown for each $t \in [-1, 1]$ that

$$L \cap \{(t, 0, 0) + H\} = conv\{(t, 1 - \sqrt{1 - t^2}, 0),\ (t, 1, 0),\ (t, 1, \sqrt{1 - t^2})\},$$

which is a right-angled triangle. With this more explicit representation for L, we can calculate

$$\frac{\int_K x\,\phi\,dx}{\int_K \phi\,dx} \approx (0,\ 0.705,\ 0) \quad \textit{and} \quad \frac{\int_K x\,\Phi\,dx}{\int_K \Phi\,dx} \approx (0,\ 0.697,\ 0).$$

7.3 Main Results

Theorem 7.3.1. *Let K be a convex body in $\mathbb{R}^n$, and $p > 0$. Let $\psi : K \to \mathbb{R}^+$ be a concave function, not identically zero, with $g(\psi^p, K)$ at the origin. Then*

$$\frac{\mathrm{vol}_n(K \cap \xi^+)}{\mathrm{vol}_n(K)} \geq \left(\frac{n}{n + 1 + p}\right)^n \quad \forall\ \xi \in S^{n-1};$$

there is equality for some ξ if and only if

$$\begin{cases} K = conv\{y_1 + L, y_2\}; \\ L \subset \xi^\perp \text{ is an } (n-1)\text{-dimensional convex body}; \\ \langle y_1, \xi\rangle < 0 < \langle y_2, \xi\rangle; \\ \psi(x) = \tau[\langle x, \xi\rangle + h_K(-\xi)] \quad \forall\ x \in K, \quad \text{for some } \tau > 0; \\ g(\psi^p, K) = 0. \end{cases}$$

Proof. Put $G = G(K, 0, \xi)$. Define $\Phi : G \to \mathbb{R}^+$ by

$$\Phi(x) := \langle x, \xi \rangle + h_G(-\xi) \qquad \forall\, x \in G.$$

By Lemma 7.2.2 and Lemma 7.2.4,

$$0 = \frac{\int_K \langle x, \xi \rangle \psi^p \, dx}{\int_K \psi^p \, dx} \leq \frac{\int_G \langle x, \xi \rangle \Phi^p \, dx}{\int_G \Phi^p \, dx} =: c; \tag{7.12}$$

equality implies K and ψ satisfy the equality conditions given in the theorem statement. Given the definition of G and Lemma 7.2.1,

$$\frac{\mathrm{vol}_n(K \cap \xi^+)}{\mathrm{vol}_n(K)} = \frac{\mathrm{vol}_n(G \cap \xi^+)}{\mathrm{vol}_n(G)} \geq \frac{\mathrm{vol}_n(\{x \in G : \langle x, \xi \rangle \geq c\})}{\mathrm{vol}_n(G)};$$

equality implies equality in (7.12).

Now suppose K and ψ satisfy these equality conditions, but without the requirement that the centroid of ψ^p is at the origin. Assume without loss of generality that $h_K(-\xi) = 0$ and $b := h_K(\xi) > 0$. For some $\tau > 0$, we have

$$\begin{aligned}\int_K \langle x, \xi \rangle \psi^p \, dx &= \int_0^b t(\tau \cdot t)^p \left(\mathrm{vol}_{n-1}(L) \left(1 - \frac{t}{b}\right)^{n-1} \right) dt \\ =& b^{2+p} \tau^p \mathrm{vol}_{n-1}(L) \frac{\Gamma(2+p)\Gamma(n)}{\Gamma(n+2+p)}\end{aligned}$$

and

$$\begin{aligned}\int_K \psi^p \, dx &= \int_0^b (\tau \cdot t)^p \left(\mathrm{vol}_{n-1}(L) \left(1 - \frac{t}{b}\right)^{n-1} \right) dt \\ &= b^{1+p} \tau^p \mathrm{vol}_{n-1}(L) \frac{\Gamma(1+p)\Gamma(n)}{\Gamma(n+1+p)},\end{aligned}$$

where Γ is the gamma function. So

$$d := \frac{\int_K \langle x, \xi \rangle \psi^p \, dx}{\int_K \psi^p \, dx} = \left(\frac{1+p}{n+1+p} \right) b.$$

We can then calculate

$$\frac{\mathrm{vol}_{n-1}\left(\{x \in K : \langle x, \xi \rangle \geq d\}\right)}{\mathrm{vol}_n(K)} = \left(\frac{n}{n+1+p} \right)^n.$$

$\square$

Corollary 7.3.2. *Let K be a convex body in $\mathbb{R}^n$ with its centroid at the origin, and let $k \in \mathbb{Z}$ be such that $1 \leq k \leq n$. Then*

$$\frac{\mathrm{vol}_k\Big((K|E) \cap \xi^+\Big)}{\mathrm{vol}_k\Big(K|E\Big)} \geq \left(\frac{k}{n+1} \right)^k \quad \forall\ E \in G(n,k), \quad \forall\ \xi \in S^{n-1} \cap E;$$

there is equality for some E and ξ if and only if $K = conv\{y_1 + L_1, y_2 + L_2\}$ where

$$\begin{cases} L_1 \subset \xi^\perp \text{ and } L_1|(E \cap \xi^\perp) \text{ are } (k-1)\text{-dimensional convex bodies;} \\ L_2 \subset E^\perp \text{ is an } (n-k)\text{-dimensional convex body;} \\ \langle y_1, \xi \rangle < 0 < \langle y_2, \xi \rangle; \\ g(K) = 0. \end{cases}$$

Proof. Suppose $1 \leq k < n$. For $E \in G(n,k)$, define $\psi : K|E \to \mathbb{R}^+$ by

$$\psi(x) := \Big[\mathrm{vol}_{n-k}(K \cap \{x + E^\perp\})\Big]^{\frac{1}{n-k}}.$$

By the Brunn-Minkowski inequality, ψ is concave. For all $\xi \in S^{n-1} \cap E$

$$\frac{\int_{K|E} \langle x, \xi \rangle \psi^{n-k}\, dx}{\int_{K|E} \psi^{n-k}\, dx} = \frac{\int_K \langle x, \xi \rangle\, dx}{\mathrm{vol}_n(K)} = 0,$$

so the centroid of ψ^{n-k} is at the origin. Therefore, by Theorem 7.3.1,

$$\frac{\mathrm{vol}_k\Big((K|E) \cap \xi^+\Big)}{\mathrm{vol}_k\Big(K|E\Big)} \geq \left(\frac{k}{k+1+(n-k)}\right)^k = \left(\frac{k}{n+1}\right)^k \quad \forall\, \xi \in S^{n-1} \cap E;$$

there is equality for some ξ if and only if

$$\begin{cases} K|E = \mathrm{conv}\{y_1 + L, y_2\}; \\ L \subset E \cap \xi^{\perp} \in G(n, k-1) \text{ is a } (k-1)\text{-dimensional convex body;} \\ \langle y_1, \xi \rangle < 0 < \langle y_2, \xi \rangle; \\ \psi(x) = \tau[\langle x, \xi \rangle + h_K(-\xi)] \quad \forall\, x \in K|E, \quad \text{for some } \tau > 0; \\ g(\psi^{n-k}, K|E). \end{cases} \tag{7.13}$$

The conditions (7.13) are equivalent to the equality conditions in the corollary statement. $\square$

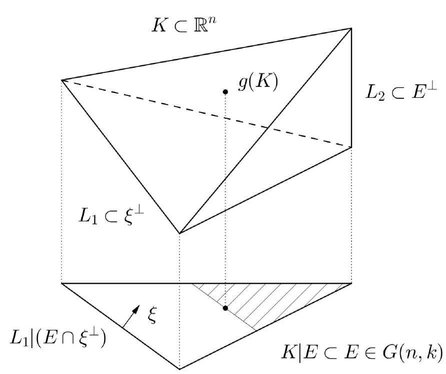

Figure 7.1: The equality conditions for Corollary 7.3.2.

Remark 7.3.3. *Observe that for k close to n, the constant of Meyer et al. given in inequality (1.2) is asymptotically equivalent to our constant $c = \left(\frac{k}{n+1}\right)^k$. We conjecture that c is also the best constant for Problem 1.3.1, with equality conditions similar to those given in Corollary 7.3.2.*

Bibliography

[1] M. Alfonseca, M. Cordier, and D. Ryabogin, *On bodies with directly congruent projections and sections*, Israel J. Math. (2) **215** (2016) 765–799.

[2] A. Barvinok, *Integer Points in Polyhedra*, European Mathematical Society (EMS), Zürich, 2008.

[3] J. A. Barker, D. G. Larman, *Determination of convex bodies by certain sets of sectional volumes,* Discrete Mathematics **241** (2001), 79–96.

[4] M. Beck, S. Robins, *Computing the Continuous Discretely, Integer-point Enumeration in Polyhedra*, Undergraduate Texts in Mathematics, Springer, New York, 2007.

[5] T. Bonnesen and W. Fenchel, *Theory of convex bodies*, BCS Associates, Moscow ID, 1987.

[6] C. De Concini, C. Procesi, *Topics in Hyperplane Arrangements, Polytopes and Box-splines*, Universitext, Springer, New York, 2011.

[7] P. Dulio, R. Gardner, and C. Peri, *Discrete point X-rays of convex lattice sets*, Electronic Notes in Discrete Mathematics **20** (2005) 1-–13.

[8] K. Falconer, *X-ray problems for point sources,* Proc. London Math. Soc. (3) **46** (1983), 241–262.

[9] M. Fradelizi, M. Meyer, and V. Yaskin, *On the volume of sections of a convex body by cones*, Proc. Amer. Math. Soc., to appear.

[10] R. J. Gardner, *Symmetrals and X-rays of planar convex bodies*, Arch. Math. **41** (1983), 183–189.

[11] R. J. Gardner, *Geometric Tomography,* second edition, Cambridge University Press, New York, 2006.

[12] R. J. Gardner, P. Gronchi, C. Zong, *Sums, projections, and sections of lattice sets, and the discrete covariogram*, Discrete Comput. Geom. **34**, (2005), no. 3, 391–409.

[13] R. J. Gardner, D. Ryabogin, V. Yaskin, and A. Zvavitch, *A problem of Klee on inner section functions of convex bodies*, J. Differential Geom. **91** (2012), 261–279.

[14] P. Goodey, V. Yaskin, and M. Yaskina, *Fourier transforms and the Funk-Hecke theorem in convex geometry,* J. London Math. Soc. (2) **80** (2009), 388–404.

[15] H. Groemer,*Geometric Applications of Fourier Series and Spherical Harmonics,* Cambridge University Press, 1996.

[16] H. Groemer, *On a spherical integral transformation and sections of star bodies,* Mh. Math. **126** (1998), 117–124.

[17] P. M. Gruber, *Convex and Discrete Geometry, Grundlehren der mathematischen Wissenschaften* 336, Springer-Verlag, Berlin, Heidelberg, 2007.

[18] B. Grünbaum, *Partitions of mass-distributions and of convex bodies by hyperplanes*, Pacific J. Math. **10** (1960), 1257–1261.

[19] B. Grünbaum, *Measures of Symmetry for Convex Sets*, In: Proc. Sympos. Pure Math., Vol. VII, pp. 233–270, Amer. Math. Soc., Providence, R.I., 1963.

[20] G. Herman and A. Kube (Eds.), *Advances in discrete tomography and its applications,* Applied and Numerical Harmonic Analysis, Birkhäuser Boston, 2007.

[21] A. Koldobsky and C. Shane, *The determination of convex bodies from derivatives of section functions,* Arch. Math. **88** (2007), 279–288.

[22] A. Koldobsky, *Fourier Analysis in Convex Geometry,* Mathematical Surveys and Monographs, American Mathematical Society, Providence RI, 2005.

[23] A. Kurusa and T. Ódor, *Characterizations of balls by sections and caps*, Beitr. Algebra Geom. **56** (2015), 459–471.

[24] E. Lieb, and M. Loss, *Analysis*, second edition, Amer. Math. Soc., Providence, RI, 2001.

[25] B. Mityagin, *Two inequalities for volumes of convex bodies*, Mathematical Notes of the Academy of Sciences of the USSR (1) **5** (1969), 61–65.

[26] F. Nazarov, D. Ryabogin, and A. Zvavitch, *On the local equatorial characterization of zonoids and intersection bodies*, Adv. Math. **217** (2008), 1368–1380.

[27] F. Nazarov, D. Ryabogin, and A. Zvavitch, *An asymmetric convex body with maximal sections of constant volume*, Journal of Amer. Math. Soc. **27** (2014), 43–68.

[28] D. Ryabogin, *On the continual Rubik's cube.* Adv. Math. **231** (2012) 3429–3444.

[29] D. Ryabogin, V. Yaskin, and N. Zhang, *Unique determination of convex lattice sets*, Discrete Comput. Geom. (3) **57** (2017), 582–589.

[30] L. A. Santaló, *Two characteristic properties of circles on a spherical surface* (Spanish), Math. Notae **11** (1951) 73–78.

[31] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, 1993.

[32] S. Sobolev, *Cubature Formulas and Modern Analysis*, Gordon and Breach Science Publishers, 1992.

[33] M. Stephen and N. Zhang, *Grünbaum's inequality for projections*, J. Funct. Anal. (6) **272** (2017), 2628–2640.

[34] V. Yaskin, *Unique determination of convex polytopes by non-central sections*, Math. Ann. **349** (2011), 647–655.

[35] V. Yaskin and N. Zhang, *Non-central sections of convex bodies,* Israel J. Math. (2007), DOI:10.1007/s11856-017-1532-9.

[36] H. Xiong, *On a discrete version of Alexandrov's projection theorem*, Acta Math. Sin. (Engl. Ser.) (8) **29** (2013), 1597–1606.

[37] N. Zhang, *An analogue of the Aleksandrov projection theorem for convex lattice polygons*, Proc. Amer. Math. Soc., **145** (2017), 2305–2310.

[38] N. Zhang, *On bodies with congruent sections by cones or non-central planes,* preprint.

[39] J. Zhou, *On the projections of convex lattice sets*, Acta Math. Sin. (Engl. Ser.) (10) **26** (2010), 1969–1980.

[40] G. Ziegler, *Lectures on Polytopes*, Springer-Verlag, Berlin, 1995.