University of Alberta

Representability of Algebraic CHOW Groups of Complex Projective Complete Intersections and Applications to Motives

by

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To Kalaycı and Tuncer Families,

for being there all the time giving support and love.

Abstract

Let $X \subset \mathbb{P}^{n+r}$ be an n dimensional smooth complex projective complete intersection of type $(d_1, \ldots, d_r)$. $CH^r(X)$ denotes the Chow group of codimension r algebraic cycles on X modulo rational equivalence and $A^r(X)$ denotes the subgroup of the codimension r algebraic cycles that are algebraically equivalent to zero. In 1990 James D. Lewis made a conjecture on the representability of $A^r(X)$. We will show that his conjecture holds for smooth complex complete intersections satisfying a numerical condition and consider some applications to motives.

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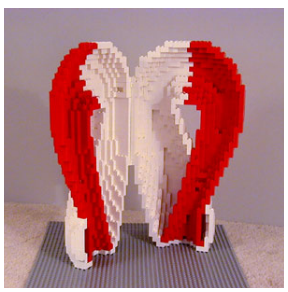
Contents

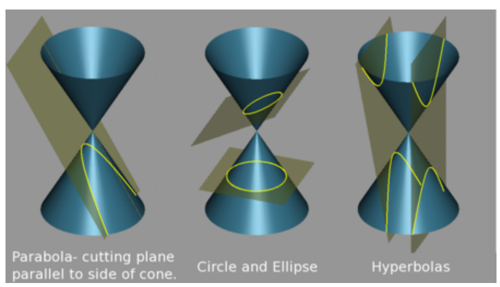
Chapter 1

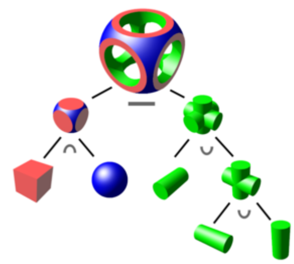
Introduction

1.1 Introduction (for the layperson)

In nature we can find different geometric objects like points, lines, spheres, etc. We can express these objects mathematically as the zero sets of some polynomials. Classifiying these geometric objects is one of the main problems of algebraic geometry. Examining the smaller parts of an object can give information that can be used to compare with other objects and hence to classify. Let us consider the pictures below.







Picture of Klein bottle is taken from http://www.maa.org/features/lego.html.
Picture of conic section is taken from http://en.wikipedia.org/wiki/Conic_section.
Third picture is taken from http://en.wikipedia.org/wiki/Constructive_solid_geometry.

The first picture shows a Klein bottle that is built from rectangle shapes like legos. This is an example of obtaining new objects from existing ones by gluing method. Intersecting two objects is another way to get a new object as in the second picture that shows conic sections or we can use these methods together as in the third picture. Analogues of these ideas can be found in algebraic geometry; a manifold can be built by gluing together the corresponding coordinate charts, subvariety of an algebraic variety can be obtained by intersecting appropriate number of hyperplanes with the variety.

In this thesis we will work with complex projective algebraic manifolds and use special subvarieties called algebraic cycles to get information about our objects. We will specially focus on Chow groups that are formed from algebraic cycles by introducing adequate equivalence relations that enables one to introduce ring structures (e.g. multiplication via intersection pairings). Of particular relevance in our situation is rational equivalence. By the equivalence relations one cuts algebraic cycles into smaller parts so that we can at least understand parts of these huge groups.

One interesting thing about Chow groups is that although they are formed from geometric objects, algebraic cycles, they may not be represented as a geometric object. There is the well-known Abel-Jacobi map from algebraic Chow groups of a projective variety to the Jacobian of that projective variety, which in some cases represents algebraic Chow groups as tori. For example, for a projective algebraic manifold X, $A^1(X) \simeq J^1(X)$, where $A^1(X)$ is a certain divisor subgroup of a Chow group of algebraic cycles of codimension one on X, and $J^1(X)$ is a complex torus called the (first) Jacobian of X (also called the Picard variety of X), and where the identification is given by an integration over cycle map called the Abel-Jacobi map. This situation forces one to explore the question; under which conditions is it possible to represent algebraic Chow groups of any codimension as a geometric object? In the path of answering this question, the range of the Abel-Jacobi map is modified to yield a surjective map from the analogous higher codimension Chow groups $A^r(X)$ to the algebraic $J^r_{\text{alg}}(X)$ of $J^r(X)$, namely a certain compact complex torus called an abelian variety. Hence, our first enquiry boils down to the question of under what conditions is it possible to ensure that the Abel-Jacobi map $A^r(X) \twoheadrightarrow J^r_{\text{alg}}(X)$ is injective, say for all $r > 0$?

We will explore this question for complex projective complete intersections and find a numerical condition on complex projective complete intersections that will imply the injectivity of the Abel-Jacobi map and hence the representability of algebraic Chow groups of any codimension. The motivation for looking at complete intersections came from a result of James D. Lewis. In 1990 he made a conjecture on the representability of algebraic Chow groups generalizing a conjecture of Bloch. We will explore the Lewis conjecture for smooth complex projective complete intersections satisfying a numerical condition with the guidance of [Lewis1].

1.2 Precise Results (for the expert)

As we mentioned in the introduction our main interest is to understand algebraic cycles. In 1950 William V.D. Hodge introduced a conjecture, namely the Hodge conjecture, that relates codimension r algebraic cycles to (r,r)-forms with integral periods via the cycle class map. He conjectured that the cycle class map is surjective, that is every (r,r)-form with integral periods is supported on a codimension r algebraic cycle. This raised another issue about the kernel of the cycle class map. There are attempts to understand this kernel and subgroups of the kernel of the cycle class map, that lie in a certain filtration level.

In the first chapter we will give the construction of the cycle class map and (Generalized) Hodge Conjecture. We will also give the construction of Abel-Jacobi map that is induced from the kernel of the cycle class map, which will be used to study the Algebraic Chow groups.

In the second chapter we will give the statement of the Lewis conjecture which coincides with Bloch's conjecture in the case of a smooth complex projective surface. The importance of this conjecture is that it tells us that the Algebraic Chow groups $A^\bullet(X) \subset CH^\bullet(X)$ for projective algebraic manifolds X with Level$(H^*(X)) \leq 1$ are representable, that is they can be identified with compact complex tori. We will prove Lewis conjecture for smooth complex complete intersections, say X, by using Ω_X, the Fano variety of k-planes inside X, where the calculation of k is defined in [SGA7] with Level$(H^*(X)) = n - 2k$. The relation between the dimension of Ω_X and Level$(H^*(X))$ will give us a numerical condition on page 21 that is needed to show surjectivity of cohomology of Ω_X and X. We state this result as Corollary **??** which is a generalization of Corollary 3.8 in [Lewis1] to complete intersections. The surjectivity of the cohomology groups will be then used to show that a special case of General Hodge Conjecture holds for smooth complex complete intersections satisfying a numerical condition (Proposition **??**). This result is needed for proving one part of the Lewis Conjecture that we stated as Corollary **??**.

In section 2.3 we generalized the Theorem 1.1 (ii) of [Lewis-Sertoz] to complete intersections and get a short exact sequence of Chow groups (Theorem **??**). In section 2.4 Theorem **??** is used as a main tool for the proof of Theorem **??** which states that Lewis conjecture holds for smooth complete intersection satisfying a numerical condition. In Proposition **??** we give a list of all possible complete intersections that satisfy the numerical condition and have representable Algebraic Chow groups.

Lastly in Chapter 3, we generalize Theorem 1.1 (i) of [Lewis-Sertoz] to complete intersections, that gives an isomorphism between the motive of Ω_X and X (Theorem **??**). For the proof we make two reasonable assumptions which are consequences of the existence of a Bloch-Beilinson filtration.

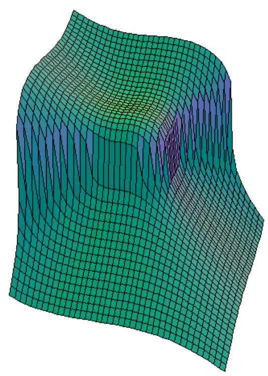
1.3 Preliminaries

Throughout the thesis our main object will be a smooth complex projective complete intersection. So let us start with the definition of a complete intersection:

Definition 1 *Let $z = (z_0, \ldots, z_{n+r})$ and assume given $\{f_1, \ldots, f_r\}$ homogeneous polynomials of degrees $(d_1, \ldots, d_r)$ in z such that the corresponding ideal $\mu = (f_1, \ldots, f_r)$ is prime and that $X = V(\mu) \subset \mathbb{P}^{n+r}$ is of dimension n. Then X is called a complete intersection variety of type $(d_1, \ldots, d_r)$. X is called smooth if at every point of X, its Jacobian has the full rank.*

Example 2

1. *Fermat's cubic $X = V(z_0^3+z_1^3+z_2^3+z_3^3) \subset \mathbb{P}^3$ is a projective algebraic manifold: It is given by the zeros of an irreducible cubic homogenous polynomial and $X \cap U_i = V(1 + x^3 + y^3 + z^3)$ for $U_i = \{[z_0, \ldots, z_3] | z_i \neq 0\}$ where variables x, y, z adjusted according to the coordinate charts $U_i \cong \mathbb{C}^3$. Then the Jacobian of the polynomial has rank 1 at everywhere on X hence X is smooth.*



Picture of Fermat's cubic is taken from http://en.wikipedia.org/wiki/Fermat_cubic.

2. *Let $X = V(z_0^2 - z_0z_2 - z_1z_3, z_1z_2 - z_0z_3 - z_2z_3) \subset \mathbb{P}^3$. Then we have $X \cap U_0 = V(1 - y - xz, xy - z - yz), X \cap U_1 = V(x^2 - xy - z, y - xz - yz), X \cap U_2 = V(x^2 - x - yz, y - xz - z), X \cap U_3 = V(x^2 - xz - y, yz - x - z)$. Then one can check the Jacobian of these all have rank 2, hence X is a smooth complete intersection. It is called the elliptic quartic curve in $\mathbb{P}^3$.*

3. *Let X be a complete intersection K3 surface, then X has the form*
$$X = \begin{cases} V(F) \subset \mathbb{P}^3 & d = 4 \\ V(F_1, F_2) \subset \mathbb{P}^4 & d_1 = 2, d_2 = 3 \\ V(F_1, F_2, F_3) \subset \mathbb{P}^5 & d_1 = d_2 = d_3 = 2 \end{cases}.$$

1.3.1 Algebraic Cycles

In the introduction we mentioned that algebraic cycles are the objects that we will use to get information about complex projective manifolds. Now let X be a projective algebraic manifold of dimension n. Then

Definition 3 *A codimension r algebraic cycle V on X is a $\mathbb{Z}$ formal sum of codimension r irreducible subvarieties V_i, i.e. $V = \sum_{i=1}^{k} n_i V_i$, $n_i \in \mathbb{Z}$.*
The group of codimension r algebraic cycles is denoted by $Z^r(X)$ = free abelian group generated by subvarieties of codim r in X, equivalently $Z_{n-r}(X)$= free abelian group generated by subvarieties of $\dim n - r$ *in X.*

Example 4 *Let X be the Fermat's cubic in $\mathbb{P}^3$. Let's examine algebraic cycles on X. Note that* $\dim X = 2$. *Then*

- $Z^2(X) = Z_0(X)$*=linear combination of points in X,*
- $Z^1(X) = Z_1(X)$*=linear combination of curves in X,*
- $Z^0(X) = Z_2(X)$*=linear combination of surfaces in X*

If Y is a closed subset of an irreducible finite dimensional topological space X and if $\dim Y = \dim X$ *then $Y = X$.*
For $Y \in Z^0(X)$, $Y = nX, n \in \mathbb{Z} \Rightarrow Z^0(X) = Z_2(X) = \mathbb{Z}\{X\}$.

We want to use algebraic cycles to get information about algebraic varieties. There is a commonly used tool, (co-)homology, for classifing algebraic varieties by attaching algebraic datum like rings, groups, modules, algebras to these varieties in order to compare two varieties. Now let $X \subset \mathbb{P}^N$ be a complex projective manifold with $\dim X = n$. Let $H \subset \mathbb{P}^N$ be a hyperplane. We can obtain information of the (co)homology of X by considering how a generic hyperplane intersects X.

Theorem 5 *(Lefschetz's (Weak)Hyperplane Theorem) Let $H \subset \mathbb{P}^N$ be a hyperplane for which $H \cap X$ is smooth. Then the restriction map $H^i(X, \mathbb{Z}) \longrightarrow H^i(X \cap H, \mathbb{Z})$ is an isomorphism for $i < n-1$ and is an injection for $i = n-1$.*

Note that $[X \cap H] \in H^2(X)$ represents the hyperplane class which is independent of the choice of hyperplane. It is the Poincaré dual of $X \cap H$, a hyperplane section of X. Since cohomomology has a ring structure, multiplication by the hyperplane class defines a map:
$L : H^i(X) \longrightarrow H^{i+2}(X)$. This leads us to:

Theorem 6 *(Strong Lefschetz Theorem) Multiplication by powers of the hyperplane class defines isomorphisms $L^i : H^{n-i}(X, \mathbb{Q}) \longrightarrow H^{n+i}(X, \mathbb{Q})$.*
In particular the map $L : H^k(X, \mathbb{Q}) \longrightarrow H^{k+2}(X, \mathbb{Q})$ is injective if $k < n$ so that $\dim H^k(X, \mathbb{Q}) \leq \dim H^{k+2}(X, \mathbb{Q})$ *and is surjective if $k+2 > n$ so that* $\dim H^k(X, \mathbb{Q}) \geq \dim H^{k+2}(X, \mathbb{Q})$.

Note that Lefschetz's theorems do not hold for singular spaces using singular cohomology, but there is a (co-)homology theory where they do hold, namely 'intersection (co-)homology' invented by R. MacPherson and M. Goresky.

1.3.2 Cycle Class Map and Hodge Conjecture

There is a relation between algebraic cycles and cohomology of X. Let $V \in Z^r(X)$, then $V = n_1 V_1 + \ldots + n_l V_l$ where $n_j \in \mathbb{Z}$ and V_j's are irreducible subvarieties of codimension r in X. Note that V may have singularities. In 1964, Heisuke Hironaka proved that every variety is birationally equivalent to a smooth projective variety:

Theorem 7 (Hironaka's Desingularization Theorem) *([Sim]) Let X be a complex quasi-projective variety. Then there exists a smooth quasi-projective variety $\widetilde{X}$ and a projective birational morphism $\widetilde{X} \xrightarrow{\pi} X$. Furthermore π may be assumed to be an isomorphism on the smooth locus of X, and if X is a projective variety, then so is $\widetilde{X}$.*

We have a cycle class map which gives a relation between algebraic cycles and cohomology. We will recall topological and analytic constructions of the cycle class map:

Topological Construction:

The topological definition comes from the following observation: Let $V \in Z^r(X)$, V may have singularities and by Hironaka's theorem one can find a desingularization $\sigma : \widetilde{V} \to V$ with $\widetilde{V} - \sigma^{-1}(V_{\text{sing}}) \approx V - V_{sing}$. Note that $\widetilde{V}$ is a smooth complex projective variety and $\dim_{\mathbb{R}} V = \dim_{\mathbb{R}} \widetilde{V} = 2n - 2r$. Since complex manifolds are always orientable and triangulable, one can triangulate $\widetilde{V}$ with oriented simplices such that there is a fundamental class generator $\{\widetilde{V}\} \in H^{sing}_{2n-2r}(\widetilde{V}, \mathbb{Z}) \cong \mathbb{Z}$. Now consider the composition $\widetilde{V} \xrightarrow{\sigma} V \xrightarrow{j} X$ where j is the inclusion map. Then we have the corresponding homology maps: $H^{sing}_{2n-2r}(\widetilde{V}, \mathbb{Z}) \xrightarrow{\sigma_*} H^{sing}_{2n-2r}(V, \mathbb{Z}) \xrightarrow{j_*} H^{sing}_{2n-2r}(X, \mathbb{Z})$ with $\{\widetilde{V}\} \in H^{sing}_{2n-2r}(\widetilde{V}, \mathbb{Z}) \xrightarrow{(j\circ\sigma)_*} [V] \in H^{sing}_{2n-2r}(X, \mathbb{Z}) \cong H^{2r}_{\text{sing}}(X, \mathbb{Z})$, given by the Poincaré duality, and we get our map:

Definition 8 *The cycle class map $[\ ] : Z^r(X) \to H^{2r}_{sing}(X, \mathbb{Z})$ is given by the fundamental class description above.*

Analytical Construction:

Now we will look at the analytical definition of the cycle class map: We choose a desingularization $\sigma : \widetilde{V} \to V$ with $\widetilde{V} - \sigma^{-1}(V_{\text{sing}}) \approx V - V_{sing}$. We can assume $\sigma^{-1}(V_{\text{sing}})$ is a normal crossing divisor, i.e. locally in $\widetilde{V}$ with local coordinates $(z_0, \ldots, z_{n-r})$, $\sigma^{-1}(V_{\text{sing}})$ is given by $z_{i_1} \ldots z_{i_l} = 0$. So $\sigma^{-1}(V_{\text{sing}})$ has Lebesgue measure zero. Then our analytical cycle class map is defined as taking a cycle $V \in Z^r(X)$ and sending to the fundamental class $[V] \in H^{2r}_{deR}(X, \mathbb{C})$ which is defined by the formula:

$$[V] : H^{2n-2r}_{deR}(X, \mathbb{C}) \longrightarrow \mathbb{C}, w \mapsto \int_V w = \int_{V - V_{\text{sing}}} w < \infty.$$

We have $\int_{V - V_{\text{sing}}} w = \int_{\widetilde{V} - \sigma^{-1}(V_{\text{sing}})} \sigma^*(w) = \int_{\widetilde{V}} \sigma^*(w)$. This is finite since $\widetilde{V}$ is compact and $\sigma^*(w)$ is C^∞.

Definition 9 *The cycle class map $[\ ] : Z^r(X) \to H^{2r}_{deR}(X, \mathbb{C})$ is given by the integration above.*

Note that the definition of the cycle class map obtained from topological and analytical methods agree in $H^{2r}_{\text{de R}}(X, \mathbb{C})$, i.e. up to torsion, by considering $H^{2r}_{sing}(X, \mathbb{Z}) \to H^{2r}_{sing}(X, \mathbb{C}) \cong H^{2r}_{deR}(X, \mathbb{C})$ by the De Rham isomorphism theorem and Poincaré Duality.

Definition 10 *The cycle class map $[\ ] : Z^r(X) \to H^{2r}_{sing}(X, \mathbb{Z}) \to H^{2r}_{deR}(X, \mathbb{C})$ is given by the two equivalent descriptions above.*

One has a decomposition of the DeRham cohomology into Dolbeault cohomologies by the Hodge (p,q) decomposition theorem:

Theorem 11 *Let X be a compact complex Kähler manifold. Then $H^r_{deR}(X,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X)$ with $H^{p,q}(X) = \overline{H^{q,p}}(X)$, and Dolbeault cohomology $H^{p,q}(X)$ is identified with the subspace of d-closed (p,q) forms. Under this decomposition $H^r_{deR}(X,\mathbb{R})$ corresponds to the real subspace of real valued forms in the RHS and $H^r_{deR}(X,\mathbb{C}) = H^r_{deR}(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.*

Corollary 12 *Let $V \in Z^r(X)$ then $[V] \in H^{r,r}(X)$.*

Now let us label the second map in the cycle class map

$$[\ ] : Z^r(X) \to H^{2r}_{sing}(X,\mathbb{Z}) \to H^{2r}_{deR}(X,\mathbb{C})$$

as $\lambda : H^{2r}_{sing}(X,\mathbb{Z}) \to H^{2r}_{deR}(X,\mathbb{C})$ and call $H^{r,r}(X,\mathbb{Z}) = \lambda^{-1}(H^{r,r}(X))$. Also by ignoring torsion we can identify $H^{2r}(X,\mathbb{Z})$ with its image in $H^{2r}(X,\mathbb{C})$ and then $[V] \in H^{r,r}(X,\mathbb{Z}) = H^{2r}(X,\mathbb{Z}) \cap H^{r,r}(X)$.
Then our cycle class map becomes

$$[\ ] : Z^r(X) \to H^{r,r}(X,\mathbb{Z})$$

One can also describe $H^{r,r}(X,\mathbb{Z})$ in terms of integration as

$$H^{r,r}(X,\mathbb{Z}) = \{\gamma \in H_{2n-2r}(X,\mathbb{Z}) | \int_\gamma w = 0, \forall w \in H^{p,q}(X), p+q = 2n-2r, p > n-r\}$$

Note that modulo torsion one has

$$H^{2r}(X,\mathbb{Z}) \cong \{w \in H^{2r}_{deR}(X,\mathbb{C}) | \int_\gamma w \in \mathbb{Z}, \forall \gamma \in H_{2r}(X,\mathbb{Z})\}$$

Hence every codimension r algebraic cycle corresponds to a (r,r)-form, with integral periods. What about the converse? Can we have any (r,r)-form with integral periods supported on codimension r algebraic cycles? This is the Hodge Conjecture which was introduced by William Hodge in 1950 at the International Congress of Mathematics:

Conjecture 13 *(Hodge) Let X be a projective algebraic manifold, $r \in \mathbb{Z}$. Then $[\ ] : Z^r(X) \longrightarrow H^{r,r}(X,\mathbb{Z})$ is surjective.*

This conjecture is a generalization of the Lefschetz's $(1,1)$ theorem which says the conjecture is true for the case $r = 1$:

Theorem 14 (Lefshetz (1,1) theorem) *Let X be a projective algebraic manifold. Then $H^2(X,\mathbb{Z})_{alg} = \underline{H}^{1,1}(X,\mathbb{Z})$, where $H^2(X,\mathbb{Z})_{alg} = [Z^1(X)]$= the subgroup of algebraic cocyles in $H^2(X,\mathbb{Z})$ and $\underline{H}^{1,1}(X,\mathbb{Z}) = \{\beta \in H_2(X,\mathbb{Z}) | \int_\beta w = 0, \forall$holomorphic 2-forms w on $X\}$.*

However, for $r > 1$ in 1962 Atiyah-Hirzebruch found non-algebraic torsion classes:

Theorem 15 *There exists a projective algebraic X for which $H^{2r}(X,\mathbb{Z})$ supports non-analytic torsion for certain $r > 1$*

This showed that the Hodge Conjecture is false for $r > 1$ with integer coefficients. But their example was a torsion class so there was a hope that the conjecture may hold for $H^{2r}(X,\mathbb{Z})$ which are torsion free. In 1990 Kollár (Trento) showed even for torsion-free $H^{2r}(X,\mathbb{Z})$ the conjecture is false. Hence the conjecture is revised with the rational coefficients:

Conjecture 16 $[\,]: Z^r(X)\otimes\mathbb{Q} \longrightarrow H^{r,r}(X,\mathbb{Q}) = H^{2r}(X,\mathbb{Q})\cap H^{r,r}(X)$ *is surjective for all $r \in \mathbb{Z}$.*

We denote by $\mathrm{Hodge}^{r,r}(X,\mathbb{Q})$ to mean the Hodge conjecture holds for codimension r algebraic cycles.

Some examples where the Hodge conjecture is known to hold:

Assume X is complex projective algebraic manifold and $\dim X = n$.

1. $\mathrm{Hodge}^{1,1}(X,\mathbb{Q})$ follows from the Lefschetz (1,1) theorem.
2. For $2p \leq n$ $\mathrm{Hodge}^{p,p}(X,\mathbb{Q}) \Rightarrow \mathrm{Hodge}^{n-p,n-p}(X,\mathbb{Q})$ by the Strong Lefschetz Theorem.
3. The Hodge Conjecture holds for any projective algebraic manifold X with $\dim X \leq 3$ by 1. and 2.

1.3.3 General Hodge Conjecture

For our purpose of proving Lewis' conjecture for smooth complex projective complete intersection we will need the statement of the General Hodge Conjecture. So we will explain the statement and some properties of the conjecture. For the statement we need the definition of Hodge filtrations and Hodge structures:

Definition 17

1. *Let $H_{\mathbb{R}}$ be a finite dimensional real vector space with a lattice $H_{\mathbb{Z}} \subset H_{\mathbb{R}}$, and let $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes \mathbb{C}$. Let $k \in \mathbb{Z}$. A (pure) Hodge structure of weight k is a decomposition $H_{\mathbb{C}} = \oplus_{p+q=k} H^{p,q}$ with $H^{p,q} = \overline{H^{q,p}}$(Hodge Symmetry).*
2. *$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$ are called Hodge numbers.*

Example 18 *Let X be any projetive algebraic manifold. Then $[H^k(X,\mathbb{Z}), H^k(X,\mathbb{C})]$ is a Hodge structure of weight k.*

Remark 19

1. *A variant of the above definition is obtained by replacing the lattice $H_{\mathbb{Z}}$ with a $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$.*
2. *Each Hodge structure of weight k has a decreasing filtration*
   $H_{\mathbb{C}} \supset \ldots \supset F^l \supset F^{l+1} \supset \ldots \supset \{0\}$, where $F^l H_{\mathbb{C}} = \oplus_{p+q=k, p\geq l} H^{p,q}$, satisfying:
   (i) $H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}}$ where $p+q=k$
   (ii) $H_{\mathbb{C}} = F^l H_{\mathbb{C}} \oplus \overline{F^{k-l+1} H_{\mathbb{C}}}$ for all l.

3. *Alternatively, one can define a Hodge structure of weight k as follows:* *A lattice $H_{\mathbb{Z}} \subset H_{\mathbb{R}}$ and a decreasing filtration $H_{\mathbb{C}} \supset \ldots \supset F^l \supset F^{l+1} \supset \ldots \supset \{0\}$ satisfying $H_{\mathbb{C}} = F^l \oplus \overline{F^{k-l+1}}$ for all l.*

Definition 20

1. *Let H, $H^{'}$ be two Hodge structures of weights $k, k^{'}$ respectively, such that $k^{'} = k + 2r$ for some integer r. A morphism of type (r, r) between Hodge structures H and $H^{'}$ is a rationally defined linear map $\psi : H_{\mathbb{C}} \longrightarrow H^{\prime}_{\mathbb{C}}$ satisfying either of the two equivalent conditions below:*

   (a) $\psi(F^p) \subset F^{'p+r}$

   (b) $\psi(H^{p,q}) \subset H^{'p+r,q+r}$ *for all p,q with $p + q = k$*

2. $\mathbf{H} \subset H$ *is called a sub-Hodge structure if* $\mathbf{H}$ *is a Hodge structure and the inclusion homomorphism is a morphism of the Hodge structures.*

Now we can state the motivation for the General Hodge Conjecture which comes from the following question:

- **Question:** Let X be a projective algebraic manifold of dimension n. Is it possible to describe the Hodge decomposition of $H^n(X, \mathbb{C})$ by the Hodge decomposition of $H^{n-2k}(Y, \mathbb{C})$ where Y is a codimension k subvariety of X? Furthermore is it possible to describe the sub-Hodge structures of $H^n(X, \mathbb{C})$ by looking at $H^{n-2k}(Y, \mathbb{C})$?

Remember that we started at the beginning with the idea that we want to figure out if we can get information about X from the smaller objects (subvarieties) inside X. So we consider a subvariety $Y \subset X$ and the difference $X - Y$. Also we are interested in the cohomology of X with coefficients in $\mathbb{Q}$. The statement of the General Hodge Conjecture is given as

$$GHC(p, l, X) : N^p H^l(X, \mathbb{Q}) = F_h^p H^l(X, \mathbb{Q})$$

where $F_h^p H^l(X, \mathbb{Q})$ = largest sub-Hodge structure in $\{F^p H^l(X, \mathbb{C})\} \cap \{H^l(X, \mathbb{Q})\}$ is called a rational Hodge filtration and

Proposition-Definition 21 *The coniveau filtration $\{N^p H^*(X, \mathbb{Q})\}_{p \geq 0}$ of $H^*(X, \mathbb{Q})$ is given by either of the two equivalent formulation below:*

1. $N^p H^l(X, \mathbb{Q}) = \{\gamma \in H^l(X, \mathbb{Q}) | \gamma \in \ker(i^*) : H^l(X, \mathbb{Q}) \longrightarrow H^l(X - Y, \mathbb{Q})$ *for some $Y \subset X$ of pure dimension $q \geq p$ in $X\}$.*
   [*Restatement: $N^p H^l(X, \mathbb{Q})$= those cohomology classes which vanish on the complement of a Zariski closed subset of pure codimension q, where $q \geq p$*].

2. $N^p H^l(X, \mathbb{Q}) = \{$ *Gysin images* $\sigma_* : H^{l-2q}(\tilde{Y}, \mathbb{Q}) \longrightarrow H^l(X, \mathbb{Q}) | codim_X Y = q$ *(pure) $q \geq p$ and $\tilde{Y} = desing(Y)\}$.*
   [*Restatement: $N^p H^l(X, \mathbb{Q})$= those (co)homology classes which are supported on an algebraic subset of pure codimension q, and where $q \geq p$*].

Some cases of GHC:

1. GHC$(k, 2k, X)$ is just the usual Hodge conjecture.

   - Assume we have $N^k H^{2k}(X,\mathbb{Q}) = F_h^k H^{2k}(X,\mathbb{Q})$. Then by definition $N^k H^{2k}(X,\mathbb{Q}) = \text{Gysin Image}\{\bigoplus_{\text{codim}Y=q\geq k} H^{2k-2q}(\tilde{Y},\mathbb{Q}) \longrightarrow H^{2k}(X,\mathbb{Q})\}$. Note that $q = k$ is the only possibility otherwise $2k-2q < 0$ which means $H^{2k-2q}(\tilde{Y},\mathbb{Q}) = 0$. So we get

     $$N^k H^{2k}(X,\mathbb{Q}) = \text{Gysin Image}\{\bigoplus_{\text{codim}Y=k} H^0(\tilde{Y},\mathbb{Q}) \longrightarrow H^{2k}(X,\mathbb{Q})\}$$

     We have $\bigoplus_{\text{codim}Y=k} H^0(\tilde{Y},\mathbb{Q}) \cong \bigoplus_{\text{codim}Y=k} H_{2n-2k}(\tilde{Y},\mathbb{Q}) = Z^k(X)\otimes\mathbb{Q}$. Hence $N^k H^{2k}(X,\mathbb{Q}) = [Z^k(X)\otimes\mathbb{Q}]$. Also $F_h^k H^{2k}(X,\mathbb{Q}) = H^{k,k}(X,\mathbb{Q})$ is a Hodge structure.
     Hence GHC$(k, 2k, X)$ =Hodge(k, k, X).

2. GHC$(m, 2m+1, X)$, i.e. $N^m H^{2m+1}(X,\mathbb{Q})$ = largest sub-Hodge structure in $\{H^{m+1,m}(X) \oplus H^{m,m+1}(X)\} \cap H^{2m+1}(X,\mathbb{Q})$.

3. GHC$(1, n, X)$ where $n = \dim X$ for the case X is a complete intersection with $P_g(X) = 0$. This says that $H^n(X,\mathbb{Q}) = N^1 H^n(X,\mathbb{Q})$[Lewis2, Lecture 13].

4. Let X, Y be two projective algebraic manifolds of same dimension n. If X and Y are birationally equivalent then GHC$(1, n, X)$ $\Leftrightarrow$ GHC$(1, n, Y)$.([Lewis2], Corollary 13.4)

5. Let X be a fourfold. Then GHC$(1, 4, X)$ $\Rightarrow$Hodge$(2, 2, X)$:

   - Assume we have GHC$(1, 4, X)$. That is $N^1 H^4(X,\mathbb{Q}) = F_h^1 H^4(X,\mathbb{Q})$.

     Note that $H^{2,2}(X,\mathbb{Q}) \subset F_h^1 H^4(X,\mathbb{Q})$. On the other hand

     $$N^1 H^4(X,\mathbb{Q}) := \text{Gysin Im}\{\bigoplus_{\text{codim}Y=r\geq 1} H^{4-2r}(\tilde{Y},\mathbb{Q}) \longrightarrow H^4(X,\mathbb{Q})\}$$

     $$= \{\bigoplus_{\text{codim}Y=1} H^2(\tilde{Y},\mathbb{Q}) \xrightarrow{\sigma_1} H^4(X,\mathbb{Q})\} \oplus \{\bigoplus_{\text{codim}Y=2} H^0(\tilde{Y},\mathbb{Q}) \xrightarrow{\sigma_2} H^4(X,\mathbb{Q})\}$$

     From here we have

     $$\bigoplus_{\text{codim}Y=1} H^{1,1}(\tilde{Y},\mathbb{Q}) \oplus \bigoplus_{\text{codim}Y=2} H^{0,0}(\tilde{Y},\mathbb{Q}) \overset{\sigma_1+\sigma_2}{\twoheadrightarrow} H^{2,2}(X,\mathbb{Q})$$

     and a commutative diagram:

$$
\begin{tikzcd}
\bigoplus_{\operatorname{codim} Y=1} Z^1(\tilde{Y})\otimes\mathbb{Q} \oplus \bigoplus_{\operatorname{codim} Y=2} Z^0(\tilde{Y})\otimes\mathbb{Q} \arrow[r] \arrow[d] & Z^2(X)\otimes\mathbb{Q} \arrow[d] \\
\bigoplus_{\operatorname{codim} Y=1} H^{1,1}(\tilde{Y},\mathbb{Q}) \oplus \bigoplus_{\operatorname{codim} Y=2} H^{0,0}(\tilde{Y},\mathbb{Q}) \arrow[r] & H^{2,2}(X,\mathbb{Q})
\end{tikzcd}
$$

By Lefschetz (1,1) theorem we have $Z^1(\tilde{Y})\otimes\mathbb{Q} \longrightarrow H^{1,1}(\tilde{Y},\mathbb{Q})$ is surjective, also $Z^0(\tilde{Y})\otimes\mathbb{Q} \simeq H^{0,0}(\tilde{Y},\mathbb{Q}) \simeq \mathbb{Q}\tilde{Y}$. Hence from the assumption $GHC(1,4,X)$ and Lefschetz's theorem we get $Z^2(X)\otimes\mathbb{Q} \twoheadrightarrow H^{2,2}(X,\mathbb{Q})$, that is Hodge$(2,2,X)$.

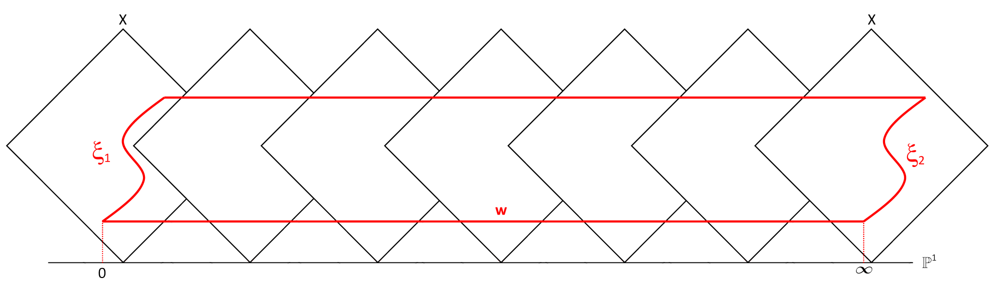
1.3.4 Abel-Jacobi map

In previous sections we examined the cycle class map $[\,]: Z^r(X)\otimes\mathbb{Q} \longrightarrow H^{2r}(X,\mathbb{Q})$ for a smooth projective algebraic manifold. We have the Hodge Conjecture which gives information about the image of this map. We want to examine this map further in order to get more information about $Z^r(X)$. Note that $Z^r(X)$ is a free abelian group which can be huge in general. Hence we want to introduce equivalence relations on it so that we can work on smaller objects, and also arrive at a ring structure.

Definition 22

1. *Let* $\xi_1, \xi_2 \in Z^r(X)$. ξ_1, ξ_2 *are* homologically equivalent *if* $\xi_1 - \xi_2 \in \ker([\,])$.

2. *Let* $\xi_1, \xi_2 \in Z^r(X)$. ξ_1, ξ_2 *are* rationally equivalent *if there exists a cycle* $w \in Z^r(\mathbb{P}^1 \times X)$ *in general position (i.e. pure* $\dim\{w \cap (\{t\}\times X)\} = n - r$*) such that* $\xi_1 - \xi_2 = w(0) - w(\infty)$ *where* $w(t) := Pr_{2,*}\{(w \cap \{t\} \times X)_{\mathbb{P}^1\times X}\}$.



3. *If we replace* $\mathbb{P}^1$ *by any irreducible curve* Γ *and* $\{0,\infty\}$ *by any two points* $P, Q \in \Gamma$*, we obtain another equivalence relation called* algebraic equivalence.

We will consider the following three subgroups of algebraic cycles (homologously, rationally, algebraically) equivalent to the zero cycle:

- $Z^r_{\text{hom}}(X) = \{\xi \in Z^r(X) | \xi \sim_{\text{hom}} 0\} = \ker([\,])$,
- $Z^r_{\text{alg}}(X) = \{\xi \in Z^r(X) | \xi \sim_{\text{alg}} 0\}$

- $Z^r_{\mathrm{rat}}(X) = \{\xi \in Z^r(X) | \xi \sim_{\mathrm{rat}} 0\}$.

Note that we have $Z^r_{\mathrm{rat}}(X) \subset Z^r_{\mathrm{alg}}(X) \subset Z^r_{\mathrm{hom}}(X)$.

As we mentioned we have the cycle class map $[\ ] : Z^r(X) \longrightarrow H^{2r}(X, \mathbb{Z})$ and a statement for the image of this map namely the Hodge Conjecture.
What about the kernel of this map? We defined above the kernel as $Z^r_{\mathrm{hom}}(X)$. To understand this group we will construct another map.
Now to define a map from $Z^r_{\mathrm{hom}}(X)$, take an element $\xi \in Z^r_{\mathrm{hom}}(X)$. This means $[\xi] = 0$ and

$$\{\xi\} \in H_{2n-2r}(X, \mathbb{Z}) = \frac{\{\ker \partial : C_{2n-2r}(X, \mathbb{Z}) \longrightarrow C_{2n-2r-1}(X, \mathbb{Z})\}}{\{\mathrm{Im}(\partial) : C_{2n-2r+1}(X, \mathbb{Z}) \longrightarrow C_{2n-2r}(X, \mathbb{Z})\}}$$

implying $\xi = \partial(\varphi)$ for some $\varphi \in C_{2n-2r+1}(X, \mathbb{Z})$. Then one can define integration of a form $w \in H^{2n-2r+1}(X, \mathbb{C})$ over φ. Hence our domain for the map will be $Z^r_{\mathrm{hom}}(X)$ and the codomain will be $H^{2r-1}(X, \mathbb{C})$ by the fact $(H^{2n-2r+1}(X, \mathbb{C}))^\vee \cong H^{2r-1}(X, \mathbb{C})$. To be able to define a well-defined map we will modify our codomain. We can write $H^{2r-1}(X, \mathbb{C})$ in terms of filtration as

$$H^{2r-1}(X, \mathbb{C}) = F^r H^{2r-1}(X, \mathbb{C}) \oplus \overline{F^r H^{2r-1}(X, \mathbb{C})}$$

By Poincaré and Kodaira-Serre dualities we have a perfect pairing:

$$H^{2r-1}(X, \mathbb{C})/F^r H^{2r-1}(X, \mathbb{C}) \times F^{n-r+1} H^{2n-2r+1}(X, \mathbb{C}) \longrightarrow \mathbb{C}$$

Hence

$$H^{2r-1}(X, \mathbb{C})/F^r H^{2r-1}(X, \mathbb{C})) \simeq (F^{n-r+1} H^{2n-2r+1}(X, \mathbb{C}))^\vee$$

By the following result of Dolbeault (see [Lewis3]):

Proposition 23 *$F^k H^l(X, \mathbb{C}) = \{\ker d : F^k E^l_X \longrightarrow F^k E^{l+1}_X\}/d(F^k E^{l-1}_X)$, where E^l_X is the vector space of C^∞ $\mathbb{C}$-valued l-forms.*

we get our map

$$\phi : Z^r_{\mathrm{hom}}(X) \longrightarrow H^{2r-1}(X, \mathbb{C})/(H^{2r-1}(X, \mathbb{Z}) \oplus F^r H^{2r-1}(X, \mathbb{C}))$$
$$\text{defined by } \xi = \partial(\varphi) \longrightarrow \phi(\xi) : \{w\} \in F^{n-r+1} H^{2n-2r+1}(X, \mathbb{C}) \longrightarrow \int_\varphi w$$

This map is well-defined as it is independent of the choice of w and φ. [Lewis2, 12.12] Independence of φ is quaranteed by taking the quotient by $H^{2r-1}(X, \mathbb{Z})$. This map also forms the well-known Abel-Jacobi map:

$$\phi : Z^r_{\mathrm{hom}}(X) \longrightarrow J^r(X)$$

Here the compact complex torus $J^r(X) = \overline{F^r H^{2r-1}(X, \mathbb{C})}/H^{2r-1}(X, \mathbb{Z})$, is called the Griffiths' intermediate Jacobian. Note that

$$\overline{F^r H^{2r-1}(X, \mathbb{C})}/H^{2r-1}(X, \mathbb{Z}) \cong H^{2r-1}(X, \mathbb{C})/(H^{2r-1}(X, \mathbb{Z}) \oplus F^r H^{2r-1}(X, \mathbb{C}))$$

1.3.5 Chow Groups

We defined three equivalence relations on the algebraic cycles of X. By taking quotients with respect to those equivalence relations we will get smaller groups, namely Chow groups, which we will define below.

Definition 24

- *The quotient of the group of codimension r algebraic cycles by the group of codimension r algebraic cycles that are rationally equivalent to zero is called the r^{th}*Chow group *denoted by $CH^r(X) := Z^r(X)/Z^r_{rat}(X)$.*
- $CH^r_{hom}(X) = Z^r_{hom}(X)/Z^r_{rat}(X)$
- $CH^r_{alg}(X) = A^r(X) = Z^r_{alg}(X)/Z^r_{rat}(X)$ *is called the continuous part of* $CH^r(X)$.

Chow groups are built directly from the algebraic subvarieties and we will see the relation to (co)homology groups via the cycle class map. In fact calculating Chow groups for arbitrary varieties would amount to solving the Hodge conjecture and other conjectures related to motives.

Example 25

- $CH^{n-r}(\mathbb{P}^n) = CH_r(\mathbb{P}^n) = \mathbb{Z}\{\mathbb{P}^r\}$
- $CH^1(X) \cong Pic(X)$
- $CH^1_{hom}(X) = CH^1_{alg}(X) \cong Pic^0(X) \cong J^1(X)$

Recall the cyle class map $[\ ] : Z^r(X) \longrightarrow H^{r,r}(X, \mathbb{Z})$. Since $[Z^r_{\text{rat}}(X)] = 0$ this induces a map on Chow groups $[\ ] : CH^r(X) \longrightarrow H^{r,r}(X, \mathbb{Z})$. Hence we have the Hodge conjecture on the level of Chow groups as well:

Conjecture 26 *(Hodge) The cycle class map $[\ ] : CH^r(X) \otimes \mathbb{Q} \longrightarrow H^{r,r}(X, \mathbb{Q})$ is surjective.*

Also we have the Abel-Jacobi map for Chow groups, induced from the Abel-Jacobi map defined above, which is also called the Abel-Jacobi map

$$\phi : CH^r_{\text{hom}}(X) \longrightarrow J^r(X)$$

This induces the map:

$$\phi : A^r(X) \longrightarrow J^r(X)$$

In general, except when $r = 1$, ϕ is neither injective nor surjective. The image of this map $\phi(A^r(X)) = J^r_{\text{alg}}(X)$ is called the r^{th} Lieberman Jacobian which is an abelian variety. This follows from a proposition [Lewis2,12.23]:

Proposition 27 *Let A be a complex torus and let $\lambda : A^r(X) \longrightarrow A$ be an analytic map. Then Im(λ) is an Abelian subvariety of A.*

In fact we know the description of $J^r_{\text{alg}}(X)$ explicitly as

$$J^r_{alg}(X) = \frac{[N^{r-1}H^{2r-1}(X, \mathbb{Q})] \otimes \mathbb{C}}{F^r(\text{numerator}) \oplus (H^{2r-1}(X, \mathbb{Z}) \cap (\text{numerator}))}$$

This comes from the observation $N^{r-1}H^{2r-1}(X, \mathbb{Q}) = \{[w]_*(H^1(\Gamma)) | \Gamma \text{ curve }, w \in CH^r(\Gamma \times X)\}$.

Chapter 2

Representability

2.1 Representability and Lewis Conjecture

In this section we will explore the representability of algebraic Chow groups of any codimension. For that purpose with the guidance of [Lewis1] we will explore the Lewis conjecture, a generalization of a conjecture of Bloch, for smooth complex projective complete intersections.

Conjecture 28 *(Lewis) Let X be a projective algebraic manifold then*

$$A^*(X) \cong J^*_{alg}(X) \Leftrightarrow Level(H^*(X)) \leq 1$$

where $A^*(X) = \bigoplus_{l=1}^{n} A^l(X), J^*_{alg}(X) = \bigoplus_{l=1}^{n} J^l_{alg}(X)$, $J^l_{alg}(X)$ is the l^{th} Lieberman Jacobian, $Level(H^*(X)) = \max_l (level(H^l(X))) = \max_l \{|p-q||H^{p,q}(X) \neq 0, p+q=l\}$.

Example 29

1. *Let E be a compact complex Riemann surface with genus g. Then $H^0(E) \cong \mathbb{Q}, H^1(E) = H^{0,1}(E) \oplus H^{1,0}(E)$ with $H^{0,1}(E) \cong H^{1,0}(E) \cong \mathbb{Q}^g \neq 0 \leftrightarrow g \geq 1$, hence $level(H^1(E)) \leq 1$. $H^2(E) = H^{0,2}(E) \oplus H^{1,1}(E) \oplus H^{2,0}(E)$ with $H^{0,2}(E) = \overline{H^{2,0}(E)} = 0$ and $H^{1,1}(E) \cong \mathbb{Q}$ hence $level(H^2(X)) = 0$.*
   So we have $Level(H^*(E)) \leq 1$ with equality if and only if $g \geq 1$.
   On the other hand $A^*(E) = A^1(E)$ and $J^*(E) = J^1(E)$. By the Abel-Jacobi map we get $A^*(E) \cong J^*(E) = J^*_{alg}(E)$

2. *Let X be a smooth complex projective surface. Then Lewis conjecture is the same as the*

   - *Bloch conjecture: Let X be a nonsingular surface with $P_g(X) = 0$. Then $A_0(X)$ is finite dimensional.*

   Note that $P_g > 0$, i.e. $H^{0,2}(X) \neq 0$, if and only if Level($H^*(E)$)=2. [$H^{3,0}(X) = H^{0,3}(X) = H^{4,0}(X) = H^{0,4}(X) = H^{1,3}(X) = H^{3,1}(X) = 0$ due to dimension].

Observation on the cohomology of complete intersections

Let $X \subset \mathbb{P}^N$ be a complete intersection of dimension n given by r homogeneous polynomials, this means X is obtained by taking exactly r hypersurface sections of $\mathbb{P}^N$, i.e. $X = V(f_1, \ldots, f_r)$. By the weak Lefschetz theorem for $i < n$ we have:

$$H^i(\mathbb{P}^N, \mathbb{Z}) \overset{\cong}{\to} H^i(X, \mathbb{Z})$$

Recall the cohomology groups of $\mathbb{P}^N$;

$$H^i(\mathbb{P}^N, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } 0 \leq i \leq 2N \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Now by the strong Lefschetz theorem: $H^{n-i}(X, \mathbb{Q}) \cong H^{n+i}(X, \mathbb{Q}), 0 \leq i \leq n$, hence the only 'nontrivial' cohomology of X is $H^n(X, \mathbb{Q})$.
Hence for $X \subset \mathbb{P}^{n+r}$ a smooth complete intersection of type $(d_1, \ldots, d_r)$ with $\dim X = n$ we have $\text{Level}(H^*(X)) = \text{level}(H^n(X))$ provided $H^n(X) \neq 0$. In fact we know the value of the level from [SGA7] where the Hodge level of $H^n(X)$ is given by $n - 2k$ with

$$k = [(n - \sum_{s \neq i}(d_i - 1) + 1)/d_s]$$

and $d_s = \max\{d_1, \ldots, d_r\}$. Note that this means we have $H^n(X, \mathbb{C}) = F^k H^n(X, \mathbb{C})$. With this observation on the cohomology of smooth complete intersections we can show one side of the Lewis Conjecture. For that purpose we need couple more definitions and results from [Lewis2].

Definition 30 *Let p, q be integers ≥ 0 such that $p + q$ is even. A linear map $\lambda : H^p(X, \mathbb{Q}) \longrightarrow H^q(Y, \mathbb{Q})$ is said to be algebraic if it is induced by an algebraic cycle $\xi \in Z^{(2n-p+q)/2}(X \times Y) \otimes \mathbb{Q}$. [Lewis2, 7.4]*

- Under the assumption of the Hodge Conjecture, λ is algebraic $\Rightarrow$ λ is a morphism of Hodge structures of type $(1/2(q-p), 1/2(q-p))$.

Conjecture 31 *(Standard Lefschetz Conjecture) Let X be a projective algebraic manifold of dimension n. Assume $0 \leq i \leq n$ and let L_X be the operator of taking cup product with the hyperplane class on X. Recall the strong Lefschetz theorem*

$$L_X^{n-i} : H^i(X, \mathbb{Q}) \overset{\cong}{\to} H^{2n-i}(X, \mathbb{Q})$$

Then for all i satisfying $0 \leq i \leq n$

$$(B(X)) \text{ The inverse } \Lambda_X^{n-i} : H^{2n-i}(X, \mathbb{Q}) \overset{\cong}{\to} H^i(X, \mathbb{Q})$$

to L_X^{n-i} is algebraic.

In fact $B(X)$ is equivalent to saying that Λ_X^{n-i} is induced by an algebraic cycle $\xi \in Z^i(X \times X) \otimes \mathbb{Q}$. $B(X)$ is known to hold for

1. For all X with $\dim X \leq 2$ by Lefschetz's (1,1) theorem

2. Flag manifolds X, using $H^*(X, \mathbb{Q})$ is generated by algebraic cocyles
3. Abelian varieties due to D. Lieberman
4. Complete Intersections by Lefschetz's theorems

Let's examine the complete intersection case. Let $X \subset \mathbb{P}^N$ be a complete intersection given by r homogeneous polynomials, this means X is obtained by taking exactly r hypersurface sections of $\mathbb{P}^N$. Hence by the weak Lefschetz theorem for $i < n$ we have:

$$H^i(\mathbb{P}^N, \mathbb{Z}) \xrightarrow{\cong} H^i(X, \mathbb{Z})$$

Then considering L_X^{n-i} for $i = n$ we see that $L_X^{n-i} = L_X^0$ is the identity map induced by the class $\Delta_X \in Z^n(X \times X)$, and $\Lambda_X^0 = L_X^0$.

Definition 32

1. *$[N^{r-\ell}H^{2r-\ell}(X, \mathbb{Q}) \otimes \mathbb{C}]^{r,r-\ell} = Pr_{r,r-\ell}(N^{r-\ell}H^{2r-\ell}(X, \mathbb{Q}) \otimes \mathbb{C})$ where $Pr_{r,r-\ell} : H^{2r-\ell}(X, \mathbb{C}) \longrightarrow H^{r,r-\ell}(X)$ is the projection map.*

$$\Rightarrow [N^{r-\ell}H^{2r-\ell}(X, \mathbb{Q}) \otimes \mathbb{C}]^{r,r-\ell} \subset H^{r,r-\ell}(X)$$

2. *$N^{r-\ell}H^{2r-\ell}(X, \mathbb{Q}) \otimes \mathbb{C} = [N^{r-\ell}H^{2r-\ell}(X, \mathbb{Q}) \otimes \mathbb{C}]^{r-\ell,r} \oplus [N^{r-\ell}H^{2r-\ell}(X, \mathbb{Q}) \otimes \mathbb{C}]^{r-\ell+1,r-1} \oplus \ldots \oplus [N^{r-\ell}H^{2r-\ell}(X, \mathbb{Q}) \otimes \mathbb{C}]^{r,r-\ell}$*

Remark 33 *level($N^{\bullet}H^*(X)$) $\leq$ level($H^*(X)$), where equality holds if GHC holds for X.*

Definition 34 *$A^*(X)$ is said to be finite dimensional if $\exists$ a (possibly reducible) smooth curve Γ and a cycle $w \in CH^r(\Gamma \times X)$ such that $w_* : A^1(\Gamma) \longrightarrow A^r(X)$ is surjective.*

That is every cycle in $A^r(X)$ comes from a cycle in $A^1(\Gamma) = A_0(\Gamma)$= degree zero points. Now we have

Lemma 35 *$A^r(X) \simeq J^r_{alg}(X) \Rightarrow A^r(X)$ finite dimensional.*

Our next step is to consider Theorem (15.36) from [Lewis2] that links $[N^{r-\ell}H^{2r-\ell}(X, \mathbb{Q}) \otimes \mathbb{C}]^{r,r-\ell}$ to the structure of $A^*(X)$.

Theorem 36 *Assume the standard Lefschetz conjecture $B(*)$. If $[N^{r-\ell}H^{2r-\ell}(X, \mathbb{Q}) \otimes \mathbb{C}]^{r,r-\ell} \neq 0$ for some $\ell \geq 2$, then $A^r(X)$ is infinite dimensional.*

For our purpose we will state a version of Corollary 15.42 from [Lewis2]:

Corollary 37 *Let $X = V(f_1, \ldots, f_r) \subset \mathbb{P}^{n+r}$ be a projective smooth complete intersection of dimension n. Assume $B(*)$ and the GHC(k, n, X) holds where $k = [(n - \sum_{s \neq i}(d_i - 1) + 1)/d_s]$ and $d_s = \max\{d_1, \ldots, d_r\}$. Then*

$$A^*(X) \cong J^*_{alg}(X) \Rightarrow \mathit{Level}(H^*(X)) \leq 1$$

Proof.
The assumption GHC(k,n,X) holds implies Level$(N^{n-2k}H^n(X,\mathbb{Q})\otimes\mathbb{C})$ =Level$(H^n(X))$ =Level$(H^*(X))$. Now assume to the contrary that Level$(H^*(X)) \geq 2$, i.e. $n-2k \geq 2$. Let $\ell = n-2k$. Then r in Theorem **??** becomes k. By Theorem **??** $A^{n-k}(X)$ is infinite dimensional, which means $A^*(X) \ncong J^*_{\text{alg}}(X)$. ■

Hence in order to show Lewis Conjecture for complete intersections we need to show that the special version of GHC holds for them. In the following section we will find a numerical condition that will guarantee that special version of GHC holds for smooth projective complete intersections.

2.2 A Special Case of General Hodge Conjecture for Complete Intersections

In this section we will prove one step towards Lewis' conjecture for certain complete intersections. This will be done by showing GHC(k,n,X) for $X \subset \mathbb{P}^{n+r}$ a smooth complete intersection of type $(d_1,\ldots,d_r)$ with dimension n and then using Corollary **??**. For that purpose we will study the cylinder homomorphism of the Fano variety of $\mathbb{P}^k$'s, k-planes in X, where k is related to the Hodge level of $H^n(X)$ described in the previous section. We will find a numerical condition depending on the dimension and degree of X to guarantee a sufficiently large family of $\mathbb{P}^k$'s lying on X. This condition then will be used to show the surjectivity of the map from the cohomology of the Fano variety to the cohomology of X, induced by the cylinder homomorphism and this surjectivity will imply GHC(k,n,X) for X.

Definition 38 *$G(k+1,N+1) = \{k+1-$ dimensional subspaces $\mathbb{C}^{k+1} \subset \mathbb{C}^{N+1}\} = \{k-$ dimensional subspaces $\mathbb{P}^k \subset \mathbb{P}^N\}$ is a Grassmannian space with dimension $(k+1)(N+1-(k+1)) = (k+1)(N-k)$.*

Definition 39 *Let $X \subset \mathbb{P}^N$ be a variety. Then $\Omega_X(k) = \{\mathbb{P}^k$'s in $X\} \subset G(k+1,N+1)$ is called the Fano variety of $\mathbb{P}^k$'s in X.*

We have the following result for $\Omega_X(k)$:

Theorem 40 (Bo) *Let $X \subset \mathbb{P}^{n+r}$ be a generic complete intersection of type $(d_1,\ldots,d_r)$. Then $\Omega_X(k)$ is non-empty and smooth of (pure) dimension $\delta = (k+1)(n+r-k) - \sum_{j=1}^{r}\binom{d_j+k}{k}$, provided $\delta \geq 0$ and X is not quadric. In the case X a quadric, we require $n \geq 2k$. Furthermore, if $\delta > 0$ or if in the case X quadric, $n-2k>0$, then $\Omega_X(k)$ is connected (hence irreducible).*

Idea of the formula: We have $\Omega_X(k) \subset G(k+1,n+r+1)$ and $\dim G(k+1,n+r+1) = (k+1)(n+r+1-(k+1)) = (k+1)(n+r-k)$. Then consider the vector bundle $E = O_{\mathbb{P}^{n+r}}(d_1) \oplus \ldots \oplus O_{\mathbb{P}^{n+r}}(d_r)$ over $\mathbb{P}^{n+r}$ where $O_{\mathbb{P}^{n+r}}(d_i)$ means the d_i^{th} power of the hyperplane line bundle over $\mathbb{P}^{n+r}$. Then $X = V(s)$ for a section $s \in \Gamma(E)$. Now consider $P(\mathbb{P}^{n+r}) = \{(p,l) | p \in l\} \subset \mathbb{P}^{n+r} \times G(k+1,n+r+1)$ with projective maps $\pi_1 : P(\mathbb{P}^{n+r}) \longrightarrow \mathbb{P}^{n+r}$, a $G(k,n+r)$ bundle over $\mathbb{P}^{n+r}$,and $\pi_2 : P(\mathbb{P}^{n+r}) \longrightarrow G(k+1,n+r+1)$, a $\mathbb{P}^k$ bundle over $G(k+1,n+r+1)$. Then we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{P}^{n+r} & P(\mathbb{P}^{n+r}) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & G(k+1,n+r+1) \\
E \arrow[u] & \pi_1^*(E) \arrow[l, "\pi_{1,*}"'] \arrow[r, "\pi_{2,*}"] \arrow[u] & \pi_{2,*}\pi_1^*(E) = \epsilon \arrow[u]
\end{tikzcd}$$

So we have $\Omega_X(k) = V(\tilde{s})$ where $\tilde{s} \in \Gamma(\epsilon)$, $\tilde{s} = \pi_{2,*}\pi_1^*(s)$ and $\text{rank}(\epsilon) = \sum_{j=1}^{r}\binom{d_j+k}{k}$.

Example 41

1. *Let $X \subset \mathbb{P}^3$ be a smooth cubic surface. That is $X = V(f)$, where f is a cubic homogenous polynomial with a Jacobian of rank 1. Then* $\dim X = 2$ *and* $\deg X = 3$. *Lets calculate δ for $k = 1, n = 2, r = 1$ and $d = 3$: $\delta = (1+1)(3-1) - \binom{3+1}{1} = 4 - 4 = 0$. By Theorem* **??**, *$\Omega_X(1)$ is non-empty smooth of dimension 0, implying $\Omega_X(1)$ consists of points. In fact, it is a well-known result that there are 27 lines on a cubic smooth surface on $\mathbb{P}^3$. Hence $\Omega_X(1)$ consists of 27 points.*

2. *Let $X \subset \mathbb{P}^5$ be a generic quintic fourfold. That is $n = \dim X = 4, d = \deg X = 5, r = 1$. Lets calculate δ for $k = 1$. We have $\delta = (1+1)(5-1) - \binom{5+1}{1} = 8 - 6 = 2$. Also $n - 2k = 4 - 2 > 0$ so by Theorem* **??**, *$\Omega_X(1)$ is smooth irreducible of dimension 2.*

Next we will define the cylinder correspondence and cylinder homomorphism which will be used to get a surjective map between the (co)homologies of $\Omega_X(k)$ and X.

Definition 42 *$P(X) = \{(c,p) \in \Omega_X(k) \times X | p \in \mathbb{P}^k_c\}$ is called the cylinder correspondence and the cylinder homomorphism map ϕ_* is induced by the intersection with $P(X)$; $\phi_* : H_{n-2k}(\Omega_X(k), \mathbb{Q}) \longrightarrow H_n(X, \mathbb{Q})$ given by $\phi_*(\gamma) = Pr_{2,*}((\gamma \times X \cap P(X))_{\Omega_X(k) \times X})$.*

Example 43 *Let $X \subset \mathbb{P}^6$ be a smooth complete intersection cut out by two quadric polynomials. That is* $\dim X = 4, r = 2, (d_1, d_2) = (2,2) \Rightarrow k = [4 - 2 + 2/2] = 2$. *Consider $\Omega_X(2) = \{\mathbb{P}^2\text{'s} \subset X\}$. By Theorem* **??** *we have* $\dim \Omega_X(2) = (2+1)(4+2-2) - \binom{2+2}{2} - \binom{2+2}{2} = 0 \Rightarrow$ *there are finitely many $\mathbb{P}^2$'s in X.*

$$H_0(\Omega_X(2), \mathbb{Q}) \longrightarrow H_0(\Omega_X(2) \times X, \mathbb{Q}) \longrightarrow H_2(\Omega_X(2) \times X, \mathbb{Q})$$

$$\begin{tikzcd}
{[c]} \arrow[r] \arrow[drr, "\phi_*"] & {[\{c\} \times X]} \arrow[r, "\cap P(X)"] & {[\{c\} \times \{p \in X | p \in \mathbb{P}^2_c\}]} \arrow[d, "Projection"] \\
& & {[\mathbb{P}^2_c] \in H_4(X, \mathbb{Q})}
\end{tikzcd}$$

Another way to see that we get a map $\phi_* : H_{n-2k}(\Omega_X(k), \mathbb{Q}) \longrightarrow H_n(X, \mathbb{Q})$ is to consider $P(X)$ and the surjective map $\pi : P(X) \longrightarrow \Omega_X(k)$ with fibers $\mathbb{P}^k$. Then $\dim P(X) = \dim \Omega_X(k) + \dim \text{fibers} \Rightarrow \dim P(X) = \delta + k$, $\dim \Omega_X(k) \times X = \delta + n \Rightarrow \text{codim}_{\Omega_X(k) \times X}(P(X)) = n - k \Rightarrow P(X) \in Z^{n-k}(\Omega_X(k) \times X)$. By proposition 7.4 of [Lewis2] $P(X)$ induces a morphism of Hodge structures of type $(n-k-\delta, n-k-\delta)$, $\rho_{P(X)} : H^{2\delta+2k-n}(\Omega_X(k), \mathbb{Q}) \longrightarrow H^n(X, \mathbb{Q})$. By Poincaré duality we have our map ϕ_*.

Now let's consider $H^{2\delta+2k-n}(\Omega_X(k), \mathbb{C})$. We have the Hodge decomposition

$$H^{2\delta+2k-n}(\Omega_X(k), \mathbb{C}) = H^{0,2\delta-(n-2k)}(\Omega_X(k)) \oplus \ldots \oplus H^{2\delta-(n-2k),0}(\Omega_X(k))$$

and under $\rho_{P(X)}$, denote this by ϕ_* as well, this is mapped into

$$H^{n-\delta-k,\delta+k}(X) \oplus \ldots \oplus H^{\delta+k,n-\delta-k}(X) \subset H^n(X,\mathbb{C})$$

Recall in [SGA7] the Hodge level of $H^n(X)$ is given by $n-2k$ where $k = [(n - \sum_{s\neq i}(d_i - 1) + 1)/d_s]$ and $d_s = \max\{d_1, \ldots, d_r\}$. Also we can assume $n - 2k \geq 0$, since otherwise $H^n(X,\mathbb{C}) = 0$. In summary ϕ_* to be surjective implies

$$\dim \Omega_X(k) = \delta \geq n - 2k. \tag{2.2.1}$$

So far we found a necessary condition for the surjectivity of the cylinder map ϕ_*. For finding the sufficent condition and actually showing it is surjective we will use another smooth complete intersection $Z \subset \mathbb{P}^{n+1+r}$ of dimension $n+1$ such that $X = Z \cap \mathbb{P}^{n+r}$. We will show that Z is covered by $\mathbb{P}^k$'s by showing through every point there passes a family of $\mathbb{P}^k$'s. The reason to use such a Z is that we may not be able to cover X by $\mathbb{P}^k$'s. That is there may be some points in X such that there is no $\mathbb{P}^k$'s passing through that point.

Example 44 *Let $X = V(F) \subset \mathbb{P}^5$ be a general quintic fourfold, i.e.* $\dim X = 4, \deg X = 5$. *We have $k = [5/5] = 1$, level $H^4(X,\mathbb{C}) = n - 2k = 2$ and* $\dim \Omega_X(1) = (2)(5-1) - \left(\begin{array}{c} 5+1 \\ 1 \end{array}\right) =$ *8-6=2. So X contains $\mathbb{P}^1$'s parametrized by a 2-dimensional manifold. Hence the family of lines in X sweeps out a threefold in X. Thus through a generic point $p \in X$ we may not find a family of $\mathbb{P}^1$'s passing through p.*
Alternatively, by a PGL action we can assume $p = [1,0,0,0,0,0] \in X \subset \mathbb{P}^5$. Then $F = z_0^4 F_1 + \ldots + F_5$ where F_i are homogenous polynomials of degree i. Lets consider the affinization of F by considering the affine coordinates $(x_1, \ldots, x_5) = (z_1/z_0, \ldots, z_5/z_0)$. We have $f = F/z_0^5 = f_1 + \ldots + f_5$, where $f_i = z_0^{5-i} F_i / z_0^5$ homogenous of degree i. Note that $p \in X \subset \mathbb{P}^5$ corresponds to $(0, \ldots, 0) \subset \mathbb{C}^5$. So any $\mathbb{P}^1 \subset \mathbb{P}^5$ containing p corresponds to $\mathbb{C}^1 \subset \mathbb{C}^5$ in affine coordinates which contains $(0, \ldots, 0)$. So we have $\Omega_X(1)_p = \{\mathbb{P}^1 \subset \mathbb{P}^5 | p \in \mathbb{P}^1 \subset V(F)\} \cong \{\mathbb{C}^1 \subset \mathbb{C}^5 | 0 \in \mathbb{C}^1 \subset V(f)\}$. Now let $\mathbb{C}^1 \subset \mathbb{C}^5$ generated by $\{v\}$. Then $\mathbb{C}^1 \subset V(f) \Leftrightarrow f(tv) = 0, \forall t \in \mathbb{C} \Leftrightarrow f_i(tv) = 0, \forall t \in \mathbb{C}, \forall i = 1, \ldots, 5$. Hence $\Omega_X(1)_p \cong \{\mathbb{P}^0 \subset \mathbb{P}^4 | \mathbb{P}^0 \subset V(f_1, \ldots, f_5)\}$. For general F, $\dim \Omega_X(1)_p = 4 - 5 < 0$. This again tells us that through a general point p on a general X we do not have a $\mathbb{P}^1$.
Alternatively, if we do the calculations for $Z \subset \mathbb{P}^6$ with $\dim Z = 5$ *and* $\deg Z = 5$ *then we have* $\dim \Omega_Z(1)_p \geq 0$, *that is through every point of Z there is at least a finite number of $\mathbb{P}^1$'s, and this will be the case for a general such point.*

From now on $X \subset \mathbb{P}^{n+r}$ will be a generic smooth complete intersection of type $(d_1, \ldots, d_r)$ and $Z \subset \mathbb{P}^{n+1+r}$ will be a generic smooth complete intersection of type $(d_1, \ldots, d_r)$ such that $X = Z \cap \mathbb{P}^{n+r}$.

Proposition 45 *Let $Z \subset \mathbb{P}^{n+1+r}$ be a generic complete intersection of type $(d_1, \ldots, d_r)$ and assume $\ell \geq 0$ where $\ell = k(n+1+r-k) + r - \sum_{j=1}^{r} \binom{d_j+k}{k}$. Then through every point of Z there passes a $\mathbb{P}^k \subset Z$. In particular, there is an ℓ-dimensional family of $\mathbb{P}^k$'s in a general Z passing through a generic point $p \in Z$. Finally we have $\ell \geq 0 \Leftrightarrow$ condition (2.2.1) holds for $X = Z \cap \mathbb{P}^{n+r}$.*

Proof. The proof is similar to that in [Lewis1], but generalized to complete intersections.
Let $Z \subset \mathbb{P}^{n+1+r}$ a complete intersection of type $(d_1, \ldots, d_r)$, i.e. $Z = V(F_1, \ldots, F_r)$ where F_i are homogenous polynomials of degree d_i in coordinates of $\mathbb{P}^{n+1+r}$. Let $[z_0, \ldots, z_{n+1+r}]$ be homogenous coordinates of $\mathbb{P}^{n+1+r}$. Let p be a generic point in Z. By a PGL action we can assume $p = [1, 0, \ldots, 0] \in Z \subset \mathbb{P}^{n+1+r}$. Then $F_i = \sum_{j=1}^{d_i} z_0^{d_i - j} F_j^i(z_1, \ldots, z_{n+r})$, where F_j^i are homogenous of degree j. Now consider the affine coordinates $(x_1, \ldots, x_{n+1+r}) = (z_1/z_0, \ldots, z_{n+1+r}/z_0)$, and note that p corresponds to $(0, \ldots, 0)$ in these coordinates. So any $\mathbb{P}^k \subset \mathbb{P}^{n+1+r}$ containing p corresponds in affine coordinates to $\mathbb{C}^k \subset \mathbb{C}^{n+1+k}$ with $(0, \ldots, 0) \in \mathbb{C}^k$.
Now write $f_i(x_1, \ldots, x_n) = F_i/z_0^{d_i} = f_i^1 + \ldots + f_i^{d_i}$, where f_i^α's are homogenous polynomials of degree $\alpha = 1, \ldots, d_i$. Let $\mathbb{C}^k \subset \mathbb{C}^{n+1+r}$ be the subspace generated by $\{v_1, \ldots, v_k\}$.
Then $\mathbb{C}^k \subset V(f_1, \ldots, f_r) \Leftrightarrow f_i(\sum_{j=1}^k t_j v_j) = 0, \forall (t_1, \ldots, t_k) \in \mathbb{C}^k, i = 1, \ldots, r.$

$\Leftrightarrow f_i^\alpha(\sum_{j=1}^k t_j v_j) = 0, \forall (t_1, \ldots, t_k) \in \mathbb{C}^k, i = 1, \ldots, r, \alpha = 1, \ldots, d_i$. Hence $\Omega_{V(F_1, \ldots, F_r), p} := \{\mathbb{P}^k \subset \mathbb{P}^{n+1+r} | p \in \mathbb{P}^k \subset V(F_1, \ldots, F_r)\}$

$$\cong \{\mathbb{P}^{k-1} \subset \mathbb{P}^{n+r} | \mathbb{P}^{k-1} \subset V(f_1^1, \ldots, f_1^{d_1}, \ldots, f_r^1, \ldots, f_r^{d_r}) \subset \mathbb{P}^{n+r}\}.$$

By genericity of Z and a deformation argument, we can reduce to the case where $V(f_1^1, \ldots, f_1^{d_1}, \ldots, f_r^1, \ldots, f_r^{d_r})$ defines a complete intersection of dimension $(n+r) - \sum_{j=1}^r d_j$ (i.e. we can choose $f_1^1, \ldots, f_1^{d_1}, \ldots, f_r^1, \ldots, f_r^{d_r}$ to set a complete intersection of dimension $(n+r) - \sum_{j=1}^r d_j$ and then construct $F_1, \ldots, F_r$ accordingly). By Theorem **??** $\Omega_{V(F_1, \ldots, F_r), p}$ is non-empty and has dimension $(k-1+1)(n+r-(k-1)) - \sum_{i=1}^r \sum_{j=1}^{d_i} \binom{j+k-1}{k-1}$.
Observe that, $\sum_{j=1}^d \binom{j+k-1}{k-1} + 1 = \dim\{$vector space of polynomials $g(y_1, \ldots, y_k) | \deg(g) \leq d\} = \dim\{$vector space of homogenous polynomials $h(x_1, \ldots, x_k) | \deg(h) = d\} = \binom{d+k}{k}$.

Hence we have $\sum_{j=1}^{d_i} \binom{j+k-1}{k-1} = \binom{d_i+k}{k} - 1 \Rightarrow \dim \Omega_{V(F_1, \ldots, F_r), p} = \ell$. Hence if $\ell \geq 0$ then through every point there passes a $\mathbb{P}^k$ and in fact there is a ℓ-dimensional family of $\mathbb{P}^k$'s in Z passing through a generic point $p \in Z$.
Lastly, observe that $\delta = \ell + (n - 2k)$. Recall the condition (2.2.1) which says that $\delta \geq n - 2k$. Hence condition (2.2.1) holds $\Leftrightarrow \ell \geq 0$. ■

Corollary 46 *Given Z in Proposition* **??**. *If the condition (2.2.1) holds, then Z is covered by a family of $\mathbb{P}^k$'s; moreover, for a general Z and a generic point $p \in Z$, there passes an ℓ-dimensional family of $\mathbb{P}^k$'s.*

Proof. If the condition (2.2.1) holds then we have $\ell \geq 0$ by proposition **??** and again by proposition **??** we have through every $p \in Z$ there passes a $\mathbb{P}^k_p$ implying $Z = \cup_{p \in Z} \mathbb{P}^k_p$. ■

We would like to see the relation between the homology of X and the homology of $\Omega_X(k)$ the Fano variety of $\mathbb{P}^k$'s in X. We already mentioned that cylinder homomorphisms induces a map $\phi_* : H_{n-2k}(\Omega_X(k), \mathbb{Q}) \longrightarrow H_n(X, \mathbb{Q})$ which has the possibility to be surjective provided condition (2.2.1) holds. We will show that condition (2.2.1) is sufficient to guarantee ϕ_* to be surjective. To do this we will get help from Z and $\Omega_Z(k)$. Note that we can view X as a hyperplane section of Z with the inclusion $j : X \hookrightarrow Z$. Then by Theorem **??** and Corollary **??** we have:

(i) $\Omega_X(k)$ is non-empty, smooth and of pure dimension $(n-2k)+\ell = \delta$, where $\ell \geq 0$ is given in Proposition **??**.

(ii) $\Omega_Z(k)$ is non-empty, smooth and of pure dimension $(n+1-k)+\ell$. In fact $\Omega_Z(k)$ is irreducible by Theorem **??** but we don't need this.

(iii) Through a generic point of Z there passes an ℓ-dimensional family of $\mathbb{P}^k$'s.

Justifications:

(i) By Theorem **??** $\dim(\Omega_X(k)) = \delta = (k+1)(n+r-k) - \sum_{j=1}^{r} \binom{d_j+k}{k}$, and by Proposition **??** $\ell = k(n+1+r-k) + r - \sum_{j=1}^{r} \binom{d_j+k}{k}$.
Hence $\ell + n - 2k = kn + k + kr - k^2 + r + n - 2k - \sum_{j=1}^{r} \binom{d_j+k}{k} = n(k+1) - (k^2 + k(1-r) - r) - \sum_{j=1}^{r} \binom{d_j+k}{k} = n(k+1) - (k-r)(k+1) - \sum_{j=1}^{r} \binom{d_j+k}{k}$ $= (k+1)(n+r-k) - \sum_{j=1}^{r} \binom{d_j+k}{k}$.

(ii) $\ell + n + 1 - k = k(n+1+r-k) + r - \sum_{j=1}^{r} \binom{d_j+k}{k} + n + 1 - k = kn + k + kr - k^2 + r + n + 1 - k - \sum_{j=1}^{r} \binom{d_j+k}{k} = n(k+1) - (k^2 - kr - (r+1)) - \sum_{j=1}^{r} \binom{d_j+k}{k}$ $= (k+1)(n+1+r-k) - \sum_{j=1}^{r} \binom{d_j+k}{k}$. Note that $\dim Z = n+1$ and by the definition of $\Omega_Z(k)$ we have $k < n+2$ which implies this expression is larger than zero.

Now consider one of the irreducible components of $\Omega_Z(k)$ which describes a covering family of $\mathbb{P}^k$'s on Z, let us denote it by $\underline{\Omega_Z}$. Let Ω_Z be a subvariety cut out by ℓ general hyperplane sections of $\underline{\Omega_Z}$ and define $\Omega_X = \Omega_X(k) \cap \Omega_Z$. We will need to make use of the following theorems;

Theorem 47 *(Bertini's theorem) Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension n. Then there is a non-empty Zariski open subset $U^* \subset \mathbb{P}^{N,*}$ such that for any $t \in U^*$:*
(a) $\mathbb{P}_t^{N-1} \cap X$ is smooth (i.e. $\mathbb{P}_t^{N-1}$ is nowhere tangent to X)
(b) if $n \geq 2$ then $\mathbb{P}_t^{N-1} \cap X$ is irreducible.

Theorem 48 *(Second Theorem of Bertini) A generic element of a linear system on an algebraic variety X cannot have singular points that are not base points of the linear system or singular points of X.*

Recall that $\mathbb{P}^{N,*} = \{\mathbb{P}^{N-1}\text{'s} \subset \mathbb{P}^N\} = \mathrm{G}(N, N+1)$, where $[a_0, \ldots, a_N] \in \mathbb{P}^{N,*}$ corresponds to $\mathbb{P}^{N-1} = V(a_0 z_0 + \ldots + a_N z_N) \subset \mathbb{P}^N$.
By Bertini's theorem we can assume:

(iv) Ω_Z is smooth and irreducible of dimension $n+1-k$. [Note that $\Omega_Z \subset \Omega_Z(k)$, $\dim(\Omega_Z(k)) = n+1-k+l$ and $\Omega_Z = V(f_1, \ldots, f_l)$, hence $\dim(\Omega_Z) = n+1-k$]

(v) Ω_X is smooth and of pure dimension $n-2k$.

Next let us recall the cylinder correspondence for X and Z, namely $P(X) = \{(c,p) \in$

$\Omega_X(k) \times X | p \in \mathbb{P}^k_c\}$ and $P(Z) = \{(d, q) \in \Omega_Z(k) \times Z | q \in \mathbb{P}^k_d\}$. We have a commutative diagram where the j's are inclusions, π's and ρ's are projections:

$$
\begin{tikzcd}
P(X) \arrow[rr, "\pi_X"] \arrow[dr, "j_1"] \arrow[ddd, "\rho_X"'] & & X \arrow[ddr, "j"] & \\
& \widetilde{X} \arrow[ur, "\pi"'] \arrow[dr, "j_2"] \arrow[dd, "\rho"'] & & \\
& & P(Z) \arrow[r, "\pi_Z"] \arrow[dl, "\rho_Z"'] & Z \\
\Omega_X \arrow[r, "j_0"] & \Omega_Z & &
\end{tikzcd}
$$

We have

(a) $\widetilde{X} := \pi_Z^{-1}(X)$ is smooth by the Second Theorem of Bertini. [$\widetilde{X}$ is a general member of a linear system on $P(Z)$ with no base points, obtained by the pullback of linear systems on Z to $P(Z)$.]

(b) π and π_Z are generically finite to one and onto of degree q say.

(c) $\rho_X : P(X) \longrightarrow \Omega_X$ and $\rho_Z : P(Z) \longrightarrow \Omega_Z$ are $\mathbb{P}^k$ bundles as being the pullback of the projectivization of a corresponding universal bundle over the Grassmanian to Ω_Z. (i.e. $\rho_X^{-1}(c) = \mathbb{P}^k_c$ since $\rho_X^{-1}(c) = \{(c, p) | p \in \mathbb{P}^k_c\}$ and $P(X)$ is a manifold.)

(d) $\widetilde{\rho} := \rho|_{\widetilde{X} - P(X)} = \rho - \rho_X : \widetilde{X} - P(X) \longrightarrow \Omega_Z - \Omega_X$ is a $\mathbb{P}^{k-1}$ bundle.
[We have $\widetilde{X} = \{(c, p) \in \Omega_Z(k) \times X | p \in \mathbb{P}^k_c\}$ and $P(X) = \{(c, p) \in \Omega_X(k) \times X | p \in \mathbb{P}^k_c\}$. Let $\mathbb{P}^k_c \subset Z$ i.e. $c \in \Omega_Z$. There are two possibilities: (1) $\mathbb{P}^k_c$ is contained in X, (2) $\mathbb{P}^k_c$ not contained in X. Now $X = Z \cap \mathbb{P}^n$. So if $\mathbb{P}^k_c$ not contained in X then $\mathbb{P}^{k-1}_c = \mathbb{P}^k_c \cap \mathbb{P}^n \subset Z \cap \mathbb{P}^n = X$. Hence if we consider the map $\widetilde{\rho} : \widetilde{X} - P(X) \longrightarrow \Omega_Z - \Omega_X$, in fact we are looking at $\mathbb{P}^k_c$'s in Z which do not lie in X, hence $\widetilde{\rho}^{-1}(c) = \mathbb{P}^{k-1}_c = \mathbb{P}^k_c \cap \mathbb{P}^n \subset Z \cap \mathbb{P}^n = X$].

(e) $\dim(X) = \dim(\widetilde{X}) = n$, $\dim(Z) = \dim(P(Z)) = n + 1, \dim(P(X)) = n - k, \dim(\Omega_X) = n - 2k, \dim(\Omega_Z) = n - k + 1$.
[Recall that all varieties here are smooth. We have $\dim(P(Z)) = \dim(\Omega_Z) +$ dim(fibers of ρ_Z)$=n-k+1+k = n+1$, also $\dim(P(X)) = \dim(\Omega_X) +$dim(fibers of ρ_X)$=n - 2k + k = n - k$].
Now let $H_Z := \mathbb{P}^{n+1} \cap Z$ be a general hyperplane section of Z, and set $H_X = H_Z \cap X$. Then let

(f) $\mu = \pi^{-1}(H_X), \widetilde{\mu} = \mu \cap \{\widetilde{X} - P(X)\}, \mu_Z = \pi_Z^{-1}(H_Z), \mu_X = \pi_X^{-1}(H_X)$. We will identify these with their respective cohomology classes.

Now lets consider $\widetilde{X} = \pi_Z^{-1}(X)$. Note that $\pi : \widetilde{X} \longrightarrow X$ is generically finite to one and onto with degree q and so $\dim \widetilde{X} = \dim X = n$. Also $H^n(\widetilde{X}, \mathbb{Q}) \longrightarrow H^n(X, \mathbb{Q})$ is surjective by the fact that $\pi^* \pi_* = \times q$. Here we will first consider $H^n(\widetilde{X}, \mathbb{Z})$ and relate it to $H^{n-2k}(\Omega_X, \mathbb{Z})$. For that purpose we will use two maps from the commutative diagram we had, $j_1 : P(X) \longrightarrow \widetilde{X}$ and $\widetilde{X} - P(X) \hookrightarrow \widetilde{X}$. Applying the Borel-Moore homology functor to these maps we get a long exact sequence

$$\ldots \longrightarrow H_n(P(X), \mathbb{Z}) \longrightarrow H_n(\widetilde{X}, \mathbb{Z}) \longrightarrow H_n(\widetilde{X} - P(X), \mathbb{Z}) \longrightarrow \ldots$$

Note that $\Omega_X \hookrightarrow G(k+1, n+r+1)$ and $P(X)$ is a smooth projective algebraic manifold as being the projectivization of the pull-back of the universal bundle $U(k+1, n+r+1)$. Since $\widetilde{X} - P(X)$ is a smooth quasi-projective algebraic manifold, we can use Poincaré Duality in our homology sequence and get

$$\ldots \longrightarrow H^{n-2k}(P(X), \mathbb{Z}) \longrightarrow H^n(\widetilde{X}, \mathbb{Z}) \longrightarrow H^n(\widetilde{X} - P(X), \mathbb{Z}) \longrightarrow \ldots$$

Now to get information about $H^n(\widetilde{X}, \mathbb{Z})$ we will look at $H^{n-2k}(P(X), \mathbb{Z})$ and $H^n(\widetilde{X} - P(X), \mathbb{Z})$.
Lets start with $H^{n-2k}(P(X), \mathbb{Z})$:
We have the surjective map $\rho_X : P(X) \longrightarrow \Omega_X$ which is a $\mathbb{P}^k$ bundle. We will use the Leray spectral sequence $E_2^{p,q} = H^p(\Omega_X, R^q\rho_X^*\mathbb{Z})$ abutting to $H^{p+q}(P(X), \mathbb{Z})$ with $R^q\rho_X^*\mathbb{Z}$ associated to ρ_X. Recall that $R^q\rho_X^*\mathbb{Z}$ is a Leray cohomology sheaf defined as the sheaf associated to the presheaf in the strong topology: for open $U \subset \Omega_X \longrightarrow H^q(\rho_X^{-1}(U), \mathbb{Z}) \simeq H^q(\mathbb{P}^k, \mathbb{Z})$ over U since $\rho_X : P(X) \longrightarrow \Omega_X$ is a $\mathbb{P}^k$ bundle. So

$$R^q\rho_X^*\mathbb{Z} = H^q(\mathbb{P}^k, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } 0 \le q = 2l \le 2k \\ 0 & \text{otherwise} \end{cases}$$

Remember we want to calculate $H^{n-2k}(P(X), \mathbb{Z})$, so here $p + q = n - 2k$.
Now for $q = 2l, 0 \le l \le k$ we have $p = n - 2k - 2l$. Hence

$$H^{n-2k}(P(X), \mathbb{Z}) \cong \bigoplus_{l=0}^{k} H^{n-2k-2l}(\Omega_X, \mathbb{Z})$$

Next we will look at $H^n(\widetilde{X} - P(X), \mathbb{Z})$:
We have a projection $\tilde{\rho} : \widetilde{X} - P(X) \longrightarrow \Omega_Z - \Omega_X$ which is a $\mathbb{P}^{k-1}$ bundle. We will use the Leray spectral sequence $E_2^{p,q} = H^p(\Omega_Z - \Omega_X, R^q\tilde{\rho}_*\mathbb{Z})$ abutting to $H^{p+q}(\widetilde{X} - P(X), \mathbb{Z})$ with $R^q\tilde{\rho}_*\mathbb{Z}$ associated to $\tilde{\rho}$. In this case

$$R^q\tilde{\rho}_*\mathbb{Z} = H^q(\mathbb{P}^{k-1}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } 0 \le q = 2l \le 2(k-1) \\ 0 & \text{otherwise} \end{cases}$$

We want to calculate $H^n(\widetilde{X} - P(X), \mathbb{Z})$ so $p + q = n$ and $p = n - 2l$. We get

$$H^n(\widetilde{X} - P(X), \mathbb{Z}) \cong \bigoplus_{l=0}^{k-1} H^{n-2l}(\Omega_Z - \Omega_X, \mathbb{Z})$$

Now we have results from [Lewis1] which also hold for general complete intersections satisfying the conditions we examined above.

Proposition 49 *There is an isomorphism*

$$\left\{\oplus_{l=0}^{k-1} H^{n-2l}(\Omega_Z, \mathbb{Z})\right\} \oplus H^{n-2k}(\Omega_X, \mathbb{Z}) \xrightarrow{\approx} H^n(\widetilde{X}, \mathbb{Z})$$

given by $(\sum_{l=0}^{k-1} \mu^l \circ \rho^*) + j_{1,*} \circ \rho_X^*$ *where* $\mu = \pi^{-1}(H_X)$.

Idea of the proof: We follow closely the idea in [Lewis1].
We have a long exact sequence

$$\ldots \longrightarrow H^{n-2k}(P(X),\mathbb{Z}) \longrightarrow H^n(\widetilde{X},\mathbb{Z}) \longrightarrow H^n(\widetilde{X}-P(X),\mathbb{Z}) \longrightarrow \ldots$$

Also we get

$$H^{n-2k}(P(X),\mathbb{Z}) \cong \bigoplus_{l=0}^{k} H^{n-2k-2l}(\Omega_X,\mathbb{Z}) \text{ and } H^n(\widetilde{X}-P(X),\mathbb{Z}) \cong \bigoplus_{l=0}^{k-1} H^{n-2l}(\Omega_Z-\Omega_X,\mathbb{Z})$$

Now we will relate $H^n(\tilde{X},\mathbb{Z})$ to cohomology of Ω_Z. For that consider the map $\rho : \tilde{X} \longrightarrow \Omega_Z$, note that ρ^* preserves codimension and ρ_* preserves dimension. Then one has $\rho_* \circ \mu^{k-1-l} : \{\rho^*(H^{n-2l}(\Omega_Z,\mathbb{Z}))\} \wedge \mu^l \cong H^{n-2l}(\Omega_Z,\mathbb{Z})$, and $\rho_* \circ \mu^{k-1} \circ \rho^*$ identity on $H^{n-2l}(\Omega_Z,\mathbb{Z})$ (1).
On the other hand consider $\rho_* \circ \mu^{k-1+r-l} \circ \rho^*(H^{n-2r}(\Omega_Z,\mathbb{Z})) \subset H^{n-2l}(\Omega_Z,\mathbb{Z})$.

Claim 50 *For $r < l$ we see that $\rho_* \circ \mu^{k-1+r-l} \circ \rho^*(H^{n-2r}(\Omega_Z,\mathbb{Z})) = 0$ (2).*

Proof of the Claim (Outline) We have $\rho : \widetilde{X} \longrightarrow \Omega_Z$ with $(c,p) \in \Omega_Z \times X \longrightarrow c \in \Omega_Z$. Now let γ be a topological cycle of real codimension $n-2r$. Roughly $\rho^*(\gamma) = \rho^{-1}(\gamma)$ counted with multiplicity. We may assume that γ meets Ω_X properly. So for general $c \in |\gamma|$, $\rho^{-1}(c) = \mathbb{P}^{k-1}$. We have $\rho_* \circ \mu^{k-1+r-l} \circ \rho^*(c) = \rho_*(\mathbb{P}^k \cap \mu^{k-1+r-l}) = \rho_*(\mathbb{P}^{l-r+1})$. Now for $r < l, l-r+1 > 1$ we have $\dim \rho(\mathbb{P}^{l-r+1}) = 0 < \dim \mathbb{P}^{l-r+1} = l-r+1$ forcing $\rho_* \circ \mu^{k-1+r-l} \circ \rho^*(\gamma)$ to be zero.
Now let

$$\lambda_1 = \bigoplus_{l=0}^{k-1} \rho_* \circ \mu^{k-1-l} : H^n(\tilde{X},\mathbb{Z}) \longrightarrow \bigoplus_{l=0}^{k-1} H^{n-2l}(\Omega_Z,\mathbb{Z})$$

$$\lambda_2 = \sum_{l=0}^{k-1} \mu^l \wedge \rho^* : \bigoplus_{l=0}^{k-1} H^{n-2l}(\Omega_Z,\mathbb{Z}) \longrightarrow H^n(\tilde{X},\mathbb{Z})$$

Let $\xi = (\xi_0,\xi_1,\ldots,\xi_{k-1}) \in \bigoplus_{l=0}^{k-1} H^{n-2l}(\Omega_Z,\mathbb{Z})$. Then

$$\lambda_2(\xi) = \rho^*(\xi_0) + \mu\rho^*(\xi_1) + \ldots + \mu^{k-1}\rho^*(\xi_{k-1})$$

$$\lambda_1(\lambda_2(\xi)) = (\rho_*\mu^{k-1}(\lambda_2(\xi)), \rho_*\mu^{k-2}(\lambda_2(\xi)), \ldots, \rho_*(\lambda_2(\xi)))$$

From the properties (1) and (2) we see that $\rho_*(\lambda_2(\xi)) = \rho_*\mu^{k-1}\rho^*(\xi_{k-1}) = \xi_{k-1}$.
So $(\lambda_1 \circ \lambda_2(\xi) - \xi) = (*,*,\ldots,*,0)$.
Now let $T = \lambda_1 \circ \lambda_2$ and consider $T(T(\xi)-\xi)-(T(\xi)-\xi)$. By the same reasoning one gets $T(T(\xi)-\xi)-(T(\xi)-\xi) = (*,*,\ldots,*,0,0)$. Hence we get $T^2(\xi)-2T(\xi)+\xi = (*,*,\ldots,*,0,0)$. By recursion one gets a polynomial $P(T) = T^k + b_{k-1}T^{k-1} + b_{k-2}T^{k-2} + \ldots + b_1T \pm I$, with $b_j \in \mathbb{Z}$ and $P(T) = 0$ on $\bigoplus_{l=0}^{k-1} H^{n-2l}(\Omega_Z,\mathbb{Z})$. From here we can write $P(T) \pm I = Tf(T)$ for some $f(T)$ and we have $\mp f(T)T = I$ on $\bigoplus_{l=0}^{k-1} H^{n-2l}(\Omega_Z,\mathbb{Z})$. Now we have a commutative diagram:

$$\begin{tikzcd}
H^{n-1}(\tilde{X}) \arrow[r] \arrow[d] & H^{n-1}(\tilde{X}-P(X)) \arrow[r] \arrow[d, "\cong"'] & H^{n-2k}(P(X)) \arrow[r, "j_{1,*}"] \arrow[d] & H^n(\tilde{X}) \arrow[r] \arrow[d, "\mp f(T)\circ\lambda_1"'] & H^n(\tilde{X}-P(X)) \arrow[d, "\cong"'] \\
\bigoplus_{l=0}^{k-1} H^{n-1-2l}(\Omega_Z) \arrow[r] & \bigoplus_{l=0}^{k-1} H^{n-1-2l}(\Omega_Z-\Omega_X) \arrow[r] & \bigoplus_{l=1}^{k} H^{n-2k-2l}(\Omega_X) \arrow[r] & \bigoplus_{l=0}^{k-1} H^{n-2l}(\Omega_Z) \arrow[r] & \bigoplus_{l=0}^{k-1} H^{n-2l}(\Omega_Z-\Omega_X)
\end{tikzcd}$$

By diagram chasing and by the following lemma we can get the isomorphism given in the proposition.

Lemma 51 *$H^n(\widetilde{X}) = j_{1,*}\circ\rho_X^*(H^{n-2k}(\Omega_X)) + \lambda_2(\oplus_{l=0}^{k-1}H^{n-2l}(\Omega_Z))$ where*

$$\lambda_2 = \sum_{l=0}^{k-1}\mu^l\wedge\rho^* : \oplus_{l=0}^{k-1}H^{n-2l}(\Omega_Z,\mathbb{Z}) \longrightarrow H^n(\widetilde{X},\mathbb{Z})$$

Now put $\phi_* = \pi_*\circ j_{1,*}\circ\rho_X^* : H^{n-2k}(\Omega_X,\mathbb{Z}) \longrightarrow H^n(X,\mathbb{Z})$.

Corollary 52

(a) If n is odd, then ϕ_* is surjective.

(b) If n is even and $n-2k>0$, then $\phi_*\otimes\mathbb{Q} : H^{n-2k}(\Omega_X,\mathbb{Q}) \longrightarrow H^n(X,\mathbb{Q})$ is surjective.

Idea of the proof:([Bloch-Murre]) We can assume X is generic hyperplane section of Z. Then we can find a Lefschetz pencil $\{X_t\}_{t\in\mathbb{P}^1}$ of hyperplane sections of Z with $X = X_{t_0}$ a generic member of that Lefschetz pencil. Let $X_{t_1}, X_{t_2},\ldots,X_{t_N}$ denote the singular members of the Lefschetz pencil [Lewis2, Lecture6]. For each X_{t_i} there is a corresponding vanishing cycle $\delta_i\in H_n(X,\mathbb{Z})$ generating the group of vanishing cycles, $H_n(X,\mathbb{Z})_v$. Then for $j : X\hookrightarrow Z$, the inclusion map we have

$$H^n(X,\mathbb{Z}) = j^*(H^n(Z,\mathbb{Z}))\oplus H^n(X,\mathbb{Z})_v$$

Recall that Z is a complete intersection hence $j^*(H^n(Z,\mathbb{Z})) = \begin{cases} 0 & \text{if } n \text{ is odd}\\ \mathbb{Z}[H_X^m] & \text{if } n=2m \text{ is even}\end{cases}$

Hence for any odd n, $H^n(X,\mathbb{Z}) = H^n(X,\mathbb{Z})_v$

Recall Proposition **??**

$$\left\{\oplus_{l=0}^{k-1}H^{n-2l}(\Omega_Z,\mathbb{Z})\right\}\oplus H^{n-2k}(\Omega_X,\mathbb{Z}) \xrightarrow{\simeq} H^n(\widetilde{X},\mathbb{Z})$$

and the map $\pi : \widetilde{X}\longrightarrow X$ that is finite to one and onto with degree q.

From here we get two maps

(i) $\phi_* = \pi_*\circ j_{1,*}\circ\rho_X^* : H^{n-2k}(\Omega_X,\mathbb{Z}) \longrightarrow H^n(X,\mathbb{Z})$ (cylinder homomorphism)

(ii) $\pi_*\circ(\sum_{l=0}^{k-1}\mu^l\circ\rho^*) : \oplus_{l=0}^{k-1}H^{n-2l}(\Omega_Z,\mathbb{Z}) \longrightarrow H^n(X,\mathbb{Z})$ with $\pi_*\circ(\sum_{l=0}^{k-1}\mu^l\circ\rho^*) = j^*(H^n(Z,\mathbb{Z}))$.

To be able to show the surjectivity in the case n is odd we need to investigate the map $\pi_* : H^n(\widetilde{X},\mathbb{Z}) \longrightarrow H^n(X,\mathbb{Z}) = H^n(X,\mathbb{Z})_v$. For that consider the map $\pi_Z : P(Z)\longrightarrow Z$, we have $\pi_Z^{-1}(X) = \widetilde{X}$. For each X_t from the Lefschetz pencil there is the corresponding $\widetilde{X_t} = \pi_Z^{-1}(X_t)$. Now let $p_i = \mathrm{Sing}(X_{t_i})$ where $t_i\in\{t_1,t_2,\ldots,t_N\}$ is a singular point over which π_Z is *é*tale. For t near t_i one can find a corresponding $\widetilde{\delta_t}\in H_n(\widetilde{X_t},\mathbb{Z})$ which maps to δ_t. Then one has $H_n(\widetilde{X},\mathbb{Z}) \twoheadrightarrow H_n(X,\mathbb{Z})$ as the vanishing cycles are conjugate under the monodromy group action [CMP, Section 4.2]. From here with the Poincaré duality and horizontal displacement one gets $H^n(\widetilde{X},\mathbb{Z}) \twoheadrightarrow H^n(X,\mathbb{Z})$. Hence we get

$$\left\{\oplus_{l=0}^{k-1}H^{n-2l}(\Omega_Z,\mathbb{Z})\right\}\oplus H^{n-2k}(\Omega_X,\mathbb{Z}) \xrightarrow{\simeq} H^n(\widetilde{X},\mathbb{Z}) \twoheadrightarrow H^n(X,\mathbb{Z})$$

From (ii) above with n being odd, the cylinder homomorphism $\phi_* : H^{n-2k}(\Omega_X, \mathbb{Z}) \longrightarrow H^n(X, \mathbb{Z})$ is surjective.

Now for the case n even we will consider Proposition **??** with rational coefficients

$$\left\{\oplus_{l=0}^{k-1} H^{n-2l}(\Omega_Z, \mathbb{Q})\right\} \oplus H^{n-2k}(\Omega_X, \mathbb{Q}) \stackrel{\approx}{\to} H^n(\widetilde{X}, \mathbb{Q})$$

and the map $\pi : \widetilde{X} \longrightarrow X$ that is finite to one and onto with degree q which implies

$$H^n(\widetilde{X}, \mathbb{Q}) \twoheadrightarrow^{\pi_*} H^n(X, \mathbb{Q})$$

as for any $\gamma \in H^n(X, \mathbb{Q})$ we have $1/q\pi^*(\gamma) \in H^n(\widetilde{X}, \mathbb{Q})$ is mapped to γ under π_*.

Hence by Proposition **??** with rational coefficients and (ii) above we get

$$H^{n-2k}(\Omega_X, \mathbb{Q}) \stackrel{\phi_*}{\to} H^n(X, \mathbb{Q}) \twoheadrightarrow H^n(X, \mathbb{Q})/j^*(H^n(Z, \mathbb{Q}))$$

Note that $n - 2k > 0 \Rightarrow H^n(X, \mathbb{Q}) \neq H^{\frac{n}{2},\frac{n}{2}}(X, \mathbb{Q})$. Now consider a Lefschetz pencil of hyperplane sections of $Z, \{X_t\}_{t\in\mathbb{P}^1}$, with $X = X_{t_0}$ for some generic $t_0 \in \mathbb{P}^1$. For generic $t \in \mathbb{P}^1$ there is the corresponding Ω_{X_t} with dimension $n - 2k$ for X_t. For a projective embedding $\Omega_Z \longrightarrow \mathbb{P}^N$, $E_t = \Omega_{X_t} \cap \mathbb{P}^{N-(\frac{n}{2}-k)}$ defines a codimension $\frac{n}{2} - k$ cycle in Ω_{X_t} for a generic $t \in \mathbb{P}^1$. Hence $[E_{t_0}] \in H^{n-2k}(\Omega_X, \mathbb{Q})$ and $\phi_*(E_{t_0})$ is an effective cycle in X of codimension $\frac{n}{2}$. We need:

Theorem 53 *(Wirtinger's theorem) Let X, Y be compact complex manifolds of dimensions n and m respectively. Assume Y is an embedded submanifold of X and that X is Kähler with Kähler form w. Then $\frac{1}{m!}\int_Y w^m =$ Vol (Y).*

By Wirtinger's theorem we have $\phi_*([E_{t_0}]) \neq 0$. Also note that $\phi_*([E_{t_0}]) \in H^n(X, \mathbb{Q})^{\pi_1(U)}$ where $U = \mathbb{P}^1 - \{t_1, \dots, t_N\}$ and $\phi_*([E_{t_0}]) \in H^{\frac{n}{2},\frac{n}{2}}(X)$. For details see [Lewis2, pages 205-206].
We have

$$H^n(X, \mathbb{Q}) = j^*(H^n(Z, \mathbb{Q})) \oplus H^n(X, \mathbb{Q})_v$$

and the fact that $\phi_*([E_{t_0}])$ is invariant under the monodromy group action and $H^n(X, \mathbb{Q}) \neq H^{\frac{n}{2},\frac{n}{2}}(X)$ implies $\phi_*([E_{t_0}]) \in j^*(H^n(Z, \mathbb{Q}))$ and hence $j^*(H^n(Z, \mathbb{Q})) \subset \phi_*(H^{n-2k}(\Omega_X, \mathbb{Q}))$.
By this we get $\phi_* : H^{n-2k}(\Omega_X, \mathbb{Q}) \longrightarrow H^n(X, \mathbb{Q})$ surjective.

Remark 54 $n - 2k = 0$
In this case we have $H^n(X, \mathbb{Q}) = H^{\frac{n}{2},\frac{n}{2}}(X, \mathbb{Q})$ and note that $k > 0$ for the case to be interesting. We can examine this case by the number of polynomials determining X;
When X is determined by one polynomial, X will be a cubic surface in $\mathbb{P}^3$ or an even dimensional quadric in $\mathbb{P}^{2k+1}$. In the cubic surface case, we have

$$H_0(\Omega_X, \mathbb{Q}) \twoheadrightarrow H_2(X, \mathbb{Q})/\mathbb{Q}H_X$$

For a cubic surface $H_X = L_1 + L_2 + L_3$ where L_1, L_2, L_3 are lines in X. By the definition of Ω_X there exists $c_1, c_2, c_3 \in \Omega_X$ such that $\mathbb{P}^1_{c_1} = L_1, \mathbb{P}^1_{c_2} = L_2, \mathbb{P}^1_{c_3} = L_3$

with $\phi_*(c_1+c_2+c_3) = H_X$ implying $H^0(\Omega_X,\mathbb{Q}) \twoheadrightarrow H^2(X,\mathbb{Q})$ via Poincáre duality. When X is determined by two polynomials then X will be an even dimensional complete intersection of two quadratics in $\mathbb{P}^{2k+2}$. When X is determined by more than two polynomials, X will be one of the above types since at least one of the polynomials will be linear. For the quadratic cases we mentioned above we have

$$H_0(\Omega_X,\mathbb{Q}) \twoheadrightarrow H_{2k}(X,\mathbb{Q})/\mathbb{Q}H_X^k$$

Note that $\dim\Omega_X = 0$ and it will consist of points c_i corresponding to k-planes in X. Also a smooth quadratic hypersurface can be identified with the Grassmanians, that is if $X \subset \mathbb{P}^{2k+1}$ with $\dim X = 2k$ a quadratic hypersurface then the Gauss map $X \longrightarrow G(2k+1,2k+2)$ sending $p \in X$ to $T_p(X)$ is an isomorphism. Then $H_X^k = \sum_j L_j$ where $L_j = \mathbb{P}^k_{c_j}$ since the fibers of the map $P(X) \longrightarrow G(k+1,2k+2)$ are isomorphic to $\mathbb{P}^k$. Hence $H_X^k = \phi_*(\sum_j c_j) \in \phi_*(H_0(\Omega_X,\mathbb{Q}))$ which gives the surjectivity we want. One can argue similarly for the intersection of two quadratics by identifying each with the corresponding Grassmanians.

By Corollary **??** we get

Proposition 55 *Let $X \subset \mathbb{P}^{n+r}$ be a generic smooth complete intersection of dimension n with degree $(d_1,\ldots,d_r)$ satisfying the condition (2.2.1). Then we have GHC(k,n,X).*

Proof. Let $Y = \pi_X(P(X)|_{\Omega_X}) = \cup_{c\in\Omega_X}\mathbb{P}^k_c$, i.e. Y is swept out by $\{\mathbb{P}^k_c\text{'s} \mid c \in \Omega_X\}$. Then $\dim Y = n-k$. Next the cylinder map factors through $H_n(Y,\mathbb{Q})$:

$$H_{n-2k}(\Omega_X,\mathbb{Q}) \longrightarrow H_n(Y,\mathbb{Q}) \longrightarrow H_n(X,\mathbb{Q}), \text{defined as}$$

$$\{\gamma\} \in H_{n-2k}(\Omega_X,\mathbb{Q}) \longrightarrow \cup_{c\in\gamma}\mathbb{P}^k_c \in H_n(Y,\mathbb{Q}) \to H_n(X,\mathbb{Q}).$$

Thus $H^n(X,\mathbb{Q}) = N^kH^n(X,\mathbb{Q})$. Remember that we start with the result saying that the Hodge level of $H^n(X,\mathbb{Q})$ is $n-2k$, this means $F^kH^n(X,\mathbb{C}) = H^n(X,\mathbb{C}) = H^{k,n-k}(X) \oplus \ldots \oplus H^{n-k,k}(X)$. Hence $F^k_hH^n(X,\mathbb{Q}) = H^n(X,\mathbb{Q}) = N^kH^n(X,\mathbb{Q})$, which means GHC$(k,n,X)$ holds for general complete intersection $X \subset \mathbb{P}^{n+r}$ of dimension n and degree $(d_1,\ldots,d_r)$ satisfying (2.2.1). ■

In fact, $GHC(k,n,X)$ holds for all smooth complete intersections X of dimension n, satisfying condition (2.2.1). This can be shown by applying deformation theory as in Chapter 13 of [Lewis2]. We have:

Corollary 56 *Let $X \subset \mathbb{P}^{n+r}$ be a generic smooth complete intersection of dimension n with degree $(d_1,\ldots,d_r)$ satisfying the condition (2.2.1). Then*

$$A^*(X) \cong J^*_{alg}(X) \Rightarrow Level(H^*(X)) \leq 1$$

Proof. (**Outline**)

We had Corollary **??** which says if $B(*)$ and $GHC(k,n,X)$ holds then $A^*(X) \cong J^*_{\text{alg}}(X) \Rightarrow \text{Level}(H^*(X)) \leq 1$. We need to show that condition (2.2.1) guarantees the assumptions of Corollary **??** holds. By Proposition **??** GHC(k,n,X) holds if condition (2.2.1) is satisfied.

Note that the role of the assumption $B(*)$ in Corollary **??** is that one can find a projective algebraic submanifold S of dimension l and an algebraic surjective map

$H^l(S,\mathbb{Q}) \longrightarrow H^{2r-l}(X,\mathbb{Q})$ where $1 \leq r \leq \dim X$. That is the surjective map is induced by an algebraic cycle from $CH^k(S \times X)$. This is then used to arrive at the result if $\text{Level}(H^*(X)) > 1$ then $A^*(X)$ will be infinite dimensional so it can not be isomorphic to $J^*_{\text{alg}}(X)$. The details of this proof can be found in the proof of Theorem 15.36 from [Lewis2].
Now note that by Corollary **??** we have a surjective map $\phi_* : H^{n-2k}(\Omega_X,\mathbb{Q}) \longrightarrow H^n(X,\mathbb{Q})$ which is induced by $P(X) \in CH^{n-k}(\Omega_X \times X)$. Hence without the assumption of $B(*)$ we have the necessary objects $S = \Omega_X$ and $P(X)$, the algebraic cyle inducing the surjective map, with $l = n-2k, r = n-k$. Then the rest of the proof follows from the proof of Theorem 15.36 of [Lewis2]. ∎

In order to show the other direction of the Lewis conjecture holds for complete intersections satisfying the condition (2.2.1) we need an analogue of the weak Lefschetz theorem for Chow groups.

2.3 Analogue of Weak Lefschetz Theorem for Chow Groups of Complete Intersections

Let $X \subset \mathbb{P}^{n+r}$ be a n-dimensional smooth complete intersection of type $(d_1, \ldots, d_r)$, i.e. $X = V(F_1, \ldots, F_r)$ where F_i are homogenous polynomials of degree d_i and consider $Z = V(G_1, \ldots, G_r) \subset \mathbb{P}^{n+r+1}$ where $G_i = F_i$ for all $i = 1, \ldots, r-1$, $G_r = F_r + z_{n+r+1}^{d_r}$.
Note that $X = Z \cap V(z_{n+r+1}) = Z \cap \mathbb{P}^{n+r}$. Now consider the projection map $v_p : Z \longrightarrow \mathbb{P}^{n+r}$ from a point $p \in \mathbb{P}^{n+r+1} - Z$ where $p = [0, \ldots, 0, 1]$. Then $v_p = v$ is given by $[z_0, \ldots, z_{n+r+1}] \xrightarrow{v} [z_0, \ldots, z_{n+r}]$. Let $W = v(Z)$ that is for $p = [p_0, \ldots, p_{n+r+1}] \in Z$, $v[p_0, \ldots, p_{n+r+1}] = [p_0, \ldots, p_{n+r}] \in W$. Note that $p = [p_0, \ldots, p_{n+r+1}] \in Z$ means $F_i(p_0, \ldots, p_{n+r}) = 0$ for $i = 1, \ldots, r-1$, and $F_r(p_0, \ldots, p_{n+r}) + p_{n+r+1}^{d_r} = 0$. There are two possibilities:

1. $p_{n+r+1} = 0 \Rightarrow [p_0, \ldots, p_{n+r}] \in X$
2. $p_{n+r+1} \neq 0 \Rightarrow [p_0, \ldots, p_{n+r}] \notin X$

Hence $X \subset W$. Also note that W is determined by the zeros of the poynomials $F_1, \ldots, F_{r-1}$. Hence $W = V(F_1, \ldots, F_{r-1})$.

Now consider the inclusion maps $j : X \hookrightarrow Z$, $i : X \hookrightarrow W$ we get:

Proposition 57 *X, Z and W be given as above. Then the following diagram is commutative:*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "i"'] & Z \arrow[d, "v"] \\ & W \end{tikzcd}$$

Proof. To check the commutativity of the diagram we will show:
$vj = i$: Let $[p_0, \ldots, p_{n+r}] \in X$, then $vj([p_0, \ldots, p_{n+r}]) = v([p_0, \ldots, p_{n+r}, 0]) = [p_0, \ldots, p_{n+r}] \in W$. Also $i([p_0, \ldots, p_{n+r}]) = [p_0, \ldots, p_{n+r}]$
■

Claim 58 *If $X = V(F_1, \ldots, F_r) \subset \mathbb{P}^{n+r}$ is a n-dimensional smooth complete intersection of type $(d_1, \ldots, d_r)$ and $W = V(F_1, \ldots, F_{r-1}) \subset \mathbb{P}^{n+r}$, described as above, is smooth complete intersection of type $(d_1, \ldots, d_{r-1})$ then $Z = V(F_1, \ldots, F_{r-1}, F_r + z_{n+r+1}^{d_r}) \subset \mathbb{P}^{n+r+1}$ is a $n+1$-dimensional smooth complete intersection of type $(d_1, \ldots, d_r)$*

Proof. Note that a variety $Y \subset \mathbb{P}^N$ is smooth if and only if $\mathrm{Cone}(Y) - \{0\} \subset \mathbb{C}^{N+1}$ is smooth.
From our assumptions $\mathrm{Cone}(X) - \{0\} = V(F_1, \ldots, F_r) \subset \mathbb{C}^{n+r+1}$ and $\mathrm{Cone}W - \{0\} = V(F_1, \ldots, F_{r-1}) \subset \mathbb{C}^{n+r}$ are smooth complete intersections. That means that Jacobian of both have full rank at every point of theirs.
Lets consider the Jacobian of $\mathrm{Cone}(Z) - \{0\}$:

Jacobian of $\mathrm{Cone}(Z)-\{0\} = \left[\begin{array}{ccc} \multicolumn{2}{c}{\text{Jacobian of } \mathrm{Cone}(W)-\{0\}} & \begin{array}{c}0\\ \vdots \\ 0\end{array} \\ \frac{\partial F_r}{\partial z_0} \quad \cdots & \frac{\partial F_r}{\partial z_{n+r}} & d_r z_{n+r+1}^{d_r} \end{array}\right]$ Now let $p = [p_0, p_1, \ldots, p_{n+r+1}]$ be any point in Z. Then

1. If $p_{n+r+1} = 0$ then from the proof of Proposition **??**, $[p_0, \ldots, p_{n+r}] \in X \cap W$ and Jacobian of $\mathrm{Cone}(Z)-\{0\} = \left[\begin{array}{cc} \text{Jacobian of } \mathrm{Cone}(X)-\{0\} & \begin{array}{c}0\\ \vdots \\ 0\end{array}\end{array}\right]$ Then in this case the Jacobian of $\mathrm{Cone}(Z)-\{0\}$ will have full rank at $(p_0, p_1, \ldots, p_{n+r+1})$ as the Jacobian of $\mathrm{Cone}(X)-\{0\}$ has rank r at $(p_0, \ldots, p_{n+r})$.

2. If $p_{n+r+1} \neq 0$ then $[p_0, \ldots, p_{n+r}] \in W - X$ from proof of Proposition **??** and the fact that $[0, 0, \ldots, 0, 1] \notin Z$. Then Jacobian of $\mathrm{Cone}(W)-\{0\}$ will have rank $r-1$ at $(p_0, \ldots, p_{n+r})$ implying Jacobian of $\mathrm{Cone}(Z)-\{0\}$ will have full rank at $(p_0, p_1, \ldots, p_{n+r+1})$.

Hence in any case Jacobian of $\mathrm{Cone}(Z)-\{0\}$ will have full rank at every point of $\mathrm{Cone}(Z)-\{0\}$, implying Z is a smooth complete intersection of dimension $n+1$ [Lewis2, Lecture 1, 1.17]. ■

Proposition 59 *Let X, Z, W be given as above, then the following diagram is commutative:*

$$\begin{tikzcd} CH^\bullet(Z) \arrow[r, "d_r j^*"] \arrow[dr, "v_*"'] & CH^\bullet(X) \\ & CH^\bullet(W) \arrow[u, "i^*"'] \end{tikzcd}$$

Proof. We already saw that for $[p_0, \ldots, p_{n+r}] \in W$ we have $v^{-1}([p_0, \ldots, p_{n+r}]) \cong \{p_{n+r+1} \in \mathbb{C} | F_r(p_0, \ldots, p_{n+r}) + p_{n+r+1}^{d_r} = 0\}$. Hence $v|_{Z-j(X)} : Z - j(X) \longrightarrow W - i(X)$ is unramified of degree d_r and $v : j(X) \longrightarrow i(X)$ completely ramifies over X. Now let $\xi_1 = \mathrm{graph}(v) \in Z \times W$ and $\xi_2 = \mathrm{graph}^T(i) \in W \times X$. Consider $\xi_1 \circ \xi_2 = Pr_{13,*}(Pr_{12}^*(\xi_1) \cap Pr_{23}^*(\xi_2)) \in Z \times X$. Then $|\xi_1 \circ \xi_2| = |\mathrm{graph}^T(j)|$ irreducible. Since $\xi_1 \circ \xi_2$ and $\mathrm{graph}^T(j)$ supports the same family we have $i^* \circ v_* = d_0 j^*$ for some integer d_0. Now $d_0\{X\} = d_0 j^*(Z) = i^* \circ v_*(Z) = i^*([\mathbb{C}(Z) : \mathbb{C}(v(Z))] v(Z)) = i^*([\mathbb{C}(Z) : \mathbb{C}(W)] W) = i^*(d_r W) = d_r i^*(W) = d_r\{X\}$. Hence we have $i^* \circ v_* = d_r j^*$. ■

Corollary 60 *Let X, Z, W be given as above, we get the following commutative diagram:*

$$\begin{tikzcd} A^\bullet(Z) \arrow[r, "d_r j^*"] \arrow[dr, "v_*"'] & A^\bullet(X) \\ & A^\bullet(W) \arrow[u, "i^*"'] \end{tikzcd}$$

Furthermore, we have $j^*(A^\bullet(Z)) = i^*(A^\bullet(W))$ from the divisibility of $A^\bullet(-)$.

Proposition 61 *Let $X \subset \mathbb{P}^{n+r}$ be a general complete intersection of type $(d_1, \ldots, d_r)$ satisfying condition (2.2.1) and let $Z \subset \mathbb{P}^{n+r+1}$ be given as above. Then $\Omega_Z(k)$ is smooth and of (pure) dimension $(n+1-k)+\ell$, where ℓ is given in proposition* **??**.

Proof. The proof is similar to that in [Lewis1]. We write out the details to account for any new details. Let $X = V(F_1, \ldots, F_r) \subset \mathbb{P}^{n+r}$ be a general complete intersection of type $(d_1, \ldots, d_r)$. Then we have $F_i(z_0, \ldots, z_{n+r}) = \sum_{[\alpha_i]=d_i} a_{\alpha_i} z^{\alpha_i}$, for $\alpha_i = (\alpha_{i0}, \ldots, \alpha_{i(n+r)}) \in \mathbb{Z}_+^{n+r}$ for all $i = 1, \ldots, r$. Here $[\alpha_i] = \alpha_{i0} + \ldots + \alpha_{i(n+r)}$ and $z^{\alpha_i} = z_0^{\alpha_{i0}} \ldots z_{n+r}^{\alpha_{i(n+r)}}$. Let $a_i = (\ldots, a_{\alpha_i}, \ldots) \in \mathbb{C}^{N_i+1}$ where $N_i = \binom{n+r+d_i}{d_i}$. Now for $\lambda \in \mathbb{C}$ let $t = [a_1, \ldots, a_{r-1}, (a_r, \lambda)] \in \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \times \ldots \times \mathbb{P}^{N_{r-1}} \times \mathbb{P}^{N_r+1}$ and consider $Z_t = V(F_1, \ldots, F_{r-1}, F_r + \lambda z_{n+r+1}^{d_r}) \subset \mathbb{P}^{n+r+1}$. By examining Z_t we will show that $\Omega_Z(k) = \{\mathbb{P}^k\text{'s} \subset Z\}$ is smooth.
Let $L = \mathbb{P}^k \subset \mathbb{P}^{n+r+1}$ be given. L is a k-plane so it has the form $L = \{\tau_0 w_0 + \ldots + \tau_k w_k | \tau_j \in \mathbb{C}, w_k \in \mathbb{P}^{n+1+r}\}$. Let $PG \subset \mathrm{Aut}(\mathbb{P}^{n+r+1})$ be the subgroup given by

$$PG = \left\{ \left[\begin{array}{ccc|c} & & & 0 \\ & [A] & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{array} \right] \Big| A \in GL(n+r+1, \mathbb{C}) \right\}$$

Under the action of PG we can write $w_0 = (\beta, 0, \ldots, 0, \eta_0)$ with either $\beta \neq 0$ or $\eta_0 \neq 0$, $w_1 = (0, 1, \ldots, 0, \eta_1), \ldots, w_k = (0, \ldots, 0, 1, 0 \ldots, 0, \eta_k)$.
So $L = \{(\tau_0\beta, \tau_1, \ldots, \tau_k, 0, \ldots, 0, \sum_{j=0}^k \tau_j \eta_j) \in \mathbb{P}^{n+r+1}\}$.

$$L \subset Z_t \Leftrightarrow F_i(\tau_0\beta, \tau_1, \ldots, \tau_k, 0, \ldots, 0) = 0 \text{ and } \lambda(\sum_{j=0}^k \tau_j\eta_j)^{d_r} + F_r(\tau_0\beta, \tau_1, \ldots, \tau_k, 0, \ldots, 0) = 0$$

for $\forall i = 1, \ldots, r-1, \forall \tau = (\tau_0, \ldots, \tau_k) \in \mathbb{C}^{k+1}$. From here we have

1. $F_i(\tau_0\beta, \tau_1, \ldots, \tau_k, 0, \ldots, 0) = 0 \Leftrightarrow \sum_{[\alpha_i]=d_i} a_{\alpha_i} (\tau_0\beta)^{\alpha_{i0}} (\tau_1)^{\alpha_{i1}} \ldots (\tau_k)^{\alpha_{ik}} = 0 \; \forall \tau \in \mathbb{C}^{k+1}$

2. $\lambda(\sum_{j=0}^k \tau_j\eta_j)^{d_r} + F_r(\tau_0\beta, \tau_1, \ldots, \tau_k, 0, \ldots, 0) = 0 \Leftrightarrow$

$$\begin{aligned} 0 &= \lambda \sum_{[\gamma]=d_r} (v_\gamma (\tau_0\eta_0)^{\gamma_0} \ldots (\tau_k\eta_k)^{\gamma_k}) + \sum_{[\gamma]=d_r} a_\gamma (\tau_0\beta)^{\gamma_0} \tau_1^{\gamma_1} \ldots \tau_k^{\gamma_k} \\ &= \sum_{[\gamma]=d_r} (\tau_0^{\gamma_0} \ldots \tau_k^{\gamma_k})(\lambda v_\gamma \eta_0^{\gamma_0} \ldots \eta_k^{\gamma_k} + a_\gamma \beta^{\gamma_0}) \\ &\Rightarrow \lambda v_\gamma \eta_0^{\gamma_0} \ldots \eta_k^{\gamma_k} + a_\gamma \beta^{\gamma_0} = 0 \end{aligned}$$

Here coefficients v_γ are coming from the expansion of $(\sum_{j=0}^k \tau_j\eta_j)^{d_r}$. Now from the last equation we have two possible cases:

- $\beta \neq 0$ then the last equation defines a linear system in $t \in \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \times \ldots \times \mathbb{P}^{N_{r-1}} \times \mathbb{P}^{N_r+1}$ of rank $\binom{k+d_r}{d_r}$. Also from the first equations, that is given by $F_i = 0$ for $i = 1, \ldots, r-1$ we have linear systems in $t \in \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \times \ldots \times \mathbb{P}^{N_{r-1}} \times \mathbb{P}^{N_r+1}$ of ranks $\binom{k+d_i}{d_i}$ for $i = 1, \ldots, r-1$ respectively.

- $\beta = 0$ this implies $\eta_0 \neq 0 \Rightarrow [0, \ldots, 0, 1] \in Z$ and so forces $\lambda = 0$ which implies Z_t is singular. Also note that $\beta = 0$ with $\eta_0 \neq 0$ implies $p = [0, \ldots, 0, 1] \in L \subset Z_t$

Observe that $p \in \mathbb{P}^{n+r+1}$ is the only point that is fixed under the PG action. Let L be given with $p \in L$. Then any k-plane L' in $\mathbb{P}^{n+r+1}$ is PG equivalent to L if and only if $p \in L'$. Now let us consider the k-planes that are PG equivalent to L. For that consider the Grassmannian variety $G(k+1, n+r+2) = \{\mathbb{P}^k \subset \mathbb{P}^{n+r+1}\}$ and then define $\Sigma = \{c \in G | p \in \mathbb{P}^k_c\}$ the parameterizing space of the k-planes PG equivalent to L. Then consider $\Lambda = \{(c,t) \in G \times \mathbb{P}^{N_1+\ldots+N_{r-1}+N_r+1} | \mathbb{P}^k \subset Z_t\}$ with the projection maps $\pi_1 : \Lambda \longrightarrow G$ and $\pi_2 : \Lambda \longrightarrow \mathbb{P}^{N_1+\ldots+N_{r-1}+N_r+1}$. Now let

- $U_1 = G - \Sigma$, parameterizing space of k- planes that that does not contain p
- $U_2 = \{t \in \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \times \ldots \times \mathbb{P}^{N_{r-1}} \times \mathbb{P}^{N_r+1} | Z_t \text{ is smooth }\}$
- $\Lambda_0 = \pi_1^{-1}(U_1) \supset \{(c,t) \in G \times \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \times \ldots \times \mathbb{P}^{N_{r-1}} \times \mathbb{P}^{N_r+1} | \mathbb{P}^k \subset Z_t, Z_t \text{ is smooth}\}$

Hence we have

1. $\pi_2^{-1}(U_2) \subset \Lambda_0$ and $\pi_2 : \Lambda_0 \longrightarrow \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \times \ldots \times \mathbb{P}^{N_{r-1}} \times \mathbb{P}^{N_r+1}$ is a dominating morphism.
2. For $c \in U_1$, $\pi_1^{-1}(c) \cong \mathbb{P}^N$ where $N = N_1 + \ldots + N_r + 1 - \sum_{i=1}^r \binom{k+d_i}{d_i}$ for all $c \in U_1$.
3. Note that Λ_0 is cut out by $\sum_{i=1}^r \binom{k+d_i}{d_i}$. Note that Λ_0 can be viewed as the zero set of $\sum_{i=1}^r \binom{k+d_i}{d_i}$ equations that are defined above. Also we have $\Lambda_0 \subset G \times \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \times \ldots \times \mathbb{P}^{N_{r-1}} \times \mathbb{P}^{N_r+1}$ and the differentials of the $\sum_{i=1}^r \binom{k+d_i}{d_i}$ equations defining Λ_0 have independent differentials in the direction of t over Λ_0. Hence by the implicit function theorem Λ_0 is smooth of dimension $N_1 + \ldots + N_r + 1 + (k+1)(n+r+1-k) - \sum_{i=1}^r \binom{k+d_i}{d_i}$ Recall from Proposition **??**, $\ell = k(n+1+r-k) + r - \sum_{i=1}^r \binom{k+d_i}{k}$. Note that $(k+1)(n+r+1-k) = n+1-k+k(n+1+r-k)+r$, hence $N_1 + \ldots + N_r + 1 + (k+1)(n+r+1-k) - \sum_{i=1}^r \binom{k+d_i}{d_i} = (N_1 + \ldots + N_r + 1) + n + 1 - k + \ell$.
   Now consider the map $\pi_2 : \Lambda_0 \longrightarrow \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \times \ldots \times \mathbb{P}^{N_{r-1}} \times \mathbb{P}^{N_r+1}$. This map is dominating and we have $\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \times \ldots \times \mathbb{P}^{N_{r-1}} \times \mathbb{P}^{N_r+1} - U_2$ has measure zero. For a general $t \in \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \times \ldots \times \mathbb{P}^{N_{r-1}} \times \mathbb{P}^{N_r+1}$, $\pi_2^{-1}(t) = \Omega_{Z_t}$ where Z_t is smooth. Then $\dim \Lambda_0 = \dim \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \times \ldots \times \mathbb{P}^{N_{r-1}} \times \mathbb{P}^{N_r+1} + \dim \text{fibers}$ implies $\dim \Omega_{Z_t} = n+1-k+\ell$ and by SARD's lemma the fibers are smooth, i.e. Ω_{Z_t} is smooth.

■

Remember we had a commutative diagram:

$$
\begin{tikzcd}
P(X) \arrow[rr, "\pi_X"] \arrow[dr, "j_1"] \arrow[ddd, "\rho_X"'] & & X \arrow[ddr, "j"] & \\
& \tilde{X} \arrow[ur, "\pi"'] \arrow[dr, "j_2"] \arrow[dd, "\rho"'] & & \\
& & P(Z) \arrow[r, "\pi_Z"] \arrow[dl, "\rho_Z"'] & Z \\
\Omega_X \arrow[r, "j_0"] & \Omega_Z & &
\end{tikzcd}
$$

where $\tilde{X} = \pi_Z^{-1}(X) = \{\mathbb{P}^k\text{'s} \subset Z | \mathbb{P}^k \subset X\} \cup \{\mathbb{P}^k\text{'s} \subset Z | \mathbb{P}^k \cap X \neq \emptyset\}$

Proposition 62 *Let $X = V(F_1, \ldots, F_r) \subset \mathbb{P}^{n+r}$ be a n-dimensional general complete intersection of type $(d_1, \ldots, d_r)$ satisfying condition (2.2.1), and consider $Z = V(G_1, \ldots, G_r) \subset \mathbb{P}^{n+r+1}$ where $G_i = F_i$ for all $i = 1, \ldots, r-1$, $G_r = F_r + z_{n+r+1}^{d_r}$. Then $\tilde{X}$ is smooth.*

Proof. Same as in [Lewis1] ∎

2.4 The cylinder homomorphism for Chow groups of complete intersections

Let $X \subset \mathbb{P}^{n+r}$ be a general complete intersection of type $(d_1, \dots, d_r)$ and dimension n. We will consider the cylinder homomorphism between the Chow group of the Fano variety of k-planes in X and the Chow group of X. Recall the cylinder correspondence $P(X) = \{(c,p) \in \Omega_X(k) \times X | p \in \mathbb{P}^k_c\}$ and the cylinder homomorphism ϕ_* induced by intersection with $P(X)$. Then we have

$$\phi_* : CH^{\bullet - k}(\Omega_X(k)) \longrightarrow CH^{\bullet}(X)$$

given by $\phi_*(\xi) = Pr_{2,*}(Pr_1^*(\xi) \cap P(X))$. Now consider $\widetilde{X}$ and recall that the surjective finite to one degree q map $\pi : \widetilde{X} \longrightarrow X$. Then we have $CH^{\bullet}(\widetilde{X}) \otimes \mathbb{Q} \longrightarrow CH^{\bullet}(X) \otimes \mathbb{Q}$ surjective by the fact that $\pi^*\pi_* = \times q$. Hence we will consider $CH^{\bullet}(\widetilde{X})$ and relate it to $CH^{n-k}(\Omega_X)$. For that we will consider the inclusion map $j_1 : P(X) \longrightarrow \widetilde{X}$ and the projection map $\tilde{\rho} : \widetilde{X} \longrightarrow \widetilde{X} - P(X)$ given in the previous sections. Remember that $\dim P(X) = n - k$ where $k = [(n + 1 - \sum_{i \neq s} d_i)/d_s]$, $d_s = \max\{d_1, \dots, d_r\}$. Note that $P(X)$ is closed in $\widetilde{X}$ with $\operatorname{codim} P(X) = k$. Now we will use the following result from [Bloch]:

Let $CH^{\bullet}(X, n), n \geq 0$ denote the higher Chow groups of X quasi-projective over a field k then

(1) (Theorem 4.1) $CH^{\bullet}(X, n)$ is covariant for proper maps, contravariant for flat map. Contravariant for arbitrary maps when X is smooth.

(2) (Theorem 3.1) For $Y \subset X$ closed, pure codimension d, we have a long exact sequence

$$\to CH^{\bullet - d}(Y, n) \to CH^{\bullet}(X, n) \to CH^{\bullet}(X - Y, n) \to CH^{\bullet - d}(Y, n-1) \to \dots$$

$$\dots \to CH^{\bullet - d}(Y, 0) \to CH^{\bullet}(X, 0) \to CH^{\bullet}(X - Y, 0) \to 0$$

(3) $CH^{\bullet}(X, 0) = CH^{\bullet}(X)$

(4) (Theorem 7.1, Projective Bundle Theorem) Let E be a rank n vector bundle on the quasi-projective X over a field k, and let ξ be the first Chern class of $O(1)$ on $\pi : \mathbb{P}(E) \longrightarrow X$. Then for any $m \geq 0$ one has

$$\bigoplus_{i=0}^{n-1} \xi^i \pi^* : \bigoplus_{i=0}^{n-1} \bigoplus_{p \geq 0} CH^p(X, m) \xrightarrow{\simeq} \bigoplus_{q \geq 0} CH^q(\mathbb{P}(E), m)$$

Now from (2) and (3) we have a long exact sequence

$$\dots \to CH^{\bullet}(\widetilde{X} - P(X), 1) \to CH^{\bullet - k}(P(X)) \xrightarrow{j_{1,*}} CH^{\bullet}(\widetilde{X}) \to CH^{\bullet}(\widetilde{X} - P(X)) \to 0$$

Recall that $P(X)$ is a $\mathbb{P}^k$ bundle over Ω_X and $\widetilde{X} - P(X)$ is a $\mathbb{P}^{k-1}$ bundle over $\Omega_Z - \Omega_X$. Hence by the Projective Bundle Theorem we have

$CH^{\bullet - k}(P(X)) \simeq \oplus_{r=0}^{k} CH^{\bullet - k - r}(\Omega_X)$ and $CH^{\bullet}(\widetilde{X} - P(X)) \simeq \oplus_{r=0}^{k-1} CH^{\bullet - r}(\Omega_Z - \Omega_X)$

Note that $\Omega_X \subset \Omega_Z$ closed and has pure codimension $k+1$, so by (2) and (3) we get a long exact sequence

$$\ldots \to \oplus_{r=0}^{k-1} CH^{\bullet-r}(\Omega_Z - \Omega_X, 1) \to \oplus_{r=1}^{k} CH^{\bullet-k-r}(\Omega_X) \to \oplus_{r=0}^{k-1} CH^{\bullet-r}(\Omega_Z) \to \oplus_{r=0}^{k-1} CH^{\bullet-r}(\Omega_Z - \Omega_X) \to 0$$

Hence we get a commutative diagram

$$\begin{tikzcd}
\ldots \to \bullet \arrow[r] \arrow[d, "\simeq"'] & CH^{\bullet-k}(P(X)) \arrow[r, "j_{1,*}"] \arrow[d, "\simeq"'] & CH^{\bullet}(\widetilde{X}) \arrow[r] \arrow[d] & CH^{\bullet}(\widetilde{X} - P(X)) \to 0 \arrow[d, "\simeq"'] \\
\ldots \to \bullet \arrow[r] & \oplus_{r=1}^{k} CH^{\bullet-k-r}(\Omega_X) \arrow[r] & \oplus_{r=0}^{k-1} CH^{\bullet-r}(\Omega_Z) \arrow[r] & \oplus_{r=0}^{k-1} CH^{\bullet-r}(\Omega_Z - \Omega_X) \to 0
\end{tikzcd}$$

Now we have results from [Lewis1] that also holds for general complete intersections satisfying the conditions we examined.

Proposition 63 $(\sum_{r=0}^{k-1} \mu^r \circ \rho^*) + j_{1,*} \circ \rho_X^* : \{\oplus_{r=0}^{k-1} CH^{\bullet-r}(\Omega_Z)\} \oplus CH^{\bullet-k}(\Omega_X) \cong CH^{\bullet}(\widetilde{X})$ *is an isomorphism.*

Remark 64 *Proof is essentially the same as the proof of Proposition* **??**. *Generally this results holds for any "good" cohomology theory.*

Corollary 65 $(\sum_{r=0}^{k-1} \mu^r \circ \rho^*) + j_{1,*} \circ \rho_X^* : \{\oplus_{r=0}^{k-1} A^{\bullet-r}(\Omega_Z)\} \oplus A^{\bullet-k}(\Omega_X) \cong A^{\bullet}(\widetilde{X})$ *is an isomorphism.*

Recall $\phi_* := \pi_{X,*} \circ \rho_X^*$. Now with these results and Proposition **??** and Corollary **??** we get

Theorem 66 *Let* $X \subset \mathbb{P}^{n+r}$ *be a general complete intersection of type* $(d_1, \ldots, d_r)$ *satisfying condition* (2.2.1) *with* $k = [(n+1-\sum_{i\neq s} d_i)/d_s]$, $d_s = \max\{d_1, \ldots, d_r\}$. *Then the following homomorphisms are surjective:*

1. $\phi_* : CH^{\bullet-k}(\Omega_X) \otimes \mathbb{Q} \longrightarrow \{CH^{\bullet}(X)/j^*(CH^{\bullet}(Z))\} \otimes \mathbb{Q}$
2. $\phi_* : A^{\bullet-k}(\Omega_X) \longrightarrow A^{\bullet}(X)/j^*(A^{\bullet}(Z))$

Corollary 67 $\phi_* : A^{\bullet-k}(\Omega_X) \longrightarrow A^{\bullet}(X)/i^*(A^{\bullet}(W))$ *is surjective.*

Proof. For $X = V(F_1, \ldots, F_r) \subset \mathbb{P}^{n+r}$ smooth complete intersection of type $(d_1, \ldots, d_r)$ satisfying condition (2.2.1) we defined $W = v(Z) = V(F_1, \ldots, F_{r-1}) \subset \mathbb{P}^{n+r}$ where v is the projection from a point $[0, \ldots, 0, 1] \in \mathbb{P}^{n+r+1} - Z$. Now the corollary follows from Proposition **??**; $d_r j^*(CH^{\bullet}(Z)) = i^*(CH^{\bullet}(W))$, together with divisibility. ■

Now we will view ϕ_* and ϕ^* in terms of the cylinder correspondence, that is $\phi_* = [P(X)]_*$ and $\phi^* = [P(X)^T]_*$. Our main aim now is to show $\ker(\phi^* \circ \phi_*) = \ker(\phi_*)$ for the map $\phi_* : A^{\bullet-k}(\Omega_X) \longrightarrow A^{\bullet}(X)/i^*(A^{\bullet}(W))$.
First let us recall the commutative diagram:

$$\begin{tikzcd}
P(X) \arrow[rr, "\pi_X"] \arrow[dr, "j_1"] \arrow[ddd, "\rho_X"'] & & X \arrow[ddr, "j"] & \\
& \tilde{X} \arrow[ur, "\pi"] \arrow[dr, "j_2"] \arrow[dd, "\rho"'] & & \\
& & P(Z) \arrow[r, "\pi_Z"] \arrow[dl, "\rho_Z"'] & Z \\
\Omega_X \arrow[r, "j_0"] & \Omega_Z & &
\end{tikzcd}$$

Recall

$$\phi_* := \pi_{X,*} \circ \rho_X^* = [P(X)]_* : CH^{\bullet - k}(\Omega_X) \longrightarrow CH^{\bullet}(X)$$

Note that

$$\phi^* : CH^{\bullet}(X) \longrightarrow CH^{\bullet - k}(\Omega_X)$$

Let us call $\sigma = \phi^* \circ \phi_*$, then

$$\sigma = \phi^* \circ \phi_* = (P(X)^T \circ P(X))_* : CH^{\bullet}(\Omega_X) \longrightarrow CH^{\bullet}(\Omega_X)$$

Proposition 68 *$\sigma(\sigma - m) = 0$ modulo $\phi^*(i^* CH^{\bullet}(W))$, where $m = (-1)^k q$.*

Proof. Consider the composition

$$\pi^* \circ \phi_* : CH^{\bullet - k}(\Omega_X) \xrightarrow{\phi_*} CH^{\bullet}(X) \xrightarrow{\pi^*} CH^{\bullet}(\widetilde{X})$$

Recall from Proposition **??**

$$(\sum_{r=0}^{k-1} \mu^r \circ \rho^*) + j_{1,*} \circ \rho_X^* : \{\oplus_{r=0}^{k-1} CH^{\bullet - r}(\Omega_Z)\} \oplus CH^{\bullet - k}(\Omega_X) \cong CH^{\bullet}(\widetilde{X})$$

Hence for $\xi \in CH^{\bullet - k}(\Omega_X)$ we get $\pi^* \circ \phi_*(\xi) = (\sum_{r=0}^{k-1} \mu^r \circ \rho^*(\xi_r)) + j_{1,*} \circ \rho_X^*(\tilde{\xi})$ for $\xi_r \in CH^{\bullet - r}(\Omega_Z)$ and for some $\tilde{\xi} \in CH^{\bullet - k}(\Omega_X)$.
From the definition of σ we have $\sigma(\xi) = \rho_{X,*} \circ j_1^* \circ \pi^* \circ \phi_*(\xi)$. Applying $\rho_{X,*} \circ j_1^*$ to $\pi^* \circ \phi_*(\xi)$ we get

$$\rho_{X,*} \circ j_1^* \circ \pi^* \circ \phi_*(\xi) = \rho_{X,*} \circ j_1^* \circ (\sum_{r=0}^{k-1} \mu^r \circ \rho^*(\xi_r)) + \rho_{X,*} \circ j_1^* \circ j_{1,*} \circ \rho_X^*(\tilde{\xi})$$

From [Lewis-Sertoz] we have the following result

- (Corollary 4.2) For any $\xi \in CH^{\bullet}(\Omega_X)$ we have $\rho_{X,*} \circ j_1^* \circ j_{1,*} \circ \rho_X^*(\xi) = (-1)^k \xi$ where k stands for the k-planes that we consider.

By this result we get

$$\sigma(\xi) = \rho_{X,*} \circ j_1^* \circ (\sum_{r=0}^{k-1} \mu^r \circ \rho^*(\xi_r)) + (-1)^k \tilde{\xi}$$

Note that since $\rho_{X,*} \circ j_1^*$ maps the cycle $(\sum_{r=0}^{k-1} \mu^r \circ \rho^*(\xi_r)) \in CH^{\bullet}(\widetilde{X})$ to smaller dimensional cycle in $CH^{\bullet - k}(\Omega_X)$, it follows that $\rho_{X,*} \circ j_1^* \circ (\sum_{r=0}^{k-1} \mu^r \circ \rho^*(\xi_r)) = 0$. Hence we have

$$\tilde{\xi} = (-1)^k \sigma(\xi) \Rightarrow \pi^* \circ \phi_*(\xi) = (\sum_{r=0}^{k-1} \mu^r \circ \rho^*(\xi_r)) + (-1)^k j_{1,*} \circ \rho_X^*(\sigma(\xi))$$

Now apply π_* to the above map on the left:

$$\pi_* \circ \pi^* \circ \phi_*(\xi) = \pi_*(\sum_{r=0}^{k-1} \mu^r \circ \rho^*)(\xi_r) + (-1)^k \pi_* \circ j_{1,*} \circ \rho_X^*(\sigma(\xi))$$

$$q\phi_*(\xi) = \pi_*(\sum_{r=0}^{k-1} \mu^r \circ \rho^*)(\xi_r) + (-1)^k \phi_*(\sigma(\xi)) \tag{2}$$

Now to conclude our proof let us consider $\pi_*(\sum_{r=0}^{k-1}\mu^r\circ\rho^*)$. From the commutative diagram above we see that

$$\pi_*\circ(\sum_{r=0}^{k-1}\mu^r\circ\rho^*) = \pi_*\circ(\sum_{r=0}^{k-1}\mu^r\circ j_2^*\circ\rho_Z^*) = \pi_*\circ j_2^*\circ(\sum_{r=0}^{k-1}\mu_Z^r\circ\rho_Z^*) = j^*\circ\pi_{Z,*}\circ(\sum_{r=0}^{k-1}\mu_Z^r\circ\rho_Z^*)$$

Hence $\mathrm{Im}(\pi_*\circ(\sum_{r=0}^{k-1}\mu^r\circ\rho^*)) = j^*(CH^\bullet(Z))$. By Proposition **??** we have $j^*(CH^\bullet(Z)) = i^*(CH^\bullet(W))$. Hence

$$\pi_*(\sum_{r=0}^{k-1}\mu^r\circ\rho^*) = 0 \text{ modulo } i^*(CH^\bullet(W))$$

Applying ϕ^* to (2) with this result we get modulo $\phi^*(i^*(CH^\bullet(W)))$

$$\begin{aligned} q\phi^*\circ\phi_*(\xi) &= (-1)^k\phi^*\circ\phi_*(\sigma(\xi)) \\ q\sigma(\xi) &= (-1)^k\sigma\circ\sigma(\xi) \\ (\sigma\circ\sigma-(-1)^k q\sigma)(\xi) &= 0 \\ \sigma\circ(\sigma-m) &= 0 \end{aligned}$$

■

One can also show that $\sigma\circ(\sigma-m)=0$ on $CH^{\bullet-k}_{\mathrm{hom}}(\Omega_X,\mathbb{Q})/\phi^*(i^*(CH^\bullet_{\mathrm{hom}}(W)))$. Then we get

Theorem 69 *There is a short exact sequence*

$$0\to\frac{(\sigma-m)CH^{\bullet-k}_{(hom)}(\Omega_X,\mathbb{Q})}{\phi^*(i^*(CH^\bullet_{(hom)}(W,\mathbb{Q})))\cap((\sigma-m)CH^{\bullet-k}_{(hom)}(\Omega_X,\mathbb{Q}))}\hookrightarrow\frac{CH^{\bullet-k}_{(hom)}(\Omega_X,\mathbb{Q})}{\phi^*(i^*(CH^\bullet_{(hom)}(W,\mathbb{Q})))}\xrightarrow{\phi_*}\frac{CH^\bullet_{(hom)}(X,\mathbb{Q})}{i^*(CH^\bullet_{(hom)}(W,\mathbb{Q}))}\to 0$$

and

$$\phi_*:\sigma\left(\frac{CH^{\bullet-k}_{(hom)}(\Omega_X,\mathbb{Q})}{\phi^*(i^*(CH^\bullet_{(hom)}(W,\mathbb{Q})))}\right)\xrightarrow{\cong}\frac{CH^\bullet_{(hom)}(X,\mathbb{Q})}{i^*(CH^\bullet_{(hom)}(W,\mathbb{Q}))}$$

Note 70 *For $X\subset\mathbb{P}^{n+1}$ a general smooth hypersurface of dimension n the short exact sequence above becomes*

$$0\longrightarrow(\sigma-m)CH^{\bullet-k}_{(hom)}(\Omega_X,\mathbb{Q})\hookrightarrow CH^{\bullet-k}_{(hom)}(\Omega_X,\mathbb{Q})\xrightarrow{\phi_*}CH^\bullet_{(hom)}(X,\mathbb{Q})\longrightarrow 0$$

and

$$\phi_*:\sigma(CH^{\bullet-k}_{(hom)}(\Omega_X,\mathbb{Q}))\xrightarrow{\cong}CH^\bullet_{(hom)}(X,\mathbb{Q})$$

Since in this case $W=V(0)=\mathbb{P}^{n+1}$ and $CH^\bullet_{(hom)}(\mathbb{P}^{n+1})=0$.

Proof.
From Theorem **??** we have the surjectivity part of the short exact sequence. The injective part is also clear. We need to show $\ker(\phi_*)=\frac{(\sigma-m)CH^{\bullet-k}_{(\mathrm{hom})}(\Omega_X,\mathbb{Q})}{\phi^*(i^*(CH^\bullet_{(\mathrm{hom})}(W,\mathbb{Q})))\cap((\sigma-m)CH^{\bullet-k}_{(\mathrm{hom})}(\Omega_X,\mathbb{Q}))}$.
From (2), which is also valid in the case we tensored the Chow groups with $\mathbb{Q}$, we get modulo $i^*(CH^\bullet_{(\mathrm{hom})}(W,\mathbb{Q}))$

$$\begin{aligned} q\phi_*(\xi) &= (-1)^k\phi_*(\sigma(\xi)) \\ (-1)^k q\phi_*(\xi) &= \phi_*(\sigma(\xi)) \\ \phi_*(\sigma-(-1)^k q)(\xi) &= 0 \end{aligned}$$

Hence $(\sigma - m)\frac{CH^{\bullet-k}_{(\mathrm{hom})}(\Omega_X,\mathbb{Q})}{\phi^*(i^*(CH^{\bullet}_{(\mathrm{hom})}(W,\mathbb{Q})))} \subset \ker(\phi_*)$. But $\sigma = \phi^* \circ \phi_*$ implies modulo $\phi^*(i^*(CH^{\bullet}_{(\mathrm{hom})}(W,\mathbb{Q})))$

$$\mathrm{Im}(\sigma - m) \subset \ker(\phi_*) \subset \ker(\sigma).$$

Now consider $(\sigma - m)(\ker(\sigma)) = -m\ker(\sigma) = m\ker(\sigma) \subset \mathrm{Im}(\sigma - m)$, the inclusion here follows from Proposition **??**. Hence we have

$$m\ker(\sigma) \subset \mathrm{Im}(\sigma - m) \subset \ker(\phi_*) \subset \ker(\sigma).$$

Since we are working with $\mathbb{Q}$-coefficients $\ker(\sigma)$ is m-divisible, hence $\ker\phi_* = \mathrm{Im}\ (\sigma - m)$ modulo $\phi^*(i^*(CH^{\bullet}_{(\mathrm{hom})}(W,\mathbb{Q})))$. ■

Corollary 71 *Let us assume that* $\ker(\sigma)$ *on* $A^{\bullet}(\Omega_X)$ *is m-divisible. Then there is a short exact sequence*

$$0 \longrightarrow \frac{(\sigma - m)A^{\bullet-k}(\Omega_X)}{\phi^*(i^*(A^{\bullet}(W))) \cap ((\sigma - m)A^{\bullet-k}(\Omega_X))} \hookrightarrow \frac{A^{\bullet-k}(\Omega_X)}{\phi^*(i^*(A^{\bullet}(W)))} \xrightarrow{\phi_*} \frac{A^{\bullet}(X)}{i^*(A^{\bullet}(W))} \longrightarrow 0\ ;$$

and $\ker(\phi_*) = \ker(\sigma)$ *in the above short exact sequence.*

2.5 Representability for Complete Intersections

Recall that in the previous section we showed that representability of $A^*(X)$ implies Level$(H^*(X)) \leq 1$. Now we will show that the converse implication holds as well.

Theorem 72 *Let $X = V(F_1, \ldots, F_r) \subset \mathbb{P}^{n+r}$ be a smooth generic complete intersection of type $(d_1, \ldots, d_r)$ satisfying condition (2.2.1). Then Level$(H^*(X)) \leq 1 \Rightarrow A^*(X) \cong J^*_{alg}(X) = J^*(X)$*

Remark 73 *Let X a smooth generic complete intersection of dimension n satisfying condition (2.2.1) and having Level$(H^*(X)) \leq 1$ then $J^*(X)_{alg} = J^*(X)$. Here is the reason:*
By definition

$$J^r(X) = \frac{H^{2r-1}(X, \mathbb{C})}{H^{2r-1}(X, \mathbb{Z}) \oplus F^r H^{2r-1}(X, \mathbb{C})} \text{ and } J^r_{alg}(X) = \frac{[N^{r-1}H^{2r-1}(X, \mathbb{Q})] \otimes \mathbb{C}}{F^r(\textit{numerator}) \oplus (H^{2r-1}(X, \mathbb{Z}) \cap (\textit{numerator}))}$$

*Since X is a complete intersection if $2r-1 \neq n$ then $J^r(X) = 0 = J^r(X)_{alg}$. So lets consider the case $2r-1 = n$; we are looking at X with Level$(H^*(X)) \leq 1$, if Level$(H^*(X)) = 0$ then $H^{2r-1}(X, \mathbb{C}) = 0 \Rightarrow J^r(X) = 0 = J^r(X)_{alg}$. Now for Level$(H^*(X)) = 1 = n - 2k$ we have $H^{2r-1}(X, \mathbb{C}) = H^{r-1,r}(X) \oplus H^{r,r-1}(X) = F_h^{r-1}H^{2r-1}(X, \mathbb{Q})$. Also note that $r-1 = k$ and by Proposition **??** we have GHC(k,n,X) implying $F_h^{r-1}H^{2r-1}(X, \mathbb{Q}) = N^{r-1}H^{2r-1}(X, \mathbb{Q}) \Rightarrow J^r(X) = J^r(X)_{alg} \Rightarrow J^*(X) = J^*(X)_{alg}$.*

Before giving the proof of the Theorem **??**, lets look at the all possible cases of a complete intersection satisfying condition (2.2.1) and have level less than or equal to 1.

Proposition 74 *All possible cases with $k \geq 0$, Level$(H^*(X)) \leq 1$ and Condition (2.2.1) satisfied other than $\mathbb{P}^N, N \geq 1$*

r	n	d_1	d_2	d_3
1	1	$d \geq 2$		
1	2	2		
1	2	3		
1	3	2		
1	3	3		
1	3	4		
1	4	2		
1	5	2		
1	5	3		
1	$n \geq 6$	2		
2	1	2	$d_2 \geq 2$	
2	1	3	$d_2 \geq 3$	
2	2	2	2	
2	3	2	2	
2	3	2	3	
2	$n \geq 4$	2	2	
3	1	2	2	$d_3 \geq 2$
3	3	2	2	2

Here r =number of the polynomial defining X, d_i = degree of the i^{th} polynomial, n = dimension of X.

Proof. Recall that Level($H^*(X)$) =level($H^n(X)$) $= n - 2k$ where

$$k = [(n - \sum_{s \neq i}(d_i - 1) + 1)/d_s] = [(n + r - \sum_{s \neq i} d_i)/d_s]$$

We assume $2 \leq d_1 \leq d_2 \leq \ldots \leq d_r$, hence $d_s = d_r$.
We want $n - 2k$ =Level($H^*(X)$) ≤ 1 and $k \geq 0$. This implies

$$\frac{n-1}{2} \leq k \leq \frac{n + r - \sum_{i=1}^{r-1} d_i}{d_r}$$

We have $2 \leq d_1 \leq d_2 \leq \ldots \leq d_{r-1} \leq d_r$ so

$$\frac{n-1}{2} \leq k \leq \frac{n + r - 2(r-1)}{d_r} \Rightarrow r \leq n - \frac{d_r}{2}(n-1) + 2 \leq n - (n-1) + 2 = 3$$

Hence Level($H^*(X)$) ≤ 1 can only be satisfied if $r \leq 3$. Now lets look at the possible n values for $r \leq 3$.

1. **r=1;** We have

$$\frac{n-1}{2} \leq k = [\frac{n+1}{d}] \leq \frac{n+1}{d} \Rightarrow n(d-2) \leq d+2$$

There are two possibilities either $d = 2$ or $d > 2$

(a) d=2; Consider $n = 2a, a \in \mathbb{N}$ then $k = a$ and the condition (2.2.1) will become $a(2a + 2 - a) + 1 - \binom{2+a}{a} = \frac{a^2+a}{2} \geq 0$

Now consider $n = 2a + 1, a \in \mathbb{N}$ then $k = a + 1$ and the condition (2.2.1) will be

$(a+1)(2a + 1 + 1 + 1 - a - 1) + 1 - \binom{2+a+1}{a+1} = \frac{a^2+a}{2} \geq 0$

(b) $d > 2$; We have $n \leq \frac{d+2}{d-2} = 1 + \frac{4}{d-2}$. In this case
- $d = 3 \Rightarrow n \leq 5$
- $d = 4 \Rightarrow n \leq 3$
- $d \geq 5 \Rightarrow n \leq 1$

(n, d) in the above range satisfies the condition (2.2.1) but $(4, 3)$ and $(2, 4)$ does not satisfy level condition.

2. **r=2;** We have

$$\frac{n-1}{2} \leq k = [\frac{n + 2 - d_1}{d_2}] \leq \frac{n + 2 - d_1}{d_2} \Rightarrow n(d_2 - 2) \leq 4 - 2d_1 + d_2$$

There are two possibilities either $d_2 = 2$ or $d_2 > 2$

(a) $d_2 = 2 \Rightarrow d_1 = 2$; Consider $n = 2a, a \in \mathbb{N}$ then $k = a$ and the condition (2.2.1) will be

$a(2a + 1 + 2 - a) + 2 - 2\binom{2+a}{a}$ =0

Now consider $n = 2a + 1, a \in \mathbb{N}$ then $k = a$ and the condition (2.2.1) will be

$a(2a + 1 + 1 + 2 - a) + 2 - 2\binom{2+a}{a} = a \geq 0$

(b) $d_2 > 2$; We have $n \leq \frac{4-2d_1+d_2}{d_2-2} = 1 + \frac{6-2d_1}{d_2-2} \leq 1 + \frac{2}{d_2-2}$ In this case
• $d_2 = 3 \Rightarrow n \leq 3$, and for each those n, $d_1 \leq 3$
• $d_2 = 4 \Rightarrow n \leq 2$, and for $n = 1, d_1 \leq 3$, $n = 2, d_1 \leq 4$
• $d_2 \geq 5 \Rightarrow n \leq 1$, and $d_1 \leq 3$
One can check easily that these (n, d_1, d_2) satisfies the condition (2.2.1) but the tuples $(2,2,4), (2,3,4), (2,4,4), (2,2,3), (2,3,3)$ and $(3,3,3)$ do not satisfy the level condition.

3. **r=3;** We have

$$\frac{n-1}{2} \leq k = [\frac{n+3-d_1-d_2}{d_3}] \leq \frac{n+3-d_1-d_2}{d_3} \Rightarrow n(d_3-2) \leq 6-2d_1-2d_2+d_3$$

There are two possibilities either $d_3 = 2$ or $d_3 > 2$

(a) $d_3 = 2 \Rightarrow d_1 = d_2 = d_3 = 2$; Consider $n = 2a, a \in \mathbb{N}$ then $k = a - 1$ and the condition (2.2.1) will be
$(a-1)(2a+1+3-(a-1)) + 3 - 3\begin{pmatrix} 2+a-1 \\ a-1 \end{pmatrix} = \frac{-(a-4)(a-1)}{2}$ and $\frac{-(a-4)(a-1)}{2} \geq 0$ for $a = 1,2,3,4$ that is $n = 2,4,6,8$. But these do not satisfy the level condition.
Now let $n = 2a+1, a \in \mathbb{N}$ then $k = a$ and the condition (2.2.1) will be $a(2a+1+1+3-a) + 3 - 3\begin{pmatrix} 2+a \\ a \end{pmatrix} = \frac{a(1-a)}{2}$ and $\frac{a(1-a)}{2} \geq 0$ only for a=0 and a=1, and this gives $n = 1$ and $n = 3$.

(b) $d_3 > 2$; We have

$$n \leq \frac{6-2d_1-2d_2+d_3}{d_3-2} = 1 + \frac{8-2d_1-2d_2}{d_3-2} \leq 1$$

So we have either $n = 0$ or $n = 1$. Now for $n = 1$ consider $k = [\frac{4-d_1-d_2}{d_3}]$. We want $k \geq 0$ hence $d_1 = d_2 = 2$ is the only possibility which makes $k = 0$ and condition (2.2.1) satisfied.

■

Proposition 75 *Let X be a smooth generic complex projective complete intersection with Level $(H^*(X)) \leq 1$. If X does not satisfy (2.2.1) then $X \subset \mathbb{P}^{n+3}$ is a complete intersection of type (2,2,2) with odd dimesion $n \geq 5$.*

Proof.
Recall that Level$(H^*(X))$ =level$(H^n(X)) = n - 2k$ where

$$k = [(n - \sum_{s \neq i}(d_i - 1) + 1)/d_s] = [(n + r - \sum_{s \neq i} d_i)/d_s]$$

We assume $2 \leq d_1 \leq d_2 \leq \ldots \leq d_r$, hence $d_s = d_r$.
We have $n - 2k$ =Level$(H^*(X)) \leq 1$. This implies

$$\frac{n-1}{2} \leq k \leq \frac{n+r-\sum_{i=1}^{r-1} d_i}{d_r}$$

$d_r n - d_r \leq 2n + 2r - 2d_1 - 2d_2 - \ldots - 2d_{r-1} \Rightarrow n(d_r - 2) \leq 2r - 2d_1 - 2d_2 - \ldots - 2d_{r-1} + d_r$

- For $d_r > 2$

$$\Rightarrow n \leq \frac{2r - 2d_1 - \ldots - 2d_{r-1} + d_r}{d_r - 2} = 1 + \frac{2(r + 1 - d_1 - d_2 - \ldots - d_{r-1})}{d_r - 2}$$

  - If $d_1 + d_2 + \ldots + d_{r-1} \geq r + 1$ then $n = 1$ or $n = 0$. In either case $k = 0$ and the condition (2.2.1) will be satisfied.

  - If $d_1 + d_2 + \ldots + d_{r-1} < r + 1$; we have $d_i \geq 2$

$$\Rightarrow 2(r - 1) \leq d_1 + \ldots + d_{r-1} < r + 1$$

    This is possible only when r=1 and r=2 (as in this case $2 < \frac{r+1}{r-1} = 1 + \frac{2}{r-1}$). So lets consider each case

    1. $r = 1$

$$\frac{n-1}{2} \leq \frac{n+1}{d} \Rightarrow d(n-1) \leq 2(n+1) \Rightarrow n \leq 1 + \frac{4}{d-2}$$

       For $d > 2$ we have $n = 1$ and $k = 0$ which implies condition (2.2.1) holds. For $d = 2$ from Proposition **??** we have condition (2.2.1) satisfied.

    2. $r = 2$

$$\frac{n-1}{2} \leq \frac{n + 2 - d_1}{d_2} \Rightarrow n \leq \frac{4 - 2d_1 + d_2}{d_2 - 2} = 1 + \frac{2(3 - d_1)}{d_2 - 2}$$

       For $d_1 = d_2 = 2$ we have from the Proposition **??** that condition (2.2.1) will be satisfied.
       For $d_2 > 2$ we can have

       $d_1 \geq 3 \Rightarrow n = 1 \Rightarrow k = 0$ so the condition will be satisfied.
       $d_1 = 2, d_2 = 3 \Rightarrow n = 3 \Rightarrow k = 1$ and condition will be satisfied.
       $d_1 = 2, d_2 \geq 4 \Rightarrow n = 1, k = 0$ and condition will be satisfied.

- For $d_r = 2 \Rightarrow d_1 = d_2 = \ldots = d_r = 2$

$$\frac{n-1}{2} \leq k \leq \frac{n + r - \sum_{i=1}^{r-1} d_i}{d_r} = \frac{n + r - (r-1)2}{2} = \frac{n - r + 2}{2} = \frac{n-1}{2} + \frac{3-r}{2}$$

This is possible when $r \leq 3$. For $r = 1$ and $r = 2$ we covered them above. Now for r=3; if we consider $n = 2a, a \in \mathbb{N}$ then $n - 2k = 2$ so it will not satisfy the level condition. Then when we consider $n = 2a + 1$ then $k = a$ and the condition (2.2.1) will not be satisfied for a values such that $\frac{a(1-a)}{2} < 0$ this implies $a > 1 \Rightarrow n \geq 5$ odd number. ■

Example 76 *In the Appendix one can find the examples of hypersurfaces and complete intersections which satisfy condition (2.2.1) but has Level($H^*(X)$) > 1.*

From the list given in Proposition **??** we can see that for X satisfying condition (2.2.1) and having level less than or equal to 1, one also has condition (2.2.1) holds for the W obtained from that X. Now we can give the proof of Theorem **??**.

Proof. of Theorem ??
We will use induction on the number of polynomials defining X. Let us first show the case for $r = 1$

Proposition 77 *Let $X = V(F_1) \subset \mathbb{P}^{n+1}$ a general hypersurface satisfying condition* (2.2.1). *Then*

$$\mathit{Level}(H^*(X)) \leq 1 \Rightarrow A^*(X) \cong J^*(X).$$

Proof. For $X = V(F_1) \subset \mathbb{P}^{n+1}$ a general hypersurface, then $W = \mathbb{P}^{n+1} \Rightarrow A^\bullet(W) = 0$. Then from Theorem **??** (2) we have surjective homomorphism

$$\phi_* : A^{\bullet - k}(\Omega_X) \longrightarrow A^\bullet(X)$$

Note that $\dim \Omega_X = n - 2k =$Level$(H^*(X)) \leq 1$ by our assumption.

1. For $\bullet - k > 1$ or $\bullet - k \leq 0$; $A^{\bullet - k}(\Omega_X) = 0 \Rightarrow A^\bullet(X) = 0$. Also $A^\bullet(X) = 0 \Rightarrow J^\bullet(X) = 0$ since by Abel-Jacobi map $A^\bullet(X) \longrightarrow J^\bullet(X)$ is surjective.

2. For $\bullet - k = 1$; whe have the surjective map $\phi_* : A^1(\Omega_X) \longrightarrow A^{k+1}(X)$. Note that $A^1(\Omega_X) \cong J^1(\Omega_X)$ by Abel-Jacobi map ϕ_1. Then from the functoriality of $A^*(-)$ we have

$$\begin{tikzcd} A^1(\Omega_X) \arrow[r, "\phi_*"] \arrow[d, "\cong"'] & A^{k+1}(X) \arrow[d, "\phi_{k+1}"] \\ J^1(\Omega_X) \arrow[r] & J^{k+1}(X) \end{tikzcd}$$

We want to show ϕ_{k+1} is an isomorphism and for that we will consider a special version of the short exact sequence given in the Corollary **??** with the Abel-Jacobi maps:

Corollary 78 *There exists a short exact sequence*

$$0 \longrightarrow (\sigma - m)A^1(\Omega_X) \hookrightarrow A^1(\Omega_X) \xrightarrow{\phi_*} A^{k+1}(X) \longrightarrow 0$$

and $\ker(\phi_*) = \ker(\sigma)$.

Proof.
Injectivity is clear, surjectivity follows from Corollary **??** . We will show $\ker(\phi_*) = \ker(\sigma) = \text{ Im } (\sigma - m)$.

1. $\text{Im}(\sigma - m) \subset \ker(\phi_*) \subset \ker(\sigma)$:
   We have $\sigma = \phi^* \circ \phi_*$, so $\ker(\phi_*) \subset \ker(\sigma)$. On the other hand from (2) we get $\phi_*(\sigma - m) = 0$ on $A^1(\Omega_X) \Rightarrow \text{ Im } (\sigma - m) \subset \ker(\phi^*)$.

2. $\ker(\sigma) = \text{ Im } (\sigma - m)$:
   We have the quadratic relation $\sigma \circ (\sigma - m) = 0$ and this is the assumption I in [Bloch-Murre](7.1) and by Lemma 7.2 in [Bloch-Murre] we get

(a) $\{\ker\sigma\}^0 = \text{ Im }(\sigma - m)$ and $\{\ker(\sigma - m)\}^0 = \text{ Im }(\sigma)$, where $\{\cdots\}^0$ means connected component of identity.

(b) $\{\ker\sigma\}^0 + \{\ker(\sigma - m)\}^0 = J^1(\Omega_X)$ and $\{\ker\sigma\}^0 \cap \{\ker(\sigma - m)\}^0 \subset J^1(\Omega_X)_m$ where $J^1(\Omega_X)_m$ is the $(m)-$torsion subgroup of $J^1(\Omega_X)$.

Now let $\tau : \{\ker\sigma\}^0 \times \{\ker(\sigma - m)\}^0 \longrightarrow J^1(\Omega_X)$ defined by $(x, y) \longrightarrow x + y$ then one has $\ker\tau = \{(x, -x) \in \{\ker\sigma\}^0 \times \{\ker(\sigma - m)\}^0 | x \in \{\ker\sigma\}^0 \cap \{\ker(\sigma - m)\}^0 \subset J^1(\Omega_X)_m\}$. Also consider the projection map $pr_2 : \{\ker\sigma\}^0 \times \{\ker(\sigma - m)\}^0 \longrightarrow \{\ker(\sigma - m)\}^0$ then by lemma 7.7 in [Bloch-Murre] we have the following equivalent statements

(a) $\{\ker\sigma\}^0 = \ker\sigma$

(b) $Pr_2(\ker\tau) = \{\ker(\sigma - m)\}^0_m$

(c) $j_2^*(\tau^*\theta) = mE$ with E a divisor on $\{\ker(\sigma - m)\}^0$, $j_2 : \{\ker(\sigma - m)\}^0 \longrightarrow \{\ker\sigma\}^0 \times \{\ker(\sigma - m)\}^0 \longrightarrow \{\ker(\sigma - m)\}^0$ and θ is a principal polarization of $J^1(\Omega_X)$.

Lemma 7.11 in [Bloch-Murre] implies this equivalent conditions holds for σ and hence $\ker(\sigma) = \text{ Im }(\sigma - m) \Rightarrow \ker(\phi_*) = \ker(\sigma) = \text{ Im }(\sigma - m)$. So we get the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & (\sigma - m)A^1(\Omega_X) \arrow[r] \arrow[d, "\cong"'] & A^1(\Omega_X) \arrow[r, "\phi_*"] \arrow[d, "\cong"'] & A^{k+1}(X) \arrow[r] \arrow[d, "\phi_{k+1}"'] & 0 \\
0 \arrow[r] & (\sigma - m)J^1(\Omega_X) \arrow[r] & J^1(\Omega_X) \arrow[r] & J^{k+1}(X) \arrow[r] & 0
\end{tikzcd}$$

Now from five lemma we get $A^{k+1}(X) \cong J^{k+1}(X) =$ and hence $A^*(X) \cong J^*(X)$. Note that $J^*_{\text{alg}}(X) = J^*(X)$ as we have surjective map from $J^1(\Omega_X) \longrightarrow J^*(X)$. ■
This proves Proposition **??**. ■
Now with having $r = 1$ case covered, assume our result is true for complete intersections determined by $r - 1$ polynomials, satisfying condition (2.2.1) and having level less than or equal to 1. Note that $W = V(F_1, \ldots, F_{r-1}) \subset \mathbb{P}^{n+1+(r-1)}$ has dimension $n + 1$. And as X satisfies condition (2.2.1) W also satisfies condition (2.2.1) from the observation we had above. To apply our induction hypothesis to W we should show it also have Level $(H^*(W)) \leq 1$.

Proposition 79 *Level*$(H^*(X)) \geq$ *Level*$(H^*(W))$.

Proof.
Note that W is also a complete intersection with $\dim W = n + 1$ and dimension of X is n. Let's consider Level$(H^*(X)) = \max\{0, n - 2k_X\}$, where $n - 2k_X$ =level$(H^n(X))$, and $k_X = [(n + r - \sum_{i\neq s} d_i)/d_s]$, $d_s = \max d_1, \ldots, d_r$. Also Level$(H^*(W)) = \max\{0, n + 1 - 2k_W\}$ where $n + 1 - 2k_W$ =level$(H^{n+1}(W))$ and $k_W = [(n+r-\sum_{i\neq s'} d_i)/d_{s'}]$, $d_{s'} = \max d_1, \ldots, d_{r-1}$. We will show that $n+1-2k_w \leq n - 2k_X$ which is equivalent to showing $k_W - \frac{1}{2} \geq k_X$.
As before assume $d_s = d_r$ and $d_{s'} = d_{r-1}$. Then

$$k_X = [\frac{n + r - (d_1 + \ldots + d_{r-1})}{d_r}], k_W = [\frac{n + r - (d_1 + \ldots + d_{r-2})}{d_{r-1}}]$$

$$\Rightarrow k_W - \frac{1}{2} = [\frac{n+r-(d_1+\ldots+d_{r-2})}{d_{r-1}}] - \frac{1}{2} \geq [\frac{n+r-(d_1+\ldots+d_{r-2})}{d_{r-1}} - 1]$$

$$\Rightarrow k_W - \frac{1}{2} \geq [\frac{n+r-(d_1+\ldots+d_{r-1})}{d_{r-1}}]$$

We have $d_{r-1} \leq d_r$

$$\Rightarrow k_W - \frac{1}{2} \geq [\frac{n+r-(d_1+\ldots+d_{r-1})}{d_{r-1}}] \geq [\frac{n+r-(d_1+\ldots+d_{r-1})}{d_r}] = k_X$$

$$\Rightarrow k_W - \frac{1}{2} \geq k_X \Rightarrow n+1-2k_W \leq n-2k_X$$

Hence $\mathrm{Level}(H^*(X)) \geq \mathrm{Level}(H^*(W))$. ∎

From this proposition we have for X satisfying condition (2.2.1), $\mathrm{Level}(H^*(X)) \leq 1 \Rightarrow \mathrm{Level}(H^*(W)) \leq 1$. Now by induction hypothesis $A^*(W) \cong J^*(W)$.

From Theorem **??** (2) we have surjective homomorphism

$$\phi_* : A^{\bullet-k}(\Omega_X) \longrightarrow A^\bullet(X)/j^*(A^\bullet(Z)) = A^\bullet(X)/i^*(A^\bullet(W))$$

Recall $\dim \Omega_X = n-2k$, and with our assumption $\mathrm{Level}(H^*(X)) = n-2k \leq 1$, $\dim \Omega_X \leq 1$. Then we have the following cases:

1. $\bullet - k > 1 \Rightarrow A^{\bullet-k}(\Omega_X) = 0 \Rightarrow A^\bullet(X) = i^*(A^\bullet(W))$
2. $\bullet - k \leq 1$ then we have two possibilities
   (a) $\bullet - k = 1$ we get a surjective map $A^1(\Omega_X) \longrightarrow A^{k+1}(X)/i^*(A^{k+1}(W))$ and by induction $A^{k+1}(W) \cong J^{k+1}(W)$. Since W is also a complete intersection we have
      i. $J^{k+1}_{\mathrm{alg}}(W) = 0$ for $k \neq \frac{n}{2}$ or n odd $\Rightarrow i^*(A^{k+1}(W)) = 0$
      ii. for $k = \frac{n}{2}$ and n even, $\dim \Omega_X = n-2k = 0 \Rightarrow A^1(\Omega_X) = 0 \Rightarrow A^{\frac{n}{2}+1}(X) = i^*(A^{\frac{n}{2}+1}(W)) \cong i^*(J^{\frac{n}{2}+1}(W))$. Also $0 = A^1(\Omega_X) \cong J^1(\Omega_X) \Rightarrow J^{\frac{n}{2}+1}(X) \cong i^*(J^{\frac{n}{2}+1}(W))$. Now by the weak Lefschetz theorem, $J^{\frac{n}{2}+1}(X) = 0$ which implies $i^*(J^{\frac{n}{2}+1}(W)) = 0$ and hence $A^{\frac{n}{2}+1}(X) = i^*(A^{\frac{n}{2}+1}(W)) = 0$.
   (b) $\bullet - k < 1 \Rightarrow \bullet - k \leq 0 \Rightarrow A^{\bullet-k}(\Omega_X) = 0 \Rightarrow A^\bullet(X) = i^*(A^\bullet(W))$

As a summary

- For $\bullet \neq k+1$ we have a commutative diagram with i^* and ϕ surjective

$$\begin{array}{ccc} A^\bullet(W) & \xrightarrow{i^*} & A^\bullet(X) \\ \simeq\downarrow & & \phi\downarrow \\ J^\bullet(W) & \xrightarrow{i^*} & J^\bullet(X) \end{array}$$

  If we have $J^\bullet(W) = 0$ then we can conclude $A^\bullet(X) \simeq J^\bullet(X)$. Since W is a complete intersection there is a possibility of $J^\bullet(W) \neq 0$ if $(2\bullet-1) = \dim W = n+1 \Rightarrow n$ must be an even number. This forces $\mathrm{Level}(H^*(X)) = n-2k = 0$ (Otherwise $n-2k = 1$). By Proposition **??** we get $0 = \mathrm{Level}(H^*(W)) = \mathrm{level}(H^{2\bullet-1}(W)) \Rightarrow H^{2\bullet-1}(W) = 0 \Rightarrow J^\bullet(W) = 0$. Hence $A^\bullet(X) \simeq J^\bullet(X)$.

- For $\bullet = k+1$ we have either $A^{k+1}(X) = J^{k+1}(X) = 0$ or a surjective map

$$A^1(\Omega_X) \longrightarrow A^{k+1}(X)$$

Hence to show $A^\bullet(X) \cong J^\bullet(X)$ we only need to consider the case when $\bullet - k = 1$. The proof of this case is the same as the case $\bullet - k = 1$ in the proof of Proposition **??**. ■

With this result we can state the following theorem proving Lewis' conjecture for complete intersections satisfying a certain condition:

Theorem 80 *Let $X \subset \mathbb{P}^{n+r}$ be a general smooth complete intersection of type $(d_1, \ldots, d_r)$ satisfying condition (2.2.1). Then Level$(H^*(X)) \leq 1 \Leftrightarrow A^*(X) \cong J^*(X)$.*

Corollary 81 *If $X \subset \mathbb{P}^{n+1}$ is a general smooth hypersurface satisfying condition (2.2.1). Then Level$(H^*(X)) \leq 1 \Leftrightarrow A^*(X) \cong J^*(X)$.*

Chapter 3

Motives

Algebraic cycles of codimension higher than one is a mysterious subject in that we do not have a complete description of these objects. In the previous sections we gave a partial description of algebraic cycles of complete intersection (satisfying certain property). For a complete understanding of algebraic cycles and Chow groups, a conjectural filtration on the Chow groups was introduced by Bloch and Beilinson [Jann2]. One approach to constructing such a (candidate) filtration is via the work of J. Murre [Mur, page 149], where a decomposition of the Chow motive of projective algebraic manifolds, on the level of rational equivalence, is used to construct such a filtration. In this section our goal is to compute the motive of a smooth projective complete intersection in terms of the motive of its Fano variety, as a way of arriving at partial results for the aforementioned filtration. In particular, we will examine one of Murre's conjectures on Chow motives of complex projective complete intersections in terms of a corresponding statement about its Fano variety.

3.1 Pure Chow Motives

The theory of motives was created by Grothendieck in mid 1960's and these objects carry all different cohomology groups associated to a projective variety. Motives arise from the phenomena of what is in common with all cohomology theories. They can be thought as a universal cohomology theory carrying properties of other cohomology theories. We have an explicit definition for the construction of motives however, the properties that it satisfies are conjectural.
We will work with pure motives constructed from smooth projective varieties that are defined over $\mathbb{C}$. For the construction we need to fix an equivalence relation $\sim$ on the algebraic cycles of X.

Definition 82 *The group of correspondences between two smooth projective varieties X and Y with respect to $\sim$ is defined as*

$$Corr_{\sim}(X,Y) := C_{\sim}(X\times Y,\mathbb{Q}) = \bigoplus_{i=0}^{d+e} C^i_{\sim}(X\times Y,\mathbb{Q})$$

where $d=\dim X, e=\dim Y, C^i_{\sim}=\frac{Z^i(X\times Y)}{Z^i_{\sim}(X\times Y)}$.

If this $\sim$ is the rational equivalence we write $Corr(X\times Y) = CH(X\times Y,\mathbb{Q})$.

Definition 83

1. *Let $f \in Corr_\sim(X,Y)$ and $g \in Corr_\sim(Y,Z)$ then $g \bullet f \in Corr_\sim(X,Z)$ where*
$$g \bullet f = pr_{XZ}\{(gr(f) \times Z) \cap (X \times gr(g))\}$$

2. *The category of pure motives is denoted by $\mathrm{M}_\sim(\mathbb{C})$ with*
   - *Objects: $M = (X,p,m)$ where X is a smooth projective variety defined over $\mathbb{C}$, p is a projector of X and $m \in \mathbb{Z}$*
   - *Morphisms: If $M = (X,p,m)$ and $N = (Y,q,n)$ then*
$$Hom_{\mathrm{M}_\sim}(M,N) = q \bullet Corr_\sim^{n-m}(X,Y) \bullet p$$

3. *Objects $M = (X,p,m)$ are called motives with respect to $\sim$. If $\sim$ is the rational equivalence then M is called a Chow motive.*

Definition 84 *A correspondence $p \in Corr^0_\sim(X,X)$ is called a projector of X with respect to $\sim$ if $p \bullet p = p$. Two projectors are called orthogonal if $p \bullet q = q \bullet p = 0$.*

Example 85 *Let X be a smooth complex projective variety of dimension n. Let $\sim$ be the rational equivalence then*

1. *$Corr^0_\sim(X,X) = CH^n(X \times X, \mathbb{Q})$ and $Id = \Delta_X = \{(x,x)|x \in X\} \in CH^n(X \times X)$ is a projector of X.*

2. *If p is a projector on X then $\Delta_X - p$ is also a projector on X. As $(\Delta_X - p)^2 = \Delta_X^2 - \Delta_X \bullet p - p \bullet \Delta_X + p^2 = \Delta_X - p - p + p = \Delta_X - p$.*

3. *We can write $(X, Id) = (X,p) \oplus (X, Id - p)$ since p and $Id - p$ are orthogonal as $p \circ (Id - p) = p - p^2 = p - p = 0$.*

Now we can state our main theorem for this section:

Theorem 86 *Let $X \subset \mathbb{P}^{n+r}$ be a smooth complete intersection of type $(d_1, \dots, d_r)$ and $W \subset \mathbb{P}^{n+r}$ be the projection from $[0,\dots,0,1]$ of $Z \subset \mathbb{P}^{n+r+1}$ where $X = Z \cap \mathbb{P}^{n+r}$. Assume (k,n,d) is given with $k = [\frac{n - \sum_{i \neq s} d_i + 1}{d_s}]$ where $d_s = \max\{d_1, \dots, d_r\}$ satisfying*
$$k(n+1+r-k) + r - \sum_{j=1}^{r} \binom{d_j + k}{k} \geq 0$$
*Then modulo two assumptions **??** and **??** below, one can find a motivic decomposition for Ω_X*
$$(\Omega_X, Id) \quad = \quad (\Omega_X, \tilde{\tau}) \oplus (\Omega_X, Id - \tilde{\tau})$$
where
$$(\Omega_X, \tilde{\tau}, 0) \simeq (X, \tilde{\pi}_n^X, -k)$$
as pure motives, $\tilde{\pi}_n^X$ is a certain primitive projector associated to the middle dimensional cohomology of X.

For the proof of our theorem we need to clarify the aforementioned assumptions and construct projectors on Ω_X and X which will give an isomorphism between the motive of X and the motive of Ω_X. We will do these in the following sections.

3.2 Chow-Künneth Decomposition for Complete Intersections

Recall that for $X \subset \mathbb{P}^{n+r}$ an n-dimensional smooth complete intersection of type $(d_1, \ldots, d_r)$ the corresponding object in the category of pure motives is (X, p, m), where $p \in CH^n(X \times X, \mathbb{Q})$ is a projector and $m \in \mathbb{Z}$. For the construction of projectors on X and Ω_X we will use Chow-Künneth Decomposition.

Definition 87 *Let X be a smooth projective algebraic manifold of dimension n. X is said to have a Chow-Künneth decomposition if there exists a projector $\pi_i \in CH^n(X \times X, \mathbb{Q})$ for $0 \le i \le 2n$ such that*

(i) $\pi_i \bullet \pi_j = \begin{cases} \pi_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

(ii) $\Delta_X = \sum_i \pi_i$

(iii) $(\Delta_X)_{2n-i,i} = \pi_i$ *(modulo homological equivalence), i.e. the usual i^{th} Künneth component.*

The existence of a Chow-Künneth decomposition for any smooth projective algebraic manifold was conjectured by Jacob P. Murre in [Mur], where a number of other conjectures are stated:

Conjecture 88 *([Mur]) Let X be any smooth projective algebraic manifold of dimension n. Then*

1. *Conjecture I: There exists a Chow-Künneth decomposition for X.*
2. *Conjecture II: The $\pi_{2n}, \pi_{2n-1}, \ldots, \pi_{2j+1}$ and $\pi_0, \ldots, \pi_{j-1}$ operate as zero on $CH^j(X, \mathbb{Q})$ where $0 \le j \le n$.*
3. *Conjecture III: The filtration defined below is independent of the ambiguity in the choices of π_i:*
   Assuming Conjecture I and II define the filtration on $CH^j(X, \mathbb{Q})$:

$$F^0 := CH^j(X, \mathbb{Q})$$

$$F^1 := \ker(\pi_{2j})$$

$$F^2 := \ker(\pi_{2j-1}|F^1) = \ker(\pi_{2j}) \cap \ker(\pi_{2j-1})$$

   inductively

$$F^v := \ker(\pi_{2j-v+1}|F^{v-1}) = \ker(\pi_{2j}) \cap \ker(\pi_{2j-1}) \cap \ldots \cap \ker(\pi_{2j-v+1})$$

4. *Conjecture IV: $F^1 := CH^j_{hom}(X, \mathbb{Q})$*

It was shown in [Jann2] that if all of Murre's Conjectures hold then Bloch-Beilinson filtration exists and vice versa. For certain manifolds some of the Murre's Conjectures are known to hold. Here is a list of manifolds and the Murre's conjectures that are known to hold:

(a) Curves ; Conjectures I-IV (Manin)

(b) Surfaces ; Conjectures I-IV (Murre)

(c) Complete intersections; Conjecture I (Lewis)

(d) Threefolds that are product of a surface and a curve; Conjectures I-IV (Murre)

(e) Abelian varieties; Conjecture I and part of Conjectures II-IV (Shermenev, Deninger-Murre, K*ü*nnemann)

(f) Some modular varieties ; Conjecture I (Gordon-Murre, Gordon-Hanamura-Murre, Miller-Muller-Stach-Wortmann-Yang-Zuo)

(g) Product of two surfaces; Conjecture II and part of Conjecture IV (Murre)

(h) Product of two curves and a surface; Conjecture IV (Kimura)

Conjecture I holds for smooth complete intersections. Let us give an explicit construction of a Chow-Künneth decomposition of X. Let H_X be a hyperplane section of X and recall that by Lefschetz's theorems for $i \neq n$

$$H^i(X,\mathbb{Q}) = \begin{cases} \mathbb{Q}\cdot(\mathbb{P}^{n+r-m}\cap X) = \mathbb{Q}\cdot H_X^m & \text{if } i = 2m \text{ for } 0 \leq m \leq n \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

For $p+q = 2n$, we set

$$\Delta_X(p,q) = \begin{cases} \frac{1}{\deg X}(H_X^l \otimes H_X^{n-l}) & \text{if } (p,q) = (2l, 2n-2l) \neq (n,n) \\ 0 & \text{if } p \text{ or } q \text{ is odd} \end{cases}$$

and we have

$$\Delta_X(n,n) = \Delta_X - \sum_{(p,q)\neq(n,n)} \Delta_X(p,q)$$

In $CH^n(X\times X,\mathbb{Q})$ put,

$$\pi_l^X = \begin{cases} \frac{1}{\deg(X)}(H_X^{n-l/2}\times H_X^{l/2}) & \text{if } l \neq n \text{ is even} \\ 0 & \text{if } l \neq n \text{ is odd} \\ \Delta_X(n,n) & \text{if } l = n \end{cases}$$

Then we have

$$\pi_m^X \circ \pi_m^X = \pi_m^X \text{ and } \pi_m^X \circ \pi_l^X = 0 \text{ for } m \neq l$$

[Justification: Consider $X\times X\times X$ with the corresponding projections then for

- $m \neq n = \dim X$

$$\begin{aligned} \pi_m^X \circ \pi_l^X &= Pr_{13,*}(Pr_{12}^*(\pi_l^X)\cap Pr_{23}^*(\pi_m^X)) \\ &= Pr_{13,*}(\deg X^{-1} H_X^{n-l/2}\times H_X^{l/2}\times X \cap \deg X^{-1} X\times H_X^{n-m/2}\times H_X^{m/2}) \\ &= \deg X^{-1} Pr_{13,*}(H_X^{n-l/2}\times (H_X^{l/2}\cap H_X^{n-m/2})\times H_X^{m/2}) \end{aligned}$$

Note that $H_X^{l/2} \cap H_X^{n-m/2} \neq 0$ if $n - m/2 + l/2 \leq n \Rightarrow l \leq m$, if this is the case we have

$$\pi_m^X \circ \pi_l^X = \deg X^{-1} H_X^{n-l/2} \times H_X^{m/2}$$

and this is non zero if $n - l/2 + m/2 \leq n \Rightarrow m \leq l$. Hence $\pi_m^X \circ \pi_m^X = \pi_m^X$ and $\pi_m^X \circ \pi_l^X = 0$ for $m \neq l$.

- For m=n, we have $\pi_n^X \circ \pi_n^X = \pi_n^X$.

Hence we get

Proposition 89 *Let $X \subset \mathbb{P}^{n+r}$ be a smooth complete intersection. Then we have a Chow-Künneth decomposition via the projectors $\{\pi_l^X\}$*

$$\Delta_X = \pi_0^X + \ldots + \pi_{2n}^X$$

Remark 90 *Let X and W be two complete intersections with Chow-Künneth decomposition via the projectors $\{\pi_i^X\}$ and $\{\pi_j^W\}$. Then the product $X \times W$ also has a Chow-Künneth decomposition with projectors $\{\pi_l^{X \times W}\}$ where $\pi_l^{X\times W} = \bigoplus_{i+j=l} \pi_i^X \times \pi_j^W$.*

Now let us examine Murre's II Conjecture on X smooth complete intersection of dimension n with Chow-Künneth projectors $\pi_l^X, l = 0, 1, \ldots, 2n$:

- For $l \neq n$ we have

  $$\pi_{l,*} CH^r(X, \mathbb{Q}) = \Delta_X(2n - l, l)_* CH^r(X, \mathbb{Q}) = Gr_F^{2r-l} CH^r(X, \mathbb{Q})$$

  From here we see that for $2r - l < 0 \Rightarrow l > 2r$ and $2r - l > r \Rightarrow l < r$, $\pi_{l,*}$ operate as zero on $CH^r(X, \mathbb{Q})$.
  In addition to this for $\xi \in CH^r(X, \mathbb{Q})$ and $l \neq n$ we have

  $$\begin{aligned} \pi_{l,*} CH^r(X, \mathbb{Q}) = \Delta_X(2n - l, l)_* CH^r(X, \mathbb{Q}) &= Pr_{2,*}(Pr_1^*(\xi) \cap \frac{1}{\deg X} H_X^{n-l/2} \times H_X^{l/2}) \\ &= Pr_{2,*}(\xi \times X \cap \frac{1}{\deg X} H_X^{n-l/2} \times H_X^{l/2}) \end{aligned}$$

  Note that codim$(\xi \cap H^{n-l/2}) = n + r - l/2$, $\pi_{l,*}$ will operate as zero on $CH^r(X, \mathbb{Q})$ for l such that $r - l/2 > 0 \Rightarrow l < 2r$. Hence $\pi_{l,*}$ operate as zero on $CH^r(X, \mathbb{Q})$ for $l \neq n$ and $l \neq 2r$. So to check if Murre's II Conjecture holds for X we need to check only the case $l = n$.

- For $l = n < r$ we have $CH^r(X, \mathbb{Q}) = 0$ because of the dimension. For $l = n > 2r$ consider the diagram

  $$\begin{tikzcd} CH^r(X,\mathbb{Q}) \arrow[r, "\pi_{n,*}"] \arrow[d, "cl_r"'] & CH^r(X,\mathbb{Q}) \arrow[d, "cl_r"'] \\ H^{2r}(X,\mathbb{Q}) \arrow[r, "\pi_{n,*}"] & H^{2r}(X,\mathbb{Q}) \end{tikzcd}$$

  Since $n > 2r$ we have $\pi_{n,*} = 0$ on $H^{2r}(X, \mathbb{Q})$, hence $\pi_{n,*}(CH^r(X, \mathbb{Q})) \subset CH^r_{\text{hom}}(X, \mathbb{Q})$.
  The question of whether Murre's II Conjecture holds for X translates to the question of whether $CH^r_{\text{hom}}(X, \mathbb{Q}) = 0$ for $r < n/2$, which is the question of Hartshorne in [Har2](page 142).

Assumption 91 *There exists a Künneth formula*

$$CH^n(X \times W, \mathbb{Q}) = \bigoplus_{l=0}^{n} CH^{n-l}(X, \mathbb{Q}) \otimes CH^l(W, \mathbb{Q})$$

for our X and W defined above.

Example 92 *In the case $r = 1$, we have $W = \mathbb{P}^{n+1}$ hence we get such a decomposition from the projective bundle theorem on page 35.*

Assumption 93 *For any such $0 \leq l \leq n$ either*

$$CH^{n-l}(X, \mathbb{Q}) = \mathbb{Q}H_X^{n-l}$$

or

$$CH^l(W, \mathbb{Q}) = \mathbb{Q}H_W^l$$

Example 94 *Recall Hartshorne's conjecture: For a smooth compelete intersection Z of dimension m, $CH^r_{hom}(Z) = 0$ for $r < \frac{m}{2}$. This means $Z^r_{hom}(Z) = Z^r_{rat}(Z)$, and with the cycle class map being surjective by the Hodge conjecture, $CH^r(Z, \mathbb{Q}) \simeq H^{2r}(Z, \mathbb{Q}) = \mathbb{Q}H_Z^r$.*

Proposition 95 *If Hartshorne's conjecture holds, then Assumption* **??** *holds.*

Proof. Let X be a smooth complete intersection of dimension n and W be a smooth complete intersection of dimension $n+1$, then for $n-l < \frac{n}{2}$ we have $CH^{n-l}(X, \mathbb{Q}) = \mathbb{Q}H_X^{n-l}$ and for $l < \frac{n+1}{2}$ we have $CH^l(W, \mathbb{Q}) = \mathbb{Q}H_W^l$ by Hartshorne's conjecture. Now lets consider the other possible values of $n - l$ and l.
For $n - l > \frac{n}{2} \Rightarrow 2l < n$ and $l > \frac{n+1}{2} \Rightarrow 2l > n + 1$. This implies $n > 2l > n + 1$ which is not possible. Hence Assumption **??** follows. ■

3.3 Chow Motive of X and Its Fano Variety

In this section we will prove Theorem **??**. For that we will construct projectors of X, Ω_X and W that will relate the motives of these objects.
Lets start with X;
Consider $\tilde{\pi}_n^X = \pi_n^X - h_n^X$ where $\pi_n^X = \Delta_X(n,n)$ and $h_n^X = \begin{cases} \frac{1}{\deg(X)}(H_X^{n/2} \times H_X^{n/2}) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$.
Note that

$$h_n^X \circ h_n^X = h_n^X \text{ and } \pi_n^X \circ h_n^X = h_n^X \circ \pi_n^X = h_n^X$$

Then we have $\tilde{\pi}_n^X$ is a projector of X which is orthogonal to h_n^X since

$$[\tilde{\pi}_n^X]^2 = (\pi_n^X)^2 - \pi_n^X \circ h_n^X - h_n^X \circ \pi_n^X + (h_n^X)^2 = \pi_n^X - h_n^X = \tilde{\pi}_n^X$$

and

$$\tilde{\pi}_n^X \circ h_n^X = h_n^X \circ \tilde{\pi}_n^X = 0$$

To be able to give a relation between the motive of X and Ω_X we will consider the following results:

Proposition 96 *Let $X = V(F_1, \dots, F_r) \subset \mathbb{P}^{n+r}$ be a smooth projective complete intersection of type $(d_1, \dots, d_r)$ satisfying condition (2.2.1) introduced in section 2.2. Then $\phi_* \circ \phi^* = \times m$ on $CH^\bullet(X,\mathbb{Q})/i^*(CH^\bullet(W,\mathbb{Q}))$ where $W = V(F_1, \dots, F_{r-1}) \subset \mathbb{P}^{n+r}$ is a projective algebraic manifold.*

Proof. Recall that for X satisfying the above properties we get a surjective map

$$\phi_* : CH^{\bullet - k}(\Omega_X, \mathbb{Q}) \longrightarrow CH^\bullet(X,\mathbb{Q})/i^*(CH^\bullet(W,\mathbb{Q}))$$

with $\phi^* \circ \phi_* = \times(-1)^k q = \times m$ where q is the order of the map $\pi : \widetilde{X} \longrightarrow X$.
Now consider $\phi_* \circ \phi^* \circ \phi_* = \phi_* \circ (\times m) = m\phi_*$. By the surjectivity of the map ϕ_* we get our result. ■
Next we will consider the diagrams obtained by applying $X\times$:

$$\begin{tikzcd}
X \times X \arrow[r, "1\times j"] \arrow[dr, "1\times i"'] & X \times Z \arrow[d, "1\times v"] \\
 & X \times W
\end{tikzcd}$$

$$\begin{tikzcd}
X \times P(X) \arrow[rr, "1\times\pi_X"] \arrow[dr, "1\times j_1"] \arrow[ddd, "1\times\rho_X"'] & & X \times X \arrow[ddr, "1\times j"] & \\
 & X \times \tilde{X} \arrow[ur, "1\times\pi"] \arrow[dr, "1\times j_2"] \arrow[dd, "1\times\rho"'] & & \\
 & & X \times P(Z) \arrow[r, "1\times\pi_Z"] \arrow[dl, "1\times\rho_Z"] & X \times Z \\
X \times \Omega_X \arrow[r, "1\times j_0"] & X \times \Omega_Z & &
\end{tikzcd}$$

Using the same methods we did for the previous diagrams we will get the results:

1. $1\times\phi_* : CH^{\bullet-k}(X\times\Omega_X, \mathbb{Q}) \longrightarrow CH^\bullet(X\times X, \mathbb{Q})/i^*(CH^\bullet(X\times W, \mathbb{Q}))$ is surjective
2. $1 \times (\phi_* \circ \phi^*) = \times m$ on $CH^\bullet(X \times X, \mathbb{Q})/i^*(CH^\bullet(X \times W, \mathbb{Q}))$

Hence for $\bullet = n$ we have

$$[1 \times (\phi_* \circ \phi^*) - \times m] = 0 \text{ on } CH^n(X \times X, \mathbb{Q})/i^*(CH^n(X \times W, \mathbb{Q}))$$

Proposition 97

$$P(X) \circ P(X)^T - m\Delta_X = 0 \textit{ in } CH^n(X \times X, \mathbb{Q})/i^*(CH^n(X \times W, \mathbb{Q}))$$

Hence

$$\tilde{\pi}_n^X \circ P(X) \circ P(X)^T = m\tilde{\pi}_n^X \textit{ in } CH^n(X \times X)$$

Proof. Recall that $\phi_* = [P(X)]$, $\phi^* = [P(X)^T]$ and from Proposition **??** we have $1 \times \phi_* \circ \phi^* = \times m$ on $CH^n(X \times X, \mathbb{Q})/i^*(CH^n(X \times W, \mathbb{Q}))$. Now the fact that for a smooth projective variety Y and a correspondence $E \subset Y \times Y$, $(\Delta_Y \times E)_*(\Delta_Y) = E$ gives

$$P(X) \circ P(X)^T - m\Delta_X = 0 \text{ in } CH^n(X \times X, \mathbb{Q})/i^*(CH^n(X \times W, \mathbb{Q}))$$

To show the second part let's examine $i^*(CH^n(X \times W, \mathbb{Q}))$ and $\tilde{\pi}_n^X(i^*(CH^n(X \times W, \mathbb{Q})))$. From Assumption **??** and Assumption **??** we have

$$CH^n(X \times W, \mathbb{Q}) = \bigoplus_{l < \frac{n+1}{2}}^{n} CH^{n-l}(X, \mathbb{Q}) \otimes \mathbb{Q}\{H_W^l\} \oplus \bigoplus_{l \geq \frac{n+1}{2}}^{n} \mathbb{Q}\{H_X^{n-l}\} \otimes CH^l(W, \mathbb{Q})$$

Note that $i^*(H_W^\bullet) = H_X^\bullet$. Then

$$i^*(CH^n(X \times W, \mathbb{Q})) = \bigoplus_{l < \frac{n+1}{2}}^{n} CH^{n-l}(X, \mathbb{Q}) \otimes \mathbb{Q}\{H_X^l\} \oplus \bigoplus_{l \geq \frac{n+1}{2}}^{n} \mathbb{Q}\{H_X^{n-l}\} \otimes i^*(CH^l(W, \mathbb{Q}))$$

We have $i^*(CH^\bullet(W, \mathbb{Q})) \subset CH^\bullet(X, \mathbb{Q})$ so

$$i^*(CH^n(X \times W, \mathbb{Q})) \subset \bigoplus_{l < \frac{n+1}{2}}^{n} CH^{n-l}(X, \mathbb{Q}) \otimes \mathbb{Q}\{H_X^l\} \oplus \bigoplus_{l \geq \frac{n+1}{2}}^{n} \mathbb{Q}\{H_X^{n-l}\} \otimes CH^l(X, \mathbb{Q})$$

Now lets apply $\tilde{\pi}_n^X = \Delta_X - \frac{1}{\deg X} \bigoplus_{p=0}^{n} \mathbb{Q}\{H_X^p \otimes H_X^{n-p}\}$ to $i^*(CH^n(X \times W, \mathbb{Q}))$

$$\tilde{\pi}_n^X(i^*(CH^n(X \times W, \mathbb{Q}))) \subset \tilde{\pi}_n^X(\bigoplus_{l < \frac{n+1}{2}}^{n} CH^{n-l}(X, \mathbb{Q}) \otimes \mathbb{Q}\{H_X^l\} \oplus \bigoplus_{l \geq \frac{n+1}{2}}^{n} \mathbb{Q}\{H_X^{n-l}\} \otimes CH^l(X, \mathbb{Q}))$$

We will first consider the sum over $l < \frac{n+1}{2}$:

$$\tilde{\pi}_n^X(\bigoplus_{l < \frac{n+1}{2}}^{n} CH^{n-l}(X, \mathbb{Q}) \otimes \mathbb{Q}\{H_X^l\}) = \bigoplus_{l < \frac{n+1}{2}}^{n} CH^{n-l}(X, \mathbb{Q}) \otimes \mathbb{Q}\{H_X^l\}$$

$$- \bigoplus_{l < \frac{n+1}{2}}^{n} \bigoplus_{r=0}^{n} Pr_{13,*}(CH^{n-l}(X, \mathbb{Q}) \otimes \mathbb{Q}\{H_X^l\} \times X \cap X \times \mathbb{Q}\{H_X^r \otimes H_X^{n-r}\})$$

$$= \bigoplus_{l<\frac{n+1}{2}}^{n} CH^{n-l}(X,\mathbb{Q}) \otimes \mathbb{Q}\{H_X^l\} - CH^{n-l}(X,\mathbb{Q}) \otimes \mathbb{Q}\{H_X^l\} = 0$$

Now for the second sum over $l \geq \frac{n+1}{2}$ we will take the transpose of the first sum as $(\tilde{\pi_n^X})^T$ in $CH^n(X \times X)$ with $(\tilde{\pi_n^X})^T = \tilde{\pi_n^X}$. Hence

$$\tilde{\pi}_n^X \circ P(X) \circ P(X)^T \circ (\tilde{\pi_n^X})^T = m\tilde{\pi}_n^X \circ (\tilde{\pi_n^X})^T = m\tilde{\pi}_n^X \Rightarrow$$

$$\tilde{\pi}_n^X \circ P(X) \circ P(X)^T \circ \tilde{\pi_n^X} = m\tilde{\pi_n^X} \text{ in } CH^n(X \times X, \mathbb{Q})$$

■

Next we will construct a projector for Ω_X in order to show the relation between the motives of X and Ω_X. For that let

$$\tilde{\tau} = \frac{1}{m} P(X)^T \circ \tilde{\pi_n^X} \circ P(X) \in CH^{n-2k}(\Omega_X \times \Omega_X, \mathbb{Q})$$

We have $\tilde{\tau}$ is a projector on Ω_X since

$$\begin{aligned}
\tilde{\tau} \circ \tilde{\tau} &= \frac{1}{m} P(X)^T \circ \tilde{\pi_n^X} \circ P(X) \circ \frac{1}{m} P(X)^T \circ \tilde{\pi_n^X} \circ P(X) \\
&= \frac{1}{m} P(X)^T \circ \frac{1}{m} [\tilde{\pi_n^X} \circ P(X) \circ P(X)^T \circ \tilde{\pi_n^X}] \circ P(X) \\
&= \frac{1}{m} P(X)^T \circ \frac{1}{m} m\tilde{\pi_n^X} \circ P(X) = \frac{1}{m} P(X)^T \circ \tilde{\pi_n^X} \circ P(X) \\
&= \tilde{\tau}
\end{aligned}$$

$$\Rightarrow \tilde{\tau} \circ \tilde{\tau} = \tilde{\tau}$$

Now we can prove our main Theorem **??**

Proof. We want to show that the motive $(\Omega_X, \tilde{\tau}, 0)$ is isomorphic to the motive $(X, \tilde{\pi_n^X}, -k)$. For that we will consider the correspondences $\alpha = \frac{1}{m} P(X)^T \in CH^{n-k}(X \times \Omega_X, \mathbb{Q})$ and $\beta = P(X) \in CH^{n-k}(\Omega_X \times X, \mathbb{Q})$ which give the morphisms

$$\tilde{\pi_n^X} \circ \alpha \circ \tilde{\tau} : (\Omega_X, \tilde{\tau}, 0) \longrightarrow (X, \tilde{\pi_n^X}, -k)$$

and

$$\tilde{\tau} \circ \beta \circ \tilde{\pi_n^X} : (X, \tilde{\pi}, -k) \longrightarrow (\Omega_X, \tilde{\tau}, 0)$$

By the relations we have for $\tilde{\tau}$ and $\tilde{\pi_n^X}$ we get

$$\tilde{\pi_n^X} \circ \beta \circ \tilde{\tau} \circ \alpha \circ \tilde{\pi_n^X} = \tilde{\pi_n^X}$$

and

$$\tilde{\tau} \circ \alpha \circ \tilde{\pi_n^X} \circ \beta \circ \tilde{\tau} = \tilde{\tau}$$

which gives the isomorphism between our motives. ■

3.4 Bloch-Beilinson Filtration

In this section we will show that the existence of Bloch-Beilinson filtration implies Assumption **??** and also Hartshorne's conjecture by [Jann2]. This also shows the usefulness of the assumption of Bloch-Beilinson filtration in our main Theorem **??** whose proof depends on both Assumption **??** and Hartshorne's conjecture.

Definition 98 *(Bloch-Beilinson Filtration)[Jann2] Let X be a smooth complex projective variety of dimension n. There exists a descending filtration F on $CH^j(X,\mathbb{Q})$ with the following properties:*

1. $F^0CH^j(X,\mathbb{Q}) = CH^j(X,\mathbb{Q})$, $F^1CH^j(X,\mathbb{Q}) = CH^j(X,\mathbb{Q})_{hom}$ *for some fixed Weil cohomology theory.*
2. $F^rCH^j(X,\mathbb{Q}) \cdot F^sCH^j(X,\mathbb{Q}) \subset F^{r+s}CH^j(X,\mathbb{Q})$ *under the intersection product*
3. *Assuming the algebraicity of the Künneth components of the diagonal,* $Gr^v_F CH^j(X,\mathbb{Q}) = F^vCH^j(X,\mathbb{Q})/F^{v+1}CH^j(X,\mathbb{Q})$ *depends only on the motive* $(X, \Delta_{2n-2j+v,2j-v}, 0)$ *modulo homological equivalence (i.e.* $CH^d(X\times X)_{hom}$*)*
4. $F^vCH^j(X,\mathbb{Q}) = 0$ *for* $v >> 0$

Let $X \subset \mathbb{P}^{n+r}$ be a smooth projective complete intersection of type $(d_1,\ldots,d_r)$ satisfying condition (2.2.1) introduced in section 2.2. Recall that $X = Z \cap \mathbb{P}^{n+r+1}$ where $Z \subset \mathbb{P}^{n+r+1}$ was a complete intersection of type $(d_1,\ldots,d_r)$ and we obtained $W \subset \mathbb{P}^{n+r}$ a complete intersection of type $(d_1,\ldots,d_{r-1})$ as the projection of Z to $\mathbb{P}^{n+r}$. Let us show that the existence of Bloch-Beilinson filtration implies Assumption **??**:

Claim 99 *(Assumption **??**) We have a Künneth formula*

$$CH^n(X\times W,\mathbb{Q}) = \bigoplus_{l=0}^{n} CH^{n-l}(X,\mathbb{Q}) \otimes CH^l(W,\mathbb{Q})$$

for our X and W defined above.

Proof. Recall that $X \subset \mathbb{P}^{n+r}$ a complete intersection and $W \subset \mathbb{P}^{n+r}$ given by the projection from $Z \subset \mathbb{P}^{n+r+1}$. Hence $\dim W = n+1$ and W is a complete intersection. By the Bloch-Beilinson filtration (equivalently if Murre's Conjectures hold) one gets a (noncanonical) decomposition for any smooth projective variety V

$$CH^k(V,\mathbb{Q}) \cong \bigoplus_{l=0}^{k} Gr^l_F CH^k(V,\mathbb{Q})$$

With the assumption that the Künneth components of the diagonal class are algebraic one gets

$$CH^k(V,\mathbb{Q}) = \bigoplus_{l=0}^{k} \Delta(2\dim V - 2k + l, 2k - l)_* CH^k(V,\mathbb{Q})$$

For our purpose, $k = n$ and $V = X \times W$, if Δ_X and Δ_W have algebraic Künneth components then so has $\Delta_{X\times W}$. Hence we get

$$\begin{aligned} CH^n(X \times W, \mathbb{Q}) &= \bigoplus_{l=0}^{n} \Delta(2(2n+1) - 2n + l, 2n - l)_* CH^n(X \times W, \mathbb{Q}) \\ &= \bigoplus_{l=0}^{n} \Delta(2n + 2 + l, 2n - l)_* CH^n(X \times W, \mathbb{Q}) \end{aligned}$$

Note that

$$[\Delta(2n+2+l, 2n-l)] \in (H^{2n+2+l}(X\times W, \mathbb{Q}) \otimes H^{2n-l}(X\times W, \mathbb{Q})) \cap H^{2n+1,2n+1}(X\times W\times X\times W)$$

Lets consider $H^{2n+2+l}(X \times W, \mathbb{Q}) = \bigoplus_{p+q=2n+2+l} H^p(X, \mathbb{Q}) \otimes H^q(W, \mathbb{Q})$ and $H^{2n-l}(X \times W, \mathbb{Q}) = \bigoplus_{p+q=2n-l} H^p(X, \mathbb{Q}) \otimes H^q(W, \mathbb{Q})$. The only nontrivial cohomologies are $H^n(X, \mathbb{Q})$ and $H^{n+1}(W, \mathbb{Q})$ as X, W are complete intersections. Hence the only terms needed to be considered are

- $H^n(X, \mathbb{Q}) \otimes H^{n+2+l}(W, \mathbb{Q}) \oplus H^{n+1+l}(X, \mathbb{Q}) \otimes H^{n+1}(W, \mathbb{Q}) \subset H^{2n+2+l}(X \times W, \mathbb{Q})$
- $H^n(X, \mathbb{Q}) \otimes H^{n-l}(W, \mathbb{Q}) \oplus H^{n-1-l}(X, \mathbb{Q}) \otimes H^{n+1}(W, \mathbb{Q}) \subset H^{2n-l}(X \times W, \mathbb{Q})$

Now let us consider the nontrivial parts of $H^{2n+2+l}(X \times W, \mathbb{Q}) \otimes H^{2n-l}(X \times W, \mathbb{Q})$. We know that odd degree cohomologies other than middle dimensions of complete intersections are zero. Hence

1. if n, l are both even or both odd we get $H^n(X, \mathbb{Q}) \otimes H^{n+2+l}(W, \mathbb{Q}) \otimes H^n(X, \mathbb{Q}) \otimes H^{n-l}(W, \mathbb{Q}) \subset H^{2n+2+l}(X \times W, \mathbb{Q}) \otimes H^{2n-l}(X \times W, \mathbb{Q})$
2. otherwise we get $H^{n+1+l}(X, \mathbb{Q}) \otimes H^{n+1}(W, \mathbb{Q}) \otimes H^{n-1-l}(X, \mathbb{Q}) \otimes H^{n+1}(W, \mathbb{Q}) \subset H^{2n+2+l}(X \times W, \mathbb{Q}) \otimes H^{2n-l}(X \times W, \mathbb{Q})$

The cycle corresponding to the case (1) is given by $Pr_{13}^*(\Delta_X(n, n)) \cap Pr_{24}^*(\Delta_W(n + 2 + l, n - l))$, and to the case (2) is given by $Pr_{13}^*(\Delta_X(n + 1 + l, n - 1 - l)) \cap Pr_{24}^*(\Delta_W(n + 1, n + 1))$.
We will show $CH^n(X \times W, \mathbb{Q}) = \bigoplus_{p+q=n} CH^p(X, \mathbb{Q}) \otimes CH^q(W, \mathbb{Q})$.

- Case (1):

Let $\xi \in CH^n(X \times W)$, consider

$$Pr_{14,*}[Pr_{12}^*(\xi) \cap Pr_{13}^*(\Delta_X(n, n)) \cap Pr_{24}^*(\Delta_W(n + 2 + l, n - l))]$$

$$\begin{aligned} &= Pr_{14,*}[(\xi \times X \times W) \cap (X \times H_W^{\frac{n+2+l}{2}} \times X \times H_W^{\frac{n-l}{2}}) \cap Pr_{13}^*(\Delta_X(n, n))] \\ &= Pr_{14,*}(\xi \cap (X \times H_W^{\frac{n+2+l}{2}}) \times X \times H_W^{\frac{n-l}{2}} \cap Pr_{13}^*(\Delta_X(n, n))) \end{aligned}$$

Now $\operatorname{codim}_{X\times W}(\xi)=n$, $\operatorname{codim}_{X\times W}(X\times H_W^{\frac{n+2+l}{2}})=\frac{n+2+l}{2}$, so $\operatorname{codim}_{X\times W}(\xi\cap(X\times H_W^{\frac{n+2+l}{2}}))=\frac{3n+2+l}{2}$. We have $\dim(\xi\cap(X\times H_W^{\frac{n+2+l}{2}}))=\frac{n-l}{2}$ in $X\times W$, under the projection it is mapped to a cycle ξ_X in X of codimension $n-\frac{n-l}{2}=\frac{n+l}{2}$.
We have the following diagram

$$
\begin{tikzcd}
 & \xi\cap(X\times H_W^{\frac{n+2+l}{2}})\times X\times H_W^{\frac{n-l}{2}}\in X\times W\times X\times W \arrow[dl] \arrow[dr] & \\
\xi_X\times X\in X\times X \arrow[d, "\Delta_X(n,n)_*"'] & & \xi_W\times H_W^{\frac{n-l}{2}}\in W\times W \arrow[d, "pr_2"'] \\
\tilde{\xi}\in X & & H_W^{\frac{n-l}{2}}\in W
\end{tikzcd}
$$

Note that $\operatorname{codim}(\tilde{\xi})=\frac{n+l}{2}$, so we have

$$Pr_{14,*}[Pr_{12}^*(\xi)\cap Pr_{13}^*(\Delta_X(n,n))\cap Pr_{24}^*(\Delta_W(n+2+l,n-l))]=\tilde{\xi}\times H_W^{\frac{n-l}{2}}\in CH^{\frac{n+l}{2}}(X,\mathbb{Q})\otimes CH^{\frac{n-l}{2}}(W,\mathbb{Q})$$

Also with the trivial cases we get

$$CH^n(X\times W,\mathbb{Q})=\bigoplus_{p+q=n}CH^p(X,\mathbb{Q})\otimes CH^q(W,\mathbb{Q})$$

- Case (2):

Let $\xi\in CH^n(X\times W)$, consider

$$Pr_{14,*}[Pr_{12}^*(\xi)\cap Pr_{13}^*(\Delta_X(n+1+l,n-1-l))\cap Pr_{24}^*(\Delta_W(n+1,n+1))]$$

$$
\begin{aligned}
&= Pr_{14,*}[(\xi\times X\times W)\cap(H_X^{\frac{n+1+l}{2}}\times W\times H_X^{\frac{n-1-l}{2}}\times W)\cap Pr_{24}^*(\Delta_W(n+1,n+1))]\\
&= Pr_{14,*}(\xi\cap(H_X^{\frac{n+1+l}{2}}\times W)\times H_X^{\frac{n-1-l}{2}}\times W\cap Pr_{24}^*(\Delta_W(n+1,n+1)))
\end{aligned}
$$

Now $\operatorname{codim}_{X\times W}(\xi)=n$, $\operatorname{codim}_{X\times W}(H_W^{\frac{n+1+l}{2}}\times W)=\frac{n+1+l}{2}$, so $\operatorname{codim}_{X\times W}(\xi\cap(H_X^{\frac{n+1+l}{2}}\times W))=\frac{3n+1+l}{2}$. We have $\dim(\xi\cap(H_X^{\frac{n+1+l}{2}}\times W))=\frac{n+1-l}{2}$ in $X\times W$, under the projection it is mapped to a cycle ξ_W in W of codimension $n+1-\frac{n+1-l}{2}=\frac{n+1+l}{2}$.

We have the following diagram

$$\begin{tikzcd}
 & \xi \cap H_X^{\frac{n+1+l}{2}} \times W \times H_X^{\frac{n-1-l}{2}} \times W \in X \times W \times X \times W \arrow[dl] \arrow[dr] & \\
\xi_X \times H_X^{\frac{n-1-l}{2}} \in X \times X \arrow[d, "pr_2"] & & \xi_W \times W \in W \times W \arrow[d, "\Delta_W(n+1,n+1))_*"'] \\
H_X^{\frac{n-1-l}{2}} \in X & & \tilde{\xi} \in W
\end{tikzcd}$$

Note that codim $(\tilde{\xi}) = \frac{n+1+l}{2}$, so we have

$$\begin{aligned} Pr_{14,*}[Pr_{12}^*(\xi) \cap Pr_{13}^*(\Delta_X(n+1+l, n-1-l)) \cap Pr_{24}^*(\Delta_W(n+1, n+1))] \\ = H_X^{\frac{n-1-l}{2}} \times \tilde{\xi} \in CH^{\frac{n-1-l}{2}}(X, \mathbb{Q}) \otimes CH^{\frac{n+1+l}{2}}(W, \mathbb{Q}) \end{aligned}$$

Also with the trivial cases we get $CH^n(X \times W, \mathbb{Q}) = \bigoplus_{p+q=n} CH^p(X, \mathbb{Q}) \otimes CH^q(W, \mathbb{Q})$. Hence for X and W complete intersections with dimension $n, n+1$ respectively we have K$\ddot{u}$nneth formula

$$CH^n(X \times W, \mathbb{Q}) = \bigoplus_{p+q=n} CH^p(X, \mathbb{Q}) \otimes CH^q(W, \mathbb{Q})$$

■

In [Jann2] it was shown that existence of a Bloch-Beilinson filtration implies Murre's conjectures. So if we assume the existence of Bloch-Beilinson filtration we will have Murre's conjectures and in section 3.2 we saw that for smooth complete intersections Murre's II Conjecture is equivalent to Hartshorne's conjecture. Hence the existence of Bloch-Beilinson filtration implies Hartshorne's conjecture for smooth complete intersections.

Chapter 4

Appendix

Let $X \subset \mathbb{P}^{n+r}$ be a n-dimensional smooth complex projective complete intersection of type $(d_1, \ldots, d_r)$. Assume $d_i \leq d_j$ whenever $i \leq j$.
We have $k = [(n - \sum_{s \neq i}(d_i - 1) + 1)/d_s] = [(n + r - \sum_{s \neq i} d_i)/d_s]$ where $d_s = \max\{d_1, \ldots, d_r\}$.

- Note that $k < 0$ for $n + r < \sum_{s \neq i} d_i$ which implies condition

  (1) $k(n+1+r-k) + r - \sum_{j=1}^{r} \binom{d_j + k}{k} \geq 0$ does not hold.

Now lets look at examples where $k \geq 0$.

1. Examples with $k \geq 0$ and $\mathrm{Level}(H^*(X)) \leq 1$ other than $\mathbb{P}^N, N \geq 1$

r	n	d_1	d_2	d_3
1	1	$d \geq 2$		
1	2	2		
1	2	3		
1	3	2		
1	3	3		
1	3	4		
1	4	2		
1	5	2		
1	5	3		
1	$n \geq 6$	2		
2	1	2	$d \geq 2$	
2	1	3	$d \geq 3$	
2	2	2	2	
2	3	2	2	
2	3	2	3	
2	5	2	2	
2	$n \geq 6$	2	2	
3	1	2	2	$d_3 \geq 2$
3	$n =$ odd ≥ 3	2	2	2

2. All possible cases with $k \geq 0$, Level$(H^*(X)) \leq 1$ and Condition (1) satisfied other than $\mathbb{P}^N, N \geq 1$

r	n	d_1	d_2	d_3
1	1	$d \geq 2$		
1	2	2		
1	2	3		
1	3	2		
1	3	3		
1	3	4		
1	4	2		
1	5	2		
1	5	3		
1	$n \geq 6$	2		
2	1	2	$d_2 \geq 2$	
2	1	3	$d_2 \geq 3$	
2	2	2	2	
2	3	2	2	
2	3	2	3	
2	$n \geq 4$	2	2	
3	1	2	2	$d_3 \geq 2$
3	3	2	2	2

3. Examples with $k \geq 0$ but Condition (1) is not satisfied

r	n	d
1	9	5
1	11	4
1	11	6
1	12	4
1	12	6
1	13	6
1	13	7
1	14	5
1	14	7
1	15	4
1	15	5
1	15	7
1	15	8

r	n	d
1	16	4
1	16	5
1	16	7
1	16	8
1	17	3
1	17	4
1	17	5
1	17	6
1	17	7
1	17	8
1	17	9
1	18	4
1	18	5
1	18	6
1	18	8
1	18	9
1	19	4
1	19	5
1	19	6
1	19	8
1	19	9
1	19	10
1	20	3
1	20	4
1	20	5
1	20	6
1	20	7
1	20	8
1	20	9
1	20	10
1	21	3
1	21	4
1	21	5
1	21	6
1	21	7
1	21	8
1	21	9
1	21	10
1	21	11
1	22	4
1	22	5
1	22	6
1	22	7
1	22	9
1	22	10

r	n	d
1	22	11
1	23	3
1	23	4
1	23	5
1	23	6
1	23	7
1	23	8
1	23	9
1	23	10
1	23	11
1	23	12
1	24	3
1	24	4
1	24	5
1	24	6
1	24	7
1	24	8
1	24	9
1	24	10
1	24	11
1	24	12
1	25	3
1	25	4
1	25	5
1	25	6
1	25	7
1	25	8
1	25	9
1	25	10
1	25	11
1	25	12
1	25	13
1	26	3
1	26	4
1	26	5
1	26	6
1	26	7
1	26	8
1	26	9
1	26	10
1	26	11
1	26	12
1	26	13
1	27	3
1	27	4

r	n	d
1	27	5
1	27	6
1	27	7
1	27	8
1	27	9
1	27	10
1	27	11
1	27	12
1	27	13
1	27	14
1	28	3
1	28	4
1	28	5
1	28	6
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1	28	13
1	28	14
1	29	3
1	29	4
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1	29	10
1	29	11
1	29	12
1	29	13
1	29	14
1	29	15
1	30	3
1	30	4
1	30	5
1	30	6
1	30	7
1	30	8
1	30	9
1	30	10
1	30	11
1	30	12
1	30	13

r	n	d_1	d_2
1	30	14	
1	30	15	
2	7	3	3
2	8	2	4
2	9	2	3
2	9	3	4
2	10	2	5
2	10	3	3
2	10	3	4
2	10	4	4
2	11	2	5
2	11	3	3
2	11	3	5
2	11	4	4
2	12	2	3
2	12	2	4
2	12	2	6
2	12	3	3
2	12	3	5
2	12	4	4
2	12	4	5
2	13	2	4
2	13	2	6
2	13	3	3
2	13	3	4
2	13	3	5
2	13	3	6
2	13	4	5
2	13	5	5
2	14	2	4
2	14	2	6
2	14	2	7
2	14	3	3
2	14	3	4
2	14	3	6
2	14	4	4
2	14	4	5
2	14	4	6
2	14	5	5
2	15	2	3
2	15	2	5
2	15	2	7
2	15	3	3
2	15	3	4
2	15	3	6
2	15	3	7

$\underline{r}$	$\underline{n}$	$\underline{d_1}$	$\underline{d_2}$	$\underline{d_3}$
2	15	4	4	
2	15	4	5	
2	15	4	6	
2	15	5	5	
2	15	5	6	
3	$n =$ odd ≥ 5	2	2	2
3	7	2	2	3
3	8	2	3	3
3	9	2	2	4
3	9	2	3	3
3	9	3	3	3
3	10	2	2	2
3	10	2	2	3
3	10	2	3	4
3	10	3	3	3
3	11	2	2	3
3	11	2	2	5
3	11	2	3	3
3	11	2	3	4
3	11	2	4	4
3	11	3	3	3
3	11	3	3	4
3	12	2	2	2
3	12	2	2	5
3	12	2	3	3
3	12	2	3	5
3	12	2	4	4
3	12	3	3	3
3	12	3	3	4
3	12	3	4	4
3	13	2	2	2
3	13	2	2	3
3	13	2	2	4
3	13	2	2	6
3	13	2	3	3
3	13	2	3	5
3	13	2	4	4
3	13	2	4	5
3	13	3	3	3
3	13	3	3	4
3	13	3	3	5
3	13	3	4	4
3	13	4	4	4
3	14	2	2	2
3	14	2	2	3
3	14	2	2	4

$\underline{r}$	$\underline{n}$	$\underline{d_1}$	$\underline{d_2}$	$\underline{d_3}$
3	14	2	2	6
3	14	2	3	3
3	14	2	3	4
3	14	2	3	5
3	14	2	3	6
3	14	2	4	4
3	14	2	4	5
3	14	2	5	5
3	14	3	3	3
3	14	3	3	5
3	14	3	4	4
3	14	3	4	5
3	14	4	4	4
3	15	2	2	3
3	15	2	2	4
3	15	2	2	6
3	15	2	2	7
3	15	2	3	3
3	15	2	3	4
3	15	2	3	6
3	15	2	4	4
3	15	2	4	5
3	15	2	4	6
3	15	2	5	5
3	15	3	3	3
3	15	3	3	4
3	15	3	3	5
3	15	3	3	6
3	15	3	4	4
3	15	3	4	5
3	15	3	5	5
3	15	4	4	4
3	15	4	4	5

4. Examples with $k \geq 0$ and Condition (1) satisfied but Level$(H^*(X)) > 1$

$\underline{r}$	$\underline{n}$	$\underline{d_1}$	$\underline{d_2}$
1	2	$d \geq 4$	
1	3	$d \geq 5$	
1	4	$d \geq 3$	
1	5	$d \geq 4$	
1	6	$d \geq 3$	
1	7	$d \geq 3$	
1	8	$d \geq 3$	
1	8	$d \geq 3, d \neq 5$	

r	n	d_1	d_2	d_3
1	10	$d \geq 3$		
1	11	$d \geq 3, d \neq 4, 6$		
1	12	$d \geq 3, d \neq 4, 6$		
1	13	$d \geq 3, d \neq 6, 7$		
1	14	$d \geq 3, d \neq 5, 7$		
1	15	$d \geq 3, d \neq 4, 5, 7, 8$		
1	16	$d \geq 3, d \neq 4, 5, 7, 8$		
1	17	$d \geq 10$		
1	18	3		
1	18	7		
1	18	$d \geq 10$		
1	19	3		
1	19	7		
1	19	$d \geq 11$		
1	20	$d \geq 11$		
1	21	$d \geq 12$		
1	22	3		
1	22	8		
1	22	$d \geq 12$		
1	23	$d \geq 13$		
1	24	$d \geq 13$		
1	25	$d \geq 14$		
1	26	$d \geq 14$		
1	27	$d \geq 15$		
1	28	$d \geq 15$		
1	29	$d \geq 16$		
1	30	$d \geq 16$		
2	$n \geq 2$	2	$d_2 \geq 3$	
2	$n \geq 2$	3	$d_2 \geq 3$	
2	$n \geq 2$	4	$d_2 \geq 4$	
2	$n \geq 2$	$\vdots$	$d_2 \geq \vdots$	
2	$n \geq 2$	$n+2$	$d_2 \geq n+2$	
3	$n =$ even ≥ 2	2	2	$d_3 \geq 2$
3	$n =$ even ≥ 2	2	3	$d_3 \geq 3$
3	$n =$ even ≥ 2	2	$\vdots$	$d_3 \geq \vdots$
3	$n =$ even ≥ 2	2	n	$d_3 \geq n$
3	$n =$ odd ≥ 3	2	2	$d_3 \geq 3$
3	$n =$ odd ≥ 3	2	3	$d_3 \geq 3$
3	$n =$ odd ≥ 3	2	$\vdots$	$d_3 \geq \vdots$
3	$n =$ odd ≥ 3	2	n	$d_3 \geq n$
3	$n \geq 3$	3	3	$d_3 \geq 3$
3	$n \geq 3$	3	$\vdots$	$\vdots$
3	$n \geq 3$	3	n	$d_3 \geq n$

r	n	d_1	d_2	d_3
3	$n \geq 5$	4	4	$d_3 \geq 4$
3	$n \geq 5$	4	$\vdots$	$\vdots$
3	$n \geq 5$	4	$n-1$	$d_3 \geq n-1$
3	7	5	5	$d_3 \geq 5$

5. Examples with $k \geq 0$ and Level$(H^*(X)) \leq 1$ but Condition (1) is not satisfied

r	n	d_1	d_2	d_3
3	$n =$ odd ≥ 5	2	2	2

6. Examples with $k \geq 0$ and Condition (1) satisfied.

r	n	d_1	d_2	d_3
1	$n \geq 5$	2	2	2

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