The Geometry of Landau-Ginzburg models

by

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Abstract

In this thesis we address several questions around mirror symmetry for Fano manifolds and Calabi-Yau varieties. Fano mirror symmetry is a relationship between a Fano manifold X and a pair (Y, w) called a Landau-Ginzburg model, which consists of a manifold Y and a regular function w on Y. The goal of this thesis is to study of Landau-Ginzburg models as geometric objects, using toric geometry as a tool, and to understand how K3 surface fibrations on Calabi-Yau varieties behave under mirror symmetry. These two problems are very much interconnected and we explore the relationship between them.

As in the case of Calabi-Yau varieties, there is a version of Hodge number mirror symmetry for Fano varieties and Landau-Ginzburg models. We study the Hodge numbers of Landau-Ginzburg models and prove that Hodge number mirror symmetry holds in a number of cases, including the case of weak Fano toric varieties with terminal singularities and for many quasi-Fano hypersurfaces in toric varieties.

We describe the structure of a specific class of degenerations of a d-dimensional Fano complete intersection X in toric varieties to toric varieties. We show that these degenerations are controlled by combinatorial objects called amenable collections, and that the same combinatorial objects produce birational morphisms between the

Landau-Ginzburg model of X and $(\mathbb{C}^\times)^d$. This proves a special case of a conjecture of Przyjalkowski. We use this to show that if X is "Fano enough", then we can obtain a degeneration to a toric variety. An auxiliary result developed in the process allows us to find new Fano manifolds in dimension 4 which appear as hypersurfaces in smooth toric Fano varieties.

Finally, we relate so-called Tyurin degenerations of Calabi-Yau threefolds to K3 fibrations on their mirror duals and speculate as to the relationship between these K3 surface fibrations and Landau-Ginzburg models, giving a possible answer to a question of Tyurin [141]. We show that this speculative relationship holds in the case of Calabi-Yau threefold hypersurfaces in toric Fano varieties. We show that if V is a hypersurface in a Fano toric variety associated to a polytope Δ, then a bipartite nef partition of Δ defines a degeneration of V to the normal crossings union of a pair of smooth quasi-Fano varieties and that the same data describes a K3 surface fibration on its Batyrev-Borisov mirror dual. We relate the singular fibers of this fibration to the quasi-Fano varieties involved in the degeneration of V. We then classify all Calabi-Yau threefolds which admit fibrations by mirror quartic surfaces and show that their Hodge numbers are dual to the Hodge numbers of Calabi-Yau threefolds obtained from smoothings of unions of specific blown up Fano threefolds.

Preface

Chaper 4 contains work done in collaboration with Charles F. Doran, which will appear in the Canadian Journal of Mathematics. Chapters 7 and 8 of this thesis contain work done with Charles F. Doran, Andrey Novoseltsev and Alan Thompson. Chapter 8 will appear in International Mathematics Research Notices Volume 2015, pp. 12265-12318. The rest of this thesis is my own work.

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Chapter 1

Introduction

The goal of this thesis is to use tools from complex algebraic geometry, toric geometry and Hodge theory to understand three different types of objects and their inter-relations under mirror symmetry: K3 surfaces, Fano threefolds, and Calabi-Yau threefolds.

1.1 Background

In what follows, we will be concerned with two separate types of algebraic varieties, which we define.

Definition 1.1.1. A smooth algebraic variety X of dimension n is a *Calabi-Yau* variety if its canonical bundle $\omega_X = \bigwedge^n \Omega^1_X$ is trivial and the cohomology groups $\mathrm{H}^i(X, \mathcal{O}_X)$ vanish for $1 \leq i \leq n-1$.

Definition 1.1.2. A smooth algebraic variety X of dimension n is a *Fano* variety if ω_X^{-1} is an ample line bundle on X. A smooth algebraic variety X will be called *quasi-Fano* if it's anticanonical divisor is effective, base-point free and $\mathrm{H}^i(X, \mathscr{O}_X) = 0$ for $i > 0$.

In dimension 1, Calabi-Yau varieties are nothing but elliptic curves and the only Fano variety is $\mathbb{P}^1$. In dimension 2, there are 9 deformation classes of Fano varieties known as del Pezzo surfaces, and just a single complex deformation class of Calabi-Yau varieties, which are known as K3 surfaces. Often, especially when we work with toric geometry, we will consider singular versions of such varieties. A variety will be called *Gorenstein* if its canonical bundle is Cartier. It is called $\mathbb{Q}$-*Gorenstein* if some integral multiple of its canonical divisor is Cartier. Replacing "smooth algebraic variety" in the above definitions with "variety with Gorenstein singularities" produces a general definition of Calabi-Yau, Fano and quasi-Fano varieties.

1.1.0.1 Mirror symmetry for Calabi-Yau varieties.

In the late 1980s and early 1990s it was noticed by physicists that Calabi-Yau varieties seem to come in pairs. Let X and $X^\vee$ be such a pair of n-dimensional Calabi-Yau varieties, then it was noticed that transcendental data associated to the periods of X seems to recover symplectic enumerative data associated to $X^\vee$ and vice versa. Eventually, this duality leads to the fact that there is an identification of cohomology groups:

$$h^p(X, \Omega^q_X) = h^{n-q}(X^\vee, \Omega^p_{X^\vee}).$$

It was quickly realized that, at least in many examples, Calabi-Yau varieties may be viewed as arising from combinatorics. In particular, if Y is a *toric variety* with only Gorenstein singularities and so that ω_Y^{-1} is ample, then a general section of ω_Y^{-1} is a Calabi-Yau variety. Such toric varieties arise from so-called reflexive polytopes.

Definition 1.1.3. A polytope $\Delta \in \mathbb{R}^d$ is called reflexive if each vertex of Δ is in $\mathbb{Z}^m$,the only integral point on the interior of Δ is the origin, and the polar polytope of Δ which is defined as

$$\Delta^\circ = \{x \in N | \langle x, y\rangle \geq -1, \forall y \in \Delta\}$$

satisfies the previous two conditions.

To each reflexive polytope, there is a toric Fano variety $\mathbb{P}_\Delta$ with at worst Gorenstein singularities, and so that a the anticanonical hypersurface in $\mathbb{P}_\Delta$ is Calabi-Yau. Batyrev [14] conjectured that if V is an anticanonical hypersurface in $\mathbb{P}_\Delta$, and W is an anticanonical hypersurface in $\mathbb{P}_{\Delta^\circ}$, and both V and W admit smooth resolutions of singularities, then V and W form a mirror pair.

Subsequent work of Batyrev and Borisov extended this to complete intersections in toric varieties. That Calabi-Yau varieties constructed in this way indeed form a mirror pair was verified on the level of Hodge numbers by Batyrev and Borisov in [17].

According to the Hochschild-Kostant-Rosenberg theorem, the Hodge filtration on the total cohomology ring of any smooth variety X can be though of as the Hochschild homology its bounded derived category of coherent sheaves. Conjecturally, the Hochschild homology of the Fukaya category of X is just the quantum cohomology ring. Thus, the classical formulations of mirror symmetry in terms of Hodge numbers or as a relationship between periods and Gromov-Witten invariants should appear as features of some sort of equivalence between a derived version of the Fukaya category of V and the bounded derived category of coherent sheaves on W. This relationship was famously proposed by Konsevich [88] in his 1994 ICM lecture.

1.1.0.2 Mirror symmetry for Fano varieties.

More recently, a similar duality was presented for Fano varieties. If we let Y be a Fano variety of dimension n, then it has been proposed that the mirror dual of Y is not a Fano variety, but a *Landau-Ginzburg model*, which in general denotes a pair $(Y^\vee, w)$ where $Y^\vee$ is a quasi-projective variety and w is a function $\mathsf{w} : Y^\vee \to \mathbb{C}$. For instance, if Y is a smooth toric Fano variety of dimension n associated to a polytope Δ, then the Landau-Ginzburg model just the pair $((\mathbb{C}^\times)^n, \mathsf{w})$ where

$$\mathsf{w} : (\mathbb{C}^\times)^n \to \mathbb{C}$$

is a generic Laurent polynomial with Newton polytope Δ. There are various ways to view mirror symmetry for Fano varieties, but they are all similar in flavour to the Calabi-Yau/Calabi-Yau mirror symmetry that we are familiar with. The most common (and most accessible) version of Fano mirror symmetry in the literature we can recover information about the quantum cohomology ring of X from periods of the fibers of the map w. There is also a form of homological mirror symmetry which relates the Fukaya-Seidel category of (Y, w) to the bounded derived category of coherent sheaves on X, or conversely relates the derived category of singularities of (Y, w) to the Fukaya category of Y. This version of mirror symmetry has been partially addressed in the case of del Pezzo surfaces [8], weighted projective planes [9], smooth toric Fano varieties [1] and hypersurfaces in $\mathbb{P}^n$ [136].

It is not clear how to construct Landau-Ginzburg models of Fano varieties. Traditionally, Landau-Ginzburg models of Fano varieties have been expressed as Laurent polynomials on subvarieities of tori $(\mathbb{C}^\times)^n$. This is enough if we are worried about relating them to Gromov-Witten invariants of Fano varieties, as in [54, 33], or for homological mirror symmetry of toric Fano varieties, but not for homological mirror symmetry of general Fano manifolds. In general, Landau-Ginzburg models of Fano varieties seem to be partial compactifications of tori. In order to obtain such compactification, one needs to find a single torus with which to begin. This "seed" torus should be obtained from degenerations of X via the following conjecture.

Conjecture 1.1.4. *Let X be a Fano variety, then X admits a degeneration to a toric variety X' so that Δ is the convex hull of the primitive generators of the Fan defining X'. The LG model of X can be expressed as a Laurent polynomial with Newton polytope Δ.*

Conjecture 1.1.4 has its roots in the work of V. Batyrev [15], though Batyrev's work only concerns the relationship between hypersurfaces in Fano varieties, and toric hypersurfaces. In the form above, this conjecture seems to first appear in the work of V. Przyjalkowski [123]. Over the past several years several authors have started to treat Landau-Ginzburg models of Fano varieties as geometric objects in their own right (see e.g. [78, 79, 6]). In [79], it is explicitly conjectured that the mirror to a Fano variety is a *Landau-Ginzburg model.*

Definition 1.1.5. If X is a variety of dimension n and D is a simple normal crossings divisor in X, then the pair (X, D) is called a *log Calabi-Yau variety* if $\omega_X^{-1}(\log D)$ admits a unique non-vanishing section up to scaling and $h^i(X; \mathcal{O}_X) = 0$ for $1 \leq i \leq n$.

We expect that if X is a d-dimensional Fano variety, then its mirror LG model Y is a smooth, quasi-projective log Calabi-Yau variety admitting a regular function $\mathsf{w} : Y \to \mathbb{A}^1$ whose general fiber is a $(d-1)$-dimensional Calabi-Yau manifold. We expect that there is a compactification Z of Y so that w extends to a morphism $\mathsf{f} : Z \to \mathbb{P}^1$ and $\mathsf{f}^{-1}(\infty)$ is a normal crossing union of smooth divisors.

As described in [68], one can construct log Calabi-Yau varieties by gluing together tori along birational maps which preserve a specific holomorphic form Ω on $(\mathbb{C}^\times)^n$. In the spirit of Conjecture 1.1.4, the image that I have in mind is that the LG model of any smooth Fano variety is nothing but a number of tori glued together by birational maps, and that each of these tori corresponds to a boundary component in the moduli space of pairs (X, W) where W is an anticanonical hypersurface in X corresponding to a degeneration of (X, W) to a toric variety $(\mathbb{P}_\Delta, W')$ where $\mathbb{P}_\Delta$ is the toric variety associated to the polytope Δ, and that Δ is also the Newton polytope of the restriction of w to the corresponding torus. Chapters 4 and 5 are motivated by this idea.

Now we assume that for a Fano variety X we have constructed a prospective Landau-Ginzburg model (Y, w) of X. How can we check, without checking homological mirror symmetry, that (Y, w) is a good candidate Landau-Ginzburg model? According to Katzarkov, Kontsevich and Pantev [79], homological mirror symmetry implies the following cohomological condition:

$$\bigoplus_{p+q=n} \mathrm{H}^{q,d-p}(X) \cong \mathrm{H}^n(Y, V; \mathbb{C}).$$

Here V is a generic fiber of w. In [79], the authors equip $\mathrm{H}^n(Y, V; \mathbb{C})$ with a filtration based upon what they call sheaves of f-adapted logarithmic forms. They call the graded pieces under this filtration $\mathrm{H}^{p,q}(Y, \mathsf{w})$. Conjecturally, we have that

$$\mathrm{H}^{q,d-p}(X) \cong \mathrm{H}^{p,q}(Y, \mathsf{w}). \tag{1.1}$$

In Chapter 2, we show that this Hodge filtration can be recovered from the natural Hodge filtration on $\mathrm{H}^{p+q}(Y, V; \mathbb{C})$ equipped with the natural mixed Hodge structure.

We prove in Theorem 2.2.2 that

$$\mathrm{H}^{p,q}(Y, \mathsf{w}) \cong \mathrm{Gr}_q^F \mathrm{H}^{p+q}(Y, V; \mathbb{C}).$$

Thus Hodge number mirror symmetry becomes a comparison between the graded pieces of the Hodge filtration on the natural mixed Hodge structure on $\mathrm{H}^{p+q}(Y, V; \mathbb{C})$ and the Hodge numbers of X. This closely resembles the naïve Hodge number mirror symmetry for Calabi-Yau threefolds, except the pure Hodge structure on $H^{p+q}(X, \mathbb{C})$ has been replaced with a mixed Hodge structure on $H^{p+q}(Y, V; \mathbb{C})$.

For a given Fano variety Y there are often natural candidate Landau-Ginzburg models for Y, coming from various techniques, but usually these Landau-Ginzburg models do not satisfy the cohomological conditions above, so we must partially compactify to produce an appropriate mirror to Y. In Chapter 2, we produce such compactifications for the Landau-Ginzburg models of smooth weak Fano toric threefolds, and in Chapter 3, we show that such compactifications exist for any complete intersection quasi-Fano variety in a weak Fano toric variety. We prove that Hodge number mirror symmetry holds in many situations in Chapters 2 and 3.

1.1.0.3 Relationship between mirror symmetry for Fano varieties and Calabi-Yau varieties.

We now consider how mirror symmetry for Fano varieties and Calabi-Yau varieties are related. Since we expect that the fibers of the Landau-Ginzburg model of X are Calabi-Yau varieties, it is natural to expect that these Calabi-Yau varieties are related by mirror symmetry to Calabi-Yau varieties of dimension $n-1$ related to X itself. Since the anticanonical bundle of X is ample, it follows that a generic anticanonical section Z of X is smooth and the adjunction formula implies that Z is Calabi-Yau. The first, and most well known relationship between mirror symmetry for Fano varieties and mirror symmetry for Calabi-Yau varieties is:

Conjecture 1.1.6. *The general anticanonical section of Z is mirror to the generic fiber of Y.*

Indeed, this conjecture seems to emerge naturally from homological mirror symmetry [6]. Next, we will try to relate mirror symmetry for an n-dimensional Fano varieties X_1 and X_2 to mirror symmetry for n-dimensional Calabi-Yau varieties built out of them. Let X_1 and X_2 be Fano varieties. Assume that there is a fixed smooth Calabi-Yau hypersurface Z which is anticanonical in both X_1 and X_2. Then let $D_1, \dots, D_k$ be smooth irreducible divisors in X so that $-K_{X_1}|_Z - K_{X_2}|_Z = D_1 + \cdots + D_k$. Let $\widetilde{X}_1$ be the blow up of X_1 in $D_1, \dots, D_k$. Then it follows from work of Kawamata and Namikawa that

Theorem 1.1.7. *if $X_1 \cup_Z X_2$ is a normal crossings union of X_1 and X_2 meeting along Z so that there is an ample class on Z which is the restriction of an ample class on X_1 and an ample class on X_2, then $\widetilde{X}_1 \cup_Z X_2$ admits a deformation to a smooth Calabi-Yau variety V*

Thus if we have a pair of Fano varieties which share an anticanonical divisor, then given any choice of smooth divisors adding up to $-K_{X_1}|_X - K_{Z_2}|_X$ there is a Calabi-Yau variety obtained by smoothing. This idea was used by Kawamata and Namikawa [83] to construct a number of Calabi-Yau threefolds and Lee [95] computed their Hodge numbers. In [141], Tyurin suggested that there should be a relationship between the Landau-Ginzburg models of X_1 and X_2 and the mirror of the smoothing of $\widetilde{X}_1 \cup_Z X_2$. Classical mirror symmetry predicts that there is a relationship between the Kähler cone of a Calabi-Yau variety W and the moduli space of complex structures on its mirror in a small neighbourhood of a maximally unipotent monodromy point. In our case, this correspondence relates the monodromy action on V associated to the degeneration of V to $\widetilde{X}_1 \cup_Z X_2$ to a boundary ray of the Kähler cone of the mirror W of V which induces a fibration on W over $\mathbb{P}^1$ (see Section 7.5.1). However, it is not clear what role the mirrors of X_1 and X_2 play in this fibration.

Question 1.1.8. *Is there a codimension 1 Calabi-Yau fibration on the mirror of the smoothing of $\widetilde{X}_1 \cup_Z X_2$? If so, how is this Calabi-Yau fibration related to the Landau-Ginzburg models of X_1 and X_2? How is this fibration related to $D_1, \dots, D_k$?*

I do not believe that an algebraic geometry construction will suffice to answer this question. In Chapter 6 we will discuss a topological construction which I believe provides an approximate answer to this question. It proceeds by gluing the LG model (Y_1, w_1) of $\widetilde{X}_1$ to the LG model (Y_2, w_2) of X_2 along a neighbourhood of the fiber at infinity to get a manifold W in a way that is consistent with the maps w_1 and w_2, and thus the superpotentials on Y_1 and Y_2 extend to a map $\pi : W \to S^2$ where S^2 is the 2-dimensional sphere as usual. Thus we obtain (at least topologically) a fibration on the prospective mirror dual of the Calabi-Yau variety V obtained by smoothing $\widetilde{X}_1 \cup_Z X_2$. We conjecture that the resulting C^∞ manifold can be equipped with a complex structure with which it is mirror to V and the fibration on W extends to a complex fibration on W. In Chapters 6 and 7, we provide evidence that this is true.

1.2 Overview of results

Here we will give a review of the main results that appear in this thesis. We will go through, chapter by chapter and review the results contained therein. There are four parts to this thesis which address roughly four different questions. The first part, made up of Chapters 2 and 3 is devoted to questions about Hodge numbers of Fano varieties and their Landau-Ginzburg models. The second part, which focusses on questions regarding degenerations of toric complete intersection quasi-Fano varieties and Laurent polynomial expressions for their Landau-Ginzburg models, is comprised of Chapters 4 ([46]) and 5. The third section discusses the relationship between K3 fibrations on Calabi-Yau threefolds and Tyurin degenerations, and is made up of Chapters 6 and 7 ([50]). The final part, which is simply Chapter 8 ([48]), addresses fundamental questions regarding the structure of K3 fibrations and their behaviour with respect to automorphisms on the fibers. This is not directly relevant to the rest of this thesis, though it includes information that relates to Chapters 6 and 7. The contents of this part have been applied directly to [49].

1.2.0.1 Chapter 2: Landau-Ginzburg models and Fano manifolds

In this chapter, we begin with a review of Hodge number mirror symmetry in formulation for Fano varieties. We will introduce the "Hodge numbers" of an LG model as defined by Katzarkov-Kontsevich-Pantev [79]. We show (Theorem 2.2.2) that they agree with the dimensions of graded pieces of the Hodge filtration on $\mathrm{H}^i(Y, V; \mathsf{w})$ for V a smooth fiber of w. We prove (Corollary 2.2.10) that in nice enough situations they form a Hodge diamond equipped which looks like the Hodge diamond of a Fano variety rotated by $\pi/2$. In particular we show that, under appropriate conditions,

$$h^{p,q}(Y, \mathsf{w}) = h^{d-p,d-q}(Y, \mathsf{w}) = h^{q,p}(Y, \mathsf{w}) = h^{d-q,d-p}(Y, \mathsf{w}).$$

This is not automatically true, since the numbers in the Hodge diamond are simply dimensions of graded pieces of a mixed Hodge structure in general.

We then suggest (Section 2.2.4) that if X is a Fano variety, then its LG model should have a specific form, extrapolating from the fact that the fiber of (Y, w) should be mirror dual to a generic anticanonical hypersurface in X. We note that the conditions required in order for the Hodge numbers to be symmetric across the horizontal and vertical axes are satisfied by the varieties we expect to be dual to Fano varieties. We compute several examples. First we compute the Hodge diamonds of LG models in dimension 2 (Section 2.2.3). We then construct smooth LG models for all toric weak Fano threefolds (Section 2.3) and show that they satisfy Hodge number mirror symmetry (Theorem 2.3.7). Then we show that if X_Δ is a general mpcp resolution of a Gorenstein toric Fano variety, then we show that a version of Hodge number mirror symmetry holds here as well (Theorem 2.3.20), though we have to be a bit careful in our definitions and statements.

Finally, we look at these results in relation to the work of Coates and collaborators [33, 34], and show that the fact that the local systems $\mathbb{L}_X$ associated to Fano threefolds in [33] are extremal is predicted by the restrictions that we expect to be placed on LG models of Fano varieties (Section 2.4). We comment on possible implications in four dimensions.

1.2.0.2 Chapter 3: Hodge numbers of Fano hypersurfaces and Landau-Ginzburg models

In this chapter, we begin our study of LG models of quasi-Fano varieties embedded as complete intersections of nef divisors in toric varieties. Very much in the spirit of Batyrev and Borisov's work on mirrors of Calabi-Yau complete intersections, we show that if X is a quasi-Fano complete intersection of nef divisors in a toric variety X_Δ so that $-K_X$ is the restriction of a nef divisor on X_Δ, then there is a naturally defined LG model for X which is relatively compact and admits a compactification to a fiber space over $\mathbb{P}^1$ (Theorem 3.2.6). The only caveat here is that the LG model of X need not be smooth, but instead has at worst terminal Gorenstein singularities, which is a much as we can reasonably expect in arbitrary dimension. We then show (Theorem 3.3.1, Corollary 3.3.3) that one can compute $h^{2d-2}(Y, \mathsf{w})$ by counting components of the fibers of (Y, w) where (Y, w) is the LG model constructed in Section 3.2.3.

Then we analyze the case of a quasi-Fano variety X which is a sufficiently ample hypersurfaces in terminal Gorenstein weak Fano toric variety. We show (Theorem 3.4.9) that $h^{1,d-2}(X) + 1$ is the number of components in the fiber over 0 of the LG model of X. We show that this implies that $h^{1,d-2}(X) \leq h^{1,1}(Y, \mathsf{w})$. If mirror symmetry holds, then this inequality is an equality, and thus we conclude that there should be no fibers of w with more than one irreducible component except for the fiber over 0. Using homological mirror symmetry, this seems to imply certain structures on the Hodge structure of a quasi-Fano variety. In turn, this implies a statement about the deformation theory of quasi-Fano varieties, which we then proceed to prove (Theorem 3.5.4).

1.2.0.3 Chapter 4: Laurent polynomials and degenerations of complete intersections

This chapter is the paper [46]. We begin with a variety X which is a complete intersection of nef Cartier divisors in a toric variety X_Δ so that $-K_X$ is the restriction of a nef $\mathbb{Q}$-Cartier divisor on X_Δ. Then we define objects called amenable collections, which consist of a sequence of elements of the lattice N which is dual to the lattice M containing Δ. An amenable collection V is shown to determine two different things (Theorem 4.2.20):

1. A degeneration of X to a binomial complete intersection X' in the homogeneous coordinate ring of X_Δ. Let Δ_V be the convex hull of the ray generators of the fan corresponding to the toric variety X'.

2. A birational map ϕ_V from $(\mathbb{C}^\times)^n$ to the LG model of X (as defined in Chapter 3) so that $\phi_V^* \mathsf{w}$ is a Laurent polynomial with Newton polytope Δ_V.

Thus we have a robust relationship between toric degenerations of X and torus charts on its LG model.

This correspondence is then shown to produce toric degenerations for all of the Fano fourfolds obtained by Coates, Kasprzyk and Prince [34] contained as complete interesctions in smooth Fano toric varieties (Theorem 4.4.3). We show that we can use rather simple combinatorics to produce toric degenerations for all complete intersections in complete flag varieties (Theorem 4.3.4). We also show that our results almost recover a result of Ilten, Lewis and Przyjalkowski [76] (Theorem 4.4.5). Finally, we present an application to geometric transitions between Calabi-Yau threefolds presented as complete intersections in toric varieties and their mirror dual geometric transitions (Theorem 4.4.6).

1.2.0.4 Chapter 5: Existence of toric degenerations

In this chapter, apply the results in Chapter 4 in the case where X is a Fano variety. We start with the case where X is a hypersurface in a Gorenstein toric Fano variety X_Δ equipped with a degeneration to a union of toric boundary divisors $D_1, \dots, D_n$ so that $-K_X$ is the restriction of a Cartier nef divisor D on X_Δ, and so that the restriction of D to the intersection of any subset of the divisors $D_1, \dots, D_n$ is ample. Then we show that in this situation, one obtains an amenable collection associated to X (Proposition 5.2.7) and hence there exists a degeneration of X to a toric variety.

We also prove that, in this situation, X is necessarily a Fano variety (Proposition 5.3.4).

Proposition 5.2.7 can be extended (Theorem 5.2.10) to the case where X is a complete intersection of nef divisors in a toric variety, and the anticanonical bundle of X is the restriction of a sufficiently ample bundle on X_Δ, then X admits a toric degeneration to a toric Fano variety. In specific, if $-K_X$ is the restriction of an ample divisor on X_Δ, then X admits a toric degeneration. This is, of course forces X to be Fano. Thus if X is, in this stronger sense, a Fano variety, then X degenerates to a toric Gorenstein Fano variety. This makes steps towards confirming the conjecture of Przyjalkowski about the existence of toric degenerations of Fano varieties.

In the course of the proof of the results above, we prove a condition for a hypersurface in a Fano variety to be Fano which is weaker than the condition that $-K_X$ is the restriction of an ample divisor on X_Δ (Proposition 5.3.4). We apply this criteria in order to produce smooth toric Fano fourfolds which do not seem to appear in the literature (Example 5.3.6). Finally, we use the main theorem of this chapter to give a brief proof that all Fano threefolds of Picard rank greater than 1 admit degenerations to toric Fano varieties (Theorem 5.4).

1.2.0.5 Chapter 6: Calabi-Yau varieties and Tyurin degenerations

Next we turn our attention to the relationship between so-called Tyurin degenerations [141, 95] and K3 surface fibrations on Calabi-Yau threefolds. A Tyurin degeneration of a Calabi-Yau threefold $\mathscr{V}$ is a projective map $\pi : \mathscr{V} \to \Delta$ where Δ here is a disc of radius r in $\mathbb{C}$ containing 0 so that the fiber $\pi^{-1}(t)$ is a smooth Calabi-Yau variety and $\pi^{-1}(0)$ is the normal crossings union of a pair of smooth quasi-Fano threefolds X_1 and X_2 whose intersection is a K3 surface S. We conjecture, following work of Tyurin [141] and Auroux [7] that the mirror to V admits a K3 surface fibration g over $\mathbb{P}^1$ in such a way that we may think of $\mathbb{P}^1$ a union of two open discs D_1 and D_2 so that for $i = 1, 2$ and $U_i = g^{-1}(D_i)$ is homeomorphic to the LG model of X_i and the restriction of g to U_i is equal to w_i.

First, we show that, assuming that X_i and (Y_i, w_i) have appropriate monodromy actions, then we may indeed glue together Y_1 and Y_2, and that the resulting C^∞ manifold has appropriate Euler number (Theorem 6.2.1). We show that in the threefold case, Hodge number mirror symmetry is equivalent to a version of Dolgachev-Nikulin mirror symmetry for certain K3 surfaces (Proposition 6.2.2).

We proceed to prove that certain approximations of this picture hold when V is a complete intersection of nef divisors in a Gorenstein toric Fano variety. In the case where V is an anticanonical hypersurface in a Gorenstein toric Fano fourfold or threefold $\mathbb{P}_\Delta$, a Tyurin degeneration is determined by a nef partition Δ_1, Δ_2 of Δ, where X_1 and X_2 are related to quasi-Fano varieities associated to Δ_1 and Δ_2. We show that if a nef partition of Δ exists, then there is a smooth Calabi-Yau threefold $\hat{W}$ birational to the Batyrev mirror of V which admits a K3 surface fibration (Corollary 6.3.6), and that the fibers of this fibration are Batyrev-Borisov mirror to the K3 surfaces associated to the nef partition Δ_1, Δ_2. We then analyze the degenerate fibers of this fibration. We prove that the number of components of these fibers carries numerical data corresponding to the quasi-Fano threefolds X_1 and X_2 in the Tyurin degeneration of V (Theorem 6.4.8).

1.2.0.6 Chapter 7: Calabi-Yau threefolds fibered by quartic mirror K3 surfaces

This chapter is the paper [51], written with coauthors C. Doran, A. Thompson and A. Novoseltsev. This chapter is simply an extended example of the phenomenon described in Chapter 6. We say that a family $f : \mathscr{S} \to U$ of smooth K3 surfaces is lattice polarized by a lattice L if there is a trivial local subsystem $\mathbb{L}$ of $R^2 f_* \mathbb{Z}$ so that for each fiber $\mathscr{S}_t = f^{-1}(t)$ the restriction of $\mathbb{L}$ to $\mathrm{H}^2(\mathscr{S}_t, \mathbb{Z})$ has image in $\mathrm{Pic}(\mathscr{S}_t)$ and is a lattice isomorphic to L.

We classify all $M_2 = E_8^2 \oplus U \oplus \langle -4 \rangle$-polarized families of K3 surfaces which admit compactifications to Calabi-Yau threefolds (Proposition 7.2.3). We compute the Hodge numbers of all smooth Calabi-Yau varieties obtained this way (Proposition 7.3.2, Corollary 7.3.7). We then show that classical mirror symmetry implies that, associated to a Tyurin degeneration of a Calabi-Yau threefold V, there should be a fibration of its mirror W over $\mathbb{P}^1$ (Section 7.5.1). Conversely, we suppose that the mirror partners of the K3 fibered Calabi-Yau threefolds of Proposition 7.2.3 can be obtained by smoothing pairs of quasi-Fano threefolds.

We show that the classification of such Calabi-Yau threefolds is almost identical to to the classification of Calabi-Yau threefolds which admit specific Tyurin degenerations. In particular, for every smooth Calabi-Yau threefold fibered by M_2-polarized K3 surfaces, there is a Calabi-Yau threefold with mirror Hodge numbers obtained by smoothing a union of a pair of quasi-Fano varieties X_1 and X_2 where X_1 and X_2 are either $\mathbb{P}^3$ a quartic double solid or a quartic hypersurface in $\mathbb{P}^4$ blown up in a union of smooth curves (Corollary 7.5.5). We observe (Section 7.5.4) that the resulting K3 fibrations seem to be exactly the LG models of X_1 and X_2 glued as described in Section 6.2.

The original version of this article mentions results that I had previously anticipated to appear in this thesis. These results have not been completed at this time, thus I have removed reference to this from Chapter 7.

1.2.0.7 Chapter 8: Families of lattice polarized K3 surfaces with monodromy

The final chapter is the paper [49], written in collaboration with C. Doran, A. Novoseltsev and A. Thompson. Herein, we address several questions regarding the relationship between K3 fibrations and automorphisms of K3 surfaces with an eye towards applications. In particular, if we have a family of K3 surfaces which are each lattice polarized by the lattice $M = E_8^2 \oplus H$, then each admits an automorphism ι so that the quotient S/ι is birational to a Kummer surface, which is dominated by a product of elliptic curves $E_1 \times E_2$. The question that we would like to answer is when this can be done in families. Precisely, if $X \to U$ is a fibration over U by K3 surfaces each admitting an M-polarization, then first, when can the automorphisms on each fiber be extended to an automorphism on all of X? In Corollary 8.2.12, we show that this automorphism extends if the action of monodromy on M inside of the Néron-Severi lattice of each K3 surface fiber is nice enough.

We then ask, if we obtain a global automorphism ι on X coming from automorphisms on each fiber, then under what conditions is the quotient X/ι dominated by a fiber product of elliptic surfaces. We show (Section 8.4.3) that this can be done under conditions on the action of monodromy on rational curves on each fiber of X. In Theorem 8.4.11 shows us that there is a specific finite group G a G-étale cover $U' \to U$ of U so that $X \times_U U'$ is the quotient of a fiber product of a pair of families of elliptic curves over U'.

In Section 8.5, we compute G in the case where each fiber of X is polarized by the lattice $M_n = \langle -2n \rangle \oplus E_8^2 \oplus H$ for $n = 1, 2, 3, 4$. Finally, in Section 8.5.4, we show that many of the mirrors of the 14 families of Doran-Morgan [52] of Calabi-Yau threefolds with hypergeometric Picard-Fuchs differential equations are compactifications of families of K3 surfaces with M_n-polarized fibers for $n = 1, 2, 3, 4$. Therefore, each of these families, in a sense, comes from a fiber product of elliptic surfaces. We comment in Section 8.6 that the structure of these fibrations seems to have some relation to whether the monodromy group of the associated family of Calabi-Yau threefolds is an arithmetic or thin subgroup of $\mathrm{Sp}_4(\mathbb{Z})$.

Part I

Mirror symmetry for Fano varieties

Chapter 2

Landau-Ginzburg models and Fano manifolds

2.1 Introduction

In this chapter, we introduce a major object of study in subsequent chapters – Landau-Ginzburg models, which we will almost uniformly call LG models. There is a quite general notion of an LG model, which is simply a variety equipped with a regular function which is sometimes used in the literature. From such an object, one can produce things like a category of matrix factorizations, which are interesting in their own right, however our goal is to present LG models as objects which correspond under mirror symmetry to Fano varieties, and for such an application, we want to be more refined in our notions. Conjecturally, this is a class of objects whose classification should relate directly to the classification of Fano manifolds. One of the goals of this chapter is to further refine the notion of an LG model corresponding to a Fano variety, as presented in [79] so that we obtain a more rigid class of objects corresponding to Fano manifolds, which we will call extremal LG models.

The most important technical results in this chapter are contained in Section 2.2, where we first reduce the computation of the Hodge numbers $h^{p,q}$ of [79] to a computation of graded pieces of the Hodge numbers of the Hodge filtration of the pair

(Y, w) for Y a smooth quasiprojective variety and w a regular function on Y. We will determine how to compute these Hodge numbers in terms of the primitive cohomology of a fiber of w and the cohomology of a smooth compactification of Y. We then prove that, despite the fact that there is no a priori reason for the usual symmetries of a Hodge diamond to hold for the Hodge numbers of (Y, w), they do in certain cases. One of these cases is the case where (Y, w) has the right structure to be the LG model of a Fano variety.

In Section 2.3, we will use toric machinery to explicitly show that Hodge number mirror symmetry holds between a weak toric Fano threefold X_Δ and its mirror dual (Y_Δ, w). A slightly weaker version of this result is included in Section 2.3.2 for general weak Fano toric varieties with terminal singularities. A byproduct of the computation in Section 2.3 is that (Y_Δ, w) is extremal if and only if the restriction map $\mathrm{H}^2(X_\Delta, \mathbb{C}) \to \mathrm{H}^2(S, \mathbb{C})$ (for S a generic anticanonical hypersurface in X_Δ) is injective, and that the kernel of this map measures how far (Y_Δ, w) is from being of extremal. Therefore, extremal LG models do not correspond to Fano varieties, directly, but instead seem to have a cohomological characterization.

Despite this somewhat disappointing negative result, one could not reasonably have hoped for more. In the end, if one wishes to define a class of complex geometric objects whose classification is mirror to the classification of Fano manifolds, then it is necessary to characterize the properties of the Fukaya category of a monotone symplectic manifold and reinterpret this in the derived category of singularities and then recast this derived characterization in terms of the geometry of the LG model. This seems, to me at least, to be a daunting task.

A much more promising and computable (but less geometric) approach to mirror classification of Fano manifolds is outlined by Coates and collaborators and carried out in dimensions 2 and 3 in [33]. The authors of *ibid.* relate the classification of Fano threefolds and fourfolds to Laurent polynomials, which are called Minkowski polynomials therein, although this notion has been corrected in subsequent work [3]. These are Laurent polynomials f with integral coefficients which satisfy other conditions that I will not describe here. For the moment, it will suffice to say that these are Laurent polynomials which should theoretically be obtained by pulling back w along an embedding $\phi_f : (\mathbb{C}^\times)^d \hookrightarrow Y$ for Y the LG model of some Fano variety. Associated to such Laurent polynomials is a local system $\mathbb{L}_f$. The Laurent polynomials that they obtain in this way have properties which [33] call *extremal* or *low ramified*. We explain in Section 2.4 how these notions intersect with the notion of an extremal LG model described above.

2.2 Landau-Ginzburg models and their Hodge numbers

Here we will perform several computations regarding the LG models of Fano varieties. We will begin, as in [79] by defining a Landau-Ginzburg model.

Definition 2.2.1. A LG model is a smooth quasiprojective variety of dimension d equipped with the following data:

1. A map $\mathsf{w} : Y \to \mathbb{A}^1$ so that Y is relatively compact and generically the fiber of w is smooth.

2. A compactification of Y to a smooth variety Z to which w extends to a map $\mathsf{f} : Z \to \mathbb{P}^1$ and so that the fiber $D_\infty := \mathsf{f}^{-1}(\infty)$ is a simple normal crossings union of smooth varieties.

3. A non-vanishing holomorphic d-form ω so that the canonical extension of ω to Z has at worst simple poles along D_∞.

4. The Hodge numbers $h^{0,n}(Z)$ vanish for all $n \neq 0$.

Note that our definition here does not coincide directly with that of [79]. In their definition, they do not require that Y be relatively compact over $\mathbb{A}^1$.

2.2.1 The $h^{p,q}(Y, \mathsf{w})$ Hodge numbers

Now let's look at a set of numbers that are canonically associated to an LG model. In [79], Katzarkov, Kontsevich and Pantev define a complex that they call $\Omega^\bullet_Z(\log D_\infty, \mathsf{f})$. Let $\Omega^\bullet_Z(\log D_\infty)$ be the usual complex of holomorphic differential forms on Z with log poles at $D_\infty := \mathsf{f}^{-1}(\infty)$, then note that df defines a holomorphic 1-form with appropriate poles along D_∞. The sheaf $\Omega^i_Z(\log D_\infty, \mathsf{f})$ is defined to be the subsheaf of $\omega \in \Omega^i_Z(\log D_\infty)$ so that $\mathsf{df} \wedge \omega$ still has log poles along D_∞. This complex is then equipped with the natural differential coming from its inclusion into $\Omega^\bullet_Z(\log D_\infty)$. The i^{th} hypercohomology group of this complex is denoted $\mathrm{H}^i(Y, \mathsf{w})$. One of the main results of [79] is that the hypercohomology spectral sequence for this complex degenerates at the E_1 term and therefore, we have a "Hodge decomposition" on $\mathrm{H}^i(Y, \mathsf{w})$, or in other words,

$$\dim \mathrm{H}^i(Y, \mathsf{w}) = \sum_{p+q=i} \dim \mathrm{H}^p(Z, \Omega^q_Z(\log D_\infty, \mathsf{f})).$$

If we let $V = \mathsf{w}^{-1}(t)$ for t a regular value of w, then it is shown in [79, Lemma 2.21] that

$$\dim_{\mathbb{C}} \mathrm{H}^i(Y, V; \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{H}^i(Y, \mathsf{w}).$$

We will let

$$h^{p,q}(Y, \mathsf{w}) = \dim \mathrm{H}^q(Z, \Omega^p_Z(\log D_\infty, \mathsf{f}))$$

and $h^{p,q}(Y, V) = \dim \mathrm{H}^q(Z, \Omega^p_Z(\log D_\infty, \operatorname{rel} V))$. Here the sheaf $\Omega^i(\log D_\infty, \operatorname{rel} V)$ is the kernel of the natural restriction map

$$\iota^* : \Omega^i_Z(\log D_\infty) \to \iota_* \Omega^i_V.$$

where $\iota : V \hookrightarrow Z$ is the embedding. It is known (see e.g. [144, pp. 220]) that the spectral sequence associated to the complex $\Omega^\bullet_Z(\log D_\infty, \operatorname{rel} V)$ degenerates at the E_1 term and thus $h^{p,q}(Y, V)$ is the dimension of the p^{th} graded piece of $\mathrm{H}^{p+q}(Y, V; \mathbb{C})$ under its natural Hodge filtration.

Theorem 2.2.2.

$$h^{i,j}(Y, V; \mathbb{C}) = h^{i,j}(Y, \mathsf{w}).$$

Proof. In the proof of [79, Claim 2.22], the authors construct the following object, which they call $\mathsf{E}^\bullet_{\mathscr{Z}/\Delta}$. In brief, this is a complex of coherent sheaves over $Z \times \Delta$ where

Z is the compactification of the LG model in question, and Δ is a small complex disc containing 0. Over the subvariety $Z \times \epsilon$ for $\epsilon \neq 0$ in Δ, this sheaf is equal to $\Omega_Z^\bullet(\log D_\infty, \mathrm{rel}\, V_{1/\epsilon})$, where $V_{1/\epsilon}$ is the fiber of f over $1/\epsilon$. The restriction of $\mathsf{E}^\bullet_{\mathscr{Z}/\Delta}$ to $Z \times 0$ is equal to $\Omega_Z^\bullet(\log D_\infty, \mathsf{f})$. If we let p be the projection of $Z \times \Delta$ onto Δ, then the hyper-derived direct image $\mathbb{R}^a p_* \mathsf{E}^\bullet_{\mathscr{Z}/\Delta}$ has fibers which are just the hyper cohomology groups of the complexes $\Omega_Z^\bullet(\log D_\infty, \mathrm{rel}\, V_{1/\epsilon})$ if $\epsilon \neq 0$ and $\Omega_Z^\bullet(\log D_\infty, \mathsf{f})$ if $\epsilon = 0$. The upshot of [79, Claim 2.22] is that the fibers of $\mathbb{R}^a p_* \mathsf{E}^\bullet_{\mathscr{Z}/\Delta}$ have constant dimension over Δ for all a.

Now the i^{th} hypercohomology group of $\Omega_Z(\log D_\infty, \mathrm{rel}\, V_{1/\epsilon})$ is simply the cohomology group $\mathrm{H}^i(Y, V_{1/\epsilon}; \mathbb{C})$, and its the spectral sequence associated to the stupid filtration on it degenerates at the E_1 term. Thus we have that

$$h^i(Y, V_{1/\epsilon}) = \sum_{p+q=i} h^p(Z, \Omega^q(\log D_\infty, \mathrm{rel}\, V_{1/\epsilon}))$$

Similarly by [79, Lemma 2.19], the same is true of $\mathbb{H}^i(Z, \Omega_Z^\bullet(\log D_\infty, \mathsf{f}))$. In other words,

$$h^i(Y, \mathsf{w}) = \sum_{p+q=i} h^p(Z, \Omega_Z^q(\log D_\infty, \mathsf{f})).$$

By Grauert's semicontinuity theorem, (see e.g. [11, Theorem 8.5(ii)]), the value of

$$\epsilon \mapsto \operatorname{rank} \mathrm{H}^p(V_{1/\epsilon}, (\mathsf{E}^q_{\mathscr{Z}/\Delta})|_{V_{1/\epsilon}})$$

is upper semicontinuous on Δ in the analytic Zariski topology. Thus it follows that for a general enough point ϵ_0 of Δ that

$$h^{p,q}(Y, V_{1/\epsilon_0}) \leq h^{p,q}(Y, \mathsf{w}).$$

However, the fact that

$$\sum_{p+q=i} h^p(Z, \Omega^q(\log D_\infty, \mathrm{rel}\, V_{1/\epsilon_0})) = h^i(Y, V_{1/\epsilon_0}) = h^i(Y, \mathsf{w}) = \sum_{p+q=i} h^p(Z, \Omega^q(\log D_\infty, \mathsf{f})).$$

implies that we must have equality between $h^{p,q}(Y, V_{1/\epsilon})$ and $h^{p,q}(Y, \mathsf{w})$ at all points. $\square$

For the moment, we will fix a smooth, relatively compact LG model $\mathsf{w} : Y \to \mathbb{A}^1$ with a compactification $\mathsf{f} : Z \to \mathbb{P}^1$ so that D_∞ is a simple normal crossings union of divisors. We will make more assumptions about Y and its fibers later on, but for now, the structure of Y is essentially unrestricted. This condition is to ensure that we may apply the Clemens-Schmid exact sequence.

Let us now compute the mixed Hodge structure on $\mathrm{H}^i(Y, V; \mathbb{C})$. This will proceed in several steps. First, we shall compute the MHS on Y itself. Define the primitive cohomology of V to be $\mathrm{PH}^i(V, \mathbb{C}) = \mathrm{coker}(\mathrm{H}^i(Z;\mathbb{C}) \to \mathrm{H}^i(V, \mathbb{C}))$ equipped with the LMHS at ∞. This exists, since the condition that D_∞ is simple normal crossings, Landman's monodromy theorem implies that the monodromy action on the fibers of w associated to a small loop around infinity is unipotent. Thus work of Schmid shows that the limit mixed Hodge structure is canonically defined. Let T_i be the monodromy automorphism on $\mathrm{PH}^i(V, \mathbb{C})$ corresponding to a small loop around ∞, let $N_i = \log T_i$ and define $\mathscr{Q}_i = \ker N_i$. Define $\mathscr{K}_i = \ker(\mathrm{H}^i(Z, \mathbb{C}) \to \mathrm{H}^i(D_\infty, \mathbb{C}))$.

Proposition 2.2.3. *There is a short exact sequence*

$$0 \to \mathscr{Q}_{i-1} \to \mathrm{H}^i_c(Y, \mathbb{C}) \to \mathscr{K}_i \to 0.$$

Proof. We have a long exact sequence in cohomology with compact supports [120, Proposition 5.54],

$$\cdots \to \mathrm{H}^{i-1}(D_\infty, \mathbb{C}) \to \mathrm{H}^i_c(Y, \mathbb{C}) \to \mathrm{H}^i(Z, \mathbb{C}) \to \mathrm{H}^i(D_\infty, \mathbb{C}) \to \dots$$

which is also an exact sequence of MHS. Note that Z and D_∞ are themselves compact, so we have dropped the subscript on their cohomology groups. Therefore it follows that if

$$\mathscr{Q}_i' = \mathrm{coker}(\mathrm{H}^i(Z, \mathbb{C}) \to \mathrm{H}^i(D_\infty, \mathbb{C}))$$

then we have a short exact sequence

$$0 \to \mathscr{Q}'_{i-1} \to \mathrm{H}^i_c(Y,\mathbb{C}) \to \mathscr{K}_i \to 0.$$

It remains to show that $\mathscr{Q}_i = \mathscr{Q}'_i$. Let U be a small disc around the point at infinity in $\mathbb{P}^1$ so that the only critical value in U is infinity itself. Then define $X = \mathsf{f}^{-1}(U)$. The Clemens contraction Theorem [28] states that X is strongly homotopic to $\mathsf{f}^{-1}(\infty)$. Let $X^\times = \mathsf{f}^{-1}(U \setminus \{\infty\})$.

The Clemens-Schmid exact sequence (see e.g. [28] or [120, Corollary 11.44]), says that there is a long exact sequence of mixed Hodge structures

$$\cdots \to \mathrm{H}^i(X, X^\times;\mathbb{C}) \to \mathrm{H}^i(X,\mathbb{C}) \xrightarrow{r_i} \mathrm{H}^i(V,\mathbb{C}) \xrightarrow{N_i} \mathrm{H}^i(V,\mathbb{C}) \to \dots$$

where $\mathrm{H}^i(V,\mathbb{C})$ is equipped with the limit mixed Hodge structure. The cokernel of r_i is $\mathrm{PH}^i(V,\mathbb{C})$ equipped with the limit mixed Hodge structure. Furthermore, it follows from the formulation of [120, Corollary 11.44] that the kernel of the map r_i is precisely the kernel of the map $\mathrm{H}^i(X,\mathbb{C}) \to \mathrm{H}^i(X^\times,\mathbb{C})$. Thus the kernel of r_i is simply the space of cohomology classes in $\mathrm{H}^i(X,\mathbb{C})$ whose dual homology classes are supported on $\mathsf{f}^{-1}(\infty)$ and in particular, the kernel of r_i is contained in the image of

$$s_i : \mathrm{H}^i(Z,\mathbb{C}) \to \mathrm{H}^i(D_\infty,\mathbb{C}) \cong \mathrm{H}^i(X,\mathbb{C})$$

and hence we get a map $\tilde{s}_i : \mathrm{H}^i(Z,\mathbb{C}) \to \ker N_i$ and the triangle

$$\begin{tikzcd}
\mathrm{H}^i(Z,\mathbb{C}) \arrow[d, "\tilde{s}_i"'] \arrow[dr] & \\
\ker N_i \arrow[r] & \mathrm{H}^i(V,\mathbb{C})
\end{tikzcd}$$

where the map from $\ker N_i$ to $\mathrm{H}^i(V,\mathbb{C})$ is the natural injection. Since this map is injective, it follows that $\mathscr{Q}_i = \mathscr{Q}'_i$. $\square$

Let us recall some facts about the action of N_i on the limit mixed Hodge structure on $\mathrm{PH}^i(V,\mathbb{C})$. We leave out the definition of the monodromy weight filtration $M^\bullet$ on $\mathrm{PH}^i(V,\mathbb{C})$, but we record the following facts.

Fact 2.2.4. *1. Each graded piece Gr_j^M admits a pure Hodge structure of weight j induced by the Hodge filtration on $\mathrm{PH}^i(V,\mathbb{C})$ and $\mathrm{Gr}_j^M = 0$ if $j > 2i$ or $j < 0$.*

2. *$N_i(M_j) \subset M_{j-2}$*

3. *N_i^j induces an isomorphism of pure Hodge structures of degree $(-j,-j)$ between Gr^M_{2i+j} and Gr^M_{2i-j}. Thus the maps $N_\ell^i : \mathrm{Gr}^M_{2i+j} \to \mathrm{Gr}^M_{2i+j-\ell}$ are injective for $\ell \leq j$ and surjective otherwise.*

Thus we can compute the values of the Hodge-Deligne numbers $i^{p,q}(\mathscr{Q}_{i-1})$.

Proposition 2.2.5. *Let $j^{p,q}_{i-1}$ be the Hodge-Deligne numbers of $\mathrm{PH}^{i-1}(V,\mathbb{C})$ equipped with the limit Mixed Hodge structure at infinity. Then*

$$\begin{aligned} i^{p,q}(\mathscr{Q}_{i-1}) &= 0 \text{ if } p+q > i-1 \\ i^{p,q}(\mathscr{Q}_{i-1}) &= j^{p,q}_{i-1} - j^{p-1,q-1}_{i-1} \text{ if } p+q \leq i-1. \end{aligned}$$

The mixed Hodge structure on $\mathrm{H}^{2d-i}(Y,\mathbb{C})$ is dual to the mixed Hodge structure on $\mathrm{H}^i_c(Y,\mathbb{C})$ shifted by (d,d). Thus if $i_c^{p,q}$ are the Hodge-Deligne numbers of $\mathrm{H}^i_c(Y,\mathbb{C})$ and $i^{p,q}$ are the Hodge-Deligne numbers of $\mathrm{H}^{2d-i}(Y,\mathbb{C})$ and let $k^{p,q}$ be the Hodge numbers of $\mathscr{K}_i$. Then

$$\begin{aligned} i_c^{p,q} &= k^{p,q} && \text{if } p+q = i \\ i_c^{p,q} &= 0 && \text{if } p+q > i \\ i_c^{p,q} &= j^{p,q}_{i-1} - j^{p-1,q-1}_{i-1} && \text{if } p+q \leq i-1. \end{aligned}$$

Duality sends $i^{p,q}$ to $i^{d-p,d-q}$. Therefore, we have that Now we may use the exact sequence above to see that

$$\begin{aligned} i^{p,q} &= k^{d-q,d-p} && \text{if } p+q = 2d-i \\ i^{p,q} &= 0 && \text{if } p+q < 2d-i \\ i^{p,q} &= j_{i-1}^{d-p,d-q} - j_{i-1}^{d-p-1,d-q-1} && \text{if } p+q \geq 2d-i+1. \end{aligned}$$

Finally, we can compute the MHS on $\mathrm{H}^i(Y,V;\mathbb{C})$. Let $\mathscr{R}_{2d-i}$ be the map

$$\mathscr{R}_{2d-i} := \mathrm{Im}(\mathrm{H}^{2d-i}(Y,\mathbb{C}) \xrightarrow{r_{2d-i}} \mathrm{H}^{2d-i}(V,\mathbb{C})).$$

Let $r^{p,q}$ be the Hodge numbers of $\mathscr{R}_{2d-i}$. From the long exact sequence for relative cohomology groups, we get a short exact sequence of mixed Hodge structures in cohomology

$$0 \to \mathrm{PH}^{2d-i-1}(V,\mathbb{C}) \to \mathrm{H}^{2d-i}(Y,V;\mathbb{C}) \to \ker(\mathrm{H}^{2d-i}(Y,\mathbb{C}) \xrightarrow{r_{2d-i}} \mathrm{H}^{2d-i}(V,\mathbb{C})) \to 0.$$

There is a pure Hodge structure on $\mathscr{R}^{2d-i}$ of weight $2d-i$, so $r^{p,q} = 0$ unless $p+q = 2d-i$. It is then clear that $k^{p,q} \geq r^{p,q}$ by strictness of morphisms of mixed Hodge structures. Furthermore, one sees that $\mathrm{PH}^{2d-i-1}(V,\mathbb{C})$ is dual (as a Hodge structure) to $\mathrm{PH}^i(V,\mathbb{C})$. This follows from the fact that the monodromy representation on $\mathrm{H}^i(V,\mathbb{C})$ is dual to the monodromy representation on $\mathrm{H}^{2d-i}(V,\mathbb{C})$ and the fact that by the global invariant cycles theorem, $\mathrm{PH}^i(V,\mathbb{C})$ is the orthogonal complement of the classes fixed by the monodromy representation. We let $ph_V^{p,q}$ be the Hodge numbers of $\mathrm{PH}^*(V,\mathbb{C})$. Since $j_{i-1}^{p,q}$ are the Deligne-Hodge numbers of the limit mixed Hodge structure on $\mathrm{PH}^i(V,\mathbb{C})$, we know that

$$\sum_p j_{i-1}^{p,q} = ph_V^{i-1-q,q} = ph_V^{2d-i+q,2d-1+q}.$$

Therefore if $f^{p,q}$ are the Hodge-Deligne numbers of $\mathrm{H}^{2d-i}(Y,V;\mathbb{C})$ then

$$\begin{aligned} f^{p,q} &= k^{d-p,d-q} - r^{p,q} && \text{if } p+q = 2d-i \\ f^{p,q} &= 0 && \text{if } p+q < 2d-i \\ f^{p,q} &= j_{i-1}^{d-p,d-q} - j_{i-1}^{d-p-1,d-q-1} && \text{if } p+q \geq 2d-i+1 \\ f^{p,q} &= ph_V^{p,q} && \text{if } p+q = 2d-i-1. \end{aligned}$$

If F is the Hodge filtration on $\mathrm{H}^{2d-i}(Y,V;\mathbb{C})$, we have that $\mathrm{gr}_F^q = \sum_p i^{p,q}$. Therefore, we have that

$$\mathrm{gr}_F^q = ph_V^{2d-i-q-1,q} + k^{d-(2d-i-q),d-q} - r^{2d-i-q,q} + \sum_{\substack{p \\ p+q \geq 2d-i+1}} \left(j_{i-1}^{d-p,d-q} - j_{i-1}^{d-p-1,d-q-1} \right)$$

We prove that the graded pieces of the Hodge filtration on $\mathrm{H}^{2d-i}(Y,V;\mathbb{C})$ satisfy the usual Hodge symmetry.

Theorem 2.2.6.

$$\mathrm{gr}_F^q = \mathrm{gr}_F^{2d-i-q}$$

Proof. It is easy to see that, since $k^{p,q}$ and $r^{p,q}$ are Hodge numbers of pure weight $2d-i$ Hodge structures, that the theorem is equivalent to the fact that

$$\begin{aligned} ph_V^{2d-i-q-1,q} + \sum_{\substack{p \\ 2d-i+1 \leq p+q}} \left(j_{i-1}^{d-p,d-q} - j_{i-1}^{d-p-1,d-q-1} \right) \\ = ph_V^{q-1,2d-i-q} + \sum_{\substack{p \\ 1 \leq p-q}} \left(j_{i-1}^{d-p,i+q-d} - j_{i-1}^{d-p-1,i+q-d-1} \right) \end{aligned}$$

We note that according to Fact 2.2.4, the Hodge-Deligne numbers $j_{i-1}^{p,q}$ satisfy:

$$j_{i-1}^{p,q} = j_{i-1}^{q,p} = j_{i-1}^{i-1-q,i-1-p} = j_{i-1}^{i-1-p,i-1-q}.$$

Thus $j_{i-1}^{d-p,d-q} = j_{i-1}^{i+p-d-1,i+q-d-1}$ and $j_{i-1}^{d-p-1,d-q-1} = j_{i-1}^{p+i-d,q+i-d}$. Therefore, small

$$\sum_{\substack{p \\ 2d-i+1 \leq p+q}} \left(j_{i-1}^{d-p,d-q} - j_{i-1}^{d-p-1,d-q-1} \right) = \sum_{\substack{p \\ 2d-i+1 \leq p+q}} \left(j_{i-1}^{i+p-d-1,i+q-d-1} - j_{i-1}^{p+i-d,q+i-d} \right) \tag{2.1}$$

Now we change the index over which we take the sum – we let $p = -p_0 + 2d - i$. Then we take the sum over all p_0 so that $2d - i + 1 \leq -p_0 + 2d - i + q$, which is equivalent to all integers p_0 so that $-1 \geq p_0 - q$ therefore, Equation (2.1) becomes

$$-\sum_{\substack{p \\ -1 \geq p-q}} \left(j_{i-1}^{d-p,i+q-d} - j_{i-1}^{d-p-1,i+q-d-1} \right)$$

after re-indexing. Therefore, we must show that

$$\begin{aligned} &ph^{2d-i-q-1,q} - ph^{q-1,2d-i-q} \\ &= \sum_{\substack{p \\ 1 \leq p-q}} \left(j_{i-1}^{d-p,i+q-d} - j_{i-1}^{d-p-1,i+q-d-1} \right) + \sum_{\substack{p \\ -1 \geq p-q}} \left(j_{i-1}^{d-p,i+q-d} - j_{i-1}^{d-p-1,i+q-d-1} \right) \end{aligned}$$

We note that N_i induces an isomorphism of pure Hodge structures between Gr_M^i and Gr_M^{i-2}, therefore if $p = q$ then $j_i^{d-p,i+q-d} = j_i^{d-p-1,i+q-d-1}$ for every p. Thus

$$\begin{aligned} \sum_{\substack{p \\ 1 \leq p-q}} \left(j_{i-1}^{d-p,i+q-d} - j_{i-1}^{d-p-1,i+q-d-1} \right) + &\sum_{\substack{p \\ -1 \geq p-q}} \left(j_{i-1}^{d-p,i+q-d} - j_{i-1}^{d-p-1,i+q-d-1} \right) \\ &= \sum_{p} \left(j_{i-1}^{d-p,i+q-d} - j_{i-1}^{d-p-1,i+q-d-1} \right) \\ &= ph^{d-1-q,i+q-d} - ph^{d-q,i+q-d-1} \\ &= ph^{2d-i-q-1,q} - ph^{q-1,2d-i-q}. \end{aligned}$$

We use the fact that $ph^{d-1-q,d-1-p} = ph^{p,q} = ph^{q,p}$. This establishes the theorem. $\square$

Corollary 2.2.7. *The Hodge numbers of an LG model are symmetric in the sense that $h^{p,q}(Y, \mathsf{w}) = h^{q,p}(Y, \mathsf{w})$.*

From the beginning, we expected that this fact would hold if we restrict the ourselves to the LG models of Katzarkov-Kontsevich and Pantev, however it is a bit surprising that Hodge symmetry holds in general.

2.2.2 Poincaré duality

We will check that Poincaré duality holds for $\mathrm{H}^i(Y, \mathsf{w})$, or in other words that $h^{2d-i}(Y, \mathsf{w}) = h^i(Y, \mathsf{w})$. First, we recall that there is a relative Mayer-Vietoris exact sequence. Let Y_1 and Y_2 be manifolds and let V_1 and V_2 be submanifolds of Y_1 and Y_2 respectively so that $Y = Y_1 \cup Y_2$ and let $S = S_1 \cup S_2$ be the intersection in Y.

$$\cdots \to \mathrm{H}^i(Y, S) \to \mathrm{H}^i(Y_1, S_1) \oplus \mathrm{H}^i(Y_2, S_2) \to \mathrm{H}^i(Y_1 \cap Y_2, S_1 \cap S_2) \to \dots.$$

Now let Σ be the set of critical values of w and let p be a base-point in $\mathbb{A}^1 = \mathbb{C}$. Take a set of open discs $\{U_s\}_{s\in\Sigma}$ so that each disc contains s and p but no other critical values of w, for any subset $S \subseteq \Sigma$, the set $U_S = \bigcap_{s\in S} U_s$ is simply connected and so that $\bigcup_{s\in\Sigma} U_s$ is a deformation retract of $\mathbb{A}^1$. Then let $Y_s = \mathsf{w}^{-1}(U_s)$ for each $s \in \Sigma$. Let $V = \mathsf{w}^{-1}(p)$. The following proposition was claimed in [78].

Proposition 2.2.8.
$$h^i(Y, V; \mathbb{C}) = \sum_{s\in\Sigma} h^i(Y_s, V; \mathbb{C}).$$

Proof. Let $s_1, s_2 \in \Sigma$, then we have chosen U_1 and U_2 so that $U_1 \cap U_2$ is simply connected, open and contains no critical points of w. Thus we have that $\mathsf{w}^{-1}(U_1 \cap U_2)$ is a deformation retract onto V by Ehresmann's theorem, thus $\mathrm{H}^i(\mathsf{w}^{-1}(U_1 \cap U_2), V) = 0$ and therefore $\mathrm{H}^i(Y_1 \cup Y_2, V) \cong \mathrm{H}^i(Y_1, V) \oplus \mathrm{H}^i(Y_2, V)$ by the relative Mayer-Vietoris sequence. Repeating this argument proves the general case. □

Now to each point, $s \in \Sigma$, we can associate a perverse sheaf of vanishing cycles $\phi_{\mathsf{w}-s}\mathbb{C}$ supported on the critical points of w in $\mathsf{w}^{-1}(s)$ (see e.g. [120, 42]), and the hypercohomology of $\phi_{\mathsf{w}-s}\mathbb{C}$ sits in a long exact sequence

$$\cdots \to \mathbb{H}^{i-1}(\mathsf{w}^{-1}(s), \phi_{\mathsf{w}-s}\mathbb{C}) \to \mathrm{H}^i(Y_s, \mathbb{C}) \xrightarrow{r_i} \mathrm{H}^i(V, \mathbb{C}) \to \mathbb{H}^i(\mathsf{w}^{-1}(s), \phi_{\mathsf{w}-s}\mathbb{C}) \to \dots$$

where the map r_i is the natural restriction map. However, this is precisely the map in the long exact sequence for relative cohomology, thus we find that

$$\mathbb{H}^{i-1}(\mathsf{w}^{-1}(s), \phi_{\mathsf{w}-s}\mathbb{C}) \cong \mathrm{H}^i(Y_s, V; \mathbb{C})$$

and therefore,

$$h^i(Y, V; \mathbb{C}) = \sum_{s \in \Sigma} \text{rank } \mathbb{H}^{i-1}(\mathsf{w}^{-1}(s), \phi_{\mathsf{w}-s}\mathbb{C}).$$

It is a well-known fact [42] that the vanishing cycles functor commutes with Verdier duality, or in other words, for any constructible complex $\mathscr{F}^\bullet$ on Y^s, if $\mathbb{D}$ denotes the Verdier duality functor, then $\mathbb{D}(\phi_{\mathsf{w}-s}\mathscr{F}^\bullet[-1]) \cong (\phi_{\mathsf{w}-s}\mathbb{D}\mathscr{F}^\bullet)[-1]$ (see [42, Proposition 4.2.10]). Since $\mathbb{D}\mathbb{C}_{Y_s} = \mathbb{C}_{Y_s}[2d]$ (see [42, Example 3.3.8]) where d is the complex dimension of Y, it follows by [42, Theorem 3.3.10] that

$$\begin{aligned}
\mathbb{H}^{m+1}(\mathsf{w}^{-1}(s), \phi_{\mathsf{w}-s}\underline{\mathbb{C}}_{Y_s}) &\cong \mathbb{H}^{m}(\mathsf{w}^{-1}(s), \phi_{\mathsf{w}-s}\underline{\mathbb{C}}_{Y_s}[-1]) \\
&\cong \mathbb{H}_c^{-m}(\mathsf{w}^{-1}(s), \mathbb{D}\phi_{\mathsf{w}-s}\underline{\mathbb{C}}_{Y_s}[-1])^\vee \\
&\cong \mathbb{H}_c^{-m}(\mathsf{w}^{-1}(s), \phi_{\mathsf{w}-s}\underline{\mathbb{C}}_{Y_s}[2d-1])^\vee \\
&\cong \mathbb{H}_c^{2d-(m+1)}(\mathsf{w}^{-1}(s), \phi_{\mathsf{w}-s}\underline{\mathbb{C}}_{Y_s})^\vee
\end{aligned}$$

Since $\mathsf{w}^{-1}(s)$ is itself compact it follows that

$$\mathbb{H}^{m}(\mathsf{w}^{-1}(s), \phi_{\mathsf{w}-s}\underline{\mathbb{C}}_{Y_s}) \cong \mathbb{H}^{2d-m}(\mathsf{w}^{-1}(s), \phi_{\mathsf{w}-s}\underline{\mathbb{C}}_{Y_s})^\vee.$$

Along with Proposition 2.2.8 this implies that:

Theorem 2.2.9. *Poincaré duality holds for LG models. In other words,*

$$h^i(Y, \mathsf{w}) = h^{2d-i}(Y, \mathsf{w}).$$

This theorem requires remarkably few assumptions. We need to have that w is a proper, relatively compact fibration with smooth total space over $\mathbb{A}^1$, but nothing more. In terms of Definition 2.2.1, we require that only Condition (1) holds.

Beyond being simply a nice fact, this allows us to conclude that a version of Lefschetz duality holds for the Hodge numbers of LG models in some situations. I expect that it holds in all situations, but I have not been able to prove it. In the following theorem, we assume that all conditions of Definition 2.2.1 hold. In [79], it is argued that if X is a Fano variety, then the limit mixed Hodge structure on $\mathrm{H}^{i-1}(V,\mathbb{C})$ at ∞ is Hodge-Tate. By this, we mean that $j^{p,q} = 0$ unless $p = q$. This implies that if $p \leq q$, then $h^{p,2d-i-p}(Y,\mathsf{w}) = ph^{p,2d-i-1-p} + k^{p,i-p} - r^{p,2d-i-p}$.

Corollary 2.2.10. *Assume the limit mixed Hodge structure on* $\mathrm{H}^{i-1}(V,\mathbb{C})$ *is Hodge-Tate for all* i. *If* $d = 3$ *or* 4 *or if* $h^{p,i-p}(\mathscr{K}_i) = 0$ *for* $p \neq i-p$ *then*

$$h^{p,i-p}(Y,\mathsf{w}) = h^{d-p,d-i+p}(Y,\mathsf{w})$$

for any i *or* p. *In other words, the Hodge numbers of* (Y,w) *form a Hodge diamond in the usual sense.*

Proof. Note that Theorem 2.2.6 ensures that Hodge duality holds, or in other words that $\mathrm{H}^{p,q}(Y,\mathsf{w}) = \mathrm{H}^{q,p}(Y,\mathsf{w})$. Using the fact that $\mathrm{H}^{0,i}(Z) = 0$ for $i \neq 0$ and the fact that V is Calabi-Yau and hence $\mathrm{H}^{i,0}(V) = 0$ for $i \neq 0, d-1$, a quick look at Theorem 2.2.6 shows that $\mathrm{H}^{p,q}(Y,\mathsf{w}) = 0$ for p or $q = 0$ or d and $(p,q) \neq (d,0)$ or $(0,d)$. In the case where Z is a threefold, we only need to check that $h^{i,4-i}(Y,\mathsf{w}) = h^{3-i,i-1}(Y,\mathsf{w})$. However, $h^{0,2} = h^{2,0} = h^{0,4} = h^{4,0} = 0$, so we only need to check that $h^{1,1} = h^{2,2}$. But we have:

$$h^{1,1}(Y,\mathsf{w}) = h^2(Y,\mathsf{w}) = h^4(Y,\mathsf{w}) = h^{2,2}(Y,\mathsf{w})$$

where the second equality comes from Poincaré duality.

The case where $d = 4$ we can show that $h^{1,1} = h^{3,3}$ in the same way. To check this duality for $h^{i,5-i}$ and $h^{i,3-i}$, note that $h^5 = 2h^{3,2}$ and $h^3 = 2h^{2,1}$ by Hodge duality, thus $h^{2,1} = h^{1,2} = h^{2,3} = h^{3,2}$.

Now assume that $h^{i,p-i}(\mathscr{K}_i) = 0$ for $i \neq p-i$. Then we have that $h^{p,q}(Y,\mathsf{w}) = h^{p,q-1}(V)$ for $q < p$ and $h^{p,q}(Y,\mathsf{w}) = h^{p-1,q}(V)$ for $q > p$. Therefore if $q < p$, then

$$h^{p,q}(Y,\mathsf{w}) = h^{p,q-1}(V) = h^{d-p-1,d-q}(V) = h^{d-p,d-q}(Y,\mathsf{w}).$$

Similarly for $q > p$. Now we have that if $i \leq d$,

$$h^i(Y, \mathsf{w}) = \sum_{j=0}^{i} h^{j,i-j}(Y, \mathsf{w}) = \sum_{j=0} h^{d-j,d-i+j}(Y, \mathsf{w}) = h^{2d-i}(Y, \mathsf{w})$$

This equality, along with the fact that $h^{j,i-j}(Y, \mathsf{w}) = h^{d-j,d-i+j}(Y, \mathsf{w})$ except if $j = i-j$ allows us to conclude equality when $j = i - j$. □

Remark 2.2.11. The fact that we have only proved that this Lefschetz-type duality holds in some cases is quite unsatisfactory to me. Given the fact that Poincaré duality holds and that we know that there is a certain amount of symmetry in the Hodge numbers of the LG model, coming from Lefschetz duality for the Hodge numbers of the primitive cohomology of V, it seems quite unlikely that a theorem like Corollary 2.2.10 holds in general.

Now we mention that in the case where (Y, w) is a LG model in the sense of Definition 2.2.1 and any of the conditions in Corollary 2.2.10 hold, then the values of $h^{p,q}(Y, \mathsf{w})$ admit horizontal and vertical symmetries. Furthermore, we may note that

- $h^{d,0}(Y, \mathsf{w}) = 1$,
- $h^{i,0}(Y, \mathsf{w}) = h^{0,i}(Y, \mathsf{w}) = 0$ for $i \neq d$,
- $h^{d,i}(Y, \mathsf{w}) = h^{i,d}(Y, \mathsf{w}) = 0$ if $i \neq 0$.

Thus if any of the conditions of Corollary 2.2.10 hold then the "Hodge diamond" of (Y, w) looks remarkably like the Hodge diamond of a Fano variety rotated by 90 degrees.

2.2.3 LG models of del Pezzo surfaces

At this point, let's work out a basic example. Surfaces Z satisfying Definition 2.2.1 are precisely elliptically fibered surfaces over $\mathbb{P}^1$ with $h^{2,0} = 0$ and a singular fiber of type I_n at ∞. For the moment, we will assume that the elliptic fibration $\mathsf{f} : Z \to \mathbb{P}^1$ admits a section and hence by the classification of elliptic surfaces, Z is a rational elliptic surface. Let us define Z_n to be an rational elliptic surface with section whose elliptic fibration given by $\mathsf{f} : Z_n \to \mathbb{P}^1$ and so that there is a singular fiber of type I_n over ∞ for $1 \leq n \leq 9$. In plain language, this is a normal crossings union of n-rational curves whose dual intersection complex is the extended Dynkin diagram $\widetilde{A}_n$ if $n > 1$ and a nodal rational curve if $n = 1$. Then let $Y_n = Z_n \setminus D_\infty$, let $\mathsf{w} = \mathsf{f}|_{Y_n}$ and let E be a smooth fiber of w. According to work of Auroux, Katzarkov and Orlov, [8], the pair (Y_n, w) is the homological mirror dual of the del Pezzo surface of degree n. We will check this against our computations.

First, we recall that $h^{p,q}(Z)$ is equal to 10 if $p = q = 1$, equal to 1 if $p = q = 0$ or $p = q = 2$ and 0 otherwise (see e.g. [99, Lecture IV, §1]). The action of monodromy around ∞ on $\mathrm{H}^2(E, \mathbb{Z})$ is given by the matrix

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

(see e.g. [99, Table VI.2.1]). We must compute the value of $\ker N_i$ for $i = 0, 1, 2$. For $i = 0, 2$, we know that $\ker N_i = \mathbb{C}$, and the weight filtration in these cases is trivial. In the case where $i = 1$, it is a standard computation to show that the LMHS has $i^{0,0} = i^{1,1} = 1$, and $\ker N_1$ is precisely the weight 0 piece of the monodromy weight filtration at ∞. It has rank 1 and F^1 is empty, so the only nonvanishing $i^{p,q}$ value of $\ker N_1$ is $i^{0,0} = 1$. Since $\mathrm{H}^1(Z_n, \mathbb{C}) = 0$ it follows that $\mathscr{Q}_1 = \ker N_1$.

Now we can compute the value of $\mathrm{H}^1(Y, \mathsf{w})$. Let $\mathsf{f}^{-1}(\infty) = D_\infty$ as usual. Using the Mayer-Vietoris spectral sequence [66], we see that $h^2(D_\infty) = n$, $h^1(D_\infty) = 1$ and $h^0(D_\infty) = 1$, and that all of the mixed Hodge structures here are pure, Hodge-Tate and of weights $2, 0$ and 0 respectively. We know that $\mathrm{H}^1(Z, \mathbb{C})$ vanishes, thus we have an exact sequence

$$0 \to \mathrm{H}^0_c(Y_n, \mathbb{C}) \to \mathrm{H}^0(Z_n, \mathbb{C}) \to \mathrm{H}^0(D_\infty, \mathbb{C}) \to \mathrm{H}^1_c(Y_n, \mathbb{C}) \to 0$$

The map $\mathrm{H}^0(Z_n, \mathbb{C}) \to \mathrm{H}^0(D_\infty, \mathbb{C})$ is an isomorphism which implies the vanishing of $h^0_c(Y_n) = h^4(Y_n)$ and $h^1_c(Y_n) = h^3(Y_n)$. Thus the sequence for relative cohomology looks like

$$\cdots \to \mathrm{H}^2(Y_n,\mathbb{C}) \to \mathrm{H}^2(E,\mathbb{C}) \to \mathrm{H}^3(Y_n,E;\mathbb{C}) \to 0 \to 0 \to \mathrm{H}^4(Y_n,E;\mathbb{C}) \to 0$$

The map $\mathrm{H}^2(Y_n,\mathbb{C}) \to \mathrm{H}^2(E,\mathbb{C})$ is also surjective, since Y_n admits a section. Therefore $h^3(Y_n,\mathsf{w}) = h^4(Y_n,\mathsf{w}) = 0$. Using the fact that Poincaré duality holds for LG models, it follows that the only non-zero $h^i(Y_n,\mathsf{w})$ occurs when $i = 2$. From the long exact sequence of compactly supported cohomolgy above, it follows that $\chi(Y_n)+\chi(D_\infty) = \chi(Z_n)$ and thus $\chi(Y_n) = 12-n$. Using the long exact sequence for the relative cohomology (Y_n,E), for E a smooth fiber of w, one finds that $\chi(Y_n,E) = h^2(Y_n,E) = 12-n$.

Continuing to work backwards, we have a long exact sequence for the relative cohomology

$$0 \to \mathrm{H}^1(Y_n,\mathbb{C}) \to \mathrm{H}^1(E,\mathbb{C}) \to \mathrm{H}^2(Y_n,E;\mathbb{C}) \to \mathrm{H}^2(Y_n,\mathbb{C}) \to \mathrm{H}^2(E,\mathbb{C}) \to 0$$

Since $\mathrm{H}^1(Y_n,\mathbb{C})$ has no weight 1 component, its image in $\mathrm{H}^1(E,\mathbb{C})$ must be 0, and hence must itself be 0. Therefore, by counting dimensions in the exact sequence above, we find that $h^2(Y_n) = 11-n$. Now we may look at the long exact sequence for the compactly supported cohomology of Y_n,

$$0 \to \mathrm{H}^1(D_\infty,\mathbb{C}) \to \mathrm{H}^2_c(Y_n,\mathbb{C}) \to \mathrm{H}^2(Z_n,\mathbb{C}) \cong \mathbb{C}^{10} \to \mathbb{C}^n \to \mathrm{H}^3_c(Y_n,\mathbb{C}) \to 0$$

The map $\mathbb{C}^{10} \to \mathbb{C}^n$ is forced to be surjective, for dimension reasons and thus $h^3_c(Y_n) = 0$. It then follows that the MHS on $\mathrm{H}^2_c(Y_n,\mathbb{C})$ has Hodge numbers

$$i^{0,0} = 1 \qquad i^{1,1} = 10-n$$

since we have noted that $\mathrm{H}^1(D_\infty,\mathbb{C})$ is pure of weight 0 and rank 1 and $\mathrm{H}^2(Z,\mathbb{C})$ is pure Hodge-Tate of weight 2. Hence $\mathrm{H}^2(Y_n,\mathbb{C})$ admits a MHS so that $i^{2,2} = 1$ and $i^{1,1} = 10-n$. Thus the Hodge numbers of $\mathrm{H}^2(Y_n,E;\mathbb{C})$ are $i^{1,0} = i^{0,1} = 1, i^{1,1} = 9-n$

and $i^{2,2} = 1$. This implies that $\mathrm{gr}_0^W = 1, \mathrm{gr}_1^W = 10 - n$ and $\mathrm{gr}_2^W = 1$. Thus the $h^{p,q}$ Hodge numbers of (Y_n, w) give us a Hodge diamond

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 10-n & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{array}$$

which is of course the Hodge diamond of a del Pezzo surface of degree n rotated by $\pi/2$.

Remark 2.2.12. There is no reason for us to restrict ourselves to the case where Z admits a section. For instance, we can let Z be an Enriques surface, whose Hodge numbers are the same as those of a rational elliptic surface, yet is not rational. Such a surface admits an elliptic fibration over $\mathbb{P}^1$ with a bi-section (an irreducible curve in Z which intersects a general fiber in 2 points).

2.2.4 LG models of Fano type

In this section, we will argue that even beyond the assumption that the limit mixed Hodge structure on $\mathrm{H}^i(V, \mathbb{C})$ at infinity be Hodge-Tate, further restrictions are necessary in order to characterize, among all LG models, which LG models correspond to Fano manifolds under mirror symmetry. We argue that a simultaneously refined and coarsened version of Dolgachev's lattice polarized mirror symmetry should extend to pairs of embedded Calabi-Yau varieties.

Topologically, a K3 surface is its own mirror, however Dolgachev [44] has formulated a subtle form of mirror symmetry which will be useful to us. He defines a *lattice polarized* K3 surface, which is simply a K3 surface S with a primitive embedding by a lattice $L \hookrightarrow \mathrm{H}^2(S, \mathbb{Z})$ so that the image of L contains a semi-ample divisor. We also assume that the orthogonal complement of L in $\mathrm{H}^2(S, \mathbb{Z})$ contains a copy of the rank 2 unimodular indefinite lattice U. Let $L^\vee \oplus \mathrm{U} = L^\perp$, then the mirror of an L-polarized K3 surface is an $L^\vee$-polarized K3 surface.

For instance, if $S \subseteq X$ is an anticanonical hypersurface in a Fano threefold X, then S comes equipped with a lattice polarization coming from the restriction map $\mathrm{H}^2(X,\mathbb{Z}) \hookrightarrow \mathrm{H}^2(S,\mathbb{Z})$. On the other hand, if $S^\vee$ is a fiber of the LG model (Y,w) of X, then it also comes equipped with a restriction map $\mathrm{H}^2(Y,\mathbb{Z}) \to \mathrm{H}^2(S^\vee,\mathbb{Z})$. It is reasonable to postulate that these induce dual lattice embeddings on S and $S^\vee$. Indeed this is true, as I have checked in joint work with Doran, Katzarkov, Lewis and Przyjalkowski [47] that this holds in many cases. A more coarse way to state this duality is as a relationship between the primitive and co-primitive cohomology of $S \subseteq X$ and $S^\vee \subseteq Y$. Define

$$\mathrm{H}^{p,q}_{\mathrm{prim}}(S) = \ker(\mathrm{H}^{p,q}(X) \to \mathrm{H}^{p,q}(S))$$
$$\mathrm{H}^{p,q}_{\mathrm{co-prim}}(S) = \mathrm{Im}(\mathrm{H}^{p,q}(X) \to \mathrm{H}^{p,q}(S)).$$

We may make similar definitions for $S^\vee \subseteq Y$, or indeed, any smooth variety embedded in another. This motivates a refined version of the mirror symmetry conjecture for Hodge numbers. If $W \subseteq X$ and $V \subseteq Y$ are Calabi-Yau manifolds of dimension d with W embedded in a Fano variety X as an anticanonical hypersurface and V the fiber of the LG model Y of X, then we expect to have

$$\mathrm{H}^{d-p,q}_{\mathrm{co-prim}}(V) = \mathrm{H}^{p,q}_{\mathrm{prim}}(W).$$

and vice-versa. Note that $\mathrm{H}^{p,q}_{\mathrm{co-prim}}(W) = \mathrm{H}^{p,q}(X)$ if $p+q \leq d-1$ since X is assumed to be Fano. This computation provides justification from mirror symmetry for the following definition:

Definition 2.2.13. A LG model is of *Fano type* if $h^{p,i-p}(\mathscr{K}_i) = 0$ unless $p = i-p$ and if the limit mixed Hodge structure on $\mathrm{PH}^{i-1}(V,\mathbb{C})$ at infinity is Hodge-Tate for all i.

Indeed, if $W \subseteq X$ is a smooth anticanonical hypersurface in a Fano variety of dimension d, then for $p+q < d-1$, we have that $\mathrm{H}^{p,q}(W,\mathbb{C}) \cong \mathrm{H}^{p,q}(X,\mathbb{C})$ by the Lefschetz hyperplane theorem. Assume that $\dim X > 3$. If V is mirror to W, then we have that $h^{d-p,q}(W) = h^{p,q}(V)$. First applying mirror symmetry for Fano varieties, then for Calabi-Yau varieties, we find that

$$h^{p,i-p}(Y,\mathsf{w}) = h^{d-p,i-p}(X) = h^{d-p,i-p}_{\text{co-prim}}(W) = h^{p-1,i-p}_{\text{prim}}(V).$$

By applying the computations preceding Theorem 2.2.6 under the condition that Y be of Fano LG type, the equality $h^{p,i-p}(Y,\mathsf{w}) = h^{p-1,i-p}_{\text{prim}}(V)$ for every $p < i$ holds if and only if (Y,w) is of Fano LG type. This implies that the only information in Y which is not carried by mirror symmetry for a generic fiber of w is contained in the ring $\bigoplus_{i=1}^{d-1} \mathrm{H}^{i,i}(Y,\mathsf{w})$. This is mirror to the statement that the only cohomological information in X that is lost when passing to W is in $\mathrm{H}^d(X,\mathbb{C})$.

Example 2.2.14 (Threefolds)**.** We can describe the MHS on $\mathrm{H}^3(Y,\mathsf{w})$ of a Landau-Ginzburg threefold. Here we will make the assumption that the action T of monodromy around ∞ on $\mathrm{H}^2(\mathsf{w}^{-1}(t),\mathbb{C})$ satisfies $(T-\mathrm{Id})^3 = 0$ and $(T-\mathrm{Id})^2 = 0$ and that D_∞ is a normal crossings union of rational surfaces. If D_∞ is semistable then this is called a type III degeneration of K3 surfaces, and the associated limit mixed Hodge structure on $\mathrm{PH}^2(S,\mathbb{C})$ has $j^{2,2} = j^{0,0} = 1$ and $ph^{1,1} = j^{1,1}_2$.

A computation by the Mayer-Vietoris spectral sequence [66] shows that the rank of $\mathrm{H}^1(\mathsf{f}^{-1}(\infty),\mathbb{C})$ is zero, thus $\mathscr{K}_1 = \mathrm{H}^1(Z,\mathbb{C})$. Furthermore, since $\mathrm{H}^2(S,\mathbb{C}) = 0$, we know that $\mathscr{R}_3 = 0$. Therefore, the MHS on $\mathrm{H}^3(Y,V;\mathbb{C})$ is described in Table 2.1, since the fact that $\mathrm{H}^3(V,\mathbb{C}) = 0$ forces $\mathscr{R}_3 = \mathscr{R}_1 = 0$ as well.

To compute the Hodge numbers $h^{1,1}(Y,\mathsf{w})$, we note that $h^{2,2}(Y,\mathsf{w})$ is just $k^{2,2} - 1$, since we have a long exact sequence in cohomology

$$\cdots \to \mathrm{H}^3(S,\mathbb{C}) = 0 \to \mathrm{H}^4(Y,S;\mathbb{C}) \to \mathscr{K}_4 \to \mathrm{H}^4(S,\mathbb{C}) = \mathbb{C} \to \dots$$

	0	1	2	3
6	0	0	0	1
5	0	0	0	0
4	0	0	$h^{1,1}_{\text{prim}} - 1$	0
3	0	$k^{1,2}$	$k^{2,1}$	0
2	1	$h^{1,1}_{\text{prim}}$	1	0

Table 2.1 The Hodge numbers of $\mathrm{H}^3(Y, \mathsf{w})$ for Y a LG model in dimension 3. The rows denote the graded pieces of the weight filtration and the columns determine grade pieces of the Hodge filtration.

Thus, using Corollary 2.2.10 the "Hodge diamond" of (Y, w) looks like

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & k^{2,2} - 1 & & 0 & \\
1 & & ph^{1,1} + k^{1,2} & & ph^{1,1} + k^{2,1} & & 1 \\
 & 0 & & k^{2,2} - 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}
$$

Therefore if Z is of Fano LG type then one must only compute the rank of $ph^{1,1}$, the value of $k^{1,2} = h^{1,2}(Z)$ and $k^{2,2}$. According to Theorem 3.3.1 if ρ_t be the number of irreducible divisorial components in $\mathsf{w}^{-1}(t)$, and Σ be the critical locus of w. Then $h^{1,1}(Y, \mathsf{w}) = \sum_{t \in \Sigma} (\rho_t - 1)$.

Thus the condition that (Y, w) be an extremal LG model is equivalent to the fact that $k^{1,2}$ vanishes. Then if X is a Fano threefold and (Y, w) is its LG model, then $h^{1,2}(Y, \mathsf{w}) = h^{1,1}(X)$ if and only if $h^{1,1}_{\text{co-prim}}(S) = ph^{1,1}(S^\vee)$ for S an anticanonical hypersurface in X and $S^\vee$ a fiber of w. We will say more about this in the following section.

2.3 LG models of smooth toric weak-Fano varieties

In the next couple of sections, we will discuss mirror symmetry for toric weak Fano varieties with at worst terminal singularities. In the case of threefolds, we will show that LG models can be constructed by direct methods, and that one can compute very directly the Hodge numbers of the LG model in order to establish that Hodge number mirror symmetry holds. In the case where X_Δ has dimension higher than 3, we are not guaranteed that there exists a smooth LG model for X_Δ in the sense of Definition 2.2.1. In the case where such a resolution does exist, we will reduce the computation of $\mathrm{H}^{p,q}(Y,\mathsf{w})$ to the computation of the Hodge-graded pieces of $\mathrm{H}^{p+q}((\mathbb{C}^\times)^d, U_f;\mathbb{C})$ where U_f is the vanishing locus of a general Laurent polynomial with Newton polytope Δ. We will show that for any polytope Δ, the $h^{p,q}((\mathbb{C}^\times)^d, U_f) = h^{d-q,p}_{st}(X_\Delta)$ where $h^{q,d-p}_{st}(X_\Delta)$ are the *stringy Hodge numbers* [19] of a maximal partial crepant projective resolution of singularities of the Gorenstein Fano toric variety associated to the polytope Δ. This should be thought of as the right analogues of Hodge number mirror symmetry for toric weak Fano varieties.

2.3.1 Explicit computation when $d = 3$

Now let us discuss mirror symmetry for smooth toric threefolds. We will show that there is no need for the hypothesis that X be Fano in order for Hodge number mirror symmetry to hold. Let Δ be a reflexive polytope embedded in $M \otimes \mathbb{R}$ for some lattice M of rank 3, and let Σ be the fan over the faces of Δ. We may choose a refinement of Σ which we call $\hat{\Sigma}$ so that each cone of $\hat{\Sigma}$ is spanned by rays which generate the lattice M, and that all of the rays of $\hat{\Sigma}$ are contained in the boundary of Δ. Such a refinement exists by [14]. Let X_Δ be the toric variety associated to the fan $\hat{\Sigma}$, then X_Δ is a smooth projective resolution of $\mathbb{P}_\Delta$. One can show that

$$h^{1,1}(X_\Delta) = \ell(\Delta) - 4.$$

where $\ell(\Delta)$ is the number of lattice points in Δ. Furthermore, since X_Δ is a number of copies of $(\mathbb{C}^\times)^k$ $(k = 0, 1, 2, 3)$ glued together, $h^{i,j}(X_\Delta) = 0$ if $i \neq j$. Let S be an anticanonical K3 hypersurface in X_Δ, then the rank of the kernel of the map

$$\mathrm{H}^2(X_\Delta, \mathbb{C}) \to \mathrm{H}^2(S, \mathbb{C})$$

is equal to the sum $\sum_{\dim F=2} \ell^*(F)$. Here the sum is over all faces of Δ of dimension 2 and $\ell^*(F)$ denotes the number of points on the relative interior of F. Thus

$$h^{1,1}_{\mathrm{co-prim}}(S) = \ell(\Delta) - 4 - \sum_{\dim F=2} \ell^*(F).$$

2.3.1.1 Building the mirror

Now let us build the mirror of X_Δ. We take first of all the polar dual polytope Δ°. If $N = \mathrm{Hom}\,(M, \mathbb{Z})$, then Δ° is a polytope in $N_\mathbb{R} = N \otimes \mathbb{R}$ which is integral with respect to the lattice N. We associate to it the toric variety X_{Δ° in the same way we constructed X_Δ above. The homogeneous coordinate ring of X_{Δ° is $\mathbb{C}[\{x_\rho\}_{\rho \in \partial\Delta^\circ \cap N}]$ and is graded by $\mathrm{Pic}(X_{\Delta^\circ})$. We may choose any generic global section s of $\omega^{-1}_{X_{\Delta^\circ}}$, and let $s_0 = \prod_{\rho \in \partial\Delta^\circ \cap N} x_\rho$. We may then produce a pencil $\mathscr{P}(s, r)$ of anticanonical hypersurfaces in X_{Δ° written as $\{rs - ts_0 = 0\}$ over $\mathbb{P}^1_{r,s}$. The base locus of this pencil is just the intersection of $S^\vee = \{s = 0\}$ with the union of all toric boundary divisors D_ρ in X_{Δ°. By the assumption that $S^\vee$ is generic, it follows that $S^\vee \cap D_\rho$ is a smooth curve ([14]). We may sequentially blow up the curves in the base locus of $\mathscr{P}(t, r)$ to resolve indeterminacy. The result is a smooth variety Z_Δ which is fibered over $\mathbb{P}^1_{t,r}$. Call this map f. We can furthermore compute that the fibers of f are sections of the anticanonical bundle of Z_Δ. Note that this construction only depends upon the fact that X_{Δ° is smooth. Therefore, it can be carried out in arbitrary dimension. We will make use of this in Section 2.3.2.

The fiber over $r = 0$ of f is a normal crossings union of smooth rational surfaces whose dual intersection complex is a triangulation of the sphere, which is precisely the triangulation of Δ° used to build the refinement of Σ_{Δ° defining X_{Δ°. Furthermore, recall that if we have a smooth blow-up $\pi : \widetilde{X} \to X$ along a codimension 2 subvariety C in X, and $E = \pi^{-1}(C)$, then $-K_{\widetilde{X}} = -\pi^* K_X - E$. Thus if $E_{C_1}, \ldots, E_{C_k}$ are the exceptional divisors of the map $\pi : Z_\Delta \to X_{\Delta^\circ}$, it follows that

$$-K_{Z_\Delta} = -\pi^* K_{X_{\Delta^\circ}} - \sum_{i=1}^{k} E_{C_i}.$$

which is just the class of a fiber in the fibration f. Therefore, since $-K_{Z_\Delta}$ is supported on fibers, the degeneration of K3 surfaces at infinity is a type III degeneration of K3 surfaces [58, 90], so a theorem of Kulikov and Persson-Pinkham [90, 118] shows that monodromy is maximally unipotent and the limit mixed Hodge structure on $\mathrm{H}^2(S^\vee, \mathbb{C})$ at the point at infinity is Hodge-Tate.

2.3.1.2 Hodge numbers

Now we compute the Hodge numbers of Z_Δ.

Proposition 2.3.1.

$$\begin{aligned} h^{2,1}(Z_\Delta) &= \sum_{F \in \Delta[2]} \ell^*(F) \\ h^{1,1}(Z_\Delta) &= 2\ell(\Delta^\circ) - 5 - \sum_{F \in \Delta^\circ[2]} \ell^*(F) + \sum_{F \in \Delta^\circ[1]} \ell^*(F)\ell^*(F^\circ). \end{aligned}$$

Proof. On the big torus $(\mathbb{C}^\times)^3_{x,y,z}$ of X_{Δ°, there is a Laurent polynomial $f(x,y,z)$ which determines $S^\vee$ and so that the Newton polytope of $f(x,y,z)$ is Δ. We can compute (see [17]) that the restriction of $S^\vee$ to the big torus $(\mathbb{C}^\times)^2$ in any D_v has Newton polytope which is computed as follows: let $v \in \Delta^\circ$, and let $\Gamma(v)$ be the smallest face of Δ° containing v. The face $\Gamma(v)$ has a dual face $\Gamma(v)^\circ$ in Δ defined to be

$$\Gamma(v)^\circ = \{\sigma \in M_\mathbb{R} : \langle v, \sigma \rangle = -1\}.$$

These faces satisfy $\dim \Gamma(v) + \dim \Gamma(v)^\circ = 2$. The restriction of $S^\vee$ to the big torus $(\mathbb{C}^\times)^2 \subseteq D_v$ has Newton polytope $\Gamma(v)^\circ$. Thus

1. If $\dim \Gamma(v)^\circ = 2$, then $D_v \cap S^\vee = \emptyset$,

2. If $\dim \Gamma(v)^\circ = 1$ then $D_v \cap S^\vee$ is a union of $1 + \ell^*(\Gamma(v)^\circ)$ smooth rational curves.

3. If $\dim \Gamma(v)^\circ = 0$ then $D_v \cap S^\vee$ is a single smooth curve whose genus is $\ell^*(\Gamma(v))^\circ$ (this follows by [38]).

These statements can be deduced from [17, Theorem 2.5] or an easy computation. Now, recall that if we let $\widetilde{X}$ be the blow up of a threefold X in a smooth irreducible curve of genus g then

$$\begin{aligned} h^{2,2}(\widetilde{X}) &= h^{1,1}(\widetilde{X}) = h^{1,1}(X) + 1 \\ h^{2,1}(\widetilde{X}) &= h^{1,2}(\widetilde{X}) = h^{1,2}(X) + g \end{aligned}$$

see e.g. [143, §7.3.3]. One may then compute without much trouble that

$$\begin{aligned} h^{2,1}(Z_\Delta) &= \sum_{F \in \Delta[2]} \ell^*(F) \\ h^{1,1}(Z_\Delta) &= 2\ell(\Delta^\circ) - 5 - \sum_{F \in \Delta^\circ[2]} \ell^*(F) + \sum_{F \in \Delta^\circ[1]} \ell^*(F)\ell^*(F^\circ). \end{aligned}$$

as claimed. □

Now we let $Y_\Delta = Z_\Delta \setminus D_\infty$ (here we consider $r = 0$ to be the point at infinity of ∞) and let $\mathsf{w} = \mathsf{f}|_Y$. We must first compute the rank of primitive cohomology of a fiber $S^\vee$ of w.

Lemma 2.3.2 (Kreuzer-Skarke, [89], Rohsiepe, [130])**.**

$$h^{1,1}_{\mathrm{prim}}(S^\vee) = 24 - \ell(\Delta^\circ) + \sum_{F \in \Delta^\circ[2]} \ell^*(F) - \sum_{F \in \Delta^\circ[1]} \ell^*(F)\ell^*(F^\circ).$$

and therefore

$$h^{2,1}(Y, \mathsf{w}) = 24 - \ell(\Delta^\circ) + \sum_{F \in \Delta^\circ[2]} \ell^*(F) - \sum_{F \in \Delta^\circ[1]} \ell^*(F)\ell^*(F^\circ) + \sum_{F \in \Delta[2]} \ell^*(F).$$

Proof. Rohsiepe [130] proves that

$$\ell(\Delta^\circ) - 4 - \sum_{F \in \Delta^\circ[2]} \ell^*(F) + \sum_{F \in \Delta^\circ[1]} \ell^*(F)\ell^*(F^\circ)$$

is the Picard rank of a general enough anticanonical hypersurface in X_{Δ° and $\mathrm{Pic}(S^\vee)$ is spanned by the curves $D_v \cap S^\vee$ for v a point in $\partial\Delta^\circ \cap N$. Since we have blown up along all of the curves in $D_v \cap S^\vee$ to obtain Z_Δ, it follows that there is indeed a divisor in Z_Δ that restricts to $S^\vee$ to give $D_v \cap S^\vee$ for any v in Δ°. Thus the rank of the image of the restriction map is at least the rank of $\mathrm{Pic}(S^\vee)$ and hence we have equality. The second statement is a result of this along with Theorem 2.2.6, Example 2.2.14 and Proposition 2.3.1. □

Remark 2.3.3. If X_Δ is itself a Fano variety, then for all $F \in \Delta[2]$, $\ell^*(F) = 0$. Thus it follows that $h^{2,1}(Z_\Delta) = 0$, or equivalently that (Y_Δ, w) is of Fano LG type. It is also clear that if $\ell^*(F) = 0$ for all $F \in \Delta[2]$, then X_Δ is of Fano LG type, but this condition on Δ is not equivalent to X_Δ being itself Fano. Therefore, it might be good to come up with another name for this class of objects.

Now we can check that:

Lemma 2.3.4.

$$\ell(\Delta) - 4 = 24 - \ell(\Delta^\circ) + \sum_{F\in\Delta^\circ[2]} \ell^*(F) - \sum_{F\in\Delta^\circ[1]} \ell^*(F)\ell^*(F^\circ) + \sum_{F\in\Delta[2]} \ell^*(F).$$

Proof. We use [89, Equation 5] to see that if $S^\vee$ is a generic hypersurface of X_{Δ°, then

$$20 = h^{1,1}(S) = \text{rank Pic}(S) + \ell(\Delta) - 4 - \sum_{v\in\Delta^\circ[1]} \ell^*(v^\circ)$$

[89, Equation 4] tells us that

$$\text{rank Pic}(S) = \ell(\Delta^\circ) - 4 - \sum_{F\in\Delta^\circ[2]} \ell^*(F) + \sum_{F\in\Delta^\circ[1]} \ell^*(F)\ell^*(F^\circ).$$

Combining these two statements and rearranging gives the proposition immediately. □

Corollary 2.3.5.

$$h^{1,1}(X_\Delta) = h^{2,1}(Y, \mathsf{w}).$$

As noted by Kreuzer and Skarke [89, pp. 8], there is a fundamental problem with trying to verify lattice polarized mirror symmetry between Batyrev dual K3 hypersurfaces in toric varieties, which is that if $S \subseteq X_\Delta$ and $S^\vee \subseteq X_{\Delta^\circ}$ are generic anticanonical hypersufaces, then

$$\text{rank Pic}(S) + \text{rank Pic}(S^\vee) = 20 + \sum_{F \in \Delta[1]} \ell^*(F)\ell^*(F^\circ)$$

while lattice polarized mirror symmetry claims [44] that we should have

$$\text{rank Pic}(S) + \text{rank Pic}(S^\vee) = 20.$$

This problem does not appear when we look at mirror symmetry for smooth toric varieties, as we have just seen. The difference between $\text{Pic}(X_\Delta)$ and $\text{Pic}(S)$ is compensated for by $h^{2,1}(Z_\Delta)$ in the mirror. This seems to suggest that Batyrev-Borisov mirror symmetry is the result of a more natural duality between a smooth toric variety and its Landau-Ginzburg mirror. Now we compute $h^{1,1}(Y_\Delta, \mathsf{w})$.

Proposition 2.3.6.

$$h^{2,2}(Y_\Delta, \mathsf{w}) = 0.$$

Proof. First note that the image of the restriction map $\mathrm{H}^2(Y_\Delta, \mathbb{C}) \to \mathrm{H}^2(S^\vee, \mathbb{C})$ has image equal to the restriction from Z_Δ. We can apply the global invariant cycles theorem to the fibers of f in Z_Δ to deduce that the image of the restriction map is precisely the monodromy invariant classes in $\mathrm{H}^2(S^\vee, \mathbb{C})$. The same is true for $S^\vee$ as a fiber of w, and the monodromy representations of f and w are identical, thus the images of the restriction maps are identical. In the proof of Lemma 2.3.2, we showed that restriction from Z_Δ to $S^\vee$ has image of rank

$$\ell(\Delta^\circ) - 4 - \sum_{F \in \Delta^\circ[2]} \ell^*(F) + \sum_{F \in \Delta^\circ[1]} \ell^*(F)\ell^*(F^\circ).$$

It then follows from Proposition 2.3.1 that the kernel of the restriction of $\mathrm{H}^2(Z_\Delta, \mathbb{C})$ to $\mathrm{H}^2(S^\vee, \mathbb{C})$ has rank $\ell(\Delta^\circ) - 1$, and in particular, since all fibers of f are linearly equivalent, the kernel of the restriction to $\mathrm{H}^2(D_\infty,\mathbb{C})$ is of rank at least $\ell(\Delta^\circ)-1$. If $\mathscr{L}$ is an ample line bundle on Z_Δ, then we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}^4(Z_\Delta,\mathbb{C}) \arrow[r] & \mathrm{H}^4(D_\infty,\mathbb{C}) \\ \mathrm{H}^2(Z_\Delta,\mathbb{C}) \arrow[r] \arrow[u, "\cup c_1(\mathscr{L})"', "\cong"] & \mathrm{H}^2(D_\infty,\mathbb{C}) \arrow[u, "\cup c_1(\mathscr{L})|_{D_\infty}"'] \end{tikzcd}$$

Hence the rank of the kernel of the map on the top of the above diagram has rank at least $\ell(\Delta^\circ)-1$. Therefore,

$$h^2(Y_\Delta) = h^4_c(Y_\Delta) \leq \ell(\Delta^\circ) - 4 - \sum_{F\in\Delta^\circ[2]} \ell^*(F) + \sum_{F\in\Delta^\circ[1]} \ell^*(F)\ell^*(F^\circ)$$

by the long exact sequence in compactly supported cohomology. We have thus argued that the image of the restriction map $\mathrm{H}^2(Y_\Delta,\mathbb{C}) \to \mathrm{H}^2(S^\vee,\mathbb{C})$ has rank at least equal to $h^2(Y_\Delta)$, hence it has rank equal to $h^2(Y_\Delta)$ and is thus injective. Therefore, by the long exact sequence in relative cohomology, $h^2(Y_\Delta, S^\vee) = 0$. □

Using Corollary 2.2.10, we find that:

Theorem 2.3.7. *Hodge number mirror symmetry holds between X_Δ and (Y,w) for Δ a reflexive 3-dimensional polytope.*

Remark 2.3.8. In light of Section 2.2.3 Hodge number mirror symmetry in the case where Δ is a reflexive polygon is a simple exercise. One checks that the only non-zero Hodge numbers of X_Δ are $h^{i,i}(X_\Delta)$ for $i = 0,1,2$, and $h^{1,1}(X_\Delta) = \ell(\Delta)-3$, and Z_Δ is an elliptically fibered surface with $\ell(\Delta^\circ)-1$ irreducible components with multiplicity 1, thus the fiber of f at infinity is of type $\mathrm{I}_{\ell(\Delta^\circ)-1}$. It is known that for any reflexive polygon, $\ell(\Delta^\circ)+\ell(\Delta) = 14$ (see e.g. [122, Theorem 1]). Therefore, $10-(\ell(\Delta^\circ)-1) = \ell(\Delta)-3 = h^{1,1}(X_\Delta)$, which is equivalent to Hodge number mirror symmetry for toric weak Fano surfaces by Section 2.2.3.

2.3.1.3 Monodromy and the fiber at infinity

Conjecturally, if t a regular value of w and $V = \mathsf{w}^{-1}(t)$, then we should have that the monodromy action on $\mathrm{H}^i(V, \mathbb{C})$ agrees, via mirror symmetry, with the action on $\bigoplus_{q=0}^{d} \mathrm{H}^{q,i-q}(W)$ by cup product with $\exp c_1(\omega_X^{-1}|_W)$ (see e.g. [7]). Friedman and Scattone [58] have shown that there is a relationship between the geometry of type III degenerations of K3 surface and the associated action of monodromy on $\mathrm{H}^2(V, \mathbb{C})$. A degeneration is Käher manifold $\mathscr{X}$ equipped with a holomorphic map $\pi : \mathscr{X} \to \Delta$ to a small disc Δ containing 0 so that $\pi^{-1}(t)$ is smooth for $t \neq 0$. We will assume that $\pi^{-1}(0)$ is a normal crossings union of smooth varieties.

The dual intersection complex of the degeneration $\mathscr{X}$ is a complex which represents each component of $\mathscr{X}_0$ with a point, each intersection between two components as an edge between two points, and each intersection of three irreducible components as a triangle with vertices corresponding to the three intersecting irreducible components, and the edges corresponding to the curves of intersection between each pair of intersecting irreducible components.

A type III degeneration of K3 surfaces is a degeneration so that $-K_{\mathscr{X}} = 0$, the fiber over 0 is simple normal crossings and its dual intersection complex is a triangulation of S^2 and the fiber over $t \neq 0$ is a smooth projective K3 surface. Examples of such a degeneration are provided by $\mathsf{f}^{-1}(U) \subset Z_\Delta$ for U a small disc around infinity for Z_Δ and f as in Section 2.3.

Assume that we have a type III degeneration of K3 surfaces so that there is an embedding of some lattice L into $\mathrm{H}^2(\mathscr{X}_t, \mathbb{Z})$ for each $t \in \Delta \setminus 0 = \Delta^\times$. Let T be the monodromy automorphism of $\mathrm{H}^2(\mathscr{X}_t, \mathbb{Z})$ associated to a small counter-clockwise loop in $\Delta^\times$ around 0. Assume that $L \subseteq \mathrm{H}^2(\mathscr{X}_t, \mathbb{Z})$ is fixed by T and the parallel transport of L is contained in $\mathrm{NS}(\mathscr{X}_t)$ for every t in $\Delta^\times$. Then, according to [90], T is unipotent, $(T - \mathrm{Id})^3 = 0$ but $(T - \mathrm{Id})^2 \neq 0$. Then T, or more properly, $N := \log T$, induces a weight filtration on the general fiber $\mathrm{H}^2(\mathscr{X}_t, \mathbb{Z})$ of π.

Lemma 2.3.9 (Friedman-Scattone [58, Lemma 1.1])**.** *There is a copy of* U *in* Λ *and* $\alpha_N \in \mathrm{U}^\perp$ *so that the automorphism* N *can be represented by the transformation*

$$N(x) = \langle x, \alpha_N \rangle e - \langle x, e \rangle \alpha_N.$$

Now we relate this to mirror symmetry. If we take a lattice L, then we say that a K3 surface S is L-polarized if there is a primitive embedding $L \hookrightarrow \mathrm{H}^2(S,\mathbb{Z})$ whose image contains a pseudo-ample class and the image of L is contained in $\mathrm{NS}(S)$. If the orthogonal complement of L in $\mathrm{H}^2(S,\mathbb{Z})$ splits as $L^\vee \oplus \mathrm{U}$ then mirror symmetry relates the moduli space of L-polarized K3 surfaces to the complexified ample cone of a K3 surface $S^\vee$ so that $\mathrm{NS}(S^\vee) = L^\vee$.

Now let α be an ample class in $\mathrm{NS}(S^\vee) = L^\vee$. We then can build an endomorphism N_α of $\mathrm{H}^2(S,\mathbb{Z})$ by letting e be a primitive isotropic element of U orthogonal to α_N in the transcendental lattice $L^\vee \oplus \mathrm{U}$ of S in $\mathrm{H}^2(S,\mathbb{Z})$ and letting

$$N_\alpha(x) = \langle x, \alpha \rangle e - \langle x, e \rangle \alpha.$$

This endomorphism is of the right sort to occur as the monodromy matrix around a type III degeneration of K3 surfaces.

Lemma 2.3.10. *We have* $N_\alpha^2 \neq 0$*,* $N_\alpha^3 = 0$ *and* $N_\alpha(L) = 0$*.*

Proof. Easy calculation. □

Thus if S is the anticanonical hypersurface in a Fano threefold X, then $\alpha_X := -K_X|_S$ induces an ample class on S. If we choose S generically so that $\mathrm{Pic}(S) \cong \mathrm{Pic}(X)$ then $\exp \alpha_X$ corresponds to an element of $\mathrm{O}(\mathrm{H}^2(S,\mathbb{Z}))$ which looks like the monodromy matrix associated to a type III degeneration of L-polarized K3 surfaces. The following conjecture is consistent with mirror symmetry.

Conjecture 2.3.11. *Let* X *be a Fano threefold and* (Y, w) *its LG model. The transformation* $T_{\alpha_X} = \exp N_{\alpha_X}$ *corresponds to the action of monodromy on the* $\mathrm{H}^2(S,\mathbb{Z})$ *for* S *a smooth fiber of* (Y, w) *associated to a small counterclockwise loop around the point at infinity.*

Friedman and Scattone [58, Proposition 1.10] prove the following result.

Proposition 2.3.12. *If N is the monodromy matrix around a type III degeneration of K3 surfaces, $\pi : \mathscr{X} \to \Delta$, and α_N is as in Lemma 2.3.9, then the number of faces of the dual intersection complex of $\mathscr{X}_0$ is equal to $\langle \alpha_N, \alpha_N \rangle$.*

Let us now assume that in Z, the compactification of Y, we have that for U a small disc around the point at infinity, $\mathsf{f}^{-1}(U)$ is a type III degeneration of K3 surfaces. Take the triangulation of S^2 determined by the dual intersection complex of $\mathsf{f}^{-1}(\infty) = D_\infty$. Since triple points correspond to faces in the triangulation of the sphere, and we have that

$$F - E + V = 2 \qquad 3F = 2E$$

for F, E, V the number of faces, edges and vertices of the dual intersection complex of D_∞. Thus $V = 2 + F/2$. If $F = \langle \alpha_X, \alpha_X \rangle$ for an ample class α_X in a smooth K3 surface in X, then the genus formula for curves on a K3 surface shows that $V = g(C) + 1$ for $g(C)$ the genus of a smooth curve associated to the ample class α_X. Thus combining mirror symmetry for K3 surfaces and Fano varieties, one arrives at the following conjecture.

Conjecture 2.3.13. *Let X be a Fano threefold and (Y, w) be its LG model. Assume there is a compactification Z of Y so that w extends to a map $\mathsf{f} : Z \to \mathbb{P}^1$ and so that $\mathsf{f}^1(U)$ is a type III degeneration of K3 surfaces for U a small disc in $\mathbb{P}^1$ containing the point at infinity. Then the number of triple points in D_∞ is equal to the anticanonical degree of X and the number of irreducible components is equal to the genus of a generic smooth curve in $(-K_X) \cap (-K_X)$ plus 1.*

Theorem 2.3.14. *Conjecture 2.3.13 holds for the LG model of a smooth toric quasi-Fano variety.*

Proof. By construction D_∞ is a normal crossings union of smooth divisors and D_∞ has components which are in bijection with points in the boundary of Δ°, thus there are $\ell(\Delta^\circ) - 1$ components in D_∞. It remains to prove that the genus of C, the intersection of two anticanonical hypersurfaces in X_Δ is equal to $\ell(\Delta) - 2$. We have the Koszul complex for C,

$$0 \to \mathscr{O}_{X_\Delta}(-2S) \to \mathscr{O}_{X_\Delta}(-S) \oplus \mathscr{O}_{X_\Delta}(-S) \to \mathscr{O}_{X_\Delta} \to 0.$$

The associated spectral sequence converges to $\mathrm{H}^i(C, \mathscr{O}_C[2])$, so it can be used to compute $\mathrm{H}^1(C, \mathscr{O}_C)$. We have that $\mathrm{H}^i(X_\Delta, \mathscr{O}_{X_\Delta}(-S)) = \mathrm{H}^i(X_\Delta, \mathscr{O}_{X_\Delta}(-2S)) = 0$ unless $i = 3$, and $\mathrm{H}^i(X_\Delta, \mathscr{O}_{X_\Delta}) = 0$ unless $i = 0$. Therefore, this spectral sequence can be seen to degenerate at $''E_2$ (simply write out the $''E_1^{p,q}$ terms). Furthermore, we know that $\mathrm{H}^i(C, \mathscr{O}_C) = 0$ if $i \neq 0, 1$. Thus the map

$$''E_1^{3,0} = \mathrm{H}^3(X_\Delta, \mathscr{O}_{X_\Delta}(-2S)) \to {''E_1^{3,1}} = \mathrm{H}^3(X_\Delta, \mathscr{O}_{X_\Delta}(-S) \oplus \mathscr{O}_{X_\Delta}(-S))$$

is surjective since $''E_2^{3,1}$ is a summand of $\mathrm{H}^2(C, \mathscr{O}_C)$. Therefore, by [17, Theorem 2.5], we have that

$$\mathrm{rank}\, {''E_2}^{3,0} = \ell^*(2\Delta^\circ) - 2\ell^*(\Delta^\circ).$$

Since Δ° is reflexive, $\ell^*(2\Delta^\circ) = \ell(\Delta^\circ)$ and $\ell^*(\Delta^\circ) = 1$, so $g(C) = \ell(\Delta^\circ) - 2$ and thus the number of irreducible components of D_∞ is equal to $g(C) + 1$ as claimed. The fact that the anticanonical degree of X equals the number of faces in the triangulation follows from the genus formula for curves on a K3 surface and the combinatorial discussion above regarding the structure of triangulations of S^2. $\square$

Remark 2.3.15. Note that, while Conjecture 2.3.13 claims to work only for Fano varieties, it seems to extend at least to weak Fano toric manifolds.

Remark 2.3.16. It is clear that for any del Pezzo surface, the number of irreducible components in D_∞ corresponds to $(-K_X)^2$, which is just $h^{0,0}(E \cap E')$ for E, E' two general smooth anticanonical hypersurfaces. A general version of Conjecture 2.3.13 relates the Hodge numbers of the intersection of two anticanonical hypersurfaces in X to the Hodge numbers of sheaves of vanishing cycles of the fiber over infinity. The fiber over the point at infinity should be mirror to the smooth intersection of two anticanonical hypersurfaces in X. See [69] for details on homological mirror symmetry for manifolds of general type.

2.3.2 Mirror symmetry for a general toric weak Fano variety

Here we extend the results of the previous section to the case where X_Δ is a weak Fano toric variety of arbitrary dimension with terminal singularities. We point out that the results in this section technically supersede the results in Theorem 2.3.7, however it seemed illustrative to include the concrete computation. Beyond the topological results in Section 2.3.2.1, our result is mainly due to work of Batyrev [13] and Batyrev-Dais [19], however we will explicitly review all of the necessary results.

2.3.2.1 Reduction of the computation when a smooth LG model exists

First of all, we deal with the issue of compactification of the Givental LG model of X_Δ. In the threefold case, we used the fact that X_{Δ° admits a smooth crepant resolution of singularities and the fact that the base locus of the pencil $\mathscr{P}$ is a simple normal crossings union of smooth varieties.

Then we may repeat the construction in Section 2.3 of the LG model of X_Δ. We choose a generic Laurent polynomial f with Newton polytope Δ. Then we have that the fibers of $f : (\mathbb{C}^\times)^d \to \mathbb{C}$ can be compactified to hypersurfaces in X_{Δ°. We assume that X_{Δ° is smooth, then the base locus of the pencil of hypersurfaces defined by f is a normal crossings union of smooth codimension 2 subvarieties $B_1, \dots, B_k$ of X_{Δ°. Then we can take a blow up of X_{Δ° along $B_1, \dots, B_k$ in order, and call the resulting variety Z_Δ. The variety Z_Δ is fibered over $\mathbb{P}^1$ by smooth Calabi-Yau $(d-1)$-folds, and we call this map f. Let $Y_\Delta = \mathsf{f}^{-1}(\mathbb{A}^1) \subseteq Z_\Delta$ and let $\mathsf{w} = \mathsf{f}|_{Y_\Delta}$.

Note that in this situation, there are exceptional divisors $E_1, \dots, E_k$ obtained as the preimage of $B_1, \dots, B_k$ under the contraction map $Z_\Delta \to X_{\Delta^\circ}$. The intersection of each E_i with each fiber of w is just B_i, thus $\mathsf{w}|_{B_i}$ is a smooth fibration over $\mathbb{A}^1$. By Ehresmann's theorem, $E_i \cap Y$ is diffeomorphic to $B_i \times \mathbb{A}^1$. It is easy to see that $Y \setminus \cup_{i=1}^k (E_i \cap Y)$ is just $(\mathbb{C}^\times)^d$ and the restriction of w to this open subset is just f itself. Now we let M be a complex manifold and let P be a smooth complex submanifold of

M of codimension 1. There is a standard quasi-isomorphism between $\Omega^\bullet_M(\log P)$ and $\Omega^\bullet_{M\setminus P}$ and a short exact sequence of complexes of sheaves,

$$0 \to \Omega^\bullet_M \to \Omega^\bullet_M(\log P) \xrightarrow{\text{res}} \Omega^\bullet_P[-1] \to 0$$

where the first map is the natural injection and the second is the Poincaré residue map. We assume that P is a divisor in M, then there are local coordinates $(x_1, \dots, x_d)$ on M so that P is expressed as as $x_1 = 0$. Then a differential k-form η on M with log poles at P can be written as

$$\eta = \alpha + \left(\frac{dx_1}{x_1}\right) \wedge \beta$$

for α a holomorphic k-form on M and β a holomorphic $(k-1)$-form on M. The map $\text{res}(\eta) = \beta|_P$. Now let Q be another smooth divisor of M, we have an exact sequence of complexes of sheaves

$$0 \to \Omega^\bullet_M(\text{rel}\, Q) \to \Omega^\bullet_M \to \Omega^\bullet_Q \to 0.$$

where $\Omega^\bullet_M(\text{rel}\, Q)$ is defined by this exact sequence. If we assume that P intersects Q smoothly, then we obtain another short exact sequence

$$0 \to \Omega^\bullet_M(\log P, \text{rel}\, Q) \to \Omega^\bullet_M(\log P) \to \Omega^\bullet_Q(\log(Q \cap P)) \to 0.$$

Again, we have that $\Omega^\bullet_M(\log P)$ is quasi-isomorphic to $\Omega^\bullet_{M\setminus P}$, $\Omega^\bullet_Q$ is quasi-isomorphic to $\Omega^\bullet_{Q\setminus(Q\cap P)}$, therefore, we have that $\Omega^\bullet_M(\log P, \text{rel}\, Q)$ is quasi-isomorphic to $\Omega^\bullet_{M\setminus P}(\text{rel}\, Q)$.

Lemma 2.3.17. *If Q and P are divisors in M which meet transversally, then there is an exact sequence of sheaves:*

$$0 \to \Omega^\bullet_M(\text{rel}\, Q) \to \Omega^\bullet_M(\log P, \text{rel}\, Q) \to \Omega^\bullet_P(\text{rel}(P \cap Q))[-1] \to 0.$$

Proof. This is a purely local computation. If we take a point $p \in M$ so that $p \in P$ but $p \notin Q$, we may choose a local chart around p not intersecting Q, and thus the first map in the sequence above is just the normal residue map and we have exactness. If p is in neither Q nor P, then the residue map is trivial. Thus we may assume that $p \in Q \cap P$. We choose coordinates $(x_1, \dots, x_d)$ around p so that $Q = \{x_1 = 0\}$ and $P = \{x_2 = 0\}$. The sheaf $\Omega^i_M(\log P, \mathrm{rel}\, Q)$ is the subsheaf of differential i-forms ω on M which vanish at Q and have logarithmic poles at P. In the local coordinates $(x_1, \dots, x_d)$ around the point p then ω can be written as

$$\sum_{\substack{I \subseteq [1,\dots,d] \\ |I|=i, 1\notin I, 2\notin I}} x_1 f_I(x) dx_I + \sum_{\substack{I \subseteq [1,\dots,d] \\ |I|=i, 1\in I, 2\notin I}} f_I(x) dx_I + \sum_{\substack{I \subseteq [1,\dots,d] \\ |I|=i, 1\notin I, 2\in I}} \frac{x_1 f_I(x)}{x_2} dx_I + \sum_{\substack{I \subseteq [1,\dots,d] \\ |I|=i, 1\in I, 2\in I}} \frac{f_I(x)}{x_2} dx_I$$

where $f_I(x)$ are arbitrary holomorphic functions and $dx_I = \bigwedge_{i\in I} dx_i$. Then the residue of this form is precisely

$$\sum_{\substack{I \subseteq [1,\dots,d] \\ |I|=i, 1\notin I, 2\in I}} x_1 f_I(x)|_{x_2=0} dx_{I\setminus 2} + \sum_{\substack{I \subseteq [1,\dots,d] \\ |I|=i, 1\in I, 2\in I}} f_I(x)|_{x_2=0} dx_{I\setminus 2}.$$

Such forms are exactly the germs of sections of $\Omega_P^{i-1}(\mathrm{rel}\,(P\cap Q))$. It is clear that the kernel of this map is the set of forms where $f_I(x)$ vanish along $x_2 = 0$ if $2 \in I$, which are just holomorphic forms. Away from points in $P \cap Q$ there is nothing to check, so we are done. □

Now returning to the situation at hand, we may recursively define $Y^1_\Delta = Y_\Delta \setminus (Y_\Delta \cap E_1)$ and $Y^i_\Delta = Y^{i-1}_\Delta \setminus (Y^{i-1}_\Delta \cap E_i)$. Clearly, $Y^k_\Delta = (\mathbb{C}^\times)^d$. Let V_f be the vanishing locus of f in Y_Δ and let V^i_f be $V_f \cap Y^i_\Delta$. We know that $V^{i-1}_f \setminus V^i_f$ is a smooth divisor in V^{i-1}_f. We can apply Lemma 2.3.17 to compute the cohomology of (Y_Δ, w) in terms of the relative cohomology groups $\mathrm{H}^i((\mathbb{C}^\times)^d, V^k_f; \mathbb{C})$. Note that both admit mixed Hodge structures, and the restriction map induces a morphism of mixed Hodge structures from $\mathrm{H}^i(Y_\Delta, V_f; \mathbb{C})$ to $\mathrm{H}^i((\mathbb{C}^\times)^d, V^k_f; \mathbb{C})$.

Theorem 2.3.18. *We have an isomorphism of mixed Hodge structures* $\mathrm{H}^i(Y_\Delta, V_f; \mathbb{C}) \cong \mathrm{H}^i((\mathbb{C}^\times)^d, V^k_f; \mathbb{C})$.

Proof. We need to show that $\mathrm{H}^j(Y^i_\Delta, V^i_f; \mathbb{C})$ is isomorphic to $\mathrm{H}^j(Y^{i+1}_\Delta, V^{i+1}_f; \mathbb{C})$. In Lemma 2.3.17, let $V = E_{i+1} \cap Y^i_\Delta$ and let $Z = V^i_f$. Then $Z \cap V = V^{i+1}_f$. Therefore, we have that there is a long exact sequence in cohomology

$$\cdots \to \mathrm{H}^{j-2}(E_{i+1} \cap Y^i_\Delta, E_{i+1} \cap V^i_f) \to \mathrm{H}^j(Y^i_\Delta, V^i_f) \to \mathrm{H}^j(Y^{i+1}_\Delta, V^{i+1}_f) \to$$
$$\to \mathrm{H}^{j-1}(E_{i+1} \cap Y^i_\Delta, E_{i+1} \cap V^i_f) \to \dots$$

However, we have that $E_{i+1} \cap Y^i_\Delta$ is diffeomorphic to $V^i_f \cap E_{i+1} \times \mathbb{A}^1$, therefore, the relative cohomolgy groups of the pair $(E_{i+1} \cap Y^i_\Delta, V^i_f \cap E_{i+1})$ are all zero. Therefore, the long exact sequence in cohomology above provides the isomorphism that we want.

We have only exhibited an isomorphism of cohomology groups, however this map underlies a morphism of mixed Hodge structures, hence we obtain an isomorphism of mixed Hodge structures. □

This is an appealing result, since it is not always possible to construct a smooth LG model Y_Δ. Theorem 2.3.18 says that in the event that such a smooth relative compactification exists, then the relative cohomology of $((\mathbb{C}^\times)^k, V^k_f)$ is the same as the cohomology of $(Y_\Delta, V_f; \mathbb{C})$.

2.3.2.2 Hodge number computations

In this section, we will replace the notation V^k_f with U_f for simplicity. It is possible to understand even the Hodge and weight filtrations on $\mathrm{H}^d((\mathbb{C}^\times)^d, U_f; \mathbb{C})$ following work of Batyrev [13]. Associated to an integral polytope Δ of dimension d, one may associate a sequence of numbers $(\psi_0(\Delta), \dots, \psi_d(\Delta))$ which we call the δ-vector of Δ. If Δ is reflexive, then it is shown by Batyrev and Dais [19, Theorem 7.2] that the $\psi_i(\Delta)$ is equal to $h^{i,i}_{\mathrm{st}}(\mathbb{P}_\Delta)$ where $\mathbb{P}_\Delta$ is the toric variety associated to Δ and $h^{i,i}_{\mathrm{st}}$ denotes the appropriate stringy Hodge number of $\mathbb{P}_\Delta$. It is known that if X_Δ is a crepant smooth resolution of $\mathbb{P}_\Delta$ then $h^{i,i}_{\mathrm{st}}(\mathbb{P}_\Delta) = h^{i,i}(X_\Delta)$, so in a very strong sense, they should be taken as a replacement for the betti numbers of a toric variety. It is also shown in [19, Theorem 7.2] that $h^{i,j}_{\mathrm{st}}(\mathbb{P}_\Delta) = 0$ if $i \neq j$. Batyrev has shown that:

Theorem 2.3.19 ([14, Corollary 3.12])**.** *If Δ is reflexive, f is a Laurent polynomial with Newton polytope Δ, and U_f is as above, then*

$$\dim F^i \mathrm{H}^{d-1}(U_f)/F^{i+1}\mathrm{H}^{d-1}(U_f)$$

is equal to $\psi_{d-i}(\Delta)$ for $i < d-1$, $\psi_1(\Delta) + d$ for $i = d-1$ and 0 otherwise.

Now by the toric Lefschetz hyperplane theorem of Danilov and Khovanskii [38], the restriction map $\mathrm{H}^i((\mathbb{C}^\times)^d, \mathbb{Q}) \to \mathrm{H}^i(U_f, \mathbb{Q})$ is an isomorphism if $i < d-1$ and an injection if $i = d-1$. Furthermore, it is clear that $h^i((\mathbb{C}^\times)^d) = \binom{d}{i}$ and it is a standard fact (see e.g. [13, Example 3.9]) that the only non-zero $i^{p,q}$s of $\mathrm{H}^i((\mathbb{C}^\times)^d, \mathbb{Q})$ are $i^{i,i}$. Therefore, we can compute the value of $\mathrm{Gr}^F_i \mathrm{H}^i((\mathbb{C}^\times)^d, U_f; \mathbb{Q})$.

Theorem 2.3.20. *If Δ is reflexive, f is a Laurent polynomial with Newton polytope Δ, and U_f is as above, then*

$$\dim F^i \mathrm{H}^d((\mathbb{C}^\times)^d, U_f)/F^{i+1}\mathrm{H}^d((\mathbb{C}^\times)^d, U_f) = \psi_{d-i}(\Delta).$$

Therefore, if there exists a smooth crepant resolution of singularities of X_{Δ°, then

$$h^{p,q}(Y_\Delta, \mathsf{w}) = h^{q,d-p}_{\mathrm{st}}(\mathbb{P}_\Delta).$$

Proof. It is clear by the toric Lefschetz hyperplane theorem of Danilov and Khovanskii that $h^i((\mathbb{C}^\times)^d, U_f) = 0$ if $i \neq d$. We have an exact sequence of cohomology groups

$$0 \to \mathrm{H}^{d-1}((\mathbb{C}^\times)^d) \to \mathrm{H}^{d-1}(U_f) \to \mathrm{H}^d((\mathbb{C}^\times)^d, U_f) \to \mathrm{H}^d((\mathbb{C}^\times)^d) \to 0.$$

It is then an elementary application of the strictness of morphisms of mixed Hodge structures and Theorem 2.3.19 to see that the statement about the dimensions of $\mathrm{gr}^i_F \mathrm{H}^d((\mathbb{C}^\times)^d, U_f)$ holds for $i < d$. If $i = d$, then the exact sequence above along with Theorem 2.3.19 and the fact that $\mathrm{H}^{d-1}((\mathbb{C}^\times)^d, \mathbb{C})$ has $i^{d-1,d-1} = d$ and $i^{p,q} = 0$ otherwise implies that $\mathrm{gr}^F_{d-1}\mathrm{H}^d((\mathbb{C}^\times)^d, U_f) = \psi_{d-1}(\Delta)$. Similarly, one sees that that $\mathrm{gr}_d^F \mathrm{H}^d((\mathbb{C}^\times)^d, U_f) = 1$. If Δ is reflexive, then it is a standard fact that $\psi_0(\Delta) = \psi_d(\Delta) = 1$ and therefore, the statement about the rank of gr_F^i is proved.

Applying Theorem 2.3.18 it follows that if X_{Δ° admits a smooth crepant resolution of singularities, then

$$h^{i,d-i}(Y_\Delta, \mathsf{w}) = h^{i,d-i}((\mathbb{C}^\times)^d, U_f) = \psi_i(\Delta) = h^{i,i}_{\mathrm{st}}(\mathbb{P}_\Delta)$$

as required, which proves the theorem, since we know that $h^{p,q}((\mathbb{C}^\times)^d, U_f) = 0$ if $p + q \neq d$ and similarly that $h^{p,q}_{\mathrm{st}}(X_\Delta) = 0$ if $p \neq q$. $\square$

In the case where $d = 3$, the existence of a smooth crepant resolution of $\mathbb{P}_\Delta$ is assured, and so in this case we recover exactly Theorem 2.3.7.

2.4 Extremal local systems

We will show that the concept of a Fano LG model seems to agree with the results of Coates, Corti, Galkin, Golyshev and Kasprzyk [32], and explain the following condition on local systems that appears in [32].

Definition 2.4.1. Let $\mathbb{L}$ be a local system over a Zariski open subset U of $\mathbb{P}^1$ where the natural injection is denoted $j : U \hookrightarrow \mathbb{P}^1$. The local system $\mathbb{L}$ is called *extremal* if $h^1(\mathbb{P}^1, j_*\mathbb{L}) = 0$.

Now, in [33], the authors compute quantum differential operators associated to Fano threefolds. Via Gromov-Witten computations, they associate to each Fano threefold a Laurent polynomial $g_X : (\mathbb{C}^\times)^3 \to \mathbb{C}$. It is expected that there is a LG model of (Y, w) of X and an injective map

$$\phi : (\mathbb{C}^\times)^3 \hookrightarrow Y$$

so that $\phi^*\mathsf{w} = g_X$. To each Laurent polynomial g_X in d variables, the authors of [33] associate a differential operator D_X. A general fiber $g_X^{-1}(t)$ is a smooth surface, and as such we can compute periods associated to it. The *classical period* associtated to

g_X is given by

$$\pi_{g_X}(t) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^d \int_{|x_1|=\cdots=|x_d|=1} \left(\frac{1}{1-tg_X}\right) \frac{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d}{x_1 \ldots x_d}$$

This period is holomorphic around $t = 0$. The operator D_X is the minimal differential operator which annihilates $\pi_{g_X}(t)$. This operator has at worst logarithmic singularities at a set of points $\Sigma \subseteq \mathbb{P}^1$, and thus defines a local system along $\mathbb{P}^1 \setminus \Sigma$. Let this local system be denoted $\mathbb{L}_X$. If the fibers of g_X can generically be compactified to K3 surfaces $\mathscr{X}$ over $\mathbb{P}^1 \setminus \Sigma$, then $\pi_{g_X}(t)$ should be, locally, a holomorphic period of this family of K3 surfaces. In other words, the operator D_X is the Picard-Fuchs operator associated to this family of K3 surfaces and $\mathbb{L}_X$ is the transcendental local system associated to $\mathscr{X}$. Recall that the local system $R^d f_* \mathbb{Q}$ of a family of smooth d-dimensional Calabi-Yau varieties splits into a pair of local systems which we call $\mathbb{T}_{\mathscr{X}}$ and $\mathbb{N}_{\mathscr{X}}$ where $\mathbb{T}_{\mathscr{X}}$ is the smallest local subsystem of $R^d f_* \mathbb{Q}$ so that $\mathbb{T}_{\mathscr{X}} \otimes \mathscr{O}_{\mathbb{P}^1 \setminus \Sigma}$ contains $\mathscr{F}^d$.

Proposition 2.4.2. *If $\mathscr{K}_{i+1}$ is Hodge-Tate, then so is $\mathrm{H}^1(\mathbb{P}^1, j_* R^i \sigma_* \mathbb{C})$.*

Proof. We use [149, Theorem 15.16 (ii)], which notes that if Σ is the set of critical values of f, then

$$\ker(\mathrm{H}^{i+1}(Z, \mathbb{C}) \to \mathrm{H}^0(\mathbb{P}^1, R^{i+1}\mathsf{f}_*\mathbb{C})) = \cap_{s \in \Sigma} \ker(\mathrm{H}^{i+1}(Z, \mathbb{C}) \to \mathrm{H}^{i+1}(\mathsf{f}^{-1}(s), \mathbb{C})).$$

Thus

$$\ker(\mathrm{H}^{i+1}(Z, \mathbb{C}) \to \mathrm{H}^0(\mathbb{P}^1, R^{i+1}\mathsf{f}_*\mathbb{C})) \subseteq \mathscr{K}_{i+1}$$

From the degeneracy of the Leray spectral sequence for f [149, Corollary 15.15], we deduce the existence of a surjective map of Hodge structures,

$$\ker(\mathrm{H}^{i+1}(Z, \mathbb{C}) \to \mathrm{H}^0(\mathbb{P}^1, R^{i+1}\mathsf{f}_*\mathbb{C})) \to \mathrm{H}^1(\mathbb{P}^1, R^i\mathsf{f}_*\mathbb{C}).$$

By [149, 15.12], we have that $\mathrm{H}^1(\mathbb{P}^1, R^i\mathsf{f}_*\mathbb{C}) \cong \mathrm{H}^1(\mathbb{P}^1, j_*R^i\sigma_*\mathbb{C})$, hence the group $\mathrm{H}^1(\mathbb{P}^1, j_*R^{i+1}\sigma_*\mathbb{C})$ is a subquotient of $\mathscr{K}_{i+1}$. If $\mathscr{K}_{i+1}$ is Hodge-Tate, then so is any sub-quotient of $\mathscr{K}_{i+1}$, and in particular, $\mathrm{H}^1(\mathbb{P}^1, j_*R^i\sigma_*\mathbb{C})$ is Hodge-Tate. □

Corollary 2.4.3. *Let (Z, f) be a $2n+1$-dimensional compactified LG model of Fano type. Let Σ be the set of critical values of f and let $\mathscr{X} = \mathsf{f}^{-1}(\mathbb{P}^1 \setminus \Sigma) \subseteq Z$. Then $\mathbb{T}_{\mathscr{X}}$ is an extremal local system.*

Proof. By Proposition 2.4.2, we know that $\mathrm{H}^1(\mathbb{P}^1, j_*R^{2n}\sigma_*\mathbb{C})$ is Hodge-Tate. It carries a weight $2n+1$ pure Hodge structure, hence it is zero. Thus

$$0 = \mathrm{H}^1(\mathbb{P}^1, j_*R^{2n}\sigma_*\mathbb{C}) = \mathrm{H}^1(\mathbb{P}^1, j_*\mathbb{T}_{\mathscr{X}} \otimes \mathbb{C}) \oplus \mathrm{H}^1(\mathbb{P}^1, j_*\mathbb{N}_{\mathscr{X}} \otimes \mathbb{C}).$$

□

We may interpret the results of Section 2.3 in light of Corollary 2.4.3.

Proposition 2.4.4. *Let Δ be a 3-dimensional reflexive polytope so that no facet of Δ contains an integral point on its relative interior. If g is a generic Laurent polynomial with Newton polytope Δ, then the local system $\mathbb{L}$ associated to g is extremal.*

It is noted in [34] that if $\dim X = 4$, then it is no longer true that $\mathbb{L}_X$ is extremal. This is of course to be expected in even dimension, since the condition that (Z, f) is of Fano type does not imply the vanishing of $\mathrm{H}^d(Z, \mathbb{C})$. Instead we have that the only non-zero Hodge numbers of Z are $h^{p,p}(Z)$. The analogue of Corollary 2.4.3 in this situation is:

Corollary 2.4.5. *Let notation be as in Corollary 2.4.3. If (Z, f) is a $2d$-dimensional LG model of Fano type then there is a natural pure Hodge structure on $\mathrm{H}^1(\mathbb{P}^1, j_*\mathbb{T}_{\mathscr{X}})$ of weight $2d$ whose only non-zero Hodge number is $h^{d,d}$.*

The existence of the pure Hodge structure described above is a consequence of [149] and the rest follows by imitating the proof of Corollary 2.4.3. This suggests a conjecture.

Conjecture 2.4.6. *Let $\mathbb{L}_X$ be a local system attached to a d-dimensional Fano manifold X by the method of [33]. Then $h^{p,d-p}(\mathrm{H}^i(\mathbb{P}^1, j_*\mathbb{L}_X)) \neq 0$ if and only if $p = d - p$.*

Checking whether a local system is Hodge-Tate does not seem like an easy problem in general, though results in some cases have been obtained by del Angel, Müller-Stach, van Straten and Zuo [40].

Problem 2.4.7. *Characterize the classes in $\mathrm{H}^d(Y, \mathsf{w})$ which should correspond to classes in $\mathrm{H}^1(\mathbb{P}^1, j_*R^{d-1}\sigma_*\mathbb{C})$ in terms of the mirror Fano variety.*

Remark 2.4.8 (Normal functions)**.** It is well known that there is a close relationship between (d, d) classes defined over $\mathbb{Q}$ in parabolic cohomology groups and admissible normal functions [40]. Thus an important role is played by the periods of normal functions in mirror symmetry for Fano varieties of even dimension – they should correspond to quantum periods of the Fano variety under mirror symmetry. Walcher [146, 145, 93] and Morrison-Walcher [103] have pointed out the importance of normal functions and their periods in open string mirror symmetry for compact Calabi-Yau varieties. It seems possible that one could identify the family of Morrison and Walcher [103] as the LG model of some quasi-Fano variety X. If this can be done, it would be interesting to determine whether the fact that these normal functions can be interpreted in two different ways is significant in terms of mirror symmetry.

Chapter 3

Hodge numbers of Fano hypersurfaces and Landau-Ginzburg models

3.1 Introduction

Here we perform several computations regarding the cohomology and geometry of Landau-Ginzburg models of hypersurfaces in toric Fano varieties. Recall that if X is a Fano variety, then mirror symmetry predicts that there is a dual variety Y equipped with a regular function $\mathsf{w} : Y \to \mathbb{A}^1$. The variety Y should have a number of properties: Y should be smooth (or at least close to it), a generic fiber V of Y should be a Calabi-Yau variety which is mirror dual to an anticanonical hypersurface in X, there should be a compactification of Y to a variety Z so that $D_\infty = Y \setminus Z$ is normal crossings, and there should be a section of ω_Z with simple poles along D_∞. The first part of this chapter will be concerned with the construction of such a pair in the case where X is a complete intersection in a toric variety. Precisely, if Δ is a reflexive polytope and $\Delta_1, \dots, \Delta_{k+1}$ is a nef partition of Δ. Let X_Δ be a mpcp resolution of the toric variety $\mathbb{P}_\Delta$ canonically associated to Δ. Then we take generic global sections $s_1, \dots, s_k$ of the line bundles $\mathscr{O}_{X_\Delta}(\Delta_i)$ on X_Δ associated to our nef partition. Let

X be the common vanishing locus of $s_1, \dots, s_k$. The anticanonical bundle of X is isomorphic to $\mathscr{O}_{X_\Delta}(\Delta_{k+1})|_X$ by the adjunction formula, thus it is effective.

We extend a construction of Givental [62] to build a prospective LG model (Y, w) for X as described in the previous paragraph. These varieties have the following properties: Y has at worst terminal Gorenstein singularities, the general fiber of w of Y is Calabi-Yau and Batyrev-Borisov mirror dual to an anticanonical hypersurface in X, there is a compactification of Y to a variety Z so that $D_\infty = Z \setminus Y$ has toroidal normal crossings and there is a section of ω_Z with simple poles along D_∞. Terminal singularities are an inextricable artifact of the toric methods used in the construction, just as they are in the case of Batyrev-Borisov mirror symmetry. Note that our construction also applies to a wide class of (possibly singular) complete intersections in toric varieties which encompasses the class of Fano complete intersections in toric varieties.

Once we have carried out this construction, we use the explicit toric description to partially address Hodge number mirror symmetry for hypersurfaces in toric varieties. According to Katzarkov, Kontsevich and Pantev, [79], mirror symmetry should imply a relationship between the Hodge numbers of X and (Y, w). They define Hodge numbers $h^{p,q}(Y, \mathsf{w})$ of Y in terms of the so-called f-adapted differential forms in order to conjecture that

$$h^{p,q}(X) = h^{d-p,q}(Y, \mathsf{w}).$$

They show that [79, Lemma 2.19, Lemma 2.21]

$$h^i(Y, \mathsf{w}^{-1}(t); \mathbb{C}) = \bigoplus_{p=0}^{i} h^{i-p,p}(Y, \mathsf{w})$$

for point $t \in \mathbb{A}^1$ so that $\mathsf{w}^{-1}(t)$ is smooth, and hence $h^2(Y, \mathsf{w}^{-1}(t))$ should be equal to $h^{1,d-1}(X)$. If we let ρ_t denote the number of irreducible components of $\mathsf{w}^{-1}(t)$, then one can show that if Σ is the set of critical values of w then

Theorem 3.3.1.

$$h^2(Y, \mathsf{w}^{-1}(t); \mathbb{C}) = \sum_{t \in \Sigma} (\rho_t - 1).$$

In [126], Przyjalkowski and Shramov have shown that if X is a Fano complete intersection in $\mathbb{P}^n$ then

$$h^{1,d-1}(X) = \rho_0 - 1.$$

So all divisors of (Y, w) corresponding to classes in $h^{1,d-1}(X)$ should, according to mirror symmetry, occur in *a single fiber* of w. We will show that this is a more general phenomenon and give some justification.

Let Δ be a d-dimensional reflexive polytope and let Δ_1, Δ_2 be a nef partition of Δ so that invertible sheaf $\mathscr{O}_{\mathbb{P}_\Delta}(\Delta_1)$ is ample. Then results of Batyrev and Borisov [17] can be modified slightly to produce a formula for $h^{1,d-2}(X)$ where X is the pullback of the vanishing locus of a generic section of $\mathscr{O}_{\mathbb{P}_\Delta}(\Delta_1)$ to a mpcp resolution of $\mathbb{P}_\Delta$ (Theorem 3.4.5). Then the results of Section 3.2.2 allow us to construct an LG model (Y, w) of X with mild singularities. We show that, in this mildly singular LG model,

Theorem 3.4.9.

$$h^{1,d-2}(X) = \rho_0 - 1.$$

This is a somewhat surprising result. In the case where X is an ample hypersurface in a smooth toric variety, then one can argue via homological mirror symmetry that this should be true (see Section 3.5), since for a general variety of this type the Hodge structure on the primitive cohomology is irreducible. However, such arguments fail when X is a a crepant resolution of an ample hypersurface in a singular toric variety.

This suggests that that for X a crepant resolution of an ample hypersurface in the d-dimensional toric variety $\mathbb{P}_\Delta$, the generic deformation should have a single sub-Hodge structure H in $\mathrm{H}^{d-1}(X, \mathbb{Q})$ so that the $(1, d-2)$ Hodge number of H is non-zero. Since there are cases where this is not true for a general hypersurface in X_Δ, this implies that the general deformation of X is no longer a hypersurface in X_Δ. We use this intuition to obtain a result about the deformations of pairs (X, E) where X is a smooth weak Fano variety and E is a divisor in X which can be crepantly contracted to a codimension 2 subvariety of a variety X' (Theorem 3.5.4).

3.1.1 Organization

This chapter is organized as follows. Section 3.2 is dedicated to constructing an appropriate version of the LG model of a quasi-Fano complete intersection in a toric variety. We also include background on toric varieties. In Section 3.3, we include a proof of a folklore theorem that states that $h^{1,1}(Y, \mathsf{w})$ may be determined by counting irreducible components in fibers of the LG model.

In Section 3.4, we prove that for X the pullback to X_Δ of an ample quasi-Fano hypersurface in $\mathbb{P}_\Delta$, the number of components in the LG model of X over 0 agrees with $h^{1,d-2}(X) + 1$. This suggests a fact about the action of monodromy on the cohomology of X, and subsequently in Section 3.5 we intuit that a result in the deformation theory of pairs should follow. We finally prove this result in a special case.

3.2 Constructing LG models

This section is devoted to the construction of an adequate version of the LG model of a quasi-Fano complete intersection in a toric variety. This section does not restrict the structure of the complete intersection. In Section 3.4.2 we will reduce our scope significantly.

3.2.1 Toric background

We will now include several results regarding mirror symmetry for complete intersections in toric varieties. In particular, let Δ be a d-dimensional reflexive polytope in $M \otimes \mathbb{R}$ for a lattice M of rank d. Let $N = \operatorname{Hom}(M, \mathbb{Z})$ as usual, and let $N_\mathbb{R} = N \otimes \mathbb{R}$. We will let Σ_Δ be the fan over faces of Δ and let $\mathbb{P}_\Delta$ denote the toric variety associated to Σ_Δ. If Δ is a lattice polytope, we will denote by $\Delta[i]$ its i-dimensional strata, and $\partial\Delta$ its boundary in $M_\mathbb{R}$. If Σ is a fan in $M_\mathbb{R}$, then a piecewise linear function on $M_\mathbb{R}$ will be called Σ-linear if it is linear on each cone of Σ.

Definition 3.2.1. A nef partition of a polytope is a partition of its vertex set $\Delta[0]$ into subsets $E_1, \dots, E_{k+1}$ so that for each $1 \leq i \leq k+1$ there exists convex, continuous

Σ_Δ-linear functions $\varphi_1, \dots, \varphi_{k+1}$ so that $\varphi_i(v) = 1$ for every $v \in E_i$ and $\varphi_i(w) = 0$ for every $w \in E_j$ and $j \neq i$.

We will let $\Delta_i = \mathrm{Conv}(E_i \cup 0_M)$ where 0 denotes the origin in M. To a nef partition, there is associated a dual nef partition, which is defined as follows. Let

$$\nabla_i = \{x \in N_\mathbb{R} : \langle x, v\rangle \geq \varphi_i(v)\}$$

then we define $\nabla = \mathrm{Conv}(\nabla_1, \dots, \nabla_k)$. It's not hard to see that n

$$\nabla_i = \left\{ x \in N_\mathbb{R} : \begin{matrix} \langle x, v\rangle \geq -1 \text{ if } v \in E_i \\ \langle x, v\rangle \geq 0 \text{ if } v \notin E_i \end{matrix} \right\}$$

We have that $\Delta^\circ = \nabla_1 + \cdots + \nabla_{k+1}$. It is shown in [23] that ∇ is again a reflexive polytope, and that $\nabla_1, \dots, \nabla_{k+1}$ forms a nef partition of ∇.

If Σ is a fan and X_Σ is a toric variety associated to the fan Σ, then there is a homogeneous coordinate ring on X_Σ which is graded by $\mathrm{A}_1(X_\Sigma)$. If we let $\Sigma[1]$ be the set of primitive generators of the 1-dimensional strata of Σ then we have an exact sequence

$$0 \to N \xrightarrow{g} \mathbb{Z}^{|\Sigma[1]|} \to \mathrm{A}_1(X_\Sigma) \to 0$$

here g is the map which sends $\sigma \in N$ to the vector $(\langle \sigma, \rho_1\rangle, \dots, \langle \sigma, \rho_{|\Sigma[1]|}\rangle)$ where $\{\rho_1, \dots, \rho_{|\Sigma[1]|}\}$ is the set $\Sigma[1]$. Then applying the functor $\mathrm{Hom}\,(-, \mathbb{C}^\times)$, we obtain a dual short exact sequence

$$0 \to G \to (\mathbb{C}^\times)^{|\Sigma[1]|} \to M \otimes \mathbb{C}^\times \to 0$$

where $G = \mathrm{Hom}\,(\mathrm{A}_1(X_\Sigma), \mathbb{C}^\times)$. Then there is a partial compactification of $(\mathbb{C}^\times)^{|\Sigma[1]|}$ depending on the combinatorial structure of Σ which we call U so that $U//G = X_\Sigma$. If we let $z_{\rho_1}, \dots, z_{\rho_{|\Sigma[1]|}}$ be variables on U, then the homogeneous coordinate ring of X_Σ is $\mathbb{C}[\{z_\sigma\}_{\rho \in \Sigma[1]}]$. The function z_ρ has vanishing locus equal to D_ρ in X_Σ.

If we have two complete fans Σ' and Σ both in $M_\mathbb{R}$ then if each cone of Σ is contained in a cone of Σ', then we get a birational map from $X_\Sigma \to X_{\Sigma'}$. If this condition holds, then we say that Σ is a *refinement* of Σ'. If Δ is a reflexive polytope, then $\mathbb{P}_\Delta$ has at worst Gorenstein singularities, but its singularities can be worse than terminal. Batyrev [14, Theorem 2.2.24] showed that there exists a refinement $\widetilde{\Sigma}_\Delta$ of Σ_Δ so that the morphism $f : X_{\widetilde{\Sigma}_\Delta} \to \mathbb{P}_\Delta$ is a crepant morphism (that is, $f^* K_{\mathbb{P}_\Delta} = K_{X_\Sigma}$), and that the singularities of $X_{\widetilde{\Sigma}_\Delta}$ are at worst terminal Gorenstein, and so that $X_{\widetilde{\Sigma}_\Delta}$ is a projective toric variety. We call such a variety an mpcp (maximal projective crepant partial) resolution of $\mathbb{P}_\Delta$. In the following sections, we will fix an mpcp resolution of $\mathbb{P}_\Delta$, and we will refer to this mpcp resolution as X_Δ. The ray generators of $\widetilde{\Sigma}_\Delta$ turn out to be all integral points in Δ not including 0_M.

The canonical divisor of a Fano Gorenstein toric variety $\mathbb{P}_\Delta$ is just the line bundle $\mathscr{O}_{\mathbb{P}_\Delta}(\sum_{\rho \in \Delta[0]} D_\rho)$. The canonical divisor of X_Δ is given by $\mathscr{O}_{X_\Delta}(\sum_{\rho \in \partial\Delta} D_\rho)$. A nef partition of Δ determines a set of line bundles $\mathscr{L}_i := \mathscr{O}_{\mathbb{P}_\Delta}(\sum_{\rho \in E_i} D_\rho)$. The condition that there exist φ_i satisfying the convexity condition ensures that these line bundles are semi-ample, and hence that their associated linear systems are base-point free. If we take the pullback of $\mathscr{L}_i$ to X_Δ (by abuse of notation, we will refer to this bundle as $\mathscr{L}_i$ as well), then this is the divisor $\mathscr{O}_{X_\Delta}(\sum_{\rho \in \partial\Delta_i \setminus 0_M} D_\rho)$. These line bundles are also semi-ample.

3.2.2 Givental's LG model

In [62], Givental gave a construction of the LG model of a toric complete intersection obtained as the complete intersection in a toric variety associated to the first k components of an $(k+1)$-partite nef partition. Here we will describe a method based on toric geometry for compactifying Givental's LG model to a relatively compact log Calabi-Yau variety with at worst terminal Gorenstein singularities.

We let $\Delta_1, \dots, \Delta_{k+1}$ be an $(k+1)$-partite nef partition of a reflexive polytope Δ, and assume that $\sum_{i=1}^k \Delta_i$ does not contain 0_M on its relative interior. Givental describes a process for producing the LG model of X as follows. Let $\mathbb{C}[x_1^\pm, \dots, x_d^\pm]$ be the coordinate ring of $(\mathbb{C}^\times)^d$ and take the complete intersection of quasi-affine hypersurfaces in $(\mathbb{C}^\times)^d$ defined as

$$Y_0 = \bigcap_{i=1}^{n-1} \left\{ \sum_{\rho \in \Delta_i \cap M} a_\rho x_\rho = 1 \right\}. \tag{3.1}$$

Here a_ρ general complex coefficients and $x_\rho = \prod_{i=1}^d x_i^{\langle v_i, \rho \rangle}$ for $v_1, \ldots, v_d$ a basis of $N = \mathrm{Hom}\,(M, \mathbb{Z})$. This complete intersection is equipped with the regular function

$$\mathsf{w}_0(x_1, \ldots, x_d) = \sum_{\rho \in \Delta_n} a_\rho x_\rho.$$

The pair (Y_0, w_0) will be called Givental's LG model for X. Givental showed that the oscillating integrals of (Y_0, w_0) recover the quantum periods of the mirror dual Fano variety. However, one can check in examples that if one is interested in behaviour of the singular fibers (Y_0, w_0) then Givental's LG model contains strictly less information than the partial compactification constructed in the following section. For this reason, we think that the partial compactification of Section 3.2.3 is the appropriate compactifcation for homological mirror symmetry.

3.2.3 A convenient partial compactification

Our goal now will be to compactify Y_0 to a pencil of hypersurfaces in a toric variety. First we will note that each fiber itself can be compactified to a hypersurface in X_∇, where X_∇ is an mpcp resolution of $\mathbb{P}_\nabla$ as described in Section 3.2.1. The fiber of w_0 over λ is compactified to

$$\sum_{\rho \in \Delta_i \cap M} a_\rho \prod_{\sigma \in \nabla \cap N} z_\rho^{\langle \sigma, \rho \rangle - \sigma^i_{\min}} = 0 \text{ if } 1 \leq i \leq k \tag{3.2}$$

$$\lambda \prod_{\sigma \in \nabla_{k+1} \cap N \setminus 0_N} z_\sigma - \sum_{\rho \in \Delta_{k+1} \cap M \setminus 0_M} a_\rho \prod_{\sigma \in \partial \nabla \cap N} z_\rho^{\langle \sigma, \rho \rangle - \sigma^{k+1}_{\min}} = 0 \tag{3.3}$$

where $\{z_\sigma\}_{\rho \in \Sigma[1]}$ are homogeneous coordinates on X_∇ and $\sigma^i_{\min} = \min\{\langle \sigma, \rho \rangle : \rho \in \nabla_i\}$. The value of σ^i is either 0 or -1 depending on whether σ is in Δ_2 or Δ_1.

Definition 3.2.2. Define $\mathscr{P}(\lambda)$ to be the subvariety of X_∇ cut out by Equations 3.2 with the parameter λ.

Note that we could just take our partial compactification to be the natural pencil of complete intersections in $X_\nabla \times \mathbb{A}^1_\lambda$ cut out by the equation above and letting λ be the coordinate on $\mathbb{A}^1$, but it is not clear a priori how bad the singularities of this complete intersection can be. We will spend the next little while trying to get around this limitation. First let $[-1,1]$ denote the 1-dimensional polytope determining $\mathbb{P}^1$ which is just the interval $[-1,1] \subseteq \mathbb{R}$, and let

$$\widetilde{\Delta} = \mathrm{Conv}(\Delta \times 0 \cup 0_M \times [-1,1])$$

in $M_\mathbb{R} \times \mathbb{R}$. Then $\widetilde{\Delta}$ is reflexive and corresponds to the Gorenstein toric Fano variety $\mathbb{P}_\Delta \times \mathbb{P}^1$.

Proposition 3.2.3. *If $\Delta_1, \dots, \Delta_{k+1}$ is a nef partition of Δ, then there is a $(k+2)$-partite nef partition of $\widetilde{\Delta}$ given by*

$$\widetilde{\Delta}_i = \Delta_i \times 0 \textit{ for } 1 \leq i \leq k$$
$$\widetilde{\Delta}_{k+1} = \mathrm{Conv}(\Delta_{k+1} \times 0, 0_M \times 1)$$
$$\widetilde{\Delta}_{k+2} = \mathrm{Conv}(0_M \times -1, 0_M \times 0).$$

Proof. It is not hard to see that the sets $\widetilde{\Delta}_i$ come from a partition of the vertices of $\widetilde{\Delta}$. It remains to check that there are piecewise linear functions with the correct properties. First, let $\widetilde{\varphi}_i$ be the extension of φ_i to $M_\mathbb{R} \times \mathbb{R}$ so that $\widetilde{\varphi}_i(a,b) = \varphi_i(a)$. Then let $v^+ : M_\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the map which is given by $v^+(b,a) = -a$ if $a \geq 0$ and $v^+(a) = 0$ otherwise. Similarly, let v^- be the map taking $v^-(b,a) = a$ if $a \leq 0$ and 0 otherwise. The maps v^+, v^- and $\widetilde{\varphi}_i$ are convex and $\Sigma_{\widetilde{\Delta}}$-linear. Note that $\widetilde{\varphi}_i$ takes value -1 on all nonzero integral points in $\widetilde{\Delta}_i$ and 0 on all other integral points of $\widetilde{\Delta}$, that $\widetilde{\varphi}_{k+1} + v^+$ takes value -1 on the nonzero integral points of $\widetilde{\Delta}_{k+1}$ and 0 on all other integral points of $\widetilde{\Delta}$, and that v^- takes value -1 on all nonzero integral points of $\widetilde{\Delta}_{k+2}$ and 0 on all other integral points of $\widetilde{\Delta}$. Thus this is a nef partition. □

We can compute the dual nef partition, which we denote $\widetilde{\nabla}_1, \ldots, \widetilde{\nabla}_{k+2}$.

Proposition 3.2.4.

$$\begin{aligned} \widetilde{\nabla}_i &= \nabla_i \times 0 \textit{ for } 1 \leq i \leq k \\ \widetilde{\nabla}_{k+1} &= \mathrm{Conv}(0_N \times -1 \cup \nabla_{k+1} \times 0) \\ \widetilde{\nabla}_{k+2} &= \mathrm{Conv}(\nabla_{k+1} \times 1 \cup 0_N \times 0). \end{aligned}$$

We will denote the polytope $\mathrm{Conv}(\widetilde{\nabla}_1 \cup \cdots \cup \widetilde{\nabla}_{k+2})$ by $\widetilde{\nabla}$. As usual, $\widetilde{\nabla}$ is a reflexive polytope and $\widetilde{\nabla}_1, \ldots, \widetilde{\nabla}_{k+2}$ forms a nef partition of $\widetilde{\nabla}$.

Remark 3.2.5. Note that the polytope $\widetilde{\nabla}$ is equal to

$$\mathrm{Conv}(0_N \times -1 \cup \nabla \times 0 \cup \nabla_{k+1} \times 1).$$

Therefore, the faces of $\nabla \times 0$ are in fact faces of $\widetilde{\nabla}$.

Theorem 3.2.6. *Let X be a complete intersection quasi-Fano variety associated to a $(k+1)$-partite nef partition of a polytope Δ. Then the partial compactification of (Y_0, w_0) to the total space of the pencil $\mathscr{P}(\lambda)$ in $X_\nabla \times \mathbb{A}^1$ has at worst terminal Gorenstein singularities, is log Calabi-Yau and admits a compactification Z where w extends to a map $\mathsf{f} : Z \to \mathbb{P}^1$ so that $\mathsf{f}^{-1}(\infty)$ is a normal crossings union of varieties with at worst toroidal singularities.*

Proof. Let us take $X_{\widetilde{\nabla}}$ to be some mpcp resolution of $\mathbb{P}_\nabla$. We note that since all faces of ∇ are faces of $\widetilde{\nabla}$, the maximal projective refinement of $\Sigma_{\widetilde{\nabla}}$ corresponding to $X_{\widetilde{\nabla}}$ induces a maximal projective refinement of Σ_∇, which we may assume is the refinement of Σ_∇ determining X_∇. Another way to say this is that if $\widetilde{\Sigma}_{\widetilde{\nabla}}$ is the maximally refined fan associated to $X_{\widetilde{\nabla}}$ then the projection of $M \times \mathbb{R}$ onto $\mathbb{R}$ induces a map $g : X_{\widetilde{\nabla}} \to \mathbb{P}^1$ so that the fiber away from 0 and infinity is X_∇.

Now let Z be the complete intersection

$$Z = \cap_{i=1}^{k+1} \{s_i = 0\}$$

for s_i generic elements of $\Gamma(\mathscr{O}_{X_{\widetilde{\nabla}}}(\widetilde{\nabla}_i))$. Then the vanishing locus of a global section s_{k+2} of $\mathscr{O}_{X_{\widetilde{\nabla}}}(\widetilde{\nabla_i})$ is an anticanonical Calabi-Yau hypersurface in Z.

We can compute that in terms of homogeneous coordinates on $X_{\widetilde{\nabla}}$, the variety Z is cut out by equations

$$s_i := \sum_{\rho \in (\partial\Delta_i \cap M \setminus 0_M)\times 0} a_\rho z^\rho = 0 \text{ for } 1 \leq i \leq k \tag{3.4}$$

If Q is a lattice polytope in $N \otimes \mathbb{R}$ then will denote by $V(Q)$ the set of points in $Q \cap (N \setminus 0_N)$. If $\sigma \in N$, then we will denote $\sigma^i_{\min} := \min\{\langle \sigma, \rho\rangle : \rho \in \Delta_i \cap M\}$. Then

$$z^\rho = \prod_{\sigma \in V(\widetilde{\nabla})} z_\sigma^{\langle \sigma,\rho\rangle - \sigma^i_{\min}} = \left(\prod_{\sigma \in V(\nabla \times 0)} z_\sigma^{\langle \sigma,\rho\rangle - \sigma^i_{\min}}\right) \times \left(\prod_{\sigma \in V(\nabla_{k+1} \times 1)} z_\sigma^{\langle \sigma,\rho\rangle - \sigma^i_{\min}}\right)$$

since $\langle 0_N \times 1, \rho\rangle = 0$ for every $\rho \in \widetilde{\Delta}_i \subseteq \Delta \times 0$. We also have

$$s_{k+1} := z^{0_M \times 0} + a_{0_M \times 1} z^{0_M \times 1} + \sum_{\rho \in V(\nabla_k \times 0)} a_\rho z^\rho \tag{3.5}$$

where

$$z^{0_M \times 1} = t \left(\prod_{\sigma \in V(\nabla_{k+1}) \times 0} z_\sigma\right) \times \left(\prod_{\sigma \in V(\nabla_{k+1}) \times 1} z_\sigma\right)$$
$$z^{0_M \times 0} = s \left(\prod_{\sigma \in V(\nabla_{k+1}) \times 0} z_\sigma\right)$$

and z^ρ is as before if $\rho \neq 0_M \times 0$ or $0_M \times 1$. Furthermore, anticanonical hypersurfaces in Z are cut out by the equation

$$s_{k+2} := s - \lambda t \left(\prod_{\sigma \in V(\nabla_{k+1}) \times 1} z_\sigma\right) = 0$$

Here $s = z_{0_N \times -1}$, $t = z_{0_N \times 1}$ and λ is some constant. The map g is expressed in terms of homogeneous coordinates as the map which sends any point to $[t : s] \in \mathbb{P}^1$.

Now we make the crucial observation that if $\widetilde{\Sigma}_{\widetilde{\nabla}}$ is the fan determining $X_{\widetilde{\nabla}}$, then $\widetilde{\Sigma}_{\widetilde{\nabla}} \cap (N_{\mathbb{R}} \times \mathbb{R}^{\leq 0})$ is the fan of $X_\nabla \times \mathbb{A}^1$. Restricting to $X_\nabla \times \mathbb{A}^1 \subseteq X_{\widetilde{\nabla}}$ is the same as excising the divisors D_σ for all $\sigma \in (\nabla_{k+1} \cap N) \times 1$. In terms of homogeneous coordinates, this is equivalent to setting $t = 1$ and $z_\sigma = 1$ for all $\sigma \in \nabla_{k+1} \times 1$. Thus $Y := Z \cap (X_\nabla \times \mathbb{A}^1)$ is Z with the divisor $s_{k+1} = 0$ excised after setting $s = 0$. Note that this is a section of the anticanonical bundle of Z. Since Z is a complete intersection of semi-ample divisors in X_∇, it has at worst terminal Gorenstein singularities. Furthermore, $Z \cap (\cup_{\sigma \in ((\nabla_{k+1} \cap N) \times 1)} D_\sigma)$ is a union of divisors with at worst toroidal simple normal crossings, thus if $u \in \Gamma(\mathscr{O}_{X_{\widetilde{\nabla}}}(\widetilde{\nabla}_{k+2})^{-1})$ with poles along $\cup_{\sigma \in (\nabla_{k+1} \times 1)} D_\sigma$, then $\Omega = u|_Y$ is a non-vanishing section of ω_Y which, when extended to Z has only simple poles along $Z \cap (\cup_{\sigma \in (\nabla_{k+1} \times 1)} D_\sigma)$. Thus Y is log Calabi-Yau.

In terms of homogeneous coordinates on $X_\nabla \times \mathbb{A}^1$, we see that Y is given by

$$\sum_{\rho \in (\Delta_i \cap M) \times 0} a_\rho \prod_{\sigma \in V(\nabla) \times 0} z_\rho^{\langle \sigma, \rho \rangle - \sigma^i_{\min}} = 0 \text{ if } 1 \leq i \leq k \tag{3.6}$$

$$(s + a_{0_M \times 1}) \prod_{\sigma \in V(\nabla_{k+1}) \times 0} z_\sigma - \sum_{\rho \in \Delta_{k+1} \cap M \setminus 0_M} a_\rho \prod_{\sigma \in V(\nabla) \times 0} z_\rho^{\langle \sigma, \rho \rangle - \sigma^{k+1}_{\min}} = 0 \tag{3.7}$$

where s is the coordinate on $\mathbb{A}^1$ and the sections cut out by s_{k+2} are just $s = \lambda$ for some constant λ. These are anticanonical hypersurfaces in Z. Thus the fibers of the map g restricted to Y are just fibers of the natural projection onto $\mathbb{A}^1$. Thus Y is the total space of the natural pencil of hypersurfaces in Equation (3.2). To be precise, this fibers of this pencil are the fibers of $\mathscr{P}(\lambda)$ with a shift by $-a_{0_M \times 1}$ on $\mathbb{A}^1$. This proves the theorem. $\square$

Remark 3.2.7. Terminal Gorenstein singularities occur in codimension 4 or greater. Therefore, if X is a threefold, then Z and Y are smooth, and thus for complete intersection quasi-Fano threefolds, we have constructed a *smooth* LG model.

3.3 Hodge numbers of LG models

In this section, we will prove a folklore theorem, showing that computing the number of irreducible components in fibers of a smooth LG model (Y, w) is equivalent to computing $h^2(Y, \mathsf{w})$. Then we will show that if (Y, w) is the slightly singular LG model constructed in Section 3.2.2 and there exists a smooth LG model of (Y', w') which is birational to (Y, w), then computing the number of irreducible components in the singular fibers of (Y, w) is equivalent to computing the number of irreducible in components of singular fibers of (Y', w'). This proves that there should be a close relationship between the number of irreducible components in $\mathscr{P}(t)$ and the Hodge numbers of the mirror dual quasi-Fano complete intersections. The precise relationship will be explored in Section 3.4 in the case where X is a hypersurface in a toric variety and (Y, w) is the LG model constructed in Section 3.2.

3.3.1 Computing $h^2(Y, \mathsf{w})$ for a smooth LG model

The follwing theorem is implicit in the work of Przyjalkowski [123] and is mentioned explicitly by Przyjalkowski and Shramov [127], though to my knowledge, no proof exists in the literature. We make heavy use of the degeneration of the Leray spectral sequence for w at the E_2 term for a projective morphism. This is proved by Zucker in [149], and an alternate proof may be found in [120, Theorem 4.24]. Note that in both of the references given, the degeneracy of the Leray spectral sequence is stated for a morphism $g : X \to S$ and for X and S smooth and projective and for g proper. However, it is easy to see that at least the proof in [120] goes through verbatim for S and X quasiprojective and for g proper.

Theorem 3.3.1. *Let ρ_t be the number of irreducible components in $\mathsf{w}^{-1}(t)$. If s is a point in $\mathbb{A}^1$ so that $\mathsf{w}^{-1}(s)$ is smooth then*

$$h^{2d-2}(Y, \mathsf{w}) = h^{2d-2}(Y, \mathsf{w}^{-1}(s)) = \sum_{t \in \mathbb{A}^1} (\rho_t - 1).$$

Proof. We begin with the case where each fiber has normal crossings singularities. Then we may apply the Mayer-Vietoris spectral sequence (see e.g. [101]) to each singular fiber X_t. If we let $X_t^{[d]}$ be the disjoint union of all intersections of $(d+1)$-dimensional components of X_t, then we have that

$$E_1^{p,q} = \mathrm{H}^q(X_t^{[p]}, \mathbb{C}).$$

Furthermore, this spectral sequence degenerates at the E_2 term to $\mathrm{H}^{p+q}(X_t, \mathbb{C})$. We compute $\mathrm{H}^{2d-2}(X_t, \mathbb{C})$ by noting that $E_1^{i,2d-2-i} = \mathrm{H}^{2d-2-i}(X^{[i]}, \mathbb{C}) = 0$ unless $i = 0$, since $\dim X^{[i]} = d - 2 - i$ and that $\mathrm{H}^{d-2}(X^{[0]}, \mathbb{C}) \cong \mathbb{C}^{\rho_t}$. Now by the Clemens contraction theorem [28], we have that $(R^{2d-2}\mathsf{w}_*\mathbb{C})_t \cong \mathbb{C}^{\rho_t}$. Let Σ be the set of critical values of w, let $j : \mathbb{A}^1 \setminus \Sigma \hookrightarrow \mathbb{A}^1$ be the natural injection and let $\sigma = \mathsf{w}|_{\mathbb{A}^1\setminus\Sigma}$. Then by [149, Proposition 15.12], we have a surjective map

$$R^n\mathsf{w}_*\mathbb{C} \to j_*R^n\sigma_*\mathbb{C}. \tag{3.8}$$

In our case, $j_*R^{2d-2}\sigma_*\mathbb{C} \cong \underline{\mathbb{C}}_{\mathbb{A}^1}$ and the kernel is $\bigoplus_{t\in\Sigma} \mathbb{C}_t^{\rho_t - 1}$ where $\mathbb{C}_t$ denotes the skyscraper sheaf supported at t with fiber $\mathbb{C}$. There is a short exact sequence in cohomology

$$0 \to \bigoplus_{t\in\Sigma} \mathbb{C}^{\rho_t-1} \to \mathrm{H}^0(\mathbb{A}^1, R^n\mathsf{w}_*\mathbb{C}) \to \mathrm{H}^0(\mathbb{A}^1, \mathbb{C}) \to 0.$$

Thus $h^0(\mathbb{A}^1, R^{2d-2}\mathsf{w}_*\mathbb{C}) = 1 + \sum_{t\in\Sigma}(\rho_t - 1)$. We now compute $\mathrm{H}^1(\mathbb{A}^1, R^{2d-1}\mathsf{w}_*\mathbb{C})$ and $\mathrm{H}^2(\mathbb{A}^1, R^{2d-2}\mathsf{w}_*\mathbb{C})$. Note that the surjection in Equation (3.8) has kernel a skyscraper sheaf supported on Σ, therefore,

$$\mathrm{H}^i(\mathbb{A}^1, R^n\mathsf{w}_*\mathbb{C}) \cong \mathrm{H}^i(\mathbb{A}^1, j_*R^n\sigma_*\mathbb{C})$$

for $i = 1, 2$. Since fibers of w are Calabi-Yau, we have $\mathrm{H}^{2d-3}(\mathsf{w}^{-1}(s); \mathbb{C}) = 0$ for $s \in \mathbb{A}^1 \setminus \Sigma$. Thus $j_*R^{2d-3}\sigma_*\mathbb{C} = 0$, and thus $\mathrm{H}^i(\mathbb{A}^1, R^n\mathsf{w}_*\mathbb{C}) = 0$. If $\mathbb{V}$ is a local system on $\mathbb{A}^1 \setminus \Sigma$, and $j_!\mathbb{V}$ is the extension by 0 sheaf, then we have a short exact sequence

$$0 \to j_!\mathbb{V} \to j_*\mathbb{V} \to i_*i^*\mathbb{V} \to 0$$

for $i : \Sigma \hookrightarrow \mathbb{A}^1$ and $i_*i^*\mathbb{V}$ a skyscrapter sheaf supported at points $t \in \Sigma$. The upshot of this is that

$$\mathrm{H}^2(\mathbb{A}^1, j_*R^{2d-4}\sigma_*\mathbb{C}) = \mathrm{H}^2(\mathbb{A}^1, j_!R^{2d-4}\sigma_*\mathbb{C}).$$

By Verdier duality, we have that $\mathrm{H}^2(\mathbb{A}^1, j_!R^{2d-4}\sigma_*\mathbb{C}) = \mathrm{H}^0_c(\mathbb{A}^1, \mathbb{D}_{\mathbb{A}^1} j_!R^{2d-4}\sigma_*\mathbb{C})^\vee$ for $\mathbb{D}_{\mathbb{A}^1}$ the Verdier duality functor on $\mathbb{A}^1$. It is a well known fact that $j_*\mathbb{D}_{\mathbb{A}^1\setminus\Sigma} = \mathbb{D}_{\mathbb{A}^1} j_!$. Furthermore, by [42, Example 3.3.8] we know that, since $R^{2d-4}\sigma_*\mathbb{C}$ is a local system, $\mathbb{D}_{\mathbb{A}^1\setminus\Sigma}(R^{2d-4}\sigma_*\mathbb{C}) = (R^{2d-4}\sigma_*\mathbb{C})^\vee[2]$, so

$$\begin{aligned} \mathrm{H}^2(\mathbb{A}^1, j_!R^{2d-4}\sigma_*\mathbb{C}) &\cong \mathrm{H}^{-2}_c(\mathbb{A}^1, \mathbb{D}_{\mathbb{A}^1} j_!R^{2d-4}\sigma_*\mathbb{C})^\vee \\ &\cong \mathrm{H}^{-2}_c(\mathbb{A}^1, j_*\mathbb{D}_{\mathbb{A}^1\setminus\Sigma}R^{2d-4}\sigma_*\mathbb{C})^\vee \\ &\cong \mathrm{H}^0_c(\mathbb{A}^1, j_*(R^{2d-4}\sigma_*\mathbb{C})^\vee)^\vee \end{aligned}$$

where $(R^{2d-4}\sigma_*\mathbb{C})^\vee$ is the local system dual to $R^{2d-4}\sigma_*\mathbb{C}$. The set of global sections of $j_*(R^{2d-4}\sigma_*\mathbb{C})^\vee$ can be identified with the global sections of $(R^{2d-4}\sigma_*\mathbb{C})^\vee$ and thus are just monodromy invariant sections. Therefore the only global section with compact support is the zero section. Thus $h^2(\mathbb{A}^1, R^{2d-4}\mathsf{f}_*\mathbb{C}) = 0$.

By the degeneration of the Leray spectral sequence at the E_2 term, we conclude that $\operatorname{rank} \mathrm{H}^{2d-2}(Y, \mathbb{C}) = 1 + \sum_{t\in\Sigma}(\rho_t - 1)$. We have a short exact sequence determining $\mathrm{H}^{2d-2}(Y, V; \mathbb{C})$,

$$0 \to \mathrm{H}^{2d-2}(Y, V; \mathbb{C}) \to \mathrm{H}^{2d-2}(Y, \mathbb{C}) \to \mathbb{C} \to 0 \tag{3.9}$$

using the fact that $\mathrm{H}^{2d-3}(V, \mathbb{C}) = \mathrm{H}^{2d-1}(Y, \mathbb{C}) = 0$ and $\mathrm{H}^{2d-2}(V, \mathbb{C}) \cong \mathbb{C}$. Therefore,

$$h^2(Y, \mathsf{w}) = h^{2d-2}(Y, \mathsf{w}) = h^{2d-2}(Y, V; \mathbb{C}) = \sum_{t\in\Sigma}(\rho_t - 1).$$

The first equality is by Theorem 2.2.9. If X_t is not normal crossings, then we can blow up repeatedly along smooth loci in X_t to obtain a variety whose fibers are normal crossings (this is possible by [71]). Let $\pi : \widetilde{Y} \to Y$ be this blow up. If π is a composition of n_t smooth blow ups in each X_t, then $h^{2d-2}(\widetilde{Y}) = h^{2d-2}(Y) + \sum_{t\in\Sigma} n_t$, and each fiber has $\rho_t + n_t$ irreducible components. The theorem then follows by comparing the short exact sequences like that in Equation (3.9) determining $\mathrm{H}^{2d-2}(Y, V; \mathbb{C})$ and $\mathrm{H}^{2d-2}(\widetilde{Y}, V; \mathbb{C})$. □

3.3.2 Relating singular and smooth LG models

Now we will show that counting components in fibers in a terminal singular model of Y is equivalent to counting components in an appropriate smooth model of Y. We assume that there is a pair (Y', w) where Y' is smooth and admits a compactification to a smooth fiber space $\mathsf{f}' : Z' \to \mathbb{P}^1$ so that $-K_{Z'} = F$, for F the class of a general fiber if f'. We assume that there is a birational map $\phi : Z \dashrightarrow Z'$ which preserves fibrations and thus sends the fiber over infinity of f to the fiber over infinity of f'. Since we have that both Z and Z' are $\mathbb{Q}$-factorial terminal, [81] states that ϕ is can be extended to a map which is an isomorphism in codimension 1. Note that according to [82, Theorem 1], ϕ is actually a series of flips.

Lemma 3.3.2. *Let $\mathsf{w}' : Z' \to \mathbb{P}^1$ for some Z' as above, and let ϕ be as in the discussion in the paragraph above. Then for any $t \in \mathbb{P}^1$ the number of irreducible components of $\mathsf{w}^{-1}(t)$ is equal to the number of irreducible components of $(\mathsf{w}')^{-1}(t)$.*

Proof. By [81] it follows that ϕ is an isomorphism in codimension 1. In other words, if D is a divisor on Y, then $\phi(D)$ is again a divisor in Y'. Since

$$\mathsf{w}(D) = (\phi \cdot \mathsf{w}')(D) = \mathsf{w}'(\phi(D)) = t$$

it follows that $\phi(D)$ is again a component of $\mathsf{w}^{-1}(t)$. Repeating this argument for ϕ^{-1} shows that the number of irreducible components of $\mathsf{w}^{-1}(t)$ is equal to the number of irreducible components of $(\mathsf{w}')^{-1}(t)$. □

Therefore, if one wants to count the number of components of the singular fibers of any smooth birational model for (Y, w), then it is enough to count the number of components of $\mathscr{P}(t)$. In the next section, we will carry this out for X the pullback to X_Δ of an ample hypersurface in $\mathbb{P}_\Delta$.Thus, combining Theorem 3.2.6 with Lemma 3.3.2 and Theorem 3.3.1 and defining ρ_t to be the number of irreducible components in $\mathscr{P}(t)$ for a point $t \in \mathbb{A}^1$, we find

Corollary 3.3.3. *If there is a smooth log Calabi-Yau model (Y', w') of (Y, w), then $h^2(Y', \mathsf{w}')$ is equal to*

$$\sum_{t \in \Sigma} (\rho_t - 1).$$

Batyrev and Dais [19] define *stringy Hodge numbers* for a variety X with mild singularities, and show that these stringy Hodge numbers agree with the Hodge numbers of a smooth crepant resolution of X if a crepant resolution exists. It is not immediately obvious to me how extend the definition of [19] to the f-adapted forms of [79], but the corollary above suggests a provisional definition,

$$h^{1,1}_{\mathrm{st}}(Y, \mathsf{w}) := \sum_{t \in \Sigma} (\rho_t - 1).$$

As noted in Remark 3.2.7, if X is a three-dimensional toric complete intersection then $h^{1,1}_{\mathrm{st}}(Y, \mathsf{w}) = h^{1,1}(Y, \mathsf{w})$.

3.4 Hodge numbers and mirror symmetry

Now that we have provided a general construction of an appropriate LG model for any quasi-Fano complete intersection, and shown that the number of components in the singular fibers of (Y, w) should reflect information about the Hodge numbers of X, our goal is to put this into action. We will compute the number of irreducible components in $\mathscr{P}(0)$ in the case where X is the pullback of an ample hypersurface in a Gorenstein Fano toric variety $\mathbb{P}_\Delta$ and show that it is actually equal to $h^{1,d-2}(X)$. It is possible that one might be able to recover the results of [126] using the same technique, but I expect that the combinatorics involved are still nontrivial. We would like to point out that the technique of [126] have the advantage that it is not wedded to the case of complete intersections in toric varieties, and thus could possibly be applied to cases which are inaccessible using my methods.

3.4.1 Some combinatorial results

Here we collect several combinatorial results which will be useful later. A first, general remark, which will be used several times in the present section, is that if v is a point on the relative interior of a polyhedron with vertices $\rho_1, \ldots, \rho_n$, then there are positive numbers $a_1, \ldots, a_n$ so that $\sum_{i=1}^n a_i = 1$ and $\sum_{i=1}^n a_i \rho_i = v$. Next, we say that a polytope Q is a Minkowski summand of P if there is some integer n and some polytope Q' so that

$$Q + Q' = P.$$

To a divisor D on $\mathbb{P}_\Delta$ there is a polytope ∇_D in N associated to D whose integral points correspond to sections of $\mathscr{O}_{\mathbb{P}_\Delta}$. For instance if $E_1, \ldots, E_{k+1}$ is a nef partition of Δ and if $D_i = \sum_{\rho \in E_i} D_\rho$, then the polytope ∇_{D_i} is just ∇_i. The divisor D is ample if and only if the polytopes ∇_D and Δ° are Minkowski summands of one another. If $\Delta_1, \ldots, \Delta_{k+1}$ is a nef partition, then ∇_i is a Minkowski summand of Δ°, so the condition that D_i be ample is just the fact that Δ° is a Minkowkski summand of ∇_i. Let Γ be a face of Δ, and take v a point on the relative interior of Γ. For a polytope Q in $N_\mathbb{R}$, define

$$v_{Q,\min} := \min\{\langle v, x\rangle : x \in Q\}, \qquad \Gamma_Q^\vee := \{x \in Q : \langle v, x\rangle = v_{Q,\min}\}.$$

If v is a point in Δ, then we define $\Gamma(v)$ to be the smallest face of Δ containing v. The following result seems to be well known, though I do not know a precise reference for it (though [17] mention it).

Proposition 3.4.1. *Assume that Δ° is a Minkowski summand of ∇_1. Then there is a combinatorial duality between ∇_1 and Δ. In other words, there is an inclusion* *reversing bijection between dimension i faces of ∇_1 and dimension $d-i-1$ faces of Δ given by the assigning $\Gamma^{\vee}_{\nabla_1}$ to a face Γ of Δ.*

If ρ is an integral point in Δ, then $\rho_{\nabla_1,\min} = 0$ if $\rho \in \Delta_2$ and -1 if ρ is in Δ_1. If Γ is a face of ∇_1, then we denote by $\Gamma^{\vee}_{\Delta}$ the face of Δ so that $\Gamma^{\vee}_{\nabla_1} = \Gamma$.

Definition 3.4.2. Define $\nabla_1^{\geq 1}$ to be the subset of integral points σ in ∇_1 satisfying $\langle \sigma, \rho \rangle \geq 1$ for each $\rho \in E_2$.

Lemma 3.4.3. *Assume that σ is an integral point in $\nabla_1^{\geq 1}$. Then $\Gamma(\sigma)^{\vee}_{\Delta_1} = \Gamma(\sigma)^{\vee}_{\Delta} = \Gamma(\sigma)^{\vee}_{\nabla^\circ}$.*

Proof. By [17, Proposition 6.3], the face $\Gamma(\sigma)$ is a face of ∇. Its dual face in ∇° is given by the set of points in ∇° so that $\langle \sigma, \rho \rangle = -1$. The face $\Gamma(\sigma)^{\vee}_{\nabla^\circ}$, in ∇° is the Minkowski sum of $\Gamma(\sigma)^{\vee}_{\Delta_1}$ and $\Gamma(\sigma)^{\vee}_{\Delta_2}$ where

$$\Gamma(\sigma)^{\vee}_{\Delta_1} = \{\rho \in \Delta_1 : \langle \sigma, \rho \rangle = -1\}$$
$$\Gamma(\sigma)^{\vee}_{\Delta_2} = \{\rho \in \Delta_2 : \langle \sigma, \rho \rangle = 0\}.$$

Since σ is in $\nabla_1^{\geq 1}$, it follows that $\Gamma(\sigma)^{\vee}_{\Delta_2}$ contains only one integral point, 0_M, and therefore is equal to 0_M. Thus $\Gamma(\sigma)^{\vee}_{\nabla^\circ} = \Gamma(\sigma)^{\vee}_{\Delta_1}$. Furthermore, if we let $\Gamma'(\circ)$ be the smallest face of Δ° containing σ. Then $\Gamma'(\sigma)^{\vee}_{\Delta}$ is precisely the set of points in Δ satisfying $\langle \sigma, \rho \rangle = -1$, which one observes is simply $\Gamma(\sigma)^{\vee}_{\Delta_1}$. □

3.4.2 Hodge numbers of hypersurfaces

Now we assume that we have a hypersurface X in X_Δ associated to a nef partition Δ_1, Δ_2. We let X be the hypersurface given by the vanishing locus of a section $s \in \mathrm{H}^0(X_\Delta, \mathscr{L}_1)$. Then by the adjunction formula, we have that $\mathscr{L}_2|_X = \omega_X^{-1}$. We may compute several Hodge numbers of X easily. The following results are minor modifications of results of Batyrev and Borisov in [17]. We state them without proof. If Q is a polytope in $M_{\mathbb{R}}$, then we let $\ell^*(Q)$ be the number of integral points on the relative interior of Q.

Theorem 3.4.4 ([17, Corollary 3.5]). *Assume that we have a nef partition Δ_1, Δ_2 so that both ∇_1 and ∇_2 have positive dimension and assume that $\ell^*(\nabla_i) = 0$ and $d - 1 \geq 2$. Then*

$$h^i(\mathscr{O}_X) = 0 \text{ for } i > 0.$$

and $h^0(\mathscr{O}_X) = 1$.

Therefore, we have identified a class of varieties which cohomology groups which look like the cohomology groups of a Fano variety.

Theorem 3.4.5 ([17, Proposition 8.6]). *Take a bipartite nef partition Δ_1, Δ_2 satisfying the conditions of Theorem 3.4.4 so that the line bundle $\mathscr{L}_1$ is ample on $\mathbb{P}_\Delta$, and let X be the pullback to X_Δ along the crepant parital resolution map $X_\Delta \to \mathbb{P}_\Delta$ of a generic hypersurface defined by a section of $\mathscr{L}_1$, then*

$$h^{1,d-2}(X) = \ell^*(2\nabla_1) - \sum_{\Gamma \in \Delta[0]} \ell^*(\Gamma)\ell^*(\Gamma^\vee_{\nabla_1}) + \sum_{\Gamma \in \Delta[1]} \ell^*(\Gamma)\ell^*(\Gamma^\vee_{\nabla_1})$$

We will proceed to simplify this computation. We fix some $0 \leq i \leq d$ for the moment. By the combinatorial duality between Δ and ∇_i (Proposition 3.4.1), it follows that

$$\sum_{\Gamma \in \Delta[i]} \ell^*(\Gamma)\ell^*(\Gamma^\vee_{\nabla_1}) = \sum_{\Gamma \in \nabla_i[d-i-1]} \ell^*(\Gamma)\ell^*(\Gamma^\vee_{\Delta})$$

If $\Gamma^\vee_\Delta$ contains an integral point on its relative interior, then it follows from [17, Proposition 6.3] that $\Gamma^\vee_\Delta$ is either in Δ_1 or Δ_2. If σ is on the relative interior of some face Γ of ∇_1 so that the dual face is in Δ_1, then σ is not in the dual of any face in Δ_2. Therefore, there is no $\rho \in E_2$ so that $\langle \sigma, \rho \rangle = \rho_{\nabla_1, \min} = 0$. Therefore, $\langle \sigma, \rho \rangle \geq 1$ for every $\rho \in E_2$. As in Definition 3.4.2, let $\nabla_1^{\geq 1}$ be the set of points in ∇_1 satisfying $\langle \sigma, \rho \rangle \geq 1$ for $\rho \in E_2$, then by definition, such a σ is in $\nabla_1^{\geq 1}$.

If $\sigma \in \nabla_1$ is on the interior of a face so that $\Gamma(\sigma)^\vee_\Delta$ is in Δ_2 and if ρ is an integral point on the relative interior of $\Gamma(\sigma)^\vee_\Delta$, then $\rho_{\nabla_1, \min} = 0$, and thus $\Gamma(\sigma)$ contains 0_N and is contained in a plane passing through the origin of $N_\mathbb{R}$. By reflexivity of Δ° any interior point of such a face is 0_N itself. Therefore

$$\sum_{\Gamma\in\nabla_i[d-i-1]} \ell^*(\Gamma)\ell^*(\Gamma^\vee_\Delta) = \sum_{\substack{\sigma\in\nabla_1^{\geq 1}\cap N\\ \dim\Gamma(\sigma)=d-i-1}} \ell^*(\Gamma(\sigma)^\vee_\Delta) + \delta_{\dim\Gamma(0_N),d-i-1}\ell^*(\Gamma(0_N)^\vee_\Delta).$$

Here $\delta_{i,j}$ is just the Kronecker delta function. If $\sigma\in\nabla_1^{\geq 1}$, then by Lemma 3.4.3 we have that $\Gamma(\sigma)^\vee_\Delta = \Gamma(\sigma)^\vee_{\Delta_1}$. If we take any $\rho\in E_2$ then $\Gamma(\rho)^\vee$ contains 0_N, thus by the combinatorial duality of Proposition 3.4.1, we have that $\Gamma(\rho)$ is contained in $\Gamma(0_N)^\vee_\Delta$. Therefore, $\Gamma(0_N)^\vee_\Delta$ contains all points in E_2 and we conclude that $\mathrm{Conv}(E_2)=\Gamma(0_N)^\vee_\Delta$. In the future, we will call this polytope Δ_2'. Thus we have that

$$\sum_{\Gamma\in\Delta[i]} \ell^*(\Gamma)\ell^*(\Gamma^\vee_{\nabla_1}) = \sum_{\substack{\sigma\in\nabla_1^{\geq 1}\cap N\\ \dim\Gamma(\sigma)=d-i-1}} \ell^*(\Gamma(\sigma)^\vee_{\Delta_1}) + \delta_{\dim\Gamma(0_N),d-i-1}\ell^*(\Delta_2').$$

In particular, we see that

$$\sum_{\Gamma\in\Delta[0]} \ell^*(\Gamma)\ell^*(\Gamma^\vee_{\nabla_1}) = \#\{\sigma\in\nabla_1^{\geq 1}\cap N, \dim\Gamma(\sigma)=d-1\} + \delta_{\dim\Gamma(0_N),d-1}$$

We further refine the statement of Theorem 3.4.5 using following lemma.

Lemma 3.4.6. *An integral point σ is on the interior of $2\nabla_1$ if and only if $\sigma\in\nabla_1^{\geq 1}$.*

Proof. The polytope $2\nabla_1$ is defined to be the set of points σ so that

$$\langle\sigma,\Delta_1'\rangle \geq -2$$
$$\langle\sigma,\Delta_2'\rangle \geq 0.$$

An integral point σ is in the *interior* of $2\nabla_1$ if and only if these inequalities are strict, in other words,

$$\langle\sigma,\Delta_1'\rangle \geq -1$$
$$\langle\sigma,\Delta_2'\rangle \geq 1.$$

Hence σ is on the interior of $2\nabla_1$ if and only if σ is in $\nabla_1^{\geq 1} \cap N$. $\square$

From Lemma 3.4.6 we obtain an awkward but ultimately useful expression for $h^{1,d-2}(X)$.

$$h^{1,d-2}(X) = \#(\nabla_1^{\geq 1} \cap N) - (\#\{\sigma \in \nabla_1^{\geq 1} \cap N, \dim \Gamma(\sigma) = d-1\} + \delta_{\dim \Gamma(0_N), d-1})$$
$$+ \sum_{\substack{\sigma \in \nabla_1^{\geq 1} \cap N \\ \dim \Gamma(\sigma) = d-2}} \ell^*(\Gamma(\sigma)^\vee_{\Delta_1}) + \delta_{\dim \Gamma(0_N), d-2} \ell^*(\Gamma(0_N)^\vee_{\Delta_1}).$$

This can be reorganized into what is essentially a weighted count of points in $\nabla_1^{\geq 1}$ by assigning weight $w_\sigma = 1 + \ell^*(\Gamma(\sigma)^\vee_{\Delta_1})$ if $\dim \Gamma(\sigma) = d-2$, $w_\sigma = 0$ if $\dim \Gamma(\sigma) = d-1$ and $w_\sigma = 1$ otherwise. We will let $w_{0_N} = \delta_{\dim \Gamma(0_N), d-2} \ell^*(\Delta_2') - \delta_{\dim \Gamma(0_N), d-1}$. We obtain the count

$$h^{1,d-2}(X) = \sum_{\sigma \in \nabla_1^{\geq 1}} w_\sigma + w_{0_N}. \tag{3.10}$$

This expression has the added bonus of being obviously non-negative if $\dim \Gamma(0_N) \neq d-1$. One can argue pretty easily that in the case where $\dim \Gamma(0_N) = d-1$ that w_σ is nonzero for some $\sigma \in \nabla_1^{\geq 1}$.

3.4.3 Counting components of $\mathscr{P}(0)$

Now we will count the irreducible components in $\mathscr{P}(0)$ and show that there are precisely $h^{1,d-1}(X) + 1$ of them. Let s_1 be a general global section of $\mathscr{L}_1$, and let $Q = \{s_1 = 0\}$. The variety $\mathscr{P}(\lambda)$ is given by the intersection of Q with the vanishing locus of

$$s_2 = \sum_{\rho \in \Delta_2' \cap N} a_\rho \prod_{\sigma \in \nabla} z_\sigma^{\langle \rho, \sigma \rangle - \sigma^2_{\min}} - \lambda \prod_{\rho \in V(\nabla_2)} z_\rho.$$

Let b_σ be the largest integer so that $\langle \sigma, \rho \rangle \geq b_\sigma$ for every $\rho \in E_2$ and let $\Delta_2' = \mathrm{Conv}(E_2)$. Then we have that $\mathscr{P}(0)$ is given by the intersection of Q with the subvariety of X_∇

cut out by the equation

$$\sum_{\rho\in\Delta_2'\cap M} a_\rho \prod_{\sigma\in\nabla\cap(N\setminus 0_N)} z_\sigma^{\langle\rho,\sigma\rangle-\sigma^2_{\min}}$$
$$= \left(\prod_{\sigma\in\nabla\cap(N\setminus 0_N)} z_\sigma^{b_\sigma}\right)\left(\sum_{\rho\in\Delta_2'\cap M} a_\rho \prod_{\sigma\in\nabla\cap(N\setminus 0_N)} z_\sigma^{\langle\rho,\sigma\rangle-b_\sigma-\sigma^2_{\min}}\right)$$

If σ is in ∇_2 then there is some integral point $\rho\in\Delta_2'$ so that $\langle\rho,\sigma\rangle=-1$, so $\sigma_{\min}+b_\sigma=0$. Thus for a generic choice of a_ρ and $\lambda=0$, the vanishing locus $s_2=0$ decomposes as a union of hypersurfaces

$$\{z_\sigma=0:\sigma\in\nabla_1^{\geq 1}\}\cup\left\{\sum_{\rho\in\Delta_2'\cap M} a_\rho \prod_{\sigma\in\nabla\cap(N\setminus 0_N)} z_\sigma^{\langle\rho,\sigma\rangle-b_\sigma-\sigma^2_{\min}}=0\right\}$$

Note that there are situations where the second term is empty – this happens precisely when Δ_2' is a single point. Otherwise, the second component, which we will call D_{nt} (the non-toric component), is the compactification of the vanishing locus a Laurent polynomial f with Newton polytope Δ_2' to a hypersurface in X_∇. It is irreducible if $\dim\Delta_2'\neq 1$ and if $\dim\Delta_2'=1$, it has $\ell^*(\Delta_2')+1$ irreducible components. We formalize this in the following lemma.

Lemma 3.4.7. *Assume that $d\geq 4$.*

1. *If $\dim\Delta_2'=0$ then $D_{\mathrm{nt}}\cap Q$ is empty.*

2. *If $\dim\Delta_2'=1$, then $D_{\mathrm{nt}}\cap Q$ has $\ell^*(\Delta_2')+1$ irreducible components.*

3. *If $\dim\Delta_2'\geq 2$ then $D_{\mathrm{nt}}\cap Q$ is irreducible.*

Proof. We may assume that $\dim\Delta_2'\geq 1$, or else D_{nt} itself is empty as noted in the paragraph above the statement of the lemma. The dimension of $\mathrm{Conv}(\Delta_2',0_M)=\Delta_2$ is equal to the dimension of the Minkowski sum $\Delta_2'+\mathrm{Conv}(p,0_M)$ for $p\in\Delta_2'$. Thus

$$4\leq d=\dim(\Delta_1+\Delta_2'+\mathrm{Conv}(p,0_M))\leq\dim(\Delta_1+\Delta_2')+1,$$

since for any pair of polytopes P_1 and P_2, we have $\dim(P_1 + P_2) \leq \dim P_1 + \dim P_2$. Thus $\dim(\Delta_1 + \Delta_2') \geq d-1$. By assumption, $d \geq 4$, so we have that $\dim(\Delta_1 + \Delta_2') \geq 3$. Now we apply the Koszul complex associated to $Q \cap D_{\mathrm{nt}}$,

$$0 \to \mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}} - Q) \to \mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}}) \oplus \mathscr{O}_{X_\nabla}(-Q) \to \mathscr{O}_{X_\nabla} \to 0$$

and its corresponding spectral sequence $''E_r^{p,q}$ to compute $\mathrm{H}^0(\mathscr{O}_{D_{\mathrm{nt}} \cap Q})$. We know that $''E_\infty^{p,q} \implies \mathrm{H}^i(\mathscr{O}_{D_{\mathrm{nt}} \cap Q}[2])$, hence we have

$$''E_\infty^{0,2} \oplus'' E_\infty^{1,1} \oplus'' E_\infty^{2,0} \cong \mathrm{H}^0(\mathscr{O}_{D_{\mathrm{nt}} \cap Q})$$

The term $''E_1^{0,2} = \mathrm{H}^2(\mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}} - Q))$ vanishes by [17, Theorem 2.5], which shows that $\mathrm{H}^i(\mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}} - Q))$ vanishes for $i \leq 3$ (here we use the fact that $\dim(\Delta_1 + \Delta_2') \geq 3$. We make note of the fact that Δ_1 has no points on its relative interior, so $\mathrm{H}^i(\mathscr{O}_{X_\nabla}(-Q)) = 0$ for all $i > 0$ and thus $''E_1^{1,i} = \mathrm{H}^i(\mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}}))$. It is well known that $\mathrm{H}^i(\mathscr{O}_{X_\nabla}) = 0$ for $i > 0$. Thus the relevant terms on the E_1 page are

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{H}^2(\mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}})) & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathrm{H}^1(\mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}})) & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathrm{H}^0(\mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}})) & \longrightarrow & \mathrm{H}^0(\mathscr{O}_{D_v}) = \mathbb{C} \end{array}$$

Thus $''E_\infty^{0,2} = 0$ and $''E_\infty^{1,1} = \mathrm{H}^1(\mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}}))$. If $\dim \Delta_2' = 1$ then $h^0(\mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}})) = 0$ and $h^1(\mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}})) = \ell^*(\Delta_2')$, thus $h^0(\mathscr{O}_{D_{\mathrm{nt}} \cap Q}) = 1 + \ell^*(\Delta_2')$. If $\dim \Delta_2' = 0$, then $h^1(\mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}})) = 0$ and $h^0(\mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}})) = 0$ and since $h^{-1}(\mathscr{O}_{D_{\mathrm{nt}} \cap Q}) = 0$, it follows that $''E_2^{2,0} ='' E_2^{1,1} = 0$. Finally, if $\dim \Delta_2' \geq 2$ then $h^i(\mathscr{O}_{X_\nabla}(-D_{\mathrm{nt}})) = 0$ for $i < 2$ and thus $h^0(\mathscr{O}_{D_{\mathrm{nt}} \cap Q}) = 1$. This proves the lemma. □

Note that this simply says that the number of irreducible components of $D_{\mathrm{nt}} \cap Q$ is equal to $w_{0_N} + 1$. Now we must compute the number of irreducible components of $\{z_\sigma = 0\} \cap Q$ for $\sigma \in \nabla_1^{\geq 1}$ in order to determine the total number of components of $\mathscr{P}(0)$. Recall that by Lemma 3.4.3 if $v \in \nabla_1^{\geq 1} \cap N$ and $\Gamma(v)$ is the minimal face of ∇_1 containing v then $\Gamma(v)_{\nabla^\circ}^\vee$ is equal to $\Gamma(v)_{\Delta_1}^\vee$ and thus has dimension $d - 1 - \dim \Gamma(v)$.

By [17, Proposition 6.6] the restriction of Q to $T_v \cong (\mathbb{C}^\times)^{d-1}$, the torus in X_∇ corresponding to v, then $Q \cap T_v$ is the vanishing of a Laurent polynomial with Newton polytope $\Gamma(v)^\vee_{\Delta_1}$.

Lemma 3.4.8. *1. If $\dim \Gamma(v) = d-1$ then $D_v \cap Q$ is empty*

2. If $\dim \Gamma(v) = d-2$ then $D_v \cap Q$ has $\ell^*(\Gamma(v)^\vee_{\Delta_1}) + 1$ components.

3. If $\dim \Gamma(v) > 1$ then $D_v \cap Q$ is irreducible.

Proof. There is a short exact sequence of sheaves whose associated long exact sequence in cohomology determines the cohomology of $\mathscr{O}_{D_v \cap Q}$,

$$0 \to \mathscr{O}_{D_v}(-Q \cap D_v) \to \mathscr{O}_{D_v} \to \mathscr{O}_{Q \cap D_v} \to 0.$$

According to [17, Theorem 2.5], $\mathrm{H}^1(D_v, \mathscr{O}_{D_v}(-Q \cap D_v)) = 0$ if $\dim \Gamma(v) \neq 1$ and $\ell^*(\Gamma(v))$ otherwise. If $\dim \Gamma(v) = 0$, then $\mathrm{H}^0(D_v, \mathscr{O}_{D_v}(-Q \cap D_v)) = \mathbb{C}$. This, along with the fact that $\mathrm{H}^0(D_v, \mathscr{O}_{D_v}) = \mathbb{C}$ and $\mathrm{H}^i(D_v, \mathscr{O}_{D_v}) = 0$ for $i \neq 0$ is enough to prove the lemma. Alternately, one could observe that if $\dim \Gamma(v) = 1$, then the polynomial determining $Q \cap T_v$ is a polynomial in one variable of degree $\ell^*(\Gamma(v)) + 1$ with general coefficients (once proper coordinates are chosen), hence it factors into $\ell^*(\Gamma(v)) + 1$ components. Similar elementary arguments can be made when $\Gamma(v)$ has dimension 0 or 2. □

By Lemma 3.4.8 and referring to Section 3.4.1 for notation, we then have that

$$\{z_\sigma = 0 : \sigma \in \nabla_1^{\geq 1}\} \cap Q$$

has

$$\sum_{\sigma \in \nabla_1^{\geq 1}} w_\sigma$$

irreducible components and that D_{nt} has $w_{0_N} + 1$ irreducible components. Therefore, by comparing this to Equation (3.10), we obtain the following result.

Theorem 3.4.9. *Let X be a partial crepant resolution of singularities of an ample hypersurface in a toric variety $\mathbb{P}_\Delta$ associated to a bipartite nef partition of Δ and let* $\dim \Delta \geq 4$. *Let (Y, w) be the LG model of X as constructed in Section 3.2.2. Then for any terminal log Calabi-Yau model of (Y, w),*

$$h^{1,d-2}(X) = \rho_0 + 1 \leq h^{1,1}_{\mathrm{st}}(Y, \mathsf{w}).$$

Remark 3.4.10. Theorem 3.4.9 is somewhat remarkable in the sense that it seems to imply that there is only one bad fiber of (Y, w) with more than one irreducible component, or else Hodge number mirror symmetry fails in the most basic case.

3.4.4 Del Pezzo examples illustrated

Here we give a visual description of how to compute the number of central fibers in the LG model of several del Pezzo surfaces. We denote for $n \leq 8$ by dP_n the del Pezzo surface obtained by blowing up $\mathbb{P}^2$ in n generic points.

Example 3.4.11 (dP_4)**.** Here we look at an example of the bidegree $(2, 1)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$. The polytope defining this toric variety is

$$\Delta = \mathrm{Conv}(\{e_1, e_2, e_3, -e_3, -e_1 - e_2\}).$$

If we choose the nef partition $E_1 = \{e_1, e_2, e_3\}$ and $E_2 = \{-e_3, -e_1 - e_2\}$, then the dual nef partition is given by

$$\nabla_1 = \mathrm{Conv}(\{-e_1^* - e_2^* - e_3^*, -e_1^* - e_2^*, -e_1^* - e_3^*, -e_2^* - e_3^*, -e_1^*, -e_2^*\})$$
$$\nabla_2 = \mathrm{Conv}(\{0_N, e_3^*, e_1^*, e_2^*, e_1^* + e_3^*, e_2^* + e_3^*\}).$$

These polytopes are given by the red and green dots respectively in Figure 3.1.

The set $\partial\nabla \cap N$ is made up of 16 points. The points in ∇_1 are given by

$$0_N, -e_1^*, -e_2^*, e_1^* - e_2^*, -e_1^* - e_2^*, -e_1^* + e_2^*$$

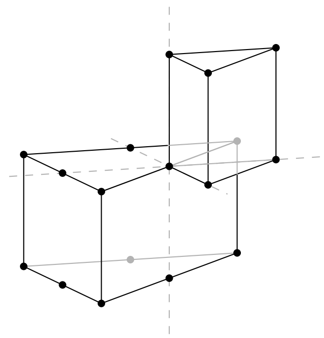

Fig. 3.1 The nef partition ∇_1, ∇_2 in Example 3.4.11

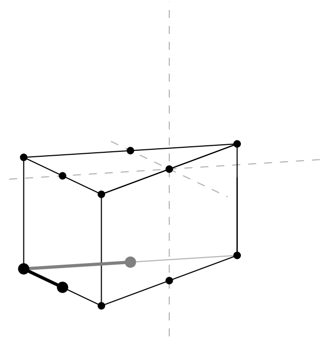

Fig. 3.2 Toric divisors contributing to the central fiber of the LG model of the del Pezzo surface of degree 4

and their translates by $-e_3^*$. We see that a point in this set satisfies $\langle \sigma, -e_3 \rangle \geq 1$ and $\langle \sigma, -e_1 - e_2 \rangle \geq 1$ if and only if $\sigma = -e_1^* - e_2^* - e_3^*, -e_1^* - e_3^*$, or $-e_2^* - e_3^*$. These points are marked in black in Figure 3.2. Since none of the faces of Δ contain interior points, and since none of the points in $\nabla_1^{\geq 1}$ in Figure 3.2 are on the relative interior of a facet of ∇_1, it follows that $w_\sigma = 1$ for all $\sigma \in \nabla_1^{\geq 1}$. Similarly, since 0_N does not lie on the interior of a facet of ∇_1 and since Δ has no faces with interior points, it follows that $w_{0_N} = 1$. Therefore $\rho_0 = 4$ for the LG model of dP_4.

We will now look at the example of the cubic in $\mathbb{P}^3$, which is nothing but dP_6. In many ways, this has been a guiding example in the writing of this chapter, since it exhibits several somewhat unexpected phenomena.

Example 3.4.12 (dP_6)**.** We begin with the polytope Δ which determines $\mathbb{P}^3$,

$$\Delta = \mathrm{Conv}(\{e_1, e_2, e_3, -e_1 - e_2 - e_3\})$$

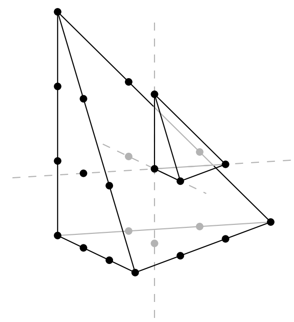

Fig. 3.3 The nef partition ∇_1, ∇_2 from Example 3.4.12

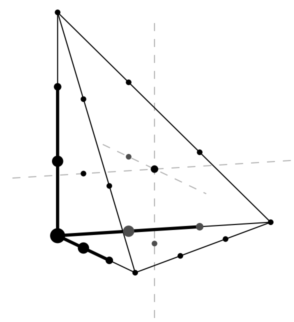

Fig. 3.4 Points of ∇_1 which are in $\nabla_1^{\geq 1}$ are marked with black dots.

and the nef partition $E_1 = \{e_1, e_2, e_3\}$ and $E_2 = \{-e_1 - e_2 - e_3\}$. The dual nef partition is given by

$$\nabla_1 = \mathrm{Conv}(\{-e_1^* - e_2^* - e_3^*, 2e_1^* - e_2^* - e_3^*, -e_1^* + 2e_2^* - e_3^*, -e_1^* - e_2^* + 2e_3^*\})$$
$$\nabla_2 = \mathrm{Conv}(\{0_N, e_1^*, e_2^*, e_3^*\}).$$

The points in the polytopes ∇_1 and ∇_2 are shown by the red and green vertices in Figure 3.3. Now we compute the number of components of $\mathsf{w}^{-1}(0)$ of the LG model of the cubic surface. The condition that $\langle \sigma, -e_1 - e_2 - e_3 \rangle \geq 1$ is simply that $\sigma = ae_1^* + be_2^* + ce_3^*$ for integers a, b, c so that $a + b + c \leq -1$. The points in ∇_1 which satisfy this criteria are marked by black dots in Figure 3.4. Now we compute the value w_σ for all of the black dots in Figure 3.4. There are three points, $-e_1^*, -e_2^*, -e_3^*$ which are contained in facets of ∇_1, thus for these points, $w_\sigma = 0$. In every other case, since Δ has no faces with integral points on their relative interiors, the value of w_σ is 1.

Thus

$$\sum_{\sigma \in \nabla_1^{\geq 1}} w_\sigma = 7.$$

Since 0_N is on a facet of ∇_1, we see that w_{0_N} is 0 and thus $\rho_0 = 7$.

Remark 3.4.13. Note that Theorem 3.4.9 is not claimed to hold when $\dim \Delta = 3$, as in the examples above. However, this is due to the fact that Lemma 3.4.7 may fail to hold in lower dimensions. One can check by hand that Lemma 3.4.7 holds in the examples above, and hence the computations that we have done are indeed valid.

Remark 3.4.14. Note that in both of these examples, we have that $\rho_0 - 1 = h^{1,1}_{\text{prim}}(X)$, where $\mathrm{H}^{1,1}_{\text{prim}}(X)$ is the orthogonal complement of the image of $\mathrm{H}^{1,1}(X_\Delta)$ in $\mathrm{H}^{1,1}(X)$. This was shown by Przyjalkowski and Shramov in [126] in the case where X is a complete intersection in $X_\Delta = \mathbb{P}^n$ of dimension 2.

Remark 3.4.15. The degeneration observed in the fiber over 0 of the LG model of the cubic surface has been observed in the literature on so-called 'tops' ([39, 24, 64, 27]). It would be interesting to know whether all of the degenerations of Calabi-Yau varieties obtained by the study of tops arise as the fibers over 0 of the LG model of some variety. This will be explored, to a certain extent, in future work.

3.5 Higher Noether-Lefschetz loci and homological mirror symmetry

As we have stated in the introduction, the general conjecture that we have set out to address is that if t is a point in $\mathbb{A}^1$ and $\mathsf{w} : Y \to \mathbb{A}^1$, and ρ_t is the number of irreducible components of $\mathsf{w}^{-1}(t)$, then

$$h^{1,n-1} = \sum_{t \in \mathbb{A}^1} (\rho_t - 1).$$

Somewhat surprisingly, we have shown that for any pullback of an ample hypersurface in a Gorenstein Fano toric variety, that

$$h^{1,n-1}(X) = \rho_0 - 1$$

which is a considerably stronger (and in a sense, weaker) statement. Under the assumption that mirror symmetry holds, this is equivalent to the fact that $\rho_t = 1$ for $t \neq 0$. In the case where $\mathbb{P}_\Delta$ is smooth, then we can justify this result using homological mirror.

3.5.1 Monodromy action on the cohomology of ample hypersurfaces

First we recall the following classical theorem:

Theorem 3.5.1 ([119, §3]). *Let $X \subseteq Z$ be a very ample hypersurface in a variety Z of dimension d and let $\mathrm{PH}^{d-1}(X, \mathbb{C})$ be the orthogonal complement of the image of the restriction map $\mathrm{H}^{d-1}(Z, \mathbb{C}) \to \mathrm{H}^{d-1}(X, \mathbb{C})$. Then the action of monodromy on $\mathrm{PH}^{d-1}(X, \mathbb{C})$ obtained by deforming X in Z is irreducible.*

Let X be an ample hypersurface in a smooth toric variety X_Σ of dimension d. Therefore, by e.g. [112, Corollary 2.15] X is actually very ample on X. Furthermore, since $h^{p,q}(X_\Sigma) = 0$ if $p \neq q$, it follows that monodromy acts irreducibly on a subspace of $\mathrm{H}^{d-1}(X, \mathbb{C})$ containing $\mathrm{H}^{1,d-2}(X)$ if $d \geq 4$.

Now we apply the features of homological mirror symmetry to understand the possible ramifications of this theorem. Assume that X is quasi-Fano, and it with the natural symplectic form ω associated to some ample class on X restricted from X_Σ, we may interpret the action of monodromy on X as a symplectomorphism of (X, ω). This in turn gives an action by a group of autoequivalences on the Fukaya category $\mathcal{F}(X, \omega)$ [59, 135]. It is expected that the Hochschild homology of $\mathcal{F}(X, \omega)$ is isomorphic to the quantum cohomology of (X, ω) and thus may be identified with the cohomology of X as a group. The action of $c_1(T_X)\star$ on $\mathrm{QH}^\bullet(X)$ where $\star$ denotes quantum product, decomposes $\mathrm{QH}^\bullet(X)$ into generalized eigenspaces $\mathrm{QH}^\bullet(X)_\lambda$ which are preserved by the action of monodromy. Thus there is at most one eigenspace of $\mathrm{QH}^\bullet(X)$ which intersects $\mathrm{H}^{1,d-2}(X)$ nontrivially.

Under homological mirror symmetry, the (derived) Fukaya category should be associated to the derived category of matrix factorizations $\mathbf{D}(\mathbf{MF}(Y,\mathsf{w}))$ [116]. This category is equivalent to $\cup_{\lambda\in\Sigma}\mathbf{D}^b_{\mathrm{sing}}(Y_\lambda)$ for Σ the critical locus of w, and $\mathbf{D}^b_{\mathrm{sing}}(Y_\lambda)$ the quotient of the derived category of coherent sheaves on $\mathsf{w}^{-1}(\lambda) = Y_\lambda$ by the derived category of perfect complexes on Y_λ. According to Katzarkov, Kontsevich and Pantev [79, pp. 10], the eigenspaces $\mathrm{QH}^\bullet(X)_\lambda$ correspond to $\mathrm{HH}_\bullet(\mathbf{D}^b_{\mathrm{sing}}(Y_\lambda))$. We know, by work of Efimov [53, Theorem 5.3] that $\mathrm{H}^\bullet(\mathbf{D}^b_{\mathrm{sing}}(Y_\lambda))$ is equal to $\mathbb{H}^\bullet(Y,(\Omega^\bullet_Y, d\mathsf{w}\wedge))$. Furthermore, we have that

$$\mathrm{HH}_\bullet(\mathbf{MF}(Y,\mathsf{w})) = \bigoplus_{\lambda\in\Sigma}\mathrm{HH}_\bullet(\mathbf{D}^b_{\mathrm{sing}}(Y_\lambda))$$

by work of Kuznetsov [92] and that

$$\mathbb{H}^\bullet(Y,(\Omega^\bullet_Y, d\mathsf{w}\wedge)) \cong \bigoplus_{\lambda\in\Sigma}\mathbb{H}^\bullet(Y_\lambda, \phi_{\mathsf{w}-\lambda}\mathbf{R}j_*\mathbb{C})$$

by Sabbah [131], Ogus and Vologodsky [115] and Barranikov and Kontsevich. Thus one expects that, for each $\lambda \in \Sigma$,

$$\mathrm{HH}_\bullet(\mathbf{D}^b_{\mathrm{sing}}(Y_\lambda)) = \mathbb{H}^\bullet(Y_\lambda, \phi_{\mathsf{w}-\lambda}\mathbf{R}j_*\mathbb{C}).$$

We can identify $\mathbb{H}^{i-1}(Y_\lambda, \phi_{\mathsf{w}-\lambda}\mathbf{R}j_*\mathbb{C})$ with a sub-space of $\mathrm{H}^i(Y,V;\mathbb{C})$ by a simple argument involving the Mayer-Vietoris sequence and the long exact sequences satisfied by both relative cohomology and the hypercohomology of sheaves of vanishing cycles. The Hodge filtration on $\mathrm{QH}^\bullet(X)$ should be identified with the filtration by degree in cohomology of $\mathrm{H}^\bullet(Y,V;\mathbb{C})$. Therefore, we expect that there is at most one $\lambda \in \Sigma$ so that $\mathbb{H}^1(Y_\lambda, \phi_{\mathsf{w}-\lambda}\mathbf{R}j_*\mathbb{C}) \neq 0$. An argument along the lines of the proof of Theorem 3.3.1 shows that $\operatorname{rank}\mathbb{H}^1(Y_\lambda, \phi_{\mathsf{w}-\lambda}\mathbf{R}j_*\mathbb{C}) = \rho_\lambda - 1$ and therefore there should exist only one point $\lambda \in \mathbb{A}^1$ so that $\mathsf{w}^{-1}(t)$ has more than one irreducible component.

3.5.2 Crepant resolutions

Thus one should expect that Theorem 3.4.9 holds in the case where $\mathbb{P}_\Delta$ is smooth or perhaps has at worst terminal singularities. It is somewhat interesting then that Theorem 3.4.9 holds when X is the pullback of an ample divisor onto a crepant resolution of $\mathbb{P}_\Delta$. In the case where $\mathbb{P}_\Delta$ is allowed to have worse than terminal singularities, the action on $\mathrm{H}^{d-1}_{\mathrm{prim}}(X, \mathbb{Q})$ obtained by letting the equation for X in X_Δ vary need not be irreducible. In fact, the term

$$\sum_{\Gamma \in \nabla_1[1]} \ell^*(\Gamma)\ell^*(\Gamma^\vee_\Delta)$$

computes the image of the Gysin homomorphism:

$$g : \bigoplus_{v \in \partial\Delta \cap M} \mathrm{H}^{0,d-3}(X \cap D_v) \to \mathrm{H}^{1,d-2}(X).$$

Since the Gysin homomorphism is a map of Hodge structures, if the term above is non-zero, then we can produce examples where the monodromy representation on $\mathrm{H}^{d-1}(X, \mathbb{C})$ obtained by letting X vary in X_Δ may have the property that there is more than one irreducible sub-representation intersecting $\mathrm{H}^{1,d-2}(X)$ nontrivially. We conjecture that the deformation space of X should contain the space of deformations of X which are hypersurfaces in X_Δ as a subspace of codimension greater than one. We show that this is true in the following example.

Example 3.5.2. Let us take the threefold X in $\mathbb{P}(1, 1, 2, 2, 2)$ of degree 6. This can be written as a toric variety with polytope whose vertices $\rho_1, \dots, \rho_5$ are given by the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

If we take the nef partition $E_1 = \{\rho_2, \rho_3, \rho_4, \rho_5\}, E_2 = \{\rho_1\}$, then sections of $\mathscr{L}_1$ are homogeneous polynomials of degree 6. We can check that the edge between ρ_4 and ρ_5

contains the integral point $(\rho_4 + \rho_5)/2$ on its relative interior, and that the dimension 2 face $\Gamma^\vee_{\nabla_1}$ of ∇_1 which is dual to the face Γ with vertices ρ_4, ρ_5 is

$$\Gamma^\vee_{\nabla_1} = \mathrm{Conv}(\{(0,2,-1,-1),(0,-1,2,-1),(3,-1,-1,-1)\})$$

which contains $(1,0,0,-1)$ on its relative interior. Therefore, $\ell^*(\Gamma)\ell^*(\Gamma^\vee_{\nabla_1}) = 1$. One can compute that if X is a resolution of singularities of the vanishing locus of a section of $\mathscr{L}_1$ then X has $h^{1,2}(X) = 18$ and $h^{1,1}(X) = 2$. If we take E to be the intersection of the divisor of X_Δ associated to the point $(-1,-1,-1,0)$ then a resolution of singularities $\widetilde{E}$ of E has $h^{1,0}(\widetilde{E}) = 1$, and the Gysin map $\mathrm{H}^{1,0}(\widetilde{E}) \to \mathrm{H}^{1,2}(X)$ is non-zero. Thus $\mathrm{H}^3(X,\mathbb{Z})$ contains as a sub-Hodge structure the Hodge structure of an elliptic curve, and hence the monodromy action on $\mathrm{H}^3(X,\mathbb{Z})$ obtained by letting the equation cutting out X in X_Δ vary cannot be absolutely irreducible.

However, just as in the case of Calabi-Yau hypersurfaces [97, 35] in $\mathbb{P}(1,1,2,2,2)$, one sees that not all complex deformations of X are obtained from deforming the equation for X in X_Δ. There are so-called non-polynomial deformations of X obtained by embedding X_Δ into a higher dimensional toric variety. In this situation, following Example 3.1 of [97], we may embed X_Δ as a complete intersection in $\mathbb{P}^5 \times \mathbb{P}^1$ cut out by equations

$$y_2 z_0 = y_0 z_1 \qquad\qquad y_1 z_1 = y_0 z_0$$

where z_0, z_1 are variables on $\mathbb{P}^1$ and $\mathbb{P}^5$ has variables $y_0, \dots, y_5$. Then X is cut out by a generic bidegree $(3,0)$ polynomial, thus X deforms to a complete intersection of bidegree $(3,0), (1,1)$ and $(1,1)$ divisors in $\mathbb{P}^5 \times \mathbb{P}^1$.

The subvariety $T = \{y_0 = y_1 = y_2 = 0\}$ is the contained in X_Δ and is the toric boundary divisor associated to $(-1,-1,-1,0)$ in X_Δ. We see that $X \cap T$ is a $(3,0)$ divisor in $\mathbb{P}^2 \times \mathbb{P}^1$, hence this subvariety is $E \times \mathbb{P}^1$ for E an elliptic curve and has $h^{1,0}(T \cap X) = 1$. We take the projection of X into $\mathbb{P}^5$ and note that a subvariety W of X is contracted if and only if $W = E \times \mathbb{P}^1$. The contracted locus of projection of X_Δ onto $\mathbb{P}^5$ is precisely the contraction of $T = \mathbb{P}^2 \times \mathbb{P}^1$ onto $\mathbb{P}^2$. A general deformation of X is the intersection of a general $(3,0)$ divisor and a variety Y that can be written in the form

$$y_2 z_0 = y_0 z_1 \qquad\qquad y_1 z_1 = z_0 \ell(y_0, \dots, y_5)$$

for some linear form $\ell(y_0, \dots, y_5)$. Thus the contracted locus of Y under the projection onto $\mathbb{P}^5$ is simply the projection of $T' = \{y_0 = y_1 = y_2 = \ell(y_0, \dots, y_5) = 0\} \cong \mathbb{P}^1 \times \mathbb{P}^1$ onto $\mathbb{P}^1$. Hence in a general deformation of X in $\mathbb{P}^5 \times \mathbb{P}^1$, the projection onto $\mathbb{P}^5$ contracts just three rational curves to ordinary double points. This is not a complete intersection of ample divisors, but we still expect that the generic complete intersection of this type has $\mathrm{PH}^3(X, \mathbb{Z})$ with an irreducible Hodge structure.

Based on this and Theorem 3.4.9, we conjecture that

Conjecture 3.5.3. *If X is a general deformation of a crepant resolution of an ample hypersurface in a d-dimensional toric Gorenstein Fano variety, then there at most one sub Hodge structure H of $\mathrm{H}^{d-1}_{\mathrm{prim}}(X, \mathbb{Z})$ so that $H \otimes \mathbb{C} \cap \mathrm{H}^{1,d-2}(X)$ is non-empty.*

3.5.3 Noether-Lefschetz loci and deformations of pairs

We will now prove a theorem which supports Conjecture 3.5.3. Let us begin with a weak Fano variety X of dimension d so that there exists a smooth section W of the anticanonical bundle of X. Then we know by a theorem of Ran [128] (extended by Sano [132] to the case where W need not be smooth) that the deformations of X are unobstructed and hence there exists a versal deformation of X over a small disc U of dimension $\mathrm{H}^1(X, \Theta_X)$. Let V be a $\mathbb{Z}$-module isomorphic to $\mathrm{H}^d_{\mathrm{prim}}(X, \mathbb{Z})/\mathrm{H}^d_{\mathrm{prim}}(X, \mathbb{Z})_{\mathrm{tors}}$ equipped with Q, the cup product pairing on V and let $h^{p,q} = h^{p,q}_{\mathrm{prim}}(X)$. According to Griffiths [65], we can write down a period space $\mathscr{P}\mathrm{er}(V, Q, h^{p,q})$ which parametrizes polarized Hodge structures of weight d on V for which Q is a polarization and whose Hodge numbers are equal to $h^{p,q}$. Then for the versal deformation $\mathscr{X} \to U$ there is a holomorphic period map $\phi : U \to \mathscr{P}\mathrm{er}(V, Q, h^{p,q})$.

If there is a sub module H of $\mathrm{H}^d_{\mathrm{prim}}(X, \mathbb{Z})$ then there is a sublocus $\mathrm{NL}(H)$ where each Hodge structure corresponding to a point of $\mathrm{NL}(H)$ restricts to induce a Hodge structure on H. This is called a (higher) Noether-Lefschetz locus of $\mathscr{P}\mathrm{er}(V, Q, h^{p,q})$. If we have an embedding $E \hookrightarrow X$ inducing a map $\mathrm{H}^{d-2}(E, \mathbb{Z}) \to \mathrm{H}^d(X, \mathbb{Z})$, then the image of this map forms a sub-Hodge structure H_E of $\mathrm{H}^d(X,\mathbb{Z})$. If we have a diagram

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[dr] & \mathscr{X} \arrow[d] \\ & U \end{tikzcd}$$

so that $\mathscr{E} \to U$ is a complex deformation of E, $\mathscr{X} \to U$ is a complex deformation of X and the map $\mathscr{E} \to \mathscr{X}$ is a smooth embedding so that over 0 this restricts to the embedding $E \to X$, then there is a map $\mathrm{H}^d(\mathscr{E}_t,\mathbb{Z}) \to \mathrm{H}^d(\mathscr{X}_t,\mathbb{Z})$ for all $t \in U$. The image of the period map $U \to \mathscr{P}\mathrm{er}(V,Q,h^{p,q})$ is thus contained in $\phi^{-1}(\mathrm{NL}(H_E))$. If Conjectue 3.5.3 holds, this means that if X is a crepant resolution of an ample quasi-Fano hypersurface in a Gorenstein toric Fano variety, then there is no Noether-Lefschetz locus with $h^{1,d-1} \geq 1$ containing $\phi(U)$ for a generic deformation of X. In particular if E is the exceptional divisor of a crepant contraction of X and the image of the map $\mathrm{H}^{1,d}(E) \to \mathrm{H}^{1,d}(X)$ has nontrivial image, we should be able to deform X so that the map $E \to X$ does not deform with X.

Theorem 3.5.4. *Let X be a smooth weak Fano variety of dimension d with smooth anticanonical section W, and let $f : X \to X'$ be a crepant map which contracts a smooth divisor E in X to a smooth subvariety B of codimension 2 in X'. Then the deformation space of the pair* $\mathrm{Def}(X,E)$ *is of codimension at least equal to the dimension of the image of the restriction map* $\mathrm{H}^1(X,\Omega_X^{d-1}) \to \mathrm{H}^1(E,\omega_E)$.

Proof. The tangent space of $\mathrm{Def}(X)$ is isomorphic to $\mathrm{H}^1(X,\Theta_X)$ and there is a sheaf $\Theta_X\langle E\rangle$ so that $\mathrm{H}^1(X,\Theta_X\langle E\rangle)$ is isomorphic to the tangent space of $\mathrm{Def}(X,E)$. By [128], the deformation space of X is unobstructed. Here, $\Theta_X\langle E\rangle$ can be described as the kernel of the map $\Theta_X \to N_{E/X}$ (see e.g. [21]). By the adjunction formula, we have that $\omega_E = (\omega_X|_E \otimes \mathscr{O}_E(E))$ which is nothing but $\omega_X|_E \otimes N_{E/X}$. Therefore, $N_{E/X} = \omega_E \otimes \omega_X^{-1}|_E$. We now write down the exact sequence

$$0 \to \Theta_X\langle E\rangle \to \Theta_X \to \omega_E \otimes \omega_X^{-1}|_E \to 0.$$

Recall that if W is a smooth effective Cartier divisor on X cut out by a section $s \in \Gamma(\mathscr{O}_X(W))$, then there is another short exact sequence of sheaves

$$0 \to \mathscr{O}_X(-W) \xrightarrow{s} \mathscr{O}_X \to \mathscr{O}_W \to 0.$$

We take the tensor product of the first short exact sequence with the short exact sequence of the second with s a section of ω_X^{-1} which determines W, then we get a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Theta_X\langle E\rangle \otimes \omega_X \arrow[r] \arrow[d] & \Theta_X \otimes \omega_X \arrow[r] \arrow[d] & \omega_E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Theta_X\langle E\rangle \arrow[r] \arrow[d] & \Theta_X \arrow[r] \arrow[d] & \omega_E \otimes (\omega_X^{-1}|_E) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Theta_X\langle E\rangle|_W \arrow[r] & \Theta_X|_W \arrow[r] \arrow[d] & \omega_E|_{E\cap W} \otimes \omega_X^{-1}|_{E\cap W} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We have that $\Theta_X \otimes \omega_X \cong \Omega_X^{d-1}$ and that the map $\Omega_X^{d-1} \to \omega_E$ is the pullback map. We obtain a commutative diagram

$$\begin{tikzcd}
 & \mathrm{H}^1(X, \Omega_X^{d-1}) \arrow[r] \arrow[d] & \mathrm{H}^1(E, \omega_E) \arrow[d] \\
\mathrm{H}^1(X, \Theta_X\langle E\rangle) \arrow[r, "r"] & \mathrm{H}^1(X, \Theta_X) \arrow[r] & \mathrm{H}^1(E, N_{E/X})
\end{tikzcd}$$

where r is the tangent map from $\mathrm{Def}(X, E)$ to $\mathrm{Def}(X)$. So we have only to show that $\mathrm{H}^1(E, \omega_E) \to \mathrm{H}^1(E, N_{E/X})$ is injective. If this is true, then the image of $\mathrm{H}^1(X, \Omega_X^{d-1})$ in $\mathrm{H}^1(E, \omega_E)$ maps into the image of $\mathrm{H}^1(X, \Theta_X)$ in $\mathrm{H}^1(E, N_{E/X})$ and thus the image of $\mathrm{H}^1(X, \Theta_X\langle E\rangle)$ in $\mathrm{H}^1(X, \Theta_X)$ has codimension as required.

Note that since f is crepant, the hypersurface W is the preimage of some $W' \subseteq X'$ where W' is a section of $\omega_{X'}$, thus $W \cap E$ is a union of fibers of $f|_E$. Furthermore, $f^*\omega_{X'}$ is, by definition, constant on fibers of $f|_E$, therefore $\omega_X^{-1}|_{V\cap E}$ is trivial on fibers of f. We have that E is a $\mathbb{P}^1$ bundle over B, thus the restriction of ω_E to each fiber of $f|_E$ is $\mathscr{O}_{\mathbb{P}^1}(-2)$. Thus the restriction of $\omega_E|_{E\cap W} \otimes \omega_X^{-1}|_{E\cap V}$ to any fiber of $f|_E$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(-2)$.

If we have a global section of $\omega_E|_{E\cap W} \otimes \omega_X^{-1}|_{E\cap W}$ then its restriction to all fibers of $f|_E$ is a section of $\mathscr{O}_{\mathbb{P}^1}(-2)$ and thus is 0. Therefore, there are no global sections of $\omega_E|_{E\cap W} \otimes \omega_X^{-1}|_{E\cap W}$. Therefore the map $\mathrm{H}^1(E, \omega_E) \to \mathrm{H}^1(E, N_{E/X})$ is injective as required. $\square$

Theorem 3.5.4 applies directly to the situation in Example 3.5.2 to show that if X is a crepant resolution of a degree 6 hypersurface in $\mathbb{P}(1,1,2,2,2)$, then the subspace of the versal deformation space of X corresponding to deformations of X in a crepant resolution of $\mathbb{P}(1,1,2,2,2)$ is of codimension at least 1. One may consider Theorem 3.5.4 as some sort of log Calabi-Yau version of [107, Theorem 6.5],[148, Proposition 4.1] or [67, Proposition 1.2].

3.5.4 Beyond crepant contractions

We used the fact that there is a crepant contraction $X \to X'$ contracting E in a very important way in Theorem 3.5.4. If X is not an ample hypersurface in a Gorenstein toric Fano variety and there is some toric boundary divisor D_v so that $\mathrm{H}^{0,d-3}(D_v \cap X) \to \mathrm{H}^{1,d-2}(X)$ is non-trivial, we should not expect that there is a deformation of X which does not extend to a deformation of $(D_v \cap X, X)$, since it may be that such divisors cannot be contracted crepantly. As we see in the following example, such situations do occur and they give rise to LG models with several fibers with more than one irreducible component.

Example 3.5.5. Let X be a cubic in $\mathbb{P}^4$. Then if we take a generic pair of hyperplanes H_1, H_2 in $\mathbb{P}^4$, then $X \cap H_1 \cap H_2$ is a smooth genus 0 curve in X, and the blow-up $\widetilde{X}$ of X along $X \cap H_1 \cap H_2$ is again a Fano variety. We may obtain X as a hypersurface in the blow up of $\mathbb{P}^4$ along $H_1 \cap H_2$ which is a toric Fano fourfold. Calling this blow-up $\widetilde{\mathbb{P}}^4$, the hypersurface $\widetilde{X}$ is nef and big in $\widetilde{\mathbb{P}}^4$ but clearly not ample, since it is the pullback of an ample divisor in $\mathbb{P}^4$. According to Mori and Mukai [100], all deformations of

$\widetilde{X}$ can be realized as a blow-up of a cubic threefold along an elliptic curve, thus the Hodge structure on the cohomology of $\widetilde{X}$ is reducible regardless of how we deform.

On the level of LG models, one can check explicitly that the fiber over 0 of the LG model of $\widetilde{X}$ constructed as in Section 3.2.2 has only 6 components, while $h^{1,2}(\widetilde{X})$ is clearly 6. A detailed computation shows that there is a fiber of (Y, w) over a point $\lambda \neq 0$ which contains two smooth components meeting in a curve of genus 0 which accounts for the remaining class in $h^{1,2}(\widetilde{X})$.

Thus Theorem 3.4.9 will undoubtedly just become an inequality if it is extended to the case where X is a semi-ample hypersurface in a Fano toric Gorenstein variety. The main obstruction to extending Theorem 3.4.9 to this case, however, is that I do not know how to compute $h^{1,d-2}(X)$.

Part II

Degenerations of Fano complete intersections and Laurent polynomials

Chapter 4

Laurent polynomials and degenerations of complete intersections

4.1 Introduction

Mirror symmetry for Fano varieties predicts that the mirror of a Fano variety X is given by a quasi-projective variety $X^\vee$ equipped with a regular function $w : X^\vee \to \mathbb{A}^1$ (with appropriate choices of symplectic and complex structure on both X and $X^\vee$) which satisfies certain conditions. In particual, homological mirror symmetry implies that there is a relationship between the bounded derived category of X and the Fukaya-Seidel category of $(X^\vee, w)$, or conversely, the Fukaya category of X is related to the derived category of singularities of $(X^\vee, w)$ (see, for instance, [78, 79] for details). The pair $(X^\vee, w)$ can be viewed as a family of varieties over $\mathbb{A}^1$. From a more classical point of view, mirror symmetry predicts that the periods of this family at infinity should be related to the Gromov-Witten invariants of X [54].

[0]This chapter is joint work with C.F. Doran. It has been accepted by the Canadian Journal of Mathematics under the title "Toric Degenerations and the Laurent polynomials related to Givental's Landau-Ginzburg models".

In the particular case where X is a smooth n-dimensional toric Fano variety, then there should be some copy of $(\mathbb{C}^\times)^n$ contained in $X^\vee$ so that on this torus, w is expressed as a Laurent polynomial

$$w : (\mathbb{C}^\times)^n \to \mathbb{C}$$

with Newton polytope equal to the polytope whose face fan from which X itself is constructed.

A folklore conjecture (stated explicitly in [61] Problem 44 and [123] Optimistic picture 38) says that for each birational map

$$\phi : (\mathbb{C}^\times)^n \dashrightarrow X^\vee$$

so that the Newton polytope of $\phi^* w$ is Δ, there is a degeneration of the Fano variety X to X_Δ. It is expected that $X^\vee$ is covered (away from a subset of codimension 2) by tori $(\mathbb{C}^\times)^n$ corresponding to toric varieties to which X degenerates, and these charts are related by a generalized type of cluster mutation. Our main result (Theorem 4.2.20) is very much in the spirit of this suggestion.

For X a complete intersection in a toric variety, Givental [62] provided a method of computing the Landau-Ginzburg model of X. This Landau-Ginzburg model is presented as complete intersection in $(\mathbb{C}^\times)^n$ which we call $X^\vee$ equipped with a function w. We call the pair $(X^\vee, w)$ obtained by Givental's method the *Givental Landau-Ginzburg model of* X.

In Section 4.2, we introduce certain types of embedded toric degenerations of Fano complete intersections in toric varieties which we call *amenable toric degenerations*, and prove that they correspond to Laurent polynomial models of Givental's Landau-Ginzburg models.

Theorem 4.1.1 (Theorem 4.2.20)**.** *Let X be a complete intersection Fano variety in a toric variety Y. Let $X \rightsquigarrow X_\Sigma$ be an amenable toric degeneration of X, then the*

Givental Landau-Ginzburg model of X can be expressed as a Laurent polynomial with Newton polytope equal to the convex hull of the primitive ray generators of Σ.

In the case where X is a smooth complete intersection in a weighted projective space, Przyjalkowski showed that there is a birational map $\phi : (\mathbb{C}^\times)^m \dashrightarrow X^\vee$ to the Givental Landau-Ginzburg model of X so that $\phi^* w$ is a Laurent polynomial. In [76], Ilten, Lewis and Przyjalkowski have shown that there is a toric variety X_Δ expressed as a binomial complete intersection in the ambient weighted projective space so that the complete intersection X admits a flat degeneration to X_Δ.

Theorem 4.2.20 generalizes both the method of Przyjalkowski in [123], and its subsequent generalization by Coates, Kasprzyk and Prince in [34]. Theorem 4.2.20 shows that there are toric degenerations corresponding to all of the Laurent polynomials associated to Fano fourfolds obtained in [34], and that the Laurent polynomials are the toric polytopes of the associated degenerations.

Using the toric degeneration techniques of [63] and [18], Przyjalkowski and Shramov have defined Givental Landau-Ginzburg models associated to complete intersection Fano varieties in partial flag varieties. They have shown that the Givental Landau-Ginzburg models of complete intersections in Grassmannians $\mathrm{Gr}(2, n)$ can be expressed as Laurent polynomials. We provide an alternate approach to this question using Theorem 4.2.20. This provides toric degenerations for most complete intersection Fano varieties in partial flag manifolds, and shows that we may express their Givental Landau-Ginzburg models as Laurent polynomials.

Theorem 4.1.2 (Theorem 4.3.4)**.** *Many Fano complete intersections in partial flag manifolds admit degenerations to toric weak Fano varieties X_Δ with at worst Gorenstein singularities and the corresponding Givental Landau-Ginzburg models may be expressed as Laurent polynomials with Newton polytope Δ.*

Of course, the word "many" will be explained in detail in Section 4.3, but as an example, this theorem encompasses all complete intersections in Grassmannians.

4.1.1 Organization

This paper will be organized as follows. In Section 4.2, we will recall facts about toric varieties and use them to prove Theorem 4.2.20. In Section 4.3, we will apply the results of Section 4.2 to exhibit toric degenerations of complete intersections Fano varieties in partial flag manifolds and show that their Givental Landau-Ginzburg models admit presentations as Laurent polynomials. In Section 4.4, we shall comment on further applications to the Przyjalkowski method of [34] and how our method seems to relate to the construction of toric geometric transitions as studied by Mavlyutov [98] and Fredrickson [56].

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4.2 General results

Here we describe the relationship between degenerations of complete intersections in toric Fano varieties with nef anticanonical divisor and their Landau-Ginzburg models. We will begin with a rapid recollection of some facts about toric varieties. A general reference for all of these facts is [36].

4.2.1 Toric facts and notation

Throughout, we will use the convention that M denotes a lattice of rank n, and N will be $\mathrm{Hom}(M, \mathbb{Z})$. We denote the natural bilinear pairing between N and M by

$$\langle -, - \rangle : N \times M \to \mathbb{Z}.$$

The symbol Σ will denote a complete fan in $M \otimes_{\mathbb{Z}} \mathbb{R}$, and X_Σ or Y_Σ will be used to denote the toric variety associated to the fan Σ. We will let Δ be a convex polytope in $M \otimes_{\mathbb{Z}} \mathbb{R}$ with all vertices of Δ at points in M, which contains the origin in its interior.

We will let Σ_Δ be the fan over the faces of the polytope Δ, and we will also denote the toric variety X_{Σ_Δ} by X_Δ. If Δ is an integral convex polytope, then we will let $\Delta[n]$ be the set of dimension n strata of Δ. In particular, denote by $\Delta[0]$ the vertices of Δ. We will abuse notation slightly and let $\Sigma[1]$ be the set of primitive ray generators of the fan Σ. Similarly, if C is a cone in Σ, then $C[1]$ will denote the set of primitive ray generators of C.

There is a bijection between primitive ray generators of Σ and the torus invariant Weil divisors on X_Σ.

If X_Σ is a toric variety, then X_Σ has a Cox homogeneous coordinate ring which is graded by $G_\Sigma = \operatorname{Hom}(\mathrm{A}_{n-1}(X_\Sigma), \mathbb{C}^\times)$. There is a short exact sequence

$$0 \to N \xrightarrow{g} \mathbb{Z}^{\Sigma[1]} \to \mathrm{A}_{n-1}(X_\Sigma) \to 0$$

where the map g assigns to a point $u \in N$ the vector

$$(\langle u, \rho \rangle)_{\rho \in \Sigma[1]}.$$

Elements of $\mathbb{Z}^{\Sigma[1]}$ are in bijection with torus invariant Weil divisors and the map g assigns to a torus invariant Weil divisor on X_Σ its class in the Chow group $\mathrm{A}_{n-1}(X_\Sigma)$.

Applying the functor $\operatorname{Hom}(-, \mathbb{C}^\times)$ to the above short exact sequence, we get a sequence

$$1 \to G_\Sigma \to (\mathbb{C}^\times)^{\Sigma[1]} \to T_M \to 1$$

where $T_M = M \otimes_{\mathbb{Z}} \mathbb{C}^\times$. Let x_ρ be a standard basis of rational functions on $(\mathbb{C}^\times)^{\Sigma[1]}$. There is a partial compactification of $(\mathbb{C}^\times)^{\Sigma[1]}$, which we may call V_Σ

$$(\mathbb{C}^\times)^{\Sigma[1]} \subseteq V_\Sigma \subseteq \mathbb{C}^{\Sigma[1]}.$$

so that there is an induced action of G_Σ on V_Σ and linearizing line bundle so that the categorical quotient $V_\Sigma//G_\Sigma$ is the toric variety X_Σ. Since we have assumed that Σ is complete, the homogeneous coordinate ring of X_Σ is $\mathbb{C}[\{x_\rho\}_{\rho\in\Sigma[1]}]$ equipped with the grading given by the action of G_Σ. A subvariety of X_Σ is a complete intersection in X_Σ if it corresponds to a quotient of a complete intersection in V_Σ.

The sublocus of X_Σ corresponding to $D_\rho = \{x_\rho = 0\}$ is exactly the torus invariant divisors associated to the ray generator ρ. Despite being given by the vanishing of a function in the homogeneous coordinate ring, these divisors need not be Cartier. A torus invariant divisor $D = \sum_{\rho\in\Sigma[1]} a_\rho D_\rho$ is Cartier if and only if there is some piecewise linear function φ on $M \otimes_{\mathbb{Z}} \mathbb{R}$ which is linear on the cones of Σ, which takes integral values on M. If ϕ is upper convex then the divisor D is nef.

The canonical divisor of a toric variety X_Σ is the divisor $K_{X_\Sigma} = -\sum_{\rho\in\Sigma[1]} D_\rho$. A toric variety X_Σ is called $\mathbb{Q}$-Gorenstein if its canonical divisor is $\mathbb{Q}$-Cartier, and Gorenstein if its canonical divisor is Cartier. In the future, we will be concerned solely with $\mathbb{Q}$-Gorenstein toric varieties.

A *nef partition* of Σ will be a partition of $\Sigma[1]$ into sets $E_1, \dots, E_{k+1}$ so that there exist integral upper convex piecewise linear functions $\varphi_1, \dots, \varphi_{k+1}$ so that

$$\varphi_i(E_j) = \delta_{ij}.$$

This means that for each maximal cone C in the fan Σ, there is some vector $u_C \in N$ so that

$$\varphi_i(v) = \max\{\langle u_C, v\rangle\}_{C\in\Sigma}$$

A $\mathbb{Q}$-*nef partition* will be a partition of $\Sigma[1]$ exactly as above, except we no longer require that the functions φ_i be integral, but only that they take rational values on $u \in M$. This is equivalent to the fact that each φ_i is determined by a vector $u_C \in N \otimes_{\mathbb{Z}} \mathbb{Q}$ for each maximal cone C of Σ.

The divisors determined by a $\mathbb{Q}$-nef partition are $\mathbb{Q}$-Cartier. Note that the existence of a $\mathbb{Q}$-nef partition implies that X_Σ is $\mathbb{Q}$-Gorenstein and the existence of a nef partition implies that X_Σ is Gorenstein.

4.2.2 Amenable collections of vectors

We begin by letting X be a complete intersection in a toric variety Y_Σ of the following type.

Definition 4.2.1. We will say that X is a *quasi-Fano complete intersection* in Y_Σ if there are divisors $Z_1, \dots, Z_k$ defined by homogeneous equations f_i in the homogeneous coordinate ring of Y_Σ so that $(f_1, \dots, f_k)$ forms a regular sequence in $\mathbb{C}[\{x_\rho\}_{\rho \in \Sigma[1]}]$, and there is a $\mathbb{Q}$-nef partition $E_1, \dots, E_{k+1}$ so that

$$Z_i \sim \sum_{\rho \in E_i} D_\rho.$$

We now define the central object of study in this paper. Fix a $\mathbb{Q}$-nef partition of Σ as $E_1, \dots, E_{k+1}$ and let X be a corresponding quasi-Fano complete intersection.

Definition 4.2.2. An *amenable collection of vectors* subordinate to a $\mathbb{Q}$-nef partition $E_1, \dots, E_{k+1}$ is a collection $V = \{v_1, \dots, v_k\}$ of vectors satisfying the following three conditions

(i) For each i, we have $\langle v_i, \rho \rangle = -1$ for every $\rho \in E_i$.

(ii) For each j so that $k+1 \geq j \geq i+1$, we have $\langle v_i, \rho \rangle \geq 0$ for every $\rho \in E_j$.

(iii) For each j so that $0 \leq j \leq i-1$, we have $\langle v_i, \rho \rangle = 0$ for every $\rho \in E_j$.

Note that this definition depends very strongly upon the order of $E_1, \dots, E_k$. Now let us show that an amenable collection of vectors may be extended to a basis of N.

Proposition 4.2.3. *An amenable set of vectors v_i spans a saturated subspace of N. In particular, there is a basis of N containing $v_1, \dots, v_k$.*

Proof. First of all, it is clear that $k \leq \operatorname{rank}(M)$, or else X would be empty. Now for each E_i, choose some $\rho_i \in E_i$. We then have a map

$$\eta : \operatorname{span}_{\mathbb{Z}}(v_1, \dots, v_k) \to \mathbb{Z}^k$$

determined by

$$\eta(v) = (\langle v, \rho_1 \rangle, \ldots, \langle v, \rho_k \rangle)$$

which, when expressed in terms of the basis $v_1, \ldots, v_k$ is upper diagonal with (-1) in each diagonal position and thus η is an isomorphism. If $\mathrm{span}_{\mathbb{Z}}(v_1, \ldots, v_k)$ were not saturated, then there would be some $v \in \mathrm{span}_{\mathbb{Q}}(v_1, \ldots, v_k) \cap N$ not in $\mathrm{span}_{\mathbb{Z}}(v_1, \ldots, v_k)$. But then $\eta(v)$ could not lie in $\mathbb{Z}^k$, which is absurd, since $v \in N$ and ρ_i are elements of M and by definition $\langle v, \rho_i \rangle \in \mathbb{Z}$.

Thus the embedding $\mathrm{span}_{\mathbb{Z}}(v_1, \ldots, v_k) \hookrightarrow N$ is primitive and there is a complementary set of vectors $v_{k+1}, \ldots, v_n$ so that $v_1, \ldots, v_n$ spans N over $\mathbb{Z}$. □

Now we will proceed to show that amenable collections of vectors lead naturally to a specific class of complete intersection toric subvarieties of Y_Σ.

4.2.3 Toric degenerations

Now let us define a toric variety associated to an amenable collection of vectors subordinate to a $\mathbb{Q}$-nef partition of a fan Σ.

Definition 4.2.4. Let V be an amenable collection of vectors subordinate to a $\mathbb{Q}$-nef partition of a fan Σ. Let M_V be the subspace of $M \otimes_{\mathbb{Z}} \mathbb{R}$ composed of elements which satisfy $\langle v_i, u \rangle = 0$ for each $v_i \in V$. Define the fan Σ_V to be the fan in M_V whose cones are obtained by intersecting the cones of Σ with M_V.

We may now look at the subvariety of Y_Σ determined by the equations

$$\prod_{\rho \in E_i} x_\rho - \prod_{\rho \notin E_i} x_\rho^{\langle v_i, \rho \rangle} = 0 \tag{4.1}$$

in its homogeneous coordinate ring for $1 \leq i \leq k$. Note that if X is a quasi-Fano complete intersection in Y_Σ determined by a $\mathbb{Q}$-nef partition $E_1, \ldots, E_k$, then the variety determined by the equations above is a degeneration of X in the sense that we can deform the equations defining X to the equations above. To see this, recall that the integral linear function which sends $\rho \in M_{\mathbb{R}}$ to $\langle v_i, \rho \rangle \in \mathbb{R}$ corresponds to a

Weil divisor

$$\sum_{\rho\in\Sigma[1]} \langle v_i,\rho\rangle D_\rho = -\sum_{\rho\in E_i} D_\rho + \sum_{\rho\notin E_i} \langle v_i,\rho\rangle D_\rho$$

which is linearly equivalent to 0 (see e.g. [36] Chapter 4, Proposition 1.2). Thus $\prod_{\rho\notin E_i} x_\rho^{\langle v_i,\rho\rangle}$ is a section of the line bundle $\mathcal{O}_{Y_\Sigma}(E_i)$ associated to E_i, and any global section of $\mathcal{O}_{Y_\Sigma}(E_i)$ can be deformed to Equation 4.1. Our goal is to show that the subvariety of Y_Σ that is determined by Equation 4.1 above is actually a complete intersection.

Following [55], we introduce a definition:

Definition 4.2.5. An integral square matrix is called mixed if each row contains both positive and negative entries. A $k \times m$ matrix is called mixed dominating if there is no square submatrix which is mixed.

Mavlyutov ([98] Corollary 8.3) packages Corollaries 2.4 and 2.10 of [55] into the following convenient form. If l is an integral vector in a lattice expressed in terms of a fixed basis, as $l = (t_1, \dots, t_n)$ then we define monomials $x^{(l_-)}$ and $x^{(l_+)}$ to be

$$x^{(l_+)} = \prod_{t_i>0} x_i^{t_i}, \quad x^{(l_-)} = \prod_{t_i<0} x_i^{t_i}.$$

Proposition 4.2.6. *Let $L = \bigoplus_{i=1}^k \mathbb{Z}l_i$ be a saturated sublattice of $\mathbb{Z}^m$ so that $L\cap\mathbb{N}^m = \{0\}$. Assume that the matrix with rows $l_1, \dots, l_k$ is mixed dominating, then the set of polynomials $(x^{(l_i)_+} - x^{(l_i)_-})$ for $i = 1, \dots, k$ forms a regular sequence in $\mathbb{C}[x_1, \dots, x_m]$*

Proposition 4.2.7. *If X is a Fano complete interesection in a toric variety Y_Σ, and V is an amenable collection of vectors associated to X, then V determine a degeneration of X to a complete intersection toric variety in the homogenous coordinate ring of Y_Σ cut out by equations*

$$\prod_{\rho\in E_i} x_\rho - \prod_{\rho\notin E_i} x_\rho^{\langle v_i,\rho\rangle} = 0.$$

Proof. The definition is clear, but what needs to be shown is that the subvariety of Y_Σ determined by these equations is a complete intersection. This is equivalent to the fact that the equations given in the statement of the proposition form a regular sequence. To check this, we prove that the conditions of Proposition 4.2.7 hold. The relevant matrix is the matrix with rows

$$(\langle v_i, \rho\rangle)_{\rho\in\Sigma[1]}.$$

Call this matrix T. If we choose some $\rho_j \in E_j$ for each $1 \le i \le k$, then the maximal submatrix $(\langle v_i, \rho_j\rangle)$ is upper triangular with (-1)s on the diagonal. Thus the rows of T form a saturated sublattice of $\mathbb{Z}^m$.

Now let us choose any square submatrix of T, or in other words, choose a set U of ℓ vertices of $\Sigma[1]$ and a set V of ℓ vectors v_i. Then we must show that the matrix $S = (\langle v_i, \rho\rangle)_{v_i\in V, \rho\in U}$ has a row without both positive and negative entries. If the row $(\langle v_i, \rho\rangle)_{\rho\in\Sigma[1]}$ contains negative entries for $V_0 = \{v_{i_1}, \ldots, v_{i_m}\} \subseteq V$, then in particular for each v_{i_j} there is some $\rho \in U$ contained in E_{i_j}. If $V_0 = V$, then it follows that for each ρ is contained in a distinct $E_{i_1}, \ldots, E_{i_m}$. If i_1 is the smallest such integer and ρ_1 is the corresponding element of U, then $\langle v_i, \rho\rangle \le 0$ for all $v_i \in V$, since $\langle v_i, E_j\rangle = 0$ for all $j < i$. Thus the corresponding row contains no positive entries.

Finally, if $L \cap \mathbb{N}^m$ is nonzero then there is some v_i so that $\langle v_i, \rho\rangle \ge 0$ for all vertices ρ of $\Sigma[1]$. If there were such a v_i, then all points ρ of $\Sigma[1]$ would be contained in the positive half-space determined by v_i, contradicting the fact that Σ is a complete fan with each cone strictly convex.

Thus applying Proposition 4.2.7, the equations in the proposition above determine a complete intersection in the homogeneous coordinate ring of Y_Σ. $\square$

Proposition 4.2.8. *The subvariety of X_{Σ_V} of Y_Σ corresponds to the complete intersection in the coordinate ring of Y_Σ cut out by equations*

$$\prod_{\rho\in E_i} x_\rho - \prod_{\rho\notin E_i} x_\rho^{\langle v_i,\rho\rangle} = 0.$$

for $1 \le i \le k$.

Proof. We recall that there is an exact sequence

$$0 \to N \xrightarrow{g} \mathbb{Z}^{\Sigma[1]} \to \mathrm{A}_{n-1}(X_\Sigma) \to 0.$$

Here g is the map which sends a point $v \in N$ to the point

$$(\langle v, \rho \rangle)_{\rho \in \Sigma[1]} \in \mathbb{Z}^{\Sigma[1]}$$

Applying the functor $\mathrm{Hom}\,(-, \mathbb{C}^\times)$ to this exact sequence, we obtain a dual exact sequence

$$0 \to G_\Sigma \to (\mathbb{C}^\times)^{\Sigma[1]} \xrightarrow{g^*} M \otimes_{\mathbb{Z}} \mathbb{C}^\times \to 0$$

where $G_\Sigma = \mathrm{Hom}\,(\mathrm{A}_{n-1}(X_\Sigma), \mathbb{C}^\times)$. For an appropriate choice of basis $(x_\rho)_{\rho \in \Sigma[1]}$ the induced action of G_Σ on $(\mathbb{C}^\times)^{\Sigma[1]} \subseteq V_\Sigma$ determines the G_Σ-action on the homogeneous coordinate ring of X_Σ.

The equations by which we have defined X_Σ can be written on the torus $(\mathbb{C}^\times)^{\Sigma[1]}$ as

$$\prod_{\rho \in \Sigma[1]} x_\rho^{\langle v_i, \rho \rangle} = 1$$

for $1 \leq i \leq k$. But this corresponds exactly to the pullback of the locus $\langle v_i, - \rangle = 0$ in M, which is simply the subspace M_V. Thus, in the homogeneous coordinate ring of Y_Σ, the toric subvariety X_{Σ_V} is cut out by the equations given in the proposition. $\square$

Thus the amenable collection V determines a degeneration of a quasi-Fano complete intersection in Y_Σ to a toric variety X_{Σ_V} where Σ_V is the fan obtained by intersecting Σ with the subspace of M orthogonal to elements of V.

Definition 4.2.9. A toric degeneration $X \rightsquigarrow X_\Sigma$ of a quasi-Fano complete intersection determined by an amenable collection of vectors subordinate to a $\mathbb{Q}$-nef partition $E_1, \ldots, E_{k+1}$ is called an amenable toric degeneration of X subordinate to the $\mathbb{Q}$-nef partition $E_1, \ldots, E_{k+1}$.

Now we define a polytope depending upon the amenable collection of vectors $V = \{v_1, \ldots, v_k\}$.

Definition 4.2.10. Let V be an amenable collection of vectors subordinate to a $\mathbb{Q}$-nef partition $E_1, \ldots, E_{k+1}$ equipped with rational convex Σ-piecewise linear functions $\varphi_1, \ldots, \varphi_{k+1}$. Then we define Δ_V to be the polytope defined by $\rho \in M \otimes \mathbb{R}$ satisfying equations

$$\begin{aligned} \langle v_i, \rho \rangle = 0 \text{ for } 1 \leq i \leq k \\ \varphi_{k+1}(\rho) \leq 1. \end{aligned}$$

This polyhedron is convex. We will refer to the subspace of $M \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying $\langle v_i, \rho \rangle = 0$ for $1 \leq i \leq k$ for $V = \{v_1, \ldots, v_k\}$ an amenable collection of vectors as M_V.

It is first of all, important to show that Δ_V is precisely the polytope whose vertices are the generating rays of the fan $M_V \cap \Sigma_\Delta$.

Lemma 4.2.11. *Let C be a sub-cone of Σ so that $C \cap M_V$ is nonempty, then there is a vertex of C which is contained in E_{k+1}.*

Proof. Let p be an element of $(C \cap M_V) \cap M$ and choose a set of vectors $U = \{u_1, \ldots, u_m\}$ contained in generating set of the 1-dimensional strata of C so that p is a strictly positive rational linear combination of a set of vectors in U. Let j be the largest integer so that $E_j \cap U \neq \emptyset$ and $j \neq k+1$. If no such integer exists, then $U \subseteq E_{k+1}$ and we are done. If not, we have that $\langle v_j, u \rangle = 0$ or (-1) for all $u \in U \setminus E_{k+1}$, since $\langle v_j, E_i \rangle = 0$ for $i < j$. If $p' = \sum_{u_i \in U \setminus E_{k+1}} a_i u_i$ for positive numbers a_i, then $\langle v_j, p' \rangle = -\sum_{u_i \in E_j} a_i < 0$, since $E_j \cap U$ is nonempty. Thus, since $p = p' + \sum_{u_i \in U \cap E_{k+1}} a_i u_i$ and $\langle v_j, p \rangle = 0$, we must have $U \cap E_{k+1}$ nonempty. $\square$

The following proposition holds in the case where $E_1, \ldots, E_{k+1}$ is any $\mathbb{Q}$-nef partition of Δ and V is amenable collection of vectors subordinate to this nef partition.

Proposition 4.2.12. *Let C be a minimal sub-cone of Σ so that $C \cap M_V$ is 1-dimensional, then there is some point ρ in $(C \cap M_V) \cap M$ so that $\varphi_{k+1}(\rho) = 1$.*

Proof. By Lemma 4.2.11, we may deduce that the set $C[1]$ of primitive integral ray generators of C must contain an element of E_{k+1}. We must show that there is some

$p \in E_{k+1}$ and $u_1, \ldots, u_m \in C[1] \setminus (C[1] \cap E_{k+1})$ so that $\rho = (p + \sum_{i=1}^{m} n_i u_i) \in M_V$, $n_i > 0$. Then since $u_i \in \Sigma[1] \setminus E_{k+1}$ and φ_{k+1} is linear on C, we must have $\varphi_{k+1}(\rho) = 1$.

Let us take p in $E_{k+1} \cap C[1]$. Then assume $\langle v_j, p \rangle = n_j > 0$. If there is no $u \in C[1]$ so that $u \in E_j$ then $\langle v_j, u \rangle \geq 0$ for all $u \in C[1]$. The subset $\langle v_j, u \rangle = 0$ contains $C \cap M_V$ by definition and must be a sub-stratum of C since C is a convex rational cone. By minimality of C, we must then have $\langle v_j, u \rangle = 0$ for all $u \in C[1]$. In particular, $n_j = 0$ for all j so that $E_j \cap C[1] = \emptyset$.

Now we will fix $u_j \in E_j$ for each E_j so that $E_j \cap C[1] \neq \emptyset$. We know that $\langle v_j, u_j \rangle = -1$. Take the largest j so that $E_j \cap C[1] \neq \emptyset$. Then $\langle v_j, p + n_j u_j \rangle = 0$. Now we have that $p + n_j u_j$ is orthogonal to v_j and since it is a positive sum of elements in $C[1]$, it is contained in C. Let j' be the next smallest integer so that $E_{j'} \cap C[1] \neq \emptyset$. Then let $\langle v_{j'}, p + n_j u_j \rangle = s_{j'}$. This is a non-negative integer since p and u_j are not contained in $E_{j'}$. We now have $\langle v_{j'}, p + n_j u_j + s_{j'} u_{j'} \rangle = 0$ and $\langle v_j, p + n_j u_j + s_{j'} u_{j'} \rangle = s_{j'} \langle v_j, u_{j'} \rangle$ which is zero by the condition that $\langle v_j, \Delta_i \rangle = 0$ for $i < j$.

We may now sequentially add positive multiples of each u_ℓ for $E_\ell \cap C[1] \neq \emptyset$ in the same way until the resulting sum ρ is orthogonal to $v_1, \ldots, v_k$. Thus we obtain $\rho \in C$ which lies in M_V and satisfies $\varphi_{k+1}(\rho) = 1$ by arguments presented in the first two paragraphs of this proof. □

We may make an even stronger claim if we make further assumptions on the divisors associated to E_i.

Recall that we have been assuming that the Weil divisors $D_i = \sum_{\rho \in E_i} D_\rho$ are $\mathbb{Q}$-Cartier, or in other words, there are rational convex Σ-linear functions φ_i and for $\rho \in D_j$, $\varphi_i(\rho) = \delta_{ij}$. We can make stronger statements about the relationship between the fan Σ_V and the polytope Δ_V.

Proposition 4.2.13. *The point ρ in the proof of Proposition 4.2.12 is a primitive lattice point under either of the following two conditions:*

(i) The divisor D_{k+1} is Cartier, or

(ii) All divisors D_i for $1 \leq i \leq k$ are Cartier.

Proof. For (1), If D_{k+1} is Cartier, then the function φ_{k+1} is integral, and thus if $\varphi_{k+1}(\rho) = 1$ implies that ρ is a primitive lattice point.

In (2), assume that there is some r so that ρ/r is still in M. Recall that $\rho = p + \sum_{i=1}^{k} n_i u_i$ for some $u_i \in E_i$. Thus $\varphi_i(\rho/r) = n_i/r$ is an integer, since φ_i is integral and ρ/r is in M. Hence $\rho/r - \sum_{i=1}^{k}(n_i/r)u_i = p/r$ is in M. Since $p \in \Sigma[1]$ are assumed to be primitive, $r = 1$. □

Finally, this shows that

Corollary 4.2.14. *Under either of the conditions of Proposition 4.2.13, the polytope Δ' in M_V obtained as the convex hull of the rays generating $\Sigma_V = M_V \cap \Sigma$ is equal to Δ_V.*

Proof. By Proposition 4.2.12, each generating ray of Δ' lies inside of Δ_V, thus the convex hull of the generating rays of Σ_V is contained inside of Δ_V. If ρ is a vertex of Δ_V, then let C in Σ be the unique cone containing ρ on its interior, C^0. If $C_V = C \cap M_V$ then ρ is in C_V^0, the interior of C_V. Since φ_{k+1} is linear on C_V^0, we must have some substratum of Δ_V containing ρ on which φ_{k+1} is a linear function, but since ρ is a vertex of Δ_V, the only such substratum is spanned by ρ itself. Thus $C \cap M_V$ is the ray generated by ρ and ρ is in $\Sigma_V[1]$. Therefore all vertices ρ of Δ_V are in $\Sigma_V[1]$, and hence are primitive, so we can conclude that $\Delta' \subseteq \Delta_V$. □

It is well known (see e.g. Remark 1.3 of [12]) that if all facets of an integral polytope Δ are of integral height 1 from the origin, then Δ is reflexive. Thus:

Theorem 4.2.15. *Let X be a quasi-Fano complete intersection in a toric variety Y_Σ, and let $E_1, \dots, E_{k+1}$ be a $\mathbb{Q}$-nef partition of $\Sigma[1]$ so that E_{k+1} corresponds to a nef Cartier divisor. If $X \rightsquigarrow X_{\Sigma_V}$ is defined by an amenable collection of vectors V subordinate to this nef partition, then X_{Σ_V} is a weak Fano partial crepant resolution of a Gorenstein Fano toric variety X_{Δ_V}.*

Proof. The polytope over the ray generators of Σ_V is Δ_V by Corollary 4.2.14, which is reflexive by Remark 1.3 of [12]. It follows that the fan Σ_V is a refinement of the fan over faces of Δ_V obtained without adding rays which are not generated by points in Δ_V. By Lemma 11.4.10 of [36], it follows that X_{Σ_V} is a projective crepant partial toric resolution of X_{Δ_V} whose anticanonical model is X_{Δ_V}, thus is weak Fano. $\square$

4.2.4 Laurent polynomials

In [62], Givental constructs the mirror of a quasi-Fano complete intersection variety X in the following way. Assume that we have a smooth toric variety Y_Σ associated to a fan Σ with ray generators $\Sigma[1]$ and a nef partition $E_1, \ldots, E_{k+1}$. Then, as in Section 4.2.1, we have an exact sequence

$$0 \to N \xrightarrow{g} \mathbb{Z}^{\Sigma[1]} \xrightarrow{h} \mathrm{Pic}(Y_\Sigma) \to 0.$$

If we choose a facet σ of Σ, then the ray generators of σ generate M as a lattice, since Y_Σ is smooth. We may label the elements of $\Sigma[1]$ as $u_1, \ldots, u_n$ and the rest of the elements of $\Sigma[1]$ as $u_{n+1}, \ldots, u_d$. Let $v_1, \ldots, v_n$ be a basis of N which is dual to $u_1, \ldots, u_n$.

Then if we express the map g as a matrix in terms of the basis $v_1, \ldots, v_n$ of N,

$$g = \begin{pmatrix} \mathrm{Id}_n & G \end{pmatrix}$$

for an $n \times (d-n)$ matrix G whose columns express the coordinates of $u_{n+1}, \ldots u_d$ in terms of $u_1, \ldots, u_n$. Furthermore, h is given by

$$h = \begin{pmatrix} -G \\ \mathrm{Id}_{(d-n)} \end{pmatrix}.$$

We let $y_1, \ldots, y_d$ be coordinates on $(\mathbb{C}^\times)^d$ and we define the complete intersection

$$\begin{aligned} &\prod_{j=1}^{d} y_\rho^{h_{j,\ell}} = q_\ell \text{ for } 1 \leq \ell \leq d-n \\ &\sum_{u_j \in E_i} y_j = 1 \text{ for } i \neq k+1 \end{aligned} \tag{4.2}$$

with parameters $q_\ell \in \mathbb{C}^\times$. This complete intersection is equipped with the superpotential

$$w = \sum_{u_j \in E_{k+1}} y_j. \tag{4.3}$$

This is the Landau-Ginzburg model of X as defined by Givental in [62] or by Hori and Vafa in [74]. We may simplify this expression further given the form of h that we have deduced above. The relations

$$\prod_{j=1}^{n} y_j^{h_{j,\ell}} = q_\ell$$

can be rewritten as

$$y_\ell = \frac{q_\ell}{\prod_{j=1}^{n} y_i^{h_{j,\ell}}} \text{ for } n+1 \leq \ell \leq d \tag{4.4}$$

where $h_{i,\ell}$ are coordinates for u_i in terms of the basis $\Sigma[1]$ of M. Thus we can express the Landau-Ginzburg model of X as a complete intersection in the torus $(\mathbb{C}^\times)^n$ with coordinates $\{y_i\}_{i=1}^n$ by substituting Equation 4.4 into Equations 4.2 and 4.3.

We are mainly interested in the case where Y_Σ is not necessarily a smooth toric variety, so we will give a different description of this construction based on the discussion above, but which does not require any smoothness properties from Y_Σ.

Let X be a quasi-Fano complete intersection in the toric variety Y_Σ obtained from a $\mathbb{Q}$-nef partition $E_1, \dots, E_{k+1}$. Let us take the usual Laurent polynomial ring in n variables, $\mathbb{C}[x_1^\pm, \dots, x_n^\pm]$ and let x^ρ be the map $\mathrm{Hom}\,(N \otimes \mathbb{C}^\times, \mathbb{C}^\times)$ associated to $\rho \in M$ which assigns to $q \otimes p \in M \otimes \mathbb{C}^\times$ the value $p^{\langle q, \rho\rangle}$. In particular, if $u_1, \dots, u_n$ is a basis for N, then we may represent it as a Laurent monomial

$$x^\rho = \prod_{i=1}^{n} x_i^{\langle u_i, \rho\rangle}.$$

The Givental Landau-Ginzburg model associated to a nef partition $E_1, \ldots, E_{k+1}$ of $\Sigma[1]$ is given by the complete intersection $X^\vee$ in $(\mathbb{C}^\times)^n$ written as

$$\sum_{\rho \in E_i} a_\rho x^\rho = 1 \tag{4.5}$$

for $1 \leq i \leq k$ equipped with the superpotential

$$w = \sum_{\rho \in E_{k+1}} a_\rho x^\rho.$$

Here a_ρ are constants in $\mathbb{C}^\times$. To be completely correct, the constants a_ρ should be chosen to correspond to complexified classes in the nef cone of X_Σ. In other words, there should be some integral Σ-piecewise linear function φ and a complex constant t so that

$$a_\rho = t^{\varphi(\rho)}.$$

(see [5] or [15] for details).

The goal of this section is to show that the existence of an amenable collection of vectors subordinate to the $\mathbb{Q}$-nef partition $E_1, \ldots, E_{k+1}$ implies that $X^\vee$ is birational to $(\mathbb{C}^\times)^{n-k}$ and that under this birational map, the superpotential w pulls back to a Laurent polynomial. In Section 4.2.5 we will determine the relationship between the Laurent polynomial w and the polytope Δ_V.

Let $v_1, \ldots, v_k$ be an amenable collection of vectors. Then by Proposition 4.2.3, we may extend $v_1, \ldots, v_k$ to a basis $v_1, \ldots v_n$ of N. Let us fix such a basis once and for all.

Now we can rewrite Equation 4.5 in terms of this basis as

$$\sum_{\rho \in E_i} a_\rho x_\rho = \sum_{\rho \in E_i} a_\rho \left(\prod_{j \geq i} x_i^{\langle v_i, \rho \rangle} \right) = 1. \tag{4.6}$$

since $\langle v_i, \rho\rangle = 0$ for $j < i$. Note that each monomial in this expression can be written as

$$\prod_{j \geq i} x_i^{\langle v_i, \rho\rangle} = \left(\frac{1}{x_i}\right) \prod_{j > i} x_i^{\langle v_i, \rho\rangle}.$$

since $\langle v_i, \rho\rangle = -1$ for any ρ in E_i. Therefore we may rearrange Equation 4.6 to get

$$\sum_{\rho \in E_i} a_\rho \left(\prod_{j > i} x_i^{\langle v_i, \rho\rangle} \right) = x_i. \tag{4.7}$$

By assumption, we have that $\langle v_i, \rho\rangle \geq 0$ for $\rho \in E_j$ and $k \geq j > i$, so $x_{i+1}, \ldots x_k$ appear to non-negative degrees in Equation 4.7. We now sequentially substitute these expressions into one another in order to get Laurent polynomial expressions for each of $x_1, \ldots, x_k$ in terms of $x_{k+1}, \ldots, x_n$. First note that in the case $i = k$, Equation 4.7 directly provides such a Laurent polynomial. Applying induction, we assume that each of $x_{i+1}, \ldots, x_k$ is given as a Laurent polynomial in $x_{k+1}, \ldots, x_n$, then direct substitution into Equation 4.7 expresses x_i as a Laurent polynomial in terms of $x_{k+1}, \ldots, x_n$. Note that the fact that the exponents of $x_{i+1}, \ldots, x_k$ are positive in Equation 4.7 is used crucially at this step.

The expressions that we obtain for each $x_i, 1 \leq i \leq k$ by this procedure will be called $f_i(x_{k+1}, \ldots, x_n)$. Now we have that, expressed as a function on $(\mathbb{C}^\times)^n$,

$$w = \sum_{\rho \in E_{k+1}} \left(\prod_{i=1}^{n} x_i^{\langle v_i, \rho\rangle} \right)$$

has $x_1, \ldots, x_k$ appearing only to positive degrees since $v_1, \ldots, v_k$ satisfy $\langle v_i, u\rangle \geq 0$ for each $u \in E_{k+1}$. Making substitutions $x_i = f_i(x_{k+1}, \ldots, x_n)$ for each $1 \leq i \leq k$ into w, we obtain a Laurent polynomial for w on the variables $x_{k+1}, \ldots, x_n$. We summarize these computations as a theorem.

Theorem 4.2.16. *Assume X is a quasi-Fano complete intersection in a toric variety Y_Σ. Then for each amenable toric degeneration $X \rightsquigarrow X_{\Sigma'}$ there is a birational map*

$$\phi_V : (\mathbb{C}^\times)^{n-k} \dashrightarrow X^\vee$$

so that $\phi_V^* w$ is a Laurent polynomial.

Proof. Let $f_i(x_{k+1}, \dots, x_n)$ be the expressions for x_i obtained by using the algorithm described above. We define the map ϕ_V as

$$\phi_V(x_{k+1}, \dots, x_n) = (f_1(x_{k+1}, \dots, x_n), \dots, f_k(x_{k+1}, \dots, x_n), x_{k+1}, \dots, x_n).$$

Of course, as a map from $(\mathbb{C}^\times)^{n-k}$ to $(\mathbb{C}^\times)^n$, this map is undefined when $f_i(x_{k+1}, \dots, x_n)$ is equal to 0 for $1 \leq i \leq k$, which is a Zariski closed subset of $(\mathbb{C}^\times)^{n-k}$. We have shown above that ϕ_V has image which lies inside of $X^\vee$, thus since $\dim(X^\vee) = n - k$, the map ϕ_V is a birational map from $(\mathbb{C}^\times)^{n-k}$ to $X^\vee$. $\square$

Thus the choice of an amenable set of vectors $v_1, \dots, v_k$ determines both a toric degeneration of X and a Laurent polynomial expression for its Landau-Ginzburg model. In Section 4.2.5 we will examine the relationship between the Laurent polynomial $\phi_V^* w$ and the polytope Δ_V.

4.2.5 Comparing polytopes

Now we will show that if $E_1, \dots, E_{k+1}$ is a $(k+1)$-partite $\mathbb{Q}$-nef partition of a fan Σ and V is an amenable collection subordinate to this $\mathbb{Q}$-nef partition, then the Newton polytope of $\varphi_V^* w$ is precisely Δ_V. Let $\Delta_{\phi_V^* w}$ be the Newton polytope of the Laurent polynomial $\phi_V^* w$.

Take any subset $S \subseteq M$, then if we choose $v \in N$, we get stratification of S by the values of $\langle v, - \rangle$. We will define subsets of S

$$S_v^b = \{ s \in S | \langle v, s \rangle = b \}.$$

If S is contained in a compact subset of $M \otimes_{\mathbb{Z}} \mathbb{R}$, then let $b_{\max}^v$ be the maximal value so that S_v^b is nonempty.

Proposition 4.2.17. *Let $E_1, \dots, E_{k+1}$ be a $\mathbb{Q}$-nef partition of a fan Σ, and let $V = \{v_1, \dots, v_k\}$ be an amenable collection of vectors in N subordinate to $E_1, \dots, E_{k+1}$.*

Then the monomials of $\varphi_V^* w$ correspond to all possible sums of points $p \in E_{k+1}$ and $u_1, \ldots, u_\ell$ in $\bigcup_{i=1}^k E_i$ (allowing for repetition in the set $u_1, \ldots, u_\ell$) so that

$$\langle v_i, p + \sum_{i=1}^{\ell} u_i \rangle = 0.$$

for all $1 \leq i \leq k$.

Proof. Let $i = 0$. It is clear then that the points in M corresponding to monomials in the Laurent polynomial

$$w = \sum_{\rho \in E_{k+1}} a_\rho x^\rho$$

are all points q in M obtained as sums of points $p \in E_{k+1}$ and $u_1, \ldots, u_m \in \cup_{j=1}^{i} E_j$ so that

$$\langle v_j, p + \sum_{i=1}^{\ell} u_i \rangle = 0.$$

for all $1 \leq j \leq i$. Now we may apply induction.

Let us define

$$h_i(x_{i+1}, \ldots, x_n) = \sum_{\rho \in E_i} a_\rho \prod_{j>i} x_j^{\langle v_j, \rho \rangle}$$

Assume that we have sequentially substituted $h_1, \ldots, h_{i-1}$ into w to get a Laurent polynomial w' in the variables $x_i, \ldots, x_n$, and that the resulting expression has monomials which correspond to all points q in M which are all sums of points p in E_{k+1} and $u_1, \ldots, u_\ell$ in $E_1, \ldots, E_{i-1}$ so that $\langle v_j, p + \sum_{s=1}^{\ell} u_s \rangle = 0$ for $1 \leq j \leq i-1$ (allowing for repetition in $u_1, \ldots, u_\ell$). Now we show that substituting the expression $h_i(x_{i+1}, \ldots, x_n)$ into w' gives a polynomial whose monomials correspond to all points $p + \sum_{j=1}^{r} u_j$ so that $p \in E_{k+1}$, $u_j \in \bigcup_{s=1}^{i+1} E_s$ and $\langle v_j, p + \sum_{s=1}^{r} u_s \rangle = 0$ for all $1 \leq j \leq i$ (again, allowing for repetition in the set $u_1, \ldots, u_r$).

We may let F be the set of integral points in M corresponding to monomials of w'. Then we have

$$w' = \sum_{b=0}^{b_{v_i}^{\max}} x_i^b g_b(x_{i+1}, \ldots, x_n)$$

where

$$g_b(x_{i+1},\ldots,x_n) = \sum_{\rho\in F^b_{v_i}} a_\rho \prod_{j>i} x_j^{\langle v_j,\rho\rangle}.$$

Substituting into w' the expression $x_i = h_i(x_{i+1},\ldots,x_n)$ gives us a Laurent polynomial in $x_{i+1},\ldots,x_n$ whose monomials correspond to points in set

$$\cup_{b=0}^{b^{v_i}_{\max}}(F^b_{v_i} + bE_i).$$

Each point in this set satisfies $\langle v_i, E^b_{v_i} + bE_i\rangle = 0$ by the condition that $\langle v_i, E_i\rangle = -1$. Furthermore, $\langle v_j, E_i\rangle = 0$ for $j < i$, and hence each set of points $F^b_{v_i} + bE_i$ is orthogonal to $v_1,\ldots,v_i$ and can be expressed as a sum of points $p + \sum_{j=1}^s u_i$ for $u_1,\ldots,u_s \in \cup_{j=1}^i E_j$ and $p \in E_{k+1}$.

Now assume that we have a point $q = p + \sum_{j=1}^s u_i$ for $u_1,\ldots,u_s \in \cup_{j=1}^i E_j$ and $p \in E_{k+1}$ which is orthogonal to v_j for $1 \le j \le i$. Then let $U = \{u_1,\ldots,u_s\} \cap E_i$, and let

$$q' = p + \sum_{i=1, u_i\notin U}^{s} u_i = \rho - \sum_{u_i\in U} u_i.$$

We see that q' is orthogonal to v_j for $1 \le j \le i-1$ since $\langle v_j, u\rangle = 0$ for $u \in U$ and $1 \le j \le i-1$, thus $q' \in F$. Note that we must have $\langle v_i, q'\rangle = \#U$. Thus the point q' is in $F^{\#U}_{v_i}$ and hence $q \in (F^{\#U}_{v_i} + (\#U)E_i)$ (since $\sum_{u_i\in U} u_i$ is clearly an element of $(\#U)$). Thus q corresponds to a monomial in w' after making the substitution $x_i = h_i(x_{i+1},\ldots,x_n)$. This completes the proof after applying induction. □

Proposition 4.2.18. *Assume that M_V intersects a cone C of Σ in a ray generated by an integral vector $\rho \in M$, then there is some multiple of ρ in the polytope $\Delta_{\phi^*_V w}$. In other words, $\Delta_V \subseteq \Delta_{\phi^*_V w}$*

Proof. This follows from Proposition 4.2.17 and Proposition 4.2.12. According to Proposition 4.2.12, there is an integral point in $M_V \cap C$ so that $\varphi_{k+1}(\rho) = 1$. In the proof of Proposition 4.2.12, it is actually shown that this point is constructed as a sum of points $p \in E_{k+1}$ and $u_1,\ldots,u_\ell \in \bigcup_{i=1}^k E_i$ (allowing for repetition in the set

$u_1, \dots, u_\ell$). According to Proposition 4.2.17 this point must correspond to a monomial of the Laurent polynomial $\phi_V^* w$. $\square$

Theorem 4.2.19. *Assume that V is an amenable collection of vectors subordinate to a $\mathbb{Q}$-nef partition $E_1, \dots, E_{k+1}$ of a fan Σ. The polytope $\Delta_{\phi_V^* w}$ is equal to Δ_V.*

Proof. We may deduce that $\Delta_V \subseteq \Delta_{\phi_V^* w}$ from Proposition 4.2.18. Thus it is sufficient to show that $\Delta_{\phi_V^* w}$ is contained in Δ_V, or in other words, each integral point $\rho \in \Delta_{\phi_V^* w}$ satisfies $\varphi_{k+1}(p) \leq 1$. However, this is reasonably easy to see. We have shown in Proposition 4.2.17 that each point in $\Delta_{\phi_V^* w}$ is a sum of a single point $p \in E_{k+1}$ and a set of points $u_1, \dots, u_\ell$ in $\Sigma[1] \setminus E_{k+1}$ (allowing for repetition in the set $u_1, \dots, u_\ell$). Recall that we have a set of vectors $w_1, \dots, w_v \in N$ for v the number of maximal dimensional faces of Σ_Δ so that

$$\varphi_{k+1}(\rho) = \max\{\langle w_i, \rho \rangle\}_{i=1}^v.$$

Now let us apply this to $\rho = p + \sum_{i=1}^\ell u_i$. We obtain

$$\max\{\langle w_i, p + \sum_{i=1}^{\ell} u_i \rangle\}_{i=1}^v \leq \max\{\langle w_i, p\rangle\}_{i=1}^v + \sum_{i=1}^{\ell} (\max\{\langle w_i, u_i \rangle\}_{i=1}^v) = \varphi_{k+1}(p) = 1$$

as required. $\square$

Note that this is actually a general description of the polytope $\Delta_{\phi^* w}$ without any restrictions on the $\mathbb{Q}$-nef partition. We summarize the results of this section as the following theorem, which follows directly from Theorem 4.2.14 and Theorem 4.2.19.

Theorem 4.2.20. *Let X be a complete intersection in a toric variety Y_Σ so that there is a $\mathbb{Q}$-nef partition $E_1, \dots, E_{k+1}$ of $\Sigma[1]$ so that X is a complete intersection of $\mathbb{Q}$-Cartier divisors determined by $E_1, \dots, E_k$, then if*

(i) E_{k+1} is a Cartier divisor or

(ii) $E_1, \dots, E_k$ are Cartier divisors,

then an amenable collection of vectors V subordinate to this $\mathbb{Q}$-nef partition determines an amenable degeneration $X \rightsquigarrow X_{\Sigma_V}$ for some fan Σ_V, and the corresponding Laurent polynomial has Newton polytope equal to the convex hull of $\Sigma_V[1]$.

A more robust geometric statement is available to us in the case where X is a Fano variety corresponding to a nef partition in a toric variety. This follows from Theorem 4.2.20 and Theorem 4.2.15

Theorem 4.2.21. *Assume X is a Fano toric complete intersction in a toric variety $Y = Y_\Delta$ cut out by the vanishing locus of sections $s_i \in H^0(Y, \mathcal{O}_Y(E_i))$ for $1 \leq i \leq k$, and $E_1, \dots, E_{k+1}$ is a nef partition of Δ, and that V is an amenable collection of vectors subordinate to this nef partition. Then V determines:*

(i) A degeneration of X to a toric variety $\widetilde{X}_{\Delta_V}$ which is a crepant partial resolution of of X_{Δ_V} and

(ii) A birational map $\phi_V : (\mathbb{C}^\times)^{n-k} \dashrightarrow X^\vee$ so that $\phi_V^* w$ has Newton polytope equal to Δ_V.

4.2.6 Mutations

Here we will analyze the relationship between Laurent polynomials obtained from the same nef partition and different amenable collections. First we recall the following definition.

Definition 4.2.22. Let f be a Laurent polynomial in n variables and let

$$\omega_n = \frac{dx_1 \wedge \cdots \wedge dx_n}{(2\pi i)^n x_1 \dots x_n}.$$

A *mutation* of f is a birational map $\phi : (\mathbb{C}^\times)^n \dashrightarrow (\mathbb{C}^\times)^n$ so that $\phi^*\omega = \omega$ and so that $\phi^* f$ is again a Laurent polynomial.

This definition is due to Galkin and Usnich [61] in the two dimensional case. The generalized form given above is due to by Akhtar, Coates, Galkin and Kasprzyk [4] and Katzarkov and Przyjalkowski [80].

Assume, first of all, that we have two different amenable collections V and V' which are subordinate to the same nef partition $E_1, \dots, E_{k+1}$ of a fan Σ. Then associated to both V and V' are two maps. The first map is

$$\phi_V : (\mathbb{C}^\times)^{n-k} \dashrightarrow X^\vee$$

and the second is a map

$$\phi_V^{-1} = \pi_V : (\mathbb{C}^\times)^k \to (\mathbb{C}^\times)^{n-k}$$

which is defined as

$$(x_1, \dots, x_n) \mapsto (x_{k+1}, \dots, x_n).$$

so that ϕ_V is a birational section of π_V. However, the map $\pi_V \cdot \phi_{V'}$ for a different amenable collection V' is simply a birational morphism of tori. If we let $y_{k+1}, \dots, y_n$ and $x_{k+1}, \dots, x_n$ be coordinates on the torus $(\mathbb{C}^\times)^{n-k}$ associated to V and V' respectively, then for each $k+1 \leq j \leq n$, there is a rational polynomial $h_j(x_{k+1}, \dots, x_n)$ so that the map $\pi_V \cdot \phi_{V'}$ is written as

$$(x_{k+1}, \dots, x_n) \mapsto (h_{k+1}, \dots, h_n).$$

In particular, to determine this map, we have Laurent polynomials $f_i(x_{k+1}, \dots, x_n)$ for each $1 \leq i \leq k$ so that

$$\phi_{V'}(x_{k+1}, \dots, x_n) = (f_1(x_{k+1}, \dots, x_n), \dots, f_k(x_{k+1}, \dots, x_n), x_{k+1}, \dots, x_n).$$

There are bases B and B' of N associated to both V and V' so that $B = \{v_1, \dots, v_n\}$ and $V = \{v_1, \dots, v_k\}$ and so that $B = \{u_1, \dots, u_n\}$ with $V' = \{u_1, \dots, u_n\}$. There is an invertible matrix Q with integral entries $q_{i,j}$ so that $v_i = \sum_{j=1}^n q_{i,j} u_j$, and torus coordinates $x_1, \dots, x_n$ and $y_1, \dots, y_n$ on $(\mathbb{C}^\times)^n$ related by

$$y_i = \prod_{j=1}^n x_j^{q_{i,j}}.$$

In particular, we have

$$h_i(x_{n-k+1}, \ldots, x_n) = \left(\prod_{j=1}^{k} f_j(x_{k+1}, \ldots, x_n)^{q_{i,j}} \right) \left(\prod_{j=k+1}^{n} x_j^{q_{i,j}} \right).$$

The map given by the polynomials $h_i(x_{n-k+1}, \ldots, x_n)$ for $k+1 \leq i \leq n$ then determine explicitly the birational morphism above associated to a pair of amenable collections subordinate to a fixed nef partition. Now it is clear that the birational map of tori $\phi_{V'}^{-1} \cdot \phi_V$ induces a birational map of tori which pulls back the Laurent polynomial $\phi_V^* w$ to a Laurent polynomial. Our goal now is to show that this map preserves the torus invariant holomorphic n form ω_{n-k} defined in Definition 4.2.22. First, we prove a lemma.

Lemma 4.2.23. *Let $\phi : (\mathbb{C}^\times)^n \dashrightarrow X^\vee \subseteq (\mathbb{C}^\times)^n$ be a birational map onto a complete intersection in $(\mathbb{C}^\times)^n$ cut out by Laurent polynomials*

$$F_i = 1 - \frac{f_i(x_{i+1}, \ldots, x_n)}{x_i}$$

which have only non-negative exponents in $x_{i+1}, \ldots, x_k$ if $i \neq k$. Then the residue

$$\mathrm{Res}_{X^\vee} \left(\frac{\omega_n}{F_1 \ldots F_k} \right)$$

agrees with the form $(2\pi i)^k \omega_{n-k}$ on the domain of definition of ϕ.

Proof. We argue by induction. We may make a birational change of variables on $(\mathbb{C}^\times)^n$ so that $x_1 = y_1 + f_1(y_{k+1}, \ldots, y_n)$ for each $1 \leq i \leq k$ and $y_i = x_i$ for $i \neq 1$. Note that

$$dx_1 = d\left(y_1 + f(y_2, \ldots, y_n)\right) = dy_1 + \rho$$

where ρ is some 1-form written as a linear combination of $dy_2, \ldots, dy_n$ with Laurent polynomial coefficients. Thus under our change of variables,

$$\begin{aligned} dx_1 \wedge \cdots \wedge dx_n &= (dy_1 + \rho) \wedge \cdots \wedge dy_n \\ &= dy_1 \wedge \cdots \wedge dy_n. \end{aligned}$$

Furthermore, under the correct choice of variables, we have

$$F_1(x_1, \ldots, x_n) = 1 - \frac{f_1(x_2, \ldots, x_n)}{x_1}$$

Thus

$$\frac{\omega_n}{F_1 \ldots, F_k} = \frac{dx_1 \wedge \cdots \wedge dx_n}{(x_1 - f_1(x_2, \ldots, x_k))F_2 \ldots F_k x_2 \ldots x_n}.$$

Changing variables to $y_1, \ldots, y_n$, we see that

$$\frac{\omega_n}{F_1 \ldots F_k} = \frac{dy_1 \wedge \cdots \wedge dy_n}{F_2 \ldots F_k y_1 y_2 \ldots, y_n}$$

whose residue along the locus $y_1 = 0$ (which is precisely the image of our torus embedding ϕ), is just $\frac{(2\pi i)\omega_{n-1}}{F_2 \ldots F_k}$ since $F_2, \ldots, F_k$ are independent of y_1. Thus locally around any point in $X^\vee$ where the birational map ϕ is well defined and the torus change of coordinates φ is well-defined, it follows that the residue of $\frac{\omega_n}{F_1 \ldots F_2}$ on $X^\vee$ agrees with $\frac{(2\pi i)\omega_{n-1}}{F_2 \ldots F_k}$. Repeating this argument for each $2 \leq i \leq k$ shows that

$$\phi_V^* \mathrm{Res}_{X^\vee} \left(\frac{\omega_n}{F_1 \ldots F_k} \right) = (2\pi i)^k \omega_{n-k}.$$

$\square$

Now this allows us to prove:

Theorem 4.2.24. *Let V and V' be two amenable collections of vectors subordinate to a nef partition $E_1, \ldots, E_{k+1}$. Then the birational map of tori $\phi_V^{-1} \cdot \phi_{V'}$ is a mutation of the Laurent polynomial $\phi_{V'}^* w$.*

Proof. It is clear that this map is birational and takes $\phi^*_{V'}w$ to a Laurent polynomial. To see that $\phi_V^{-1} \cdot \phi_{V'}$ preserves the form ω_{n-k}, we note that there is some open subset $U^\vee$ of $X^\vee$ on which both ϕ_V and $\phi_{V'}$ induce isomorphisms from open sets U_V and $U_{V'}$ in $(\mathbb{C}^\times)^{n-k}$. In other words, we have isomorphisms $\phi_V^\circ : U_V \xrightarrow{\sim} U^\vee$ and $\phi_{V'}^\circ : U_{V'} \xrightarrow{\sim} U^\vee$ between open sets. From Lemma 4.2.23, we know that

$$(\phi_V^\circ)^*\mathrm{Res}_{U^\vee}\left(\frac{\omega_n}{F_1\dots F_k}\right) = (2\pi_i)^k\omega_{n-k}|_{U_V}$$
$$(\phi_{V'}^\circ)^*\mathrm{Res}_{U^\vee}\left(\frac{\omega_n}{F_1\dots F_k}\right) = (2\pi_i)^k\omega_{n-k}|_{U_{V'}}$$

therefore we must have that on U_V, $(\phi_V \cdot \phi_{V'}^{-1})^*(\omega_{n-k}|_{U_{V'}}) = \omega_{n-k}|_V$, and thus $\phi_V \cdot \phi_V^{-1}$ is a mutation. □

Of course, Theorem 4.2.24 requires that we start with two amenable collections subordinate to the same nef partition. It is possible to have distinct nef partitions corresponding to the same quasi-Fano variety. It would be interesting to show that if we have two such nef partitions and amenable collections subordinate to each, then there is a mutation between the corresponding Laurent polynomials.

4.3 Degenerations of complete intersections in partial flag varieties

Now we discuss the question of constructing toric degenerations and Laurent polynomial expressions for Landau-Ginzburg models of complete intersections in partial flag varieties. Recall that the partial flag variety $F(n_1, \dots, n_l, n)$ is a smooth complete Fano variety which parametrizes flags in $V \cong \mathbb{C}^n$,

$$V_1 \subseteq \dots \subseteq V_l \subseteq V$$

where $\dim(V_i) = n_i$. The reader may consult [26] for general facts on partial flag varieties.

According to [18] and [63], there are small toric degenerations of the complete flag variety $F(n, n_1, \ldots, n_l)$ to Gorenstein Fano toric varieties $P(n, n_1, \ldots, n_l)$ which admit small resolutions of singularities. It is suggested in [126] that the Landau-Ginzburg models of the complete flag variety can be expressed as a Laurent polynomial whose Newton polytope is the polytope $\Delta(n_1, \ldots, n_l, n)$ whose face fan determines the toric variety $P(n, n_1, \ldots, n_l)$.

For any Fano complete intersection X in $F(n_1, \ldots, n_l, n)$, one obtains a degeneration of X to a nef Cartier complete intersection in the toric variety $P(n, n_1, \ldots, n_l)$ and hence conjectural expressions for the Landau-Ginzburg model of X can be given in terms of the Givental Landau-Ginzburg model of complete intersections in $P(n, n_1, \ldots, n_l)$. In [126], Przyjalkowski and Shramov give a method of constructing birational maps between tori and $X^\vee$ so that the superpotential pulls back to a Laurent polynomial for complete intersections in Grassmannians $\mathrm{Gr}(2, n)$. Here we will use Theorem 4.2.20 to show that most nef complete intersections X in $P(n, n_1, \ldots, n_l)$ admit an amenable toric degeneration, which express the Givental Landau-Ginzburg model of X as a Laurent polynomial.

4.3.1 The structure of $P(n_1, \ldots, n_l, n)$

In order to construct the toric variety to which $F(n_1, \ldots, n_l, n)$ degenerates, we begin with an external combinatorial construction presented in [18]. We define a graph $\Gamma(n_1, \ldots, n_l, n)$. Let us take an $n \times n$ box in the Euclidean plane with lower left corner placed at the point $(-1/2, -1/2)$. Let $k_{l+1} = n - n_l$, let $k_i = n_i - n_{i-1}$, and $k_1 = n_1$. Along the diagonal of this box moving from the bottom right corner to the top left corner, we place boxes of size $k_i \times k_i$ sequentially from 1 to $l + 1$. The region below these boxes is then divided equally into 1×1 boxes along grid lines, as shown in the first part of Figure 4.1.

From this grid, we construct a directed graph with black and white vertices. Assume that the centers of each of the 1×1 boxes beneath the diagonal are at integral points in the (x, y) plane so that the center of the bottom left box is at the origin. At the center of each 1×1 box beneath the diagonal, we place black points. In each box

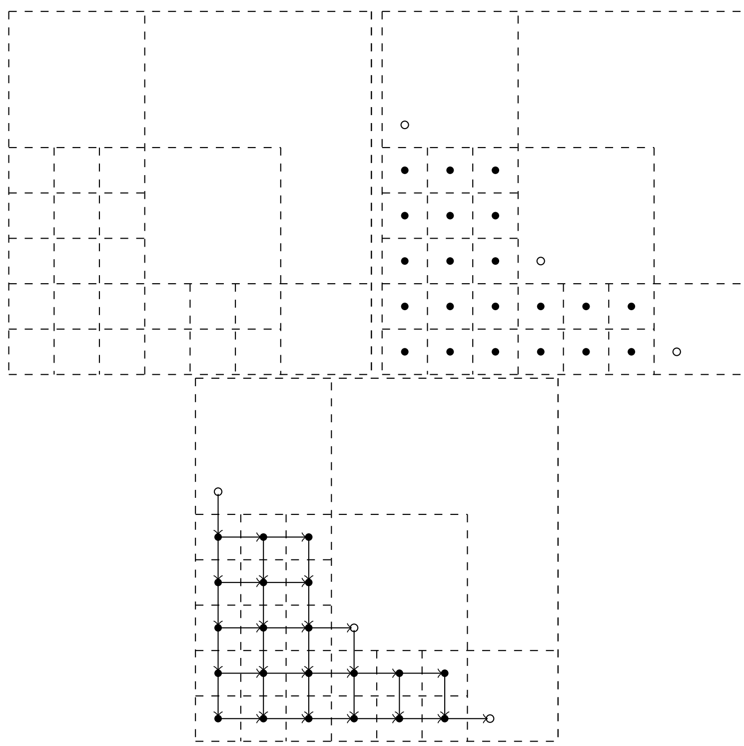

Fig. 4.1 The grid, nodes and graph of $\Gamma(2,5,8)$

B on the diagonal, we insert a white point shifted by $(1/2, 1/2)$ from the bottom left corner of B. See the second part of Figure 4.1 as an example.

We then draw arrows between each vertex u and any other vertex v which can be obtained from u by a shift of v by either $(1, 0)$ or $(0, -1)$ directed from left to right or from top to bottom, as in the third part of Figure 4.1. Let D be the set of black vertices, and let S be the set of white vertices. In the language of [18], the elements of S are called stars. Let E denote the set of edges of $\Gamma(n_1, \dots, n_l, n)$. We will denote the vertex at a point $(m, n) \in \mathbb{Z}^2_{\geq 0}$ by $v_{m,n}$ and an arrow between points v_{m_1,n_1} and v_{m_2,n_2} by $(v_{m_1,n_1} \to v_{m_2,n_2})$. We have functions

$$h : E \to D \cup S \text{ and } t : E \to D \cup S$$

assigning to an arrow the vertex corresponding to its head and tail respectively.

The polytope $\Delta(n_1, \dots, n_l, n)$ is then constructed as a polytope in the lattice $M = \mathbb{Z}^D$ as the convex hull of points corresponding to edges E, which we construct as follows. If $d \in D$, then let e_d be the associated basis vector for M, and formally define e_s to be the origin for $s \in S$. If α is an edge of $\Gamma(n_1, \dots, n_l, n)$, then to α, we associate the point in M given by

$$p_\alpha = e_{h(\alpha)} - e_{t(\alpha)}.$$

Definition 4.3.1. The polytope $\Delta(n_1, \dots, n_l, n)$ is the convex hull of the points p_α for all $\alpha \in E$.

We rapidly review properties of $\Delta(n_1, \dots, n_l, n)$. The toric variety $P(n_1, \dots, n_l, n)$ is toric variety associated to the fan over faces of $\Delta(n_1, \dots, n_l, n)$, and it has torus invariant Weil divisors associated to each vertex v, which correspond directly to the points p_α for $\alpha \in E$. We will refer to the divisor corresponding to the arrow α as D_α.

Torus invariant Cartier divisors $\sum_{\alpha \in \Delta[0]} n_\alpha D_\alpha$ correspond to piecewise linear functions φ which are Σ-linear so that $\varphi(q_\alpha) = n_\alpha$ for all q_α. In Lemma 3.2.2 of [18],

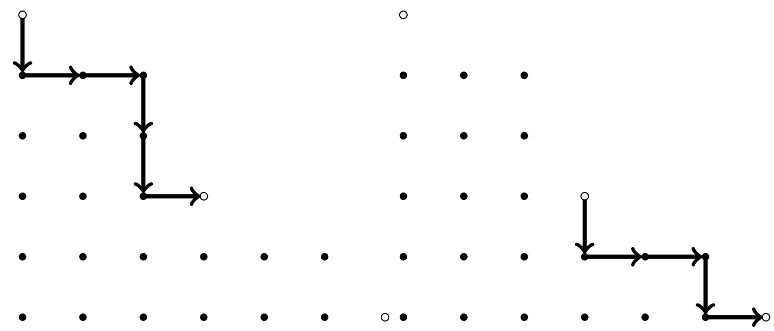

Fig. 4.2 Roof paths of $\Gamma(2,5,8)$ connecting sequential white vertices.

Cartier divisors which generate $\mathrm{Pic}(P(n_1,\ldots,n_l,n))$ are given. We will now describe these divisors.

Definition 4.3.2. A *roof* $\mathcal{R}_i$ for $i \in \{1,\ldots,l\}$ is a collection of edges which have either no edges above or to the right, and which span a path between two sequential white vertices of $\Gamma(n_1,\ldots,n_l,n)$.

Examples of roofs and the associated paths in $\Gamma(n_1,\ldots,n_l,n)$ are shown in Figure 4.2. Associated to each roof is a set of divisors. Let α be an edge in $\mathcal{R}_i$ and let $U(\alpha)$ be the collection of edges either directly below α if α is a horizontal arrow, or directly to the left of α if α is a vertical arrow. If D_β is the Weil divisor of $P(n_1,\ldots,n_l,n)$ corresponding to the arrow β then it is proven in Lemma 3.2.2 of [18] that the Weil divisor

$$H_\alpha = \sum_{\beta \in U(\alpha)} D_\beta$$

is nef and Cartier, and that if we take two edges α and α' in the same roof $\mathcal{R}_i$, then H_α is linearly equivalent to $H_{\alpha'}$. We define $\mathcal{L}_i$ to be the line bundle on $P(n_1,\ldots,n_l,n)$ associated to the divisor H_α for α any arrow in $\mathcal{R}_i$. There is an embedding

$$\psi : P(n_1,\ldots,n_l,n) \hookrightarrow \mathbb{P}^{N_1-1} \times \cdots \times \mathbb{P}^{N_l-1}$$

where $N_i = \binom{n}{n_i}$. This map is comes from the product of the morphisms determined by each $\mathcal{L}_i$ (see Theorem 3.2.13 of [18]). By work of Gonciulea and Lakshmibai [63], the Plücker embedding

$$\phi : F(n_1, \ldots, n_l, n) \hookrightarrow \mathbb{P}^{N_1-1} \times \cdots \times \mathbb{P}^{N_l-1}$$

gives a flat degeneration to the image of ψ. The divisors C_i on $F(n_1, \ldots, n_l, n)$ obtained by pulling back $\mathbb{P}^{N_1-1} \times \cdots \times h_i \times \cdots \times \mathbb{P}^{N_l-1}$ where h_i is a generic hyperplane in $\mathbb{P}^{N_i-1}$ along ϕ form the Schubert basis of the Picard group of $F(n_1, \ldots, n_k, n)$, and the ample cone is the interior of the cone generated over $\mathbb{R}_{\geq 0}$ by classes C_i (see Proposition 1.4.1 of [26]). Furthermore, the anticanonical bundle divisor of $F(n_1, \ldots, n_l, n)$ is given by

$$-K_F = \sum_{i=1}^{l} m_i C_i.$$

Here m_i is the number of edges in the i^{th} roof of $\Gamma(n, n_1, \ldots, n_l)$. We choose multi-degrees $\overline{d_i} = (d_i^{(1)}, \ldots, d_i^{(l)})$ for integers $1 \leq i \leq k$ so that $\sum_{i=1}^{k} d_i^{(j)} < m_j$. Let $\overline{d}$ denote this set of multidegrees. Then let $Z_{\overline{d_i}}$ be the intersection of $F(n_1, \ldots, n_l, n)$ with a generic divisor of multi-degree $\overline{d_i}$ in $\mathbb{P}^{N_1-1} \times \cdots \times \mathbb{P}^{N_l-1}$ under the emebedding ϕ. The complete intersection $X_{\overline{d}}$ in $F(n_1, \ldots, n_k, n)$ of the divisors $Z_{\overline{d_i}}$ is Fano, since by the adjunction formula, $-K_X = \left(\sum_{i=1}^{l} n_i C_i\right)|_X$ for $n_i = m_i - \sum_{j=1}^{k} d_j^{(i)} > 1$ is the restriction of a very ample divisor on $F(n_1, \ldots, n_l, n)$.

If we keep the divisors $Z_{\overline{d_i}}$ fixed and let $F(n_1, \ldots, n_l, n)$ degenerate to $P(n_1, \ldots, n_l, n)$, we obtain a natural degeneration of $X_{\overline{d}}$ to a generic complete intersection $X'_{\overline{d}}$ in $P(n_1, \ldots, n_l, n)$ cut out by the vanishing locus of a non-degenerate global section of $\bigoplus_{i=1}^{l} \mathcal{O}(\sum_{j=1}^{k} \mathcal{L}_j^{d_j^{(i)}})$.

We may now associate $X'_{\overline{d}}$ to a nef partition of $\Delta(n_1, \ldots, n_l, n)$. For each $\overline{d_i}$, choose a set $\mathcal{U}_{i,j}$ of $d_i^{(j)}$ vectors $\alpha \in \mathcal{R}_j$ in such a way that the sets $\mathcal{U}_{i,j}$ have pairwise empty intersection and so that no $\mathcal{U}_{i,j}$ contains an arrow α so that $h(\alpha)$ is a white vertex.

It is possible to choose sets this way since $\sum_{i=1}^{k} d_i^{(j)} < m_j$. Let $\mathcal{U}_i = \cup_{j=1}^{l} \mathcal{U}_{i,j}$. Thus we have divisors

$$H_i = \sum_{\alpha \in \mathcal{U}_i} H_\alpha$$

which are nef Cartier divisors on $P(n_1, \dots, n_l, n)$ linearly equivalent to the divisors $Z_{\overline{d_i}}$ restricted to $P(n_1, \dots, n_l, n)$ in $\mathbb{P}^{N_1-1} \times \cdots \times \mathbb{P}^{N_l-1}$. Furthermore, H_i correspond to a nef partition of $\Delta(n_1, \dots, n_l, n)$. Let $\mathcal{U}_{k+1} = (\cup_{i=1}^{l} \mathcal{R}_i) \setminus (\cup_{i=1}^{k} \mathcal{U}_i)$. Note $\mathcal{U}_{k+1}$ contains all arrows α with $h(\alpha)$ a white vertex. Then the sets

$$E_i := \bigcup_{\alpha \in \mathcal{U}_i} U(\alpha)$$

define a nef partition of $\Delta(n_1, \dots, n_l, n)$. We have the standard generating set of regular functions on $(\mathbb{C}^\times)^D$ written as $x_{m,n}$ associated to black vertices $v_{m,n}$ of $\Gamma(n_1, \dots, n_l, n)$. The monomial associated to an arrow α is

$$x^\alpha = \frac{x_{h(\alpha)}}{x_{t(\alpha)}}$$

and we define the Givental Landau-Ginzburg mirror of $X_{\overline{d}}$ to be the complete intersection $X_{\overline{d}}^\vee$

$$1 = \sum_{\alpha \in \mathcal{U}_i} a_\alpha x^\alpha$$

for $1 \leq i \leq k$ equipped with superpotential

$$w = \sum_{\alpha \in \mathcal{U}_{k+1}} a_\alpha x^\alpha.$$

Here the coefficients a_α should be chosen so that they satisfy the so-called box equations and roof equations of Section 5.1 of [18].

4.3.2 Associated amenable collections

An element ℓ of $N = \operatorname{Hom}(M, \mathbb{Z})$ is determined by the number that it assigns to each generator of M. Since we have associated to each black vertex of $\Gamma(n_1, \dots, n_l, n)$ a generator e_d, and we have formally set e_s to be the origin for $s \in S$ a white vertex, an element of N just assigns to each black vertex of $\Gamma(n_1, \dots, n_l, n)$ some integer, and assigns the value 0 uniformly to all white vertices. To the points in $\Delta(n_1, \dots, n_l, n)$

determined by edges α of $\Gamma(n_1, \ldots, n_l, n)$, the linear operator ℓ assigns the number

$$\ell(p_\alpha) = \ell(e_{h(\alpha)}) - \ell(e_{t(\alpha)}).$$

Therefore, each $\ell \in N$ is simply a rule that assigns to each black vertex of $\Gamma(n_1, \ldots, n_l, n)$ an integer so that the resulting value associated to the arrows in each E_j is (-1), takes non-negative values elsewhere, and takes the value 0 on E_k for $k < j$. Our task now is to choose carefully an amenable collection of vectors associated to a given nef partition. We will first describe this process for a single α in $\mathcal{R}_i$. There are two distinct cases to deal with:

(i) The edge α is horizontal and $t(\alpha)$ and $h(\alpha)$ are black vertices.

(ii) The edge α is vertical.

We treat these cases separately then combine them to produce the desired function. Let us take two white vertices of $\Gamma(n_1, \ldots, n_l, n)$ located at points (m_0, n_0) and (m_1, n_1) so that there is no white vertex (m_2, n_2) with $m_0 \leq m_2 \leq m_1$ and $n_1 \leq n_2 \leq n_0$, and let α be an edge in the roof between (m_1, n_1) and (m_2, n_2).

(i) Let α be a vertical arrow so that $\alpha = (v_{m,n} \to v_{m,n-1})$ for $m_0 \leq m \leq m_1 - 1$ and $n_1 - 1 \leq n \leq n_0$. Then we define the function ℓ_α so that

$$\ell_\alpha(e_{(i,j)}) = \begin{cases} -1 & \text{if } i \leq n_1 - 1 \text{ and } j \leq m - 1 \\ 0 & \text{otherwise} \end{cases}$$

We can check the value of ℓ_α on vertical arrows

$$\ell_\alpha(e_{(i,j)}) - \ell_\alpha(e_{(i,j-1)}) = \begin{cases} -1 & \text{if } j = n \\ 0 & \text{otherwise} \end{cases}$$

and on horizontal arrows,

$$\ell_\alpha(e_{(i,j)}) - \ell_\alpha(e_{(i+1,j)}) = \begin{cases} 1 & \text{if } i = m_1 - 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus ℓ_α takes value (-1) only on elements of $U(\alpha)$ and takes positive values only at arrows $(v_{n_1-1,j} \to v_{n_1,j})$.

(ii) Now let us take some vector $\alpha \in \mathcal{R}_i$ so that $\alpha = (v_{m,n_0-1} \to v_{m+1,n_0-1})$ for $m_0 \leq m \leq m_1 - 2$. Define ℓ_α on the basis $e_{(i,j)}$ so that

$$\ell_\alpha(e_{(i,j)}) = \begin{cases} -1 & \text{if } m+1 \leq i \leq m_1 - 1 \text{ and } j \leq n_0 - 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\ell_\alpha(e_{(i,j)}) - \ell_\alpha(e_{(i+1,j)}) = \begin{cases} -1 & \text{if } i = m \\ 1 & \text{if } i = m_1 - 1 \\ 0 & \text{otherwise} \end{cases}$$

and for any vertical arrow,

$$\ell_\alpha(e_{(i,j)}) - \ell_\alpha(e_{(i1,j)}) = 0$$

Examples of ℓ_α for both vertical and horizontal arrows α are shown in Figure 4.3. Thus we have chosen $\ell_\alpha \in N$ for each $\alpha \in \mathcal{R}_i$ so that $h(\alpha)$ is not a white vertex, in such a way that ℓ_α takes value (-1) only at arrows in $U(\alpha)$ and which takes positive values only at horizontal arrows $(v_{m_0-1,j} \to v_{m_0,j})$. Thus for any arrows $\alpha_1 \in \mathcal{R}_i$ and $\alpha_2 \in \mathcal{R}_j$ for which $h(\alpha_i)$ is not a white vertex, we have $\ell_{\alpha_1}(\alpha) = 0$ for all $\alpha \in U(\alpha_2)$ and $\ell_{\alpha_2}(\alpha) = 0$ for all $\alpha \in U(\alpha_1)$.

Now let us choose some $(k+1)$-partite nef partition of $\Delta(n_1, \dots, n_l, n)$ given by multidegrees $\overline{d}_i = (d_1^{(i)}, \dots, d_r^{(i)})$ so that $\sum_{i=1}^k d_j^{(i)} < m_i$. Then, as in Section 4.3.1, we may choose disjoint collections $\mathcal{U}_j$ of vectors in the union of all roofs $\cup_{i=1}^r \mathcal{R}_j$ so that $\mathcal{U}_j \cap \mathcal{R}_i$ is of size $d_i^{(j)}$ and so that for all $\mathcal{U}_j$ there is no $\alpha \in \mathcal{U}_j$ for which $h(\alpha)$ is a white vertex. Define

$$\ell_{\mathcal{U}_j} = \sum_{\alpha \in \mathcal{U}_j} \ell_\alpha.$$

and let $E_1, \dots, E_{k+1}$ be the nef partition described in Section 4.3.1 associated to the sets $\mathcal{U}_i$.

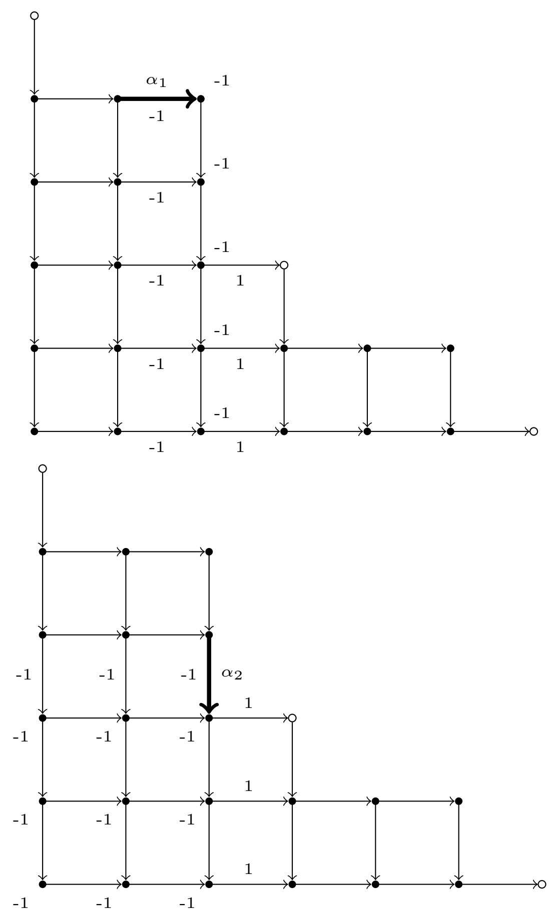

Fig. 4.3 Functions ℓ_{α_i} associated to a horizontal and vertical arrows $\alpha_1, \alpha_2 \in \mathcal{R}_1$ respectively. Vertices and arrows which have not been assigned numbers correspond to vertices and arrows to which ℓ_α assigns the number 0.

Proposition 4.3.3. *If we have a $(k+1)$-partite nef partition as described in the preceding paragraph, then the collection of vectors $V = \{\ell_{\mathcal{U}_1}, \dots, \ell_{\mathcal{U}_k}\}$ forms an amenable collection of vectors subordinate to the chosen $(k+1)$-partite nef partition.*

Proof. It is enough to show that $\ell_{\mathcal{U}_i}(\beta) = 0$ for any $\beta \in U(\alpha)$ for $\alpha \in \mathcal{U}_j$ and $j \neq i$. However, this follows easily from the fact that each ℓ_α takes the value (-1) at $\beta \in U(\alpha)$, positive values on arrows in $U(\delta)$ with $h(\delta)$ a white vertex and 0 otherwise. Thus $\ell_{\mathcal{U}_j}$ takes values (-1) only at arrows $\beta \in U(\alpha)$ for $\alpha \in \mathcal{U}_j$ and positive values on arrows in $U(\delta)$ with $h(\delta)$ a white vertex and 0 otherwise. We have that $U(\alpha) \cap U(\delta) = \emptyset$ if $\alpha \neq \delta$, thus since $\mathcal{U}_i$ contains no arrow δ with $h(\delta)$ a white vertex, $\ell_{\mathcal{U}_i}(\beta) = 0$ if $\alpha \in U(\alpha)$ and $\alpha \in \mathcal{U}_j$ with $j \neq i$. □

Therefore, we may conclude, following Theorem 4.2.20, that

Theorem 4.3.4. *Let $X'_{\overline{d}}$ be a Fano complete intersection in $P(n_1, \dots, n_l, n)$ determined by a set of multi-degrees $\overline{d}$. Then $X'_{\overline{d}}$ admits a degeneration to a weak Fano toric variety X_Σ with at worst Gorenstein singularities. Furthermore, the Givental Landau-Ginzburg model of $X'_{\overline{d}}$ admits a torus map $\phi_{\overline{d}}$ so that the pullback of the superpotential w along $\phi_{\overline{d}}$ is a Laurent polynomial with Newton polytope Δ so that $X_\Delta = X_\Sigma$.*

Example 4.3.5. We conclude with a non-trivial example of our method at work. Let us take the partial flag manifold $F(1,2,5)$, and we will compute the Laurent polynomial associated to a Fano hypersurface in this Flag manifold. First, we have variables $x_{0,1}, x_{0,0}, x_{1,1}, x_{1,0}, x_{2,1}, x_{2,0}$ and $x_{3,0}$, and we choose the nef partition of $\Delta(1,2,5)$ associated to the roof-paths of length 3 and 1 in each block (in other words the multi-degree $\overline{d}$ is just $(3,1)$). This nef partition corresponds to the following Givental Landau-Ginzburg model,

$$1 = x_{0,1} + \frac{x_{0,0}}{x_{0,1}} + \frac{x_{1,0}}{x_{1,1}} + \frac{x_{2,0}}{x_{2,1}} + x_{3,0} + \frac{x_{1,1}}{x_{0,1}} + \frac{x_{2,1}}{x_{1,1}} + \frac{x_{1,0}}{x_{0,0}} + \frac{x_{2,0}}{x_{1,0}}$$

equipped with potential

$$w = \frac{1}{x_{2,1}} + \frac{1}{x_{3,0}} + \frac{x_{3,0}}{x_{2,0}}.$$

The method described in Proposition 4.3.3 produces an amenable collection with only one element, which is given by

$$v = -e^*_{(0,1)} - 2e^*_{(0,0)} - 2e^*_{(1,1)} - 3e^*_{(1,0)} - 4e^*_{(2,0)} - e^*_{(3,0)} - 3e^*_{(2,1)}$$

which may be completed to a basis if we let $v_2 = e^*_{(0,0)}$, $v_3 = e^*_{(1,1)}$, $v_4 = e^*_{(1,0)}$, $v_5 = e^*_{(2,0)}$ $v_6 = e^*_{(3,0)}$ and $v_7 = e^*_{(2,1)}$. Then in terms of this basis, the Givental Landau-Ginzburg model looks like

$$1 = \frac{1}{y_1} + \frac{y_2}{y_1} + \frac{y_4}{y_1 y_3} + \frac{y_5}{y_1 y_7} + \frac{y_6}{y_1} + \frac{y_3}{y_1} + \frac{y_7}{y_1 y_3} + \frac{y_4}{y_1 y_2} + \frac{y_5}{y_1 y_4}$$

with potential

$$w = \frac{y_1^3}{y_7} + \frac{y_1}{y_6} + \frac{y_1^3 y_6}{y_5}.$$

Eliminating y_1 from the first equation, we obtain

$$y_1 = 1 + y_2 + \frac{y_4}{y_3} + \frac{y_5}{y_7} + y_6 + y_3 + \frac{y_7}{y_3} + \frac{y_4}{y_2} + \frac{y_5}{y_4}$$

and thus

$$\begin{aligned} w = &\left(1 + y_2 + \frac{y_4}{y_3} + \frac{y_5}{y_7} + y_6 + y_3 + \frac{y_7}{y_3} + \frac{y_4}{y_2} + \frac{y_5}{y_4}\right) \\ &\times \left(\frac{1}{y_6} + \left(1 + y_2 + \frac{y_4}{y_3} + \frac{y_5}{y_7} + y_6 + y_3 + \frac{y_7}{y_3} + \frac{y_4}{y_2} + \frac{y_5}{y_4}\right)^2 \left(\frac{y_6}{y_5} + \frac{1}{y_7}\right)\right) \end{aligned}$$

4.4 Further applications

Recently, Coates, Kasprzyk and Prince [34] have given a reasonably general method of turning a Givental Landau-Ginzburg model into a Laurent polynomial under specific conditions. We will show that all of their Laurent polynomials are cases of Theorem 4.2.16, and that all of the Laurent polynomials of Coates, Kasprzyk and Prince come from toric degenerations. We will also comment on the extent to which we recover results of Ilten Lewis and Przyjalkowski [76], and mention how our results relate to geometric transitions of toric complete intersection Calabi-Yau varieties.

4.4.1 The Przyjalkowski method

Here we recall the Przyjalkowski method as described by Coates, Kasprzyk and Prince in [34] and show that their construction can be recast in terms of amenable toric degenerations. We will conclude that if the Przyjalkowski method is applied when Y_Δ is a Fano toric variety, then results of Section 4.2.5 imply that all of the Laurent polynomials obtained in [34] correspond to amenable toric degenerations of the complete intersection X.

We begin with a smooth toric Fano variety Y_Δ obtained from a reflexive polytope $\Delta \subseteq M \otimes_{\mathbb{Z}} \mathbb{R}$ with M a lattice of rank m. Then we have an exact sequence

$$0 \to \mathrm{Hom}\,(M, \mathbb{Z}) \to \mathbb{Z}^N \xrightarrow{(m_{ij})} \mathrm{Pic}(Y_\Delta) \to 0 \tag{4.8}$$

where the vertices of Δ are given an ordering and identified with elements of the set $\{1, \dots, N\}$ and where $\mathrm{Pic}(Y_\Delta)$ is the Cartier divisor class group of Y_Δ. We make the following choices: let E be a a subset of $\{1, \dots, N\}$ corresponding to a set of torus invariant divisors which generate $\mathrm{Pic}(Y_\Delta)$ and let $S_1, \dots, S_k$ be disjoint sets subsets of $\{1, \dots, N\}$ whose corresponding divisors may be expressed as non-negative linear combinations in elements of divisors corresponding to elements of E. Assume that each S_i is disjoint from E. Torus invariant divisors of Y_Δ correspond to vertices of Δ. The method of Hori-Vafa [74] for producing Landau-Ginzburg models for X is then applied. This construction was described in Section 4.2.4, but we repeat it here using the notation of [34]. Take variables x_i for $1 \leq i \leq N$, which can be though of as coordinates on the torus $(\mathbb{C}^\times)^N$, and impose relations

$$q_\ell = \prod_{j=1}^{m} x_j^{m_{\ell j}}$$

for each $\ell \in E$ and q_ℓ a variable in $\mathbb{C}^\times$, and equip the associated toric subvariety of $(\mathbb{C}^\times)^N$ with the superpotential

$$w = \sum_{i=1}^{N} x_i$$

By assumption, we have that elements of E form a basis of $\mathrm{Pic}(Y_\Delta)$. Therefore, the matrix (m_{ij}) can be written as the identity matrix when restricted to the subspace of $\mathbb{Z}^N$ spanned by elements in E. Since the sequence in Equation 4.8 is exact the elements $\{e_1, \dots, e_n\}$ of E are part of a basis $\{e_1, \dots, e_n, u_{n+1}, \dots, u_N\}$ of $\mathbb{Z}^N$. In this basis, we have

$$q_\ell = \prod_{j=1}^{m} x_j^{m_{\ell j}} = x_\ell \prod_{j=1, i\neq \ell}^{N} x_j^{m_{\ell j}}$$

and thus we obtain the relations

$$x_\ell = \frac{q_\ell}{\prod_{j=1, j\neq \ell}^{N} x_j^{m_{\ell j}}}.$$

The superpotential for Y_Δ then becomes

$$w = \sum_{\ell \in E} \left(\frac{q_\ell}{\prod_{j=1, j\neq \ell}^{N} x_j^{m_{\ell j}}} \right) + \sum_{i \notin E} x_i \tag{4.9}$$

The monomials in w correspond to the vertices of Δ, and we have eliminated variables corresponding to elements of E. Since E has cardinality equal to $\mathrm{rank}\,(\mathrm{Pic}(Y_\Delta))$, the superpotential w is expressed in terms of n variables. All values m_{ij} involved in the expression above are non-negative if $j \in S_i$ for some $1 \leq i \leq k$, since we have chosen $S_1, \dots, S_k$ to be non-negative linear combinations in $\mathrm{Pic}(Y_\Delta)$ of elements in E.

The Givental Landau-Ginzburg model of X is then given by the subspace $X^\vee$ of $(\mathbb{C}^\times)^N$ cut out by equations

$$1 = \sum_{j \in S_i} x_j \text{ for } 1 \leq i \leq k.$$

Equipped with the superpotential obtained by restricting w to $X^\vee$. This agrees with the notion of Givental Landau-Ginzburg model presented in Section 4.2 up to a translation by the constant k.

At this point, the authors of [34] choose an element $s_i \in S_i$ for each $1 \leq i \leq k$ and then make the variable substitutions for each $\ell \in S_i$

$$x_\ell = \begin{cases} \dfrac{y_\ell}{1 + \sum_{j \in S_i, j \neq s_i} y_j} & \text{if } \ell \neq s_i \\ \dfrac{1}{1 + \sum_{j \in S_i, j \neq s_i} y_j} & \text{if } \ell = s_i \end{cases}$$

These expressions for x_ℓ in terms of y_j then parametrize the hypersurfaces defined by the equations

$$1 = \sum_{j \in S_i} x_j.$$

Since all m_{ij} in Equation 4.9 are non-negative for $j \in \cup_{i=1}^k S_i$, substitution turns w into a Laurent polynomial expressed in terms of $n - k$ variables, y_ℓ for $\ell \in \cup_{i=1}^k S_i$ and x_j for $j \in \{1, \dots, N\} \setminus (\cup_{i=1}^k S_i \cup E)$.

4.4.2 Associated amenable collections

Now we rephrase Przyjalkowski's method in terms of our discussion in Section 4.2. Since the monomials of w correspond to vertices of Δ, the conditions on $S_1, \dots, S_k$ and E restrict Δ so that we may choose m vertices of Δ which correspond to a spanning set $\{e_1, \dots, e_n\}$ of M. Then the remaining vertices of Δ, and $S_1, \dots, S_k$ correspond to subsets of this spanning set. Furthermore, the insistence on positivity of elements of $S_1, \dots, S_k$ in terms of elements of E means that every vertex of E must be a sum $-\sum_{j=1}^n m_{i,j} e_j$ so that $m_{i,j}$ is positive for j corresponding to an element of $\cup_{i=1}^k S_i$. Thus $e_1, \dots, e_n$ must actually span a maximal facet of Δ.

In other words, we have an n-dimensional polytope Δ with simplicial face with vertices $\{e_1, \dots, e_n\}$ a generating set for M so that Y_Δ is a smooth Fano toric variety. We have now chosen a partition of $\Delta[0]$ so that $E_1, \dots, E_k$ correspond to the vertices to which elements of $S_1, \dots, S_k$ correspond and are thus composed of disjoint subsets of $\{e_1, \dots, e_n\}$. The set E_{k+1} is simply the complement $\Delta[0] \setminus \cup_{i=1}^{k} E_i$. Furthermore, we have chosen E_i so that elements of $u \in E_{k+1}$ are written as $u = -\sum_{j=1}^{n} m_{i,j} e_j$ and $m_{i,j} \leq 0$ if $e_j \in E_i$ for $1 \leq i \leq k$.

Proposition 4.4.1. *The sets $E_1, \dots, E_k$ and E_{k+1} form a nef partition of Δ.*

Proof. By definition, this is a partition of vertices of Δ. It remains to show the existence of convex Σ_Δ-piecewise linear functions compatible with this partition, but this follows from the assumption that Y_Δ is a smooth Fano toric variety, hence all irreducible and reduced torus invariant Weil divisors in Y_Δ are nef and Cartier. □

The problem is then to show that there are v_i in the lattice $N = \mathrm{Hom}(M, \mathbb{Z})$ so that the method of Section 4.2 recovers the Laurent polynomial of [34].

Proposition 4.4.2. *Let $E_1, \dots, E_{k+1}$ be a nef partition chosen as above. Then there is an amenable collection of vectors V subordinate to this nef partition of Δ so that the resulting Laurent polynomial is the same as the Laurent polynomial obtained by the Przyjalkowski method.*

Proof. Let $e_1^*, \dots, e_m^*$ be the basis of N dual to $e_1, \dots, e_d$

$$v_i = -\sum_{e_j \in E_i} e_j^*.$$

This choice of v_i then satisfies $\langle v_i, e_j \rangle = -1$ if $e_j \in E_i$,$\langle v_i, e_j \rangle = 0$ if $e_j \in E_j$ for $j \neq i, k+1$, and $\langle v_i, \rho \rangle \geq 0$ for $\rho \in E_{k+1}$. Thus $v_1, \dots, v_k$ forms an amenable set of vectors. To see that this amenable collection of vectors recovers the Laurent polynomial coming from the Przyjalkowski method, we must choose vectors $v_{k+1}, \dots, v_n \in N$ so that $v_1, \dots, v_n$ form a basis of N. Here we use the choice of $s_i \in S_i$. Each s_i corresponds to some vertex of Δ represented by a basis vector of M which we may assume is given by e_i up to re-ordering of the basis of M. It is then easy to check that $\{v_1, \dots, v_k\} \cup \{v_{k+1} = e_{k+1}^*, \dots, v_n = e_n^*\}$ form a basis for the lattice N. In terms of this basis, we have

$$1 = \sum_{\rho \in E_i} \left(\prod_{j=1}^{k+1} x_i^{\langle v_j, \rho \rangle} \right) = \frac{1}{x_i} + \sum_{e_j \in E_i, j \neq i} \frac{x_j}{x_i}$$

and thus we have a torus map

$$\phi_V : (\mathbb{C}^\times)^{n-k} \dashrightarrow X^\vee$$

parametrizing $X^\vee$ given by variable assignment

$$x_i = \begin{cases} 1 + \sum_{e_j \in E_i, j \neq i} y_j \text{ if } 1 \leq i \leq k \\ y_i \text{ otherwise} \end{cases}$$

This is expressed in torus coordinates which are dual to the basis $v_1, \dots, v_n$. This is, of course different from the map used in the Przyjalkowski method, but only because we have changed to a basis dual to $v_1, \dots, v_n$ and not the basis $e_1^*, \dots, e_n^*$. Changing basis so that we return to the standard basis with which we began, we must make the toric change of variables on $(\mathbb{C}^\times)^n$

$$x_j = \begin{cases} \frac{z_j}{z_i} \text{ if } e_j \in E_i \text{ for } 1 \leq i \leq k \\ \frac{1}{z_j} \text{ if } 1 \leq j \leq k \\ z_j \text{ otherwise} \end{cases}$$

In these coordinates, ϕ is written as

$$z_j = \begin{cases} \frac{y_j}{1 + \sum_{e_j \in E_i, j \neq i} y_j} \text{ if } e_j \in E_i \text{ for } 1 \leq i \leq k \\ \frac{1}{1 + \sum_{e_j \in E_i, j \neq i} y_j} \text{ if } 1 \leq j \leq k \\ y_j \text{ otherwise} \end{cases}$$

which is precisely the embedding given by the Przyjalkowski method. $\square$

Of course, as a corollary to this, Theorem 4.2.20 allows us to conclude that the Przyjalkowski method produces toric degenerations of the complete intersection with which we began.

Theorem 4.4.3. *Let Y_Δ be a smooth toric Fano manifold and let X be a Fano complete intersection in Y. If the Givental Landau-Ginzburg model of X becomes a Laurent polynomial with Newton polytope Δ' by the Przyjalkowski method, then X degenerates to the toric variety $X_{\Delta'}$.*

4.4.3 Relation to [76]

Perhaps it now should be mentioned how this work relates to work of Przyjalkowski [123] and Ilten, Lewis and Przyjalkowski [76]. In their situation, they begin with a smooth complete intersection Fano variety X in a weighted projective space $\mathbb{WP}(w_0, \dots, w_n)$. By Remark 8 of [123], we may assume that $w_0 = 1$, and hence the polytope Δ defining $\mathbb{WP}(1, \dots, w_n)$, has vertices given by the points $e_1, \dots, e_n$ and $-\sum_{i=1}^n w_i e_i$ for $\{e_1, \dots, e_n\}$ a basis of M. Then the Przyjalkowski method may be applied, essentially verbatim, letting $S_1, \dots, S_k$ correspond to subsets of the vertices $\{e_1, \dots, e_n\}$ and $E = \{-\sum_{i=1}^n w_i e_i\}$.

Then the amenable collection constructed in the proof of Proposition 4.4.2 is given by

$$v_i = -\sum_{j \in S_i} e_j^*$$

produces a Laurent polynomial associated to the Givental Landau-Ginzburg model identical to those constructed by Przyjalkowski in [123], up to a toric change of basis. Proof of this is essentially identical to the proof of Proposition 4.4.2. Since Przyjalkowski assumes that the divisors of $\mathbb{WP}(1, w_1, \dots, w_n)$ which cut out X are Cartier, we have that X is associated to a $\mathbb{Q}$-nef partition $E_1, \dots, E_{k+1}$ where $E_1, \dots, E_k$ are Cartier. This allows us to apply Theorem 4.2.20 to show

Proposition 4.4.4. *There is a degeneration of each smooth Fano weighted projective complete intersection to a toric variety X_Σ so that the convex hull of the ray generators* *of Σ is a polytope equal to the Newton polytope of the Laurent polynomial associated to X in [123].*

This is a weaker version of the theorem proved in [76].

Theorem 4.4.5 ([76] Theorem 2.2)**.** *Let Δ_f be the Newton polytope of the Laurent polynomial associated to a smooth Fano weighted projective complete intersection X in [123]. Then there is a degeneration of X to $\widetilde{\mathbb{P}}(\Delta_f)$, as defined in Section 1.1 of [76].*

The difference between these two statements is that Proposition 4.4.4 shows that X degenerates to a toric variety which is possibly a toric blow-up of the variety to which Theorem 4.4.5 shows that X degenerates.

4.4.4 Geometric transitions of Calabi-Yau varieties

Readers interested in compact Calabi-Yau varieties, should note that we may reinterpret the work in Section 4.2 as a general description of geometric transitions of toric complete intersection Calabi-Yau varieties.

We note that there is a reinterpretation of the map $\phi_V : (\mathbb{C}^\times)^{n-k} \dashrightarrow (\mathbb{C}^\times)^n$ as a section of the toric morphism $\pi_V : (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^{n-k}$ given by

$$(x_1, \dots, x_n) \mapsto (x_{k+1}, \dots, x_n).$$

which sends the subscheme of $X^\vee$ cut out by the equations $w - \lambda$ for some complex value λ to the subscheme of $(\mathbb{C}^\times)^{n-k}$ cut out by the vanishing locus of $\phi_V^* w - \lambda$ in $(\mathbb{C}^\times)^{n-k}$. Thus we obtain a birational map between the fibers of w, and fibers of the Laurent polynomial $\phi_V^* w$ which may be compactified to anticanonical hypersurfaces in $X_{(\Delta_{\phi_V^* w})^\circ}$.

Note that if $E_1, \dots, E_{k+1}$ is a nef partition of a Fano toric variety determined by a reflexive polytope Δ, then $E_1, \dots, E_{k+1}$ determine a Calabi-Yau complete intersection Z in Y_Δ, which is precisely an anticanonical hypersurface in the complete intersection quasi-Fano variety X determined by $E_1, \dots, E_k$. According to Batyrev and Borisov [17], there is a reflexive polytope ∇ determined by $E_1, \dots, E_{k+1}$ and a dual $(k+1)$-

partite nef partition of Y_∇ which determines a complete intersection Calabi-Yau variety $Z^\vee$ which is called the Batyrev-Borisov mirror dual of Z.

It is well known [62] that the *fibers* of the Givental Landau-Ginzburg model of X may be compactified to complete intersections in Y_∇, and that these compactified fibers are the Batyrev-Borisov mirror dual to anticanonical hypersurfaces Z in X.

Now if we degenerate the homogeneous equations in the coordinate ring of Y_Δ defining X to equations defining some toric variety X_{Δ_V}, then we obtain simultaneous degenerations of anticanonical hypersurfaces Z in X to anticanonical hypersurfaces Z' of X_{Δ_V}. In general, anticanonical hypersurfaces of X_{Δ_V} are more singular than anticanonical hypersurfaces of X.

Classically, mirror symmetry predicts that there is a contraction of $Z^\vee \to (Z')^\vee$ which is mirror dual to the degeneration $Z \rightsquigarrow Z'$ where Z' and $(Z')^\vee$ are mirror dual. Since Z' is a toric hypersurface, the contracted variety $(Z')^\vee$ should be a hypersurface in the toric variety $X_{(\Delta_V)^\circ}$.

We deduce the following:

Theorem 4.4.6. *Let Z be an anticanonical hypersurface in a quasi-Fano complete intersection X in a toric Fano variety Y_Δ determined by a nef partition $E_1, \dots, E_{k+1}$ and so that $E_1, \dots, E_k$ determines the quasi-Fano variety X. Assume there is an amenable collection of vectors subordinate to the nef partition $E_1, \dots, E_{k+1}$ which determines an amenable degeneration $X \rightsquigarrow X_{\Sigma_V}$ where the convex hull of the ray generators of Σ_V is a reflexive polytope Δ_V. Then Z degenerates to a hypersurface in X_{Δ_V}, and there is a mirror birational map from $Z^\vee$ to an anticanonical hypersurface in $X_{(\Delta_V)^\circ}$*

Note that this is just a birational map, not necessarily a birational contraction. In work of Fredrickson [56], it is shown that an associated birational contraction exists in several cases, once one performs appropriate partial resolutions of singularities on both $Z^\vee$ and $(Z')^\vee$. In [98], Mavlyutov showed that any toric variety X_Δ with a fixed Minkowski decomposition of Δ° can be embedded in a Fano toric variety Y determined by the Cayley cone associated to the given Minkowski decomposition, and that anticanonical hypersurfaces in X_Δ can be deformed to nondegenerate nef complete intersections in Y. He then showed that a mirror contraction exists if the degeneration of X to X_Δ is obtained in this way.

Chapter 5

Existence of toric degenerations

5.1 Introduction

In [46], the relationship between degenerations of a complete intersection X in a toric variety to a binomial complete intersection and the structure of the Landau-Ginzburg mirror of X was investigated. A particular case of the results in [46] occurs when Δ is a reflexive polytope and $E_1, \dots, E_{k+1}$ is a partition of the vertices of Δ. Let Σ_Δ be the fan over faces of Δ. Such a partition is called a *nef partition* if there are Σ_Δ-piecewise linear upper convex functions φ_i so that $\varphi_i(\rho) = 1$ if $\rho \in E_i$ and $\varphi_i(\rho) = 0$ if $\rho \notin E_i$.

Definition 5.1.1. A collection $v_1, \dots, v_k$ of elements of $N = \mathrm{Hom}\,(M, \mathbb{Z})$ is called an amenable collection of vectors subordinate to a nef partition $E_1, \dots, E_{k+1}$ if

1. $\langle v_i, \rho \rangle = -1$ for any $\rho \in E_i$.
2. $\langle v_i, \rho \rangle = 0$ for any $\rho \in E_j$ and $j < i$.
3. $\langle v_i, \rho \rangle \geq 0$ for any $\rho \in E_j$ and $j > i$.

By general results in toric geometry, we have that each vertex ρ of Δ determines a torus invariant divisor D_ρ in $\mathbb{P}_\Delta$, and that the line bundle $\mathscr{L}_i := \mathscr{O}_{\mathbb{P}_\Delta}(\sum_{\rho \in E_i} D_\rho)$ is a semi-ample line bundle on $\mathbb{P}_\Delta$. If we let s_i be a generic global section of $\mathrm{H}^0(\mathbb{P}_\Delta, \mathscr{L}_i)$, then the complete intersection subvariety X of $\mathbb{P}_\Delta$ determined by the simultaneous vanishing of $s_1, \dots, s_k$ has effective anticanonical bundle $\mathscr{L}_{k+1}|_X$. In [46] it is proven that

Theorem 5.1.2 ([46, Theorem 2.15])**.** *If X is a complete intersection in $\mathbb{P}_\Delta$ determined by a nef partition $E_1, \dots, E_{k+1}$ of Δ as above, and there exists an amenable collection V subortinate to this nef partition, then there is a degeneration $X \rightsquigarrow X_V$ of X to a Gorenstein toric variety X_V which is a partial crepant resolution of a Gorenstein toric Fano variety.*

In Section 5.2 we will prove that

Theorem 5.1.3. *If the line bundle $\mathscr{L}_{k+1}$ is sufficiently ample then there exists an amenable collection subordinate to $E_1, \dots, E_{k+1}$.*

The phrase "sufficiently ample" will be clarified in Section 5.2, but it is a condition that includes $\mathscr{L}_{k+1}$ being ample and implies that X is Fano. Theorem 5.1.3 is consistent with a conjecture of V. Przyjalkowski [123] which states that if X is a smooth Fano variety then it admits a degeneration to a toric variety. Theorem 5.1.3 applies not just to smooth varieties, but arbitrary Fano complete intersections in toric varieties and should be thought of as support for this conjecture.

5.1.1 Organization

This chapter is organized as follows. In section 5.2, we deal with all of the relevant combinatorics and provide a proof of Theorem 5.1.3. In Section 5.3, we interpret Theorem 5.1.3 in terms of geometry. On the way to this, we provide sufficient criteria for a hypersurface in a toric variety to be Fano. We use this criteria to construct examples X of Fano fourfolds in smooth toric Fano fivefolds $\mathbb{P}_\Delta$ so that $-K_X$ is the restriction of a nef and big but not ample divisor in $\mathbb{P}_\Delta$. In Section 5.4, we use Theorem 5.1.3 to give a short proof that all Fano threefolds with Picard rank greater than 1 admit degenerations to toric varieties.

5.2 Combinatorics

In this section, we will provide a lightning (but not very enlightening) overview of some relevant combinatorial ideas and describe some of their relations to toric geometry. Fix a lattice M of rank d, and let $N = \mathrm{Hom}(M, \mathbb{Z})$. Let $M_{\mathbb{R}} = M \otimes \mathbb{R}$ and similarly, $N_{\mathbb{R}} = \mathrm{Hom}(M, \mathbb{R})$. The origin in M will be denoted 0_M and similarly, the origin in N will be denoted 0_N. We will denote the pairing between $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ by $\langle \bullet, \bullet \rangle$. If we have point sets A_1 and A_2 in M, then we denote by $A_1 + A_2$ the set

$$\{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}$$

which we call the Minkowski sum of A_1 and A_2. We say that ∇_1 is a *Minkowski summand* of ∇ if there is some integer n and a polytope ∇_2 so that $\nabla_1 + \nabla_2 = n\nabla$. We denote by $\mathrm{Conv}(A_1)$ the convex hull of points in A_1

A polytope Δ in $M_{\mathbb{R}}$ is called a lattice polytope if all of its vertices are located at points of M and reflexive if its polar polytope

$$\Delta^\circ = \{\sigma \in N_{\mathbb{R}} : \langle \sigma, \rho \rangle \geq -1 \text{ for every } \rho \in \Delta\}$$

is also integral. We will let $\Delta[i]$ be the disjoint union of all dimension i strata of Δ. A *facet* of Δ will be any element of $\Delta[d-1]$, and a *vertex* is any element of $\Delta[0]$. To a polytope Δ containing 0_M, we associate a fan Σ_Δ whose dimension n cones are cones over the elements of $\Delta[n-1]$. This is called the fan over faces of Δ.

A *nef partition* of Δ is partition of $\Delta[0]$ into subsets $E_1, \ldots, E_{k+1}$ so that there exist continuous upper convex functions φ_i which are linear on cones of Σ_Δ, which take integer values at points in M and so that $\varphi_i(\rho) = -1$ if ρ is in E_i and $\varphi_i(\rho) = 0$ if ρ is in E_j for $j \neq i$. We will let $\Delta_i = \mathrm{Conv}(E_i \cup 0_M)$.

To a nef partition, we can associated a dual nef partition which we denote $\nabla_1, \ldots, \nabla_{k+1}$ which we obtain by the inequalities

$$\nabla_i = \{v \in N_{\mathbb{R}} : \langle v, \rho \rangle \geq \varphi_i(\rho)\}.$$

It has been shown by Borisov [23] that this is in fact a nef partition of $\nabla := \mathrm{Conv}(\cup_{i=1}^{k+1}\nabla_i)$, and that ∇ is reflexive. We denote the partition of $\nabla[0]$ that we obtain this way by $F_1, \ldots, F_{k+1}$. Furthermore, Borisov has shown that $\nabla_1 + \cdots + \nabla_{k+1} = \Delta^\circ$.

Let $\mathbb{P}_\Delta$ be the toric variety associated to the fan Σ_Δ, in the sense that Σ_Δ is the fan determining $\mathbb{P}_\Delta$. There is a bijection between torus invariant divisors D_ρ of $\mathbb{P}_\Delta$ and vertices ρ of Δ. If there is a piecewise Σ_Δ-linear integral function φ, then there is a Cartier divisor $\sum_{\rho\in\Delta[0]}\varphi(\rho)D_\rho$. Thus to any nef partition $E_1, \ldots, E_{k+1}$ of Δ, we have invertible sheaves $\mathscr{L}_1, \ldots, \mathscr{L}_{k+1}$ defined as $\mathscr{L}_i = \mathscr{O}_{\mathbb{P}_\Delta}(\sum_{\rho\in E_i} D_\rho)$. Since the piecewise linear functions φ_i are convex, $\mathscr{L}_i$ are semiample. Furthermore $\bigotimes_{i=1}^{k+1}\mathscr{L}_i \cong \omega_{\mathbb{P}_\Delta}^{-1}$.

To every line bundle $\mathscr{L}$ on $\mathbb{P}_\Delta$, we can associate a polytope $\nabla_{\mathscr{L}}$ in $N_\mathbb{R}$ whose integral points correspond to global sections of $\mathscr{L}$ which are equivariant with respect to the natural action of $(\mathbb{C}^\times)^d$ on $\mathbb{P}_\Delta$. If we take the line bundles $\mathscr{L}_i$ defined above, then $\nabla_{\mathscr{L}_i}$ is just ∇_i.

Proposition 5.2.1 ([16])**.** *A line bundle $\mathscr{L}$ of $\mathbb{P}_\Delta$ is ample if and only if Δ° is a Minkowski summand of $\nabla_{\mathscr{L}}$ and $\nabla_{\mathscr{L}}$ is a Minkowski sumand of Δ°.*

Let $v \in \Delta$ and let Q be a polytope in N, then define $v_{Q,\min} = \min\{\langle\sigma, v\rangle : \sigma \in \nabla_1\}$. In the following, assume we have a bipartite nef partition Δ_1, Δ_2 of Δ with dual nef partiton ∇_1, ∇_2. If v is in Δ or Q, then we let $\Gamma(v)$ be the smallest face of Δ or Q respectively which contains v on its interior. We will define

$$\Gamma(v)_Q^\vee = \{\langle\sigma, v\rangle = v_{Q,\min} : \sigma \in Q\}$$

If we are given a face Γ, then let v be any point on the relative interior of Γ, we define

$$\Gamma_Q^\vee = \{\langle\sigma, v\rangle = v_{Q,\min} : \sigma \in Q\}.$$

The n-dimensional toric strata of $\mathbb{P}_\Delta$ correspond to the $(d-n-1)$-dimensional strata of Δ. If Γ is a $(d-n-1)$-dimensional face of Δ, then the polytope associated to the corresponding stratum is $\Gamma(v)_{\Delta^\circ}^\vee$ for some point v on the interior of Γ. Thus:

Proposition 5.2.2. *Let $\mathscr{L}$ be a line bundle on $\mathbb{P}_\Delta$, and let Y_Γ be the toric stratum associated to Γ. Then $\mathscr{L}|_{Y_\Gamma}$ is ample if and only if $\Gamma(v)^\vee_{\Delta^\circ}$ and $\Gamma(v)^\vee_{\nabla_{\mathscr{L}}}$ are summands of one another.*

Our goal will be to show that if one imposes strong enough ampleness criteria upon the invertible sheaf $\mathscr{L}_{k+1}$, then there is automatically an amenable collection associated to the nef partition $E_1, \dots, E_{k+1}$. This all hinges upon a strong criteria for the existence of an amenable collection. First, we record a lemma.

Lemma 5.2.3. *If $\nabla_1, \dots, \nabla_{k+1}$ are lattice polytopes, then every vertex of $\nabla_1 + \cdots + \nabla_{k+1}$ is a sum of vertices σ_i of ∇_i.*

Proof. The faces of $\nabla_1 + \cdots + \nabla_{k+1}$ are Minkowski sums of faces of $\nabla_1, \dots, \nabla_{k+1}$. For polytopes $Q_1, \dots, Q_n$, we have $\dim Q_j \leq \dim \sum_{i=1}^n Q_i$ for each $1 \leq j \leq n$ therefore, vertices of $\nabla_1 + \cdots + \nabla_{k+1}$ are sums of vertices of $\nabla_1, \dots, \nabla_{k+1}$. □

It is easy to prove that:

Proposition 5.2.4. *If Γ is a face of Δ_i whose vertices are all in E_i, then Γ is a face of Δ.*

Proof. First, if v is in a face of Δ_i, then there are vertices $\rho_1, \dots, \rho_m$ of Δ so that $v = \sum_{j=1}^m a_j \rho_j$, $\sum_{j=1}^m a_j = 1$ and for each j, $0 < a_j \leq 1$. We know, by convexity, that $\varphi_i(v) \geq \sum_{j=1}^m a_j \varphi_i(\rho_j) = 1$. Since $v \in \Delta$, we must have equality, and thus v is in the boundary of Δ. Thus there is a minimal face Γ' of Δ containing v.

Then if $\eta_1, \dots, \eta_n$ are vertices of Γ, then there are numbers $0 < b_1, \dots, b_n \leq 1$ so that $v = \sum_{j=1}^n b_j \eta_j$ and $\sum_{j=1}^n b_j = 1$. Since $\varphi_i(v) = 1$, it follows that $\varphi_i(\eta_j) = 1$ for all j and hence $\eta_j \in \Delta_i$ for all j and thus $\Gamma = \Gamma'$. □

Theorem 5.2.5. *Assume E_1, E_2 is a nef partition of Δ. Then an amenable collection v subordinate to E_1, E_2 exists if and only if there is a face Γ of Δ so that $\Delta_1 = \mathrm{Conv}(\Gamma \cup 0_M)$. One may choose $v \neq 0_N$ to be a vertex of ∇_1*

Proof. Assume that there exists such an amenable collection, which in the bipartite case is just a vector $v \in N$ so that $\langle v, \rho \rangle = -1$ for each $\rho \in E_1$ and $\langle v, \rho \rangle \geq 0$ for each

$\rho \in E_2$. Thus there is a hyperplane of $M_{\mathbb{R}}$ containing E_1. By convexity of Δ_1, the result follows.

Conversely, assume that there is a face Γ of Δ so that $\Delta_1 = \mathrm{Conv}(\Gamma \cup 0_M)$. Let σ be a vertex of Δ° so that $\sigma^\vee$ is a facet of Δ containing Γ. Thus $\langle \sigma, \rho \rangle = -1$ for every $\rho \in \sigma^\vee$, and in particular for every $\rho \in E_1$. We know that $\Delta^\circ = \nabla_1 + \nabla_2$, and therefore $\sigma = \sigma_1 + \sigma_2$ for σ_1 a vertex of ∇_1 and σ_2 a vertex of ∇_2 by Lemma 5.2.3. Since $\langle \sigma_1, \rho \rangle \geq -1$ for $\rho \in E_1$ and $\langle \sigma_2, \rho \rangle \geq 0$ for $\rho \in E_1$, we must have $\langle \sigma_1, \rho \rangle = -1$ for all $\rho \in E_1$ and $\langle \sigma_2, \rho \rangle = 0$ for $\rho \in E_1$. Therefore σ_1 satisfies the conditions of the theorem. □

Definition 5.2.6. Let E_1, E_2 be a bipartite nef partition of Δ so that for every face Γ of Δ_1 which is a face of Δ we have that $\Gamma^\vee_{\nabla_2}$ and $\Gamma^\vee_{\Delta^\circ}$ are Minkowski summands of one another. In this situation, we will say that E_2 is E_1-ample.

Let Θ be the unique face of ∇_2 containing 0_N on its relative interior. Note that Θ is the only face of dimension $\dim \Theta$ containing 0_N.

Proposition 5.2.7. *Assume that E_2 is E_1-ample, then there is a unique face Γ of Δ of dimension $d - 1 - \dim \Theta$ which is contained in Δ_1, which is dual to Θ. Thus $\Delta_1 = \mathrm{Conv}(\Gamma \cup 0_M)$.*

Proof. If $v_1, \dots, v_n$ are vertices of Θ, then let N_Θ be the span of $v_1, \dots, v_n$ in $N_{\mathbb{R}}$. Since 0_N is on the interior of Θ, we see that for each v_i, there are positive real numbers $a_1, \dots, a_n$ so that $-v_i = \sum_{j=1}^n a_i v_i$. Then since $\langle \nabla_2, \Delta_1 \rangle \geq 0$, we must have $\langle v_i, \rho \rangle = 0$ for all $\rho \in \Delta_1$ and all $v_1, \dots, v_n$. Since $v_1, \dots, v_n$ spans N_Θ, we have $\Delta_1 \subseteq M_\Theta$ where M_Θ is used to denote the subspace of $M_{\mathbb{R}}$ orthogonal to N_Θ. Therefore if Γ is a face of Δ_1 then $\dim \Gamma \leq d - 1 - \dim \Theta$.

Now we show that any face Γ of Δ whose dual face in ∇_2 is Θ is contained in Δ_1. Assume that there is some $v \in \Delta$ which is contained in the interior of Γ, then $\Gamma(v) = \Gamma$. Since $\Gamma(v)^\vee_{\nabla_2} = \Theta$ contains 0_N, it follows that $v_{\nabla_2,\min} = 0$. Thus v is contained in Δ_1 for any point v on the interior of Γ (by definition of Δ_1) and hence Γ is contained in Δ_1.

Since any face Γ of Δ_1 that is also a face of Δ corresponds to a toric stratum of $\mathbb{P}_\Delta$ which is contained in some D_ρ for $\rho \in E_1$, we have that $\Gamma^\vee_{\nabla_2}$ and $\Gamma^\vee_{\Delta^\circ}$ are Minkowski summands of one another. Thus any face of Δ_1 of dimension n which is a face of Δ has unique dual face in ∇_2 of dimension $d-1-n$. If Γ is a face in Δ_1 dual to Θ, then $\dim \Gamma = d - 1 - \dim \Theta$, and such a face exists by the above paragraph. If Γ' is a face of dimension $d-1-\dim\Theta$ of Δ_1, then by assumption, its dual has dimension $\dim\Theta$ and contains 0. Thus its dual is Θ. Therefore, there is exactly one face of Δ_1 which is a face of Δ of dimension $d-1-\dim\Theta$. □

Combining Theorem 5.2.5 and Proposition 5.2.7, we conclude:

Corollary 5.2.8. *If E_2 is E_1-ample, then there is an amenable collection subordinate to E_1, E_2.*

Let $I = \{1, \dots, k+1\}$. For each subset J of I, we let $E_J = \bigcup_{j\in J} E_j$. We make the following observation.

Lemma 5.2.9. *If $E_1, \dots, E_{k+1}$ is a nef partition, then for any $J \subseteq I$, the partition $\{E_i\}_{i\notin J}, E_J$ is also a nef partition with dual nef partition $\{\nabla_j\}_{j\notin J}, \sum_{j\in J}\nabla_j$.*

We will prove that under the condition that E_{k+1} is $E_{I\setminus k+1}$-ample, one may deduce the existence of an amenable collection subordinate to $E_1, \dots, E_{k+1}$.

Theorem 5.2.10. *Let $E_1, \dots, E_{k+1}$ be a nef partition of Δ, and assume that E_{k+1} is $E_{I\setminus k+1}$-ample. Then there is an amenable collection subordinate to this nef partition.*

Proof. By Proposition 5.2.7 there exists an amenable collection v subordinate to $E_{I\setminus k+1}, E_{k+1}$, and by Theorem 5.2.5, v is a vertex of $\nabla_{I\setminus k+1} = \sum_{i=1}^k \nabla_i$. By Lemma 5.2.9, we may write $v = v_1 + \cdots + v_k$ for vertices v_i of ∇_i, hence $v_i \in N$. If $\rho \in E_j$, then $\langle v_i, \rho\rangle \geq -1$ if $i = j$ and $\langle v_i, \rho\rangle \geq 0$ if $i \neq j$. Since v is an amenable collection subordinate to $E_{I\setminus k+1}, E_{k+1}$, we have

$$\langle v_1 + \cdots + v_k, \rho\rangle = -1,$$

for every $\rho \in E_{I\setminus k+1}$ and thus we must have $\langle v_j, \rho\rangle = 0$ for all $j \neq i$ and $\langle v_i, \rho\rangle = -1$. Thus $v_1, \dots, v_k$ provides an amenable collection subordinate to $E_1, \dots, E_{k+1}$. □

Remark 5.2.11. Note that the amenable collections obtained in this way do not depend upon ordering. This is not true in general.

5.3 Geometric interpretation

If X is a hypersurface in $\mathbb{P}_\Delta$ determined by the vanishing of a section s_1 of $\mathrm{H}^0(\mathbb{P}_\Delta, \mathscr{L}_1)$ then X is linearly equivalent to the union of divisors $\cup_{\rho \in E_1} D_\rho$, and thus there is a flat degeneration of X to $\cup_{\rho \in E_1} D_\rho$. It may be interesting to note that Theorem 5.2.5 can be rephrased as the fact that if there is a nef partition E_1, E_2 of Δ, then there is an amenable collection subordinate to E_1, E_2 if and only if there is a unique toric stratum of $\mathbb{P}_\Delta$ contained in $\cap_{\rho \in E_1} D_\rho$.

If E_1 is E_2-ample, then the restriction of $\mathscr{L}_2$ to $\cup_{\rho \in E_1} D_\rho$ is ample, and thus this variety is, in a sense, Fano. Note that if $\mathscr{L}_{k+1}$ is in fact an ample divisor on $\mathbb{P}_\Delta$, then $\mathscr{L}_{k+1}$ is in particular $E_{I \setminus k+1}$-ample. Furthermore, since $\mathscr{L}_{k+1}|_X = -K_X$, it follows that X itself is Fano, where X is a complete intersection associated to $E_1, \dots, E_k$. Thus it follows that:

Corollary 5.3.1 (Corollary to Theorem 5.1.3)**.** *If $\mathscr{L}_{k+1}$ is ample then X admits a degeneration to a toric Gorenstein Fano variety.*

A little thought shows that this recovers the exact form of the amenable collections constructed in Section 4.3 in relation to the results of [34].

Remark 5.3.2. Corollary 5.3.1 suffices to recover Theorem 4.3.4 ([46, Section 3]). Briefly, if Y is a partial flag variety, then the Plücker embedding of Y degenerates to a toric variety Y' in a product of projective spaces $\mathbb{P} := \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_n}$. The anticanonical divisor of Y is the restriction of a multidegree $(R_1, \dots, R_n)$ hypersurface for some integers $0 < R_1, \dots, R_n$. If X is a Fano complete intersection in Y, then X is cut out by the intersection of $Y \subseteq \mathbb{P}$ with divisors D_j in $\mathbb{P}$ with multidegree $(d_1^j, \dots, d_n^j)$ for $1 \leq j \leq \ell$, so that $\sum_{i=1}^{\ell} d_j^\ell < R_j$.

The complete intersection X degenerates to a complete intersection X' in Y' by letting the equations defining the Plücker embedding of Y degenerate to a set of binomial equations in the homogeneous coordinate ring of $\mathbb{P}$, and X' is the restriction of a multidegree $(R_1 - \sum_{i=1}^{\ell} d_1^i, \dots, R_n - \sum_{i=1}^{\ell} d_n^i)$ divisor to X'. This divisor is ample on Y' by the condition that $\sum_{i=1}^{\ell} d_j^{\ell} < R_j$ for all j, thus we may apply Theorem 5.1.3 to obtain a degeneration of X' to a parital crepant resolution of a toric Gorenstein Fano variety. We note that this proof is even more oblique than that of Section 4.3 and infinitely more oblique than the results of [126].

Remark 5.3.3. Theorem 5.1.3 covers the case where $\mathbb{P}_\Delta$ is a complete intersection of Cartier divisors in a toric Fano variety with only Gorenstein singularities and with Picard rank 1. In the case where $\mathbb{P}_\Delta$ is a weighted projective space, the existence of such degenerations is a special case of results obtained by Ilten, Lewis and Przyjalkowski [76].

The condition that E_{k+1} be $E_{I \setminus k+1}$-ample is a weakening of the condition that E_{k+1} be ample. In the case where X is a hypersurface in $\mathbb{P}_\Delta$, we are able to argue that under the condition that E_2 is E_1-ample, the restriction of $\mathscr{L}_2$ is ample when restricted to a hypersurface in $\mathbb{P}_\Delta$ determined by the vanishing of a generic section of $\mathscr{L}_1$.

Proposition 5.3.4. *If E_2 is E_1-ample, then $\mathscr{L}_2|_X = -K_X$ is an ample line bundle on X.*

Proof. Let us take a pencil of hypersurfaces over a Zarsiki open subset U of $\mathbb{A}^1$ so that the fiber over 0 is $\cup_{\rho \in E_1} D_\rho$ and the fiber over $U \setminus 0$ is generic. Then the total space $\mathscr{X}$ of this pencil maps properly onto U and $\mathscr{L}_2$ can be pulled back to a line bundle on $\mathscr{X}$ which is ample on the fiber over 0 by assumption. It is well known [70, III_1 Théorème 4.7.1] that if a line bundle on $\mathscr{X}$ is ample on the fiber $\mathscr{X}_t$ over $t \in U$, then there is an open subset V of U containing t so that this line bundle restricts to an ample line bundle on $\mathscr{X}_s$ for every $s \in V$. Thus for a generic hypersurface X linearly equivalent to $\cup_{\rho \in E_1} D_\rho$, the anticanonical bundle $-K_X$ is ample. $\square$

One should view this as saying that a nice enough Fano hypersurface in $\mathbb{P}_\Delta$ must admit a degeneration to a Gorenstein toric Fano variety. Here, by nice we mean that the natural degeneration of X to a union of toric divisors in $\mathbb{P}_\Delta$ is itself Fano, in the sense that its anticanonical bundle is ample.

Proposition 5.3.4 hints that the proper way of thinking about amenable collections is that they are ways of encoding deformations of unions of toric varieties with toroidal crossings which correspond to toric Gorenstein varieties. In terms of the homogeneous coordinate ring (see e.g. [36]) on $\mathbb{P}_\Delta$, which we denote $\mathbb{C}[\{z_\rho\}_{\rho\in\Delta[0]}]$, we have that

$$\cup_{\rho\in E_1} D_\rho = \left\{ \prod_{\rho\in E_1} z_\rho = 0 \right\}$$

and that the amenable collection v subordinate to E_1, E_2 is determines the toric subvariety of $\mathbb{P}_\Delta$ determined by the homogeneous equation

$$\prod_{\rho\in E_1} z_\rho + \prod_{\rho\in E_2} z_\rho^{\langle v,\rho\rangle} = 0.$$

If v does not form an amenable collection, then the equation above is not homogeneous.

Remark 5.3.5. The converse of Proposition 5.3.4 does not hold. Let us take the second Picard rank 3 example of Mori and Mukai [100]. According to [33], this is a hypersurface in the toric variety with vertices $\rho_0, \dots, \rho_6$ given by columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

and associated to a nef partition $E_1 = \{\rho_1, \dots, \rho_5\}$ and $E_2 = \{\rho_0, \rho_6\}$. The convex hull of E_1 is 4-dimensional, thus there can be no amenable collection subordinate to E_1, E_2 by Theorem 5.2.5, and hence E_2 is not E_1-ample. Therefore, the associated degeneration of $X_{3.2}$ to a union of toric divisors cannot be smoothed to a Gorenstein Fano toric variety in $\mathbb{P}_\Delta$.

In [34], a large number of new Fano fourfolds were constructed. To do this, the authors started with a smooth Fano toric variety $\mathbb{P}_\Delta$ of dimension $4 + k$ associated to a reflexive polytope Δ. They then take a nef partition $E_1, \dots, E_{k+1}$ so that E_{k+1} is associated to an ample divisor. They then conclude that the complete intersection of nef divisors associated to $E_1, \ldots, E_k$ is a smooth Fano fourfold.

Above, we have given a weaker criteria for a hypersurface in a toric Fano variety to be Fano itself, from which it may be possible to construct new Fano fourfolds. Indeed, in [33] several Fano threefolds were exhibited as complete intersections in toric varieties associated to nef partitions whose anticanonical bundle is not the restriction of an ample divisor of the ambient toric variety (for example, the Fano threefold discussed in Remark 5.3.5). Thus one can expect that there exist complete intersection Fano fourfolds which do not appear in the computations of [34]. Proposition 5.3.4 gives a way of determining whether a given hypersurface in a toric variety is Fano without requiring that $\mathscr{L}_2$ be itself ample. The following example carries this out in an example.

Example 5.3.6. Let us now construct a Fano fourfold hypersurface X in a smooth toric variety $\mathbb{P}_\Delta$ which was not detected by the authors of [34] since the bundle $\mathscr{L}_2$ is nef and big but not ample on $\mathbb{P}_\Delta$. We take the Fano toric fivefold associated to the polytope Δ in $\mathbb{R}^5$ with vertices $\rho_0, \ldots, \rho_7$ at the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 \end{pmatrix}$$

Then $\mathbb{P}_\Delta$ has Picard lattice of rank 3. The homogeneous coordinate ring of $\mathbb{P}_\Delta$ has variables $z_0, \ldots, z_7$, and the class of $z_i = 0$ in $\mathrm{Pic}(\mathbb{P}_\Delta) \cong \mathbb{Z}^3$ is given by the i^{th} column of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & -1 & -3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix}$$

in terms of a basis U, V, W of $\mathbb{Z}^3 = \mathrm{Pic}(\mathbb{P}_\Delta)$. The nef cone of $\mathbb{P}_\Delta$ is the cone spanned by vectors $-U+V-W, U, V$. If we take the nef partition $E_1 = \{\rho_3, \rho_5\}$ and $E_2 = \{\rho_0, \rho_1, \rho_2, \rho_4, \rho_6, \rho_7\}$, then $c_1(\mathscr{L}_1) = -U+2V-W$ and $c_1(\mathscr{L}_2) = U+V$ which is contained in a face of the nef cone of $\mathbb{P}_\Delta$, hence is nef but not ample.

Now we check that E_2 is actually E_1-ample. To see this, it is enough to show that for every $\rho \in E_1$, the dual faces $\rho^\vee_{\nabla_2}$ and $\rho^\vee_{\Delta^\circ}$ are Minkowski summands of one another. This is a somewhat laborious computation which can be made much simpler by using Sage [41]. In order to perform this computation, we compute the dual faces in the standard way, and note that $\rho^\vee_{\nabla_2}$ is automatically a summand of $\rho^\vee_{\Delta^\circ}$. To see that the converse is true, we use the fact that one can determine whether a polytope Q is a Minkowski summand of P using the fact that

$$(P - Q) + P = Q$$

where the minus sign is used to denote Minkowski difference. Both Minkowski sum and difference are functions built into Sage, hence this task can be reduced to a computational one.

Now we can compute that $(-K_X)^4$ is equal to 205 by using toric intersection theory theory functions built into Sage. One can check that there is no Fano fourfold with degree 205 occurring in the list of [34].

According to Øbro [111], there are 866 smooth toric Fano varieties of dimension 5. We have looked at all smooth toric Fano fivefolds with Picard ranks 1, 2 and 3, and have computed all hypersurfaces associated to nef partitions E_1, E_2 for which E_2 is E_1-ample, but $\mathscr{L}_2$ is not ample. In ranks 1 and 2, no such nef partitions exist, but in the case of the 91 smooth Fano toric fivefolds with Picard rank 3, there is a number of these hypersurfaces. We list them along with their $(-K_X)^4$ values in Table 5.1. We have ignored cases where E_1 is a single vertex because in this case, a section of $\mathscr{L}_1$ is a binomial hypersurface in $\mathbb{P}_\Delta$, thus it is itself a smooth Fano toric variety. Such objects are of course classified by [89].

Remark 5.3.7. Somewhat curiously, the only nef partitions that we found where E_2 was E_1-ample were in the case where E_1 is a union of two vertices or a single vertex. It would be interesting to know whether this is a structural property or simply a coincidence in low dimension.

Remark 5.3.8. We note that the 13th entry in Table 5.2 is just $\mathbb{P}^1 \times \mathbb{P}^3$, since the ambient toric variety is just $\mathbb{P}^3 \times \mathbb{F}_1$.

Remark 5.3.9. For all examples in Table 5.1, except for #4, there are known Fano varieties with the same degree as that listed below, so we cannot say for certain whether the Fano varieties that we have constructed are novel. On the other hand, to the best of my knowledge, entry 4 in Table 5.2 is new.

5.4 Toric degenerations of Fano threefolds

In this section, we will show that all Fano threefolds of Picard rank greater than 2 admit flat degenerations to toric varieties. There are, roughly, three sets of Fano varieties that we need to deal with.

1. Fano threefolds which are known smoothings of Fano toric varieties with small resolvable singularities.
2. Fano threefolds which occur as complete inetersections in toric varieties.
3. Fano threefolds which are products of $\mathbb{P}^1$ and a del Pezzo surface.
4. None of the above.

The first class is known, by work of Galkin [60]. There are 40 families of Fano threefolds with Picard rank 2, 3 or 4 which are known smoothings of small-resolvable toric Fano threefolds. According to [33], there are only six Fano threefolds of Picard rank greater than 1 which are not isomorphic to $\mathbb{P}^1 \times \mathrm{dP}$ for some del Pezzo surface dP, which cannot be obtained as a complete intersection in a toric variety. Most of these are also smoothings of Fano toric varieties with small-resolvable singularities. The fourth category listed above consists of just one Fano variety, which is called

$X_{2.14}$, the 14th Picard rank 2 example in the list of Mori-Mukai [100]. If we let B_5 be the complete intersection in $\mathrm{Gr}(2,5)$ of three hyperplane sections under the Plücker embedding, then $X_{2.14}$ is the blow up of B_5 in the union of two hyperplane sections, which is an elliptic curve. In [33], the Fano $X_{2.14}$ is also described as a hypersurface of degree $(1,1)$ in $B_5 \times \mathbb{P}^1$.

It is tautological that Fano threefolds of class (1) admit degenerations to toric Fano varieties. If X is in the third class, then there are known toric degenerations of all del Pezzo surfaces, so there is nothing to prove. If X is in the second class and it is a complete intersection of sections of line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_{d-3}$ on a toric variety of dimension d determined by a reflexive polytope Δ, then if $\omega_{\mathbb{P}_\Delta}^{-1} \otimes \mathscr{L}_1^{-1} \otimes \cdots \otimes \mathscr{L}_{d-3}^{-1}$ is ample on $\mathbb{P}_\Delta$, then we know that X admits a toric degeneration by Theorem 5.1.3. This is true in all cases of complete intersections except for:

$$2.1, 2.2, 2.3, 2.8, 3.1, 3.2, 3.4, 3.5, 3.14, 3.16, 4.2, 4.6,$$

Here the notation $x.y$ refers to the y^{th} entry in the table of rank x Fano threefolds of Mori and Mukai [100]. The last four are smoothings of toric Fano threefolds with only small resolvable singularities. Thus we have only eight Fano varieties for which we need to prove that there exist toric degenerations. In the cases listed above, one needs to show by hand that there exist appropriate amenable collections. These are listed in Table 5.2. The case of $X_{2.14}$ is more involved, but we will deal with it in much the same way. This is done in Example 5.4.2.

Remark 5.4.1. Nathan Ilten has informed me that he has independently obtained the same result by similar methods – using the expressions for Fano threefolds as complete intersections in toric varieties in a number of cases and using ad hoc methods in every other case.

Example 5.4.2 ($X_{2.14}$)**.** To see that $X_{2.14}$ admits a degeneration to a toric variety, we note that B_5 is a smoothing of the small-resolvable toric variety determined by the polytope Δ with vertices given by the columns of

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 \end{pmatrix}.$$

Therefore, there is a degeneration of $B_5 \times \mathbb{P}^1$ to $\mathbb{P}_\Delta \times \mathbb{P}^1$ and $X_{2.14}$ degenerates to a hypersurface in $\mathbb{P}_\Delta \times \mathbb{P}^1$. There is a nef partition corresponding to this hypersurface. The polytope Δ' with vertices $v_0, \dots, v_8$ determined by columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

and $X_{2.14}$ degenerates to a nef divisor corresponding to the vertices $E_1 = \{v_2, v_3, v_5, v_6, v_7\}$ of Δ'. It is then enough to show that such a hypersurface admits a subordinate amenable collection. But the line bundle $\omega^{-1}_{X_{\Delta'}} \otimes \mathscr{L}_1$ is ample, thus Theorem 5.1.3 suffices to show that such an amenable collection exists.

Finally, we state this as a theorem.

Theorem 5.4.3. *If X is a Fano threefold, then X admits a degeneration to a toric variety. If X has degree ≥ 10, then X admits a degeneration to a toric Gorenstein Fano variety.*

This of course follows from the result in rank 1 of [76] and the computations above.

Remark 5.4.4. The attentive reader will note that the results of [46] only guarantee that there is a degeneration of X to a weak Fano partial crepant resolution $\widetilde{X}'$ of a Gorenstein Fano toric variety X'. However, an argument communicated to me by Nathan Ilten shows that if X is Fano, then one indeed obtains a flat degeneration of X to the anticanonical model of $\widetilde{X}'$, which is X' itself. Roughly, one takes the relative anticanonical model of the total space of the degeneration of X to $\widetilde{X}'$ and argues that the resulting variety has X as the general fiber and X' as the special fiber.

5.5 Tables

#	Weight matrix	Weight of X	E_1	$(-K_X)^4$
1	$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$	$(0,1,1)$	$[4,6]$	341
2	$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix}$	$(0,2,-1)$	$[3,5]$	170
3	$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$	$(1,1,1)$	$[4,6]$	260
4*	$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & -1 & -3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix}$	$(-1,2,-1)$	$[3,5]$	205
5	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$	$(1,0,1)$	$[0,5]$	512
6	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & -2 & 3 \end{pmatrix}$	$(0,1,3)$	$[0,7]$	376
7	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & 2 \end{pmatrix}$	$(0,1,2)$	$[0,7]$	431
8	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$	$(1,0,1)$	$[4,6]$	376
9	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$	$(1,1,1)$	$[4,6]$	295
10	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$	$(1,2,1)$	$[5,7]$	240
11	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$	$(1,0,1)$	$[4,6]$	431
12	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$	$(1,1,1)$	$[5,7]$	350
13	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$	$(1,0,1)$	$[5,7]$	512

Table 5.1 Examples of Fano fourfold hypersurfaces in smooth toric Fano fivefolds so that $\mathscr{L}_2$ is not ample.

#	Polytope vertices	($\mathbb{Q}$)-Nef divisors	Amenable collection
2.1	$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 3 & -2 \end{pmatrix}$	$E_1 = \{1,2,5,6\}$	$v_1 = \begin{pmatrix} 0 \\ -1 \\ 6 \\ -2 \end{pmatrix}$
2.2	$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$	$E_1 = \{1,2,3,5\}$	$v_1 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ -1 \end{pmatrix}$
2.3	$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ -1 & -1 & -2 & -1 & -5 & 4 \end{pmatrix}$	$E_1 = \{1,2,4,5\}$	$v_1 = \begin{pmatrix} 2 \\ 1 \\ 6 \\ 2 \end{pmatrix}$
2.8	$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}$	$E_1 = \{0,1,2,3,4\}$	$v_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$
3.1	$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$	$E_1 = \{1,2,3,4,5,6\}$	$v_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$
3.2	$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$	$E_1 = \{2,5,6\}$	$v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$
3.4	$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$	$E_1 = \{0,2,3,4,5\}$	$v_1 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ -1 \end{pmatrix}$
3.5	$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$	$E_1 = \{3,7\}$ $E_2 = \{2,6\}$	$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

Table 5.2 Amenable collections determining degenerations for Fano threefolds for which no toric degenerations are guaranteed by other methods.

Part III

Fibrations on Calabi-Yau varieties in mirror symmetry

Chapter 6

Calabi-Yau varieties and Tyurin degenerations

6.1 Introduction

Here we will concern ourselves with a conjectural relationship between quasi-Fano varieties, LG models and compact Calabi-Yau varieties. A variant of the construction that we propose here has been described by Auroux [7] in detail in the case where V is a double cover of a Fano variety X ramified over a smooth member of $|-2K_X|$, and it is hinted at by Tyurin at the end of [141].

A Calabi-Yau manifold V can be built up from pairs of quasi-Fano manifolds X_1 and X_2. We require that Z be a smooth member of both $|-K_{X_1}|$ and $|-K_{X_2}|$, and that N_{Z/X_1} and N_{Z/X_2} be inverses of one another. By adjunction, Z is a Calabi-Yau variety of dimension $d-1$. We also require that there is some ample class D in $\mathrm{Pic}(Z)$ so that there are ample classes $D_1 \in \mathrm{Pic}(Y_1)$ and $D_2 \in \mathrm{Pic}(Y_2)$ so that $D_1|_Z = D_2|_Z = D$. Then we may take the variety which is a normal crossings union of X_1 and X_2 meeting along Z, which we denote $X_1 \cup_Z X_2$. A theorem of Kawamata and Namikawa says that there exists a complex manifold $\mathscr{V}$ equipped with a map $\pi : \mathscr{V} \to U$ to a small analytic disc U so that $\pi^{-1}(0) = X_1 \cup_Z X_2$ and $\pi^{-1}(t)$ is a smooth Calabi-Yau manifold for any $t \in U \setminus \{0\}$. Lee [95] has computed the Hodge numbers of $\pi^{-1}(t)$ in the case where X_1 and X_2 are smooth threefolds. Let us define $\rho_i : \mathrm{H}^2(X_i, \mathbb{Q}) \to \mathrm{H}^2(Z, \mathbb{Q})$ for $i = 1, 2$ and define $k = \mathrm{rank}\,(\mathrm{Im}(\rho_1) \cap \mathrm{Im}(\rho_2))$.

Theorem 6.1.1. *Let V be a Calabi-Yau threefold constructed as above. Then*

- $h^{1,1}(V) = h^2(X_1) + h^2(X_2) - k - 1$
- $h^{1,2}(V) = 21 + h^{1,2}(X_1) + h^{1,2}(X_2) - k.$

I conjecture that the degeneration of V to $X_1 \cup_Z X_2$ appears via mirror symmetry as a fibration of the mirror W by K3 surfaces. In the next few sections, I will try to justify this expectation by providing evidence from topology and toric geometry. First, we will look at the LG models of X_1 and X_2 and show that mirror symmetry suggests that the condition that $-K_{X_1}|_Z - K_{X_2}|_Z = 0$ implies that the LG model of X_1 and the LG model of X_2 may be glued together to form the mirror of V. We prove that if mirror symmetry holds for Hodge numbers of X_i and (Y_i, w) then it follows that the gluing of Y_1 to Y_2 has the correct Euler characteristic to be mirror to W, and in the threefold case, we show that if $Y_1 \cup Y_2$ admits a complex structure so that it is Calabi-Yau, then it must have Hodge numbers mirror to those of V.

Next, we will put this into practice in the case where V is a 3-dimensional anticanonical hypersurface in a Gorenstein toric Fano fourfold. We will show that if there is a nef partition F_1, F_2, then there is a Tyurin degeneration of V. We will show that there is a mirror pencil of varieties in W, the Batyrev dual to V, which are birational to K3 surfaces. We show that this induces a K3 fibration on a smooth birational model of W, and finally, we show that the singular fibers of the pencil K3 surfaces on W contain numerical information coming from the Tyurin degeneration.

6.2 Gluing LG models

Let us first define what we mean by Landau-Ginzburg model in this section. We have defined in Chapter 2 a notion of a LG model which conjecturally encapsulates the LG models of Fano varieties, and we have seen that this goes further to describe the LG model of many quasi-Fano varieties. However, there seems to be a class of quasi-Fano varieties for which this notion does not suffice. In particular, it seems forced upon us in examples that there exist quasi-Fano varieties so that the mirror LG model (Y, w) is just a pair so that Y is a Kähler manifold and w is a holomorphic function. We note that this leaves room for the image of w to be an open set in $\mathbb{C}$. However, we will insist that $\mathrm{H}^1(Y) = 0$. Since the cohomology of Y need not admit a mixed Hodge structure in this case, it seems unclear how to define Hodge numbers of (Y, w) in this case. Instead, we propose that if (Y, w) is mirror to X, then we have

$$h^i(Y, V) = \sum_j h^{d-i+j,j}(X).$$

We also expect that if (Y, w) is mirror to X, then the smooth fibers of w are mirror to anticanonical hypersurfaces in X.

Letting notation be as in previous section, we will discuss the possible relationship between the quasi-Fano varieties X_1, X_2 and the Calabi-Yau fibration on W which is conjecturally mirror dual to the degeneration of V to $X_1 \cup_Z X_2$. It seems natural to predict that the LG models of X_1 and X_2 can be somehow glued together to give W, since we are, in a topological sense, gluing X_1 and X_2 together to form V (see [141] for details on this topological construction).

First of all, we recall that we expect that if Y_i is the LG model of X_i equipped with superpotetial w_i then the monodromy symplectomorphism induced on $\mathsf{w}_i^{-1}(t)$ associated to a small loop around ∞ for t a regular value of w_i can be identified under mirror symmetry with the restriction of the the Serre functor of $\mathbf{D}^b(X_i)$ to $\mathbf{D}^b(Z)$ [79, 133]. The Serre functor is simpy $\otimes\omega_{X_i}[d]$ where $[d]$ denotes shift by $d = \dim X_i$. Thus, up to a choice of shift, we see that the action of monodromy on $\mathsf{w}_i^{-1}(t)$ should be identified with the autoequivalence of $\mathbf{D}^b(Z)$ induced by taking the tensor product with $\omega_{Y_i}^{-1}|_Z = N_{Z/X_i}$. So f $X_1 \cup_Z X_2$ can be smoothed to a Calabi-Yau manifold, then we must have that $N_{Z/X_1} \otimes N_{Z/X_2} = \mathscr{O}_Z$, which means that we should have that the monodromy symplectomorphism ϕ_1 associated to a clockwise loop around infinity on $\mathsf{w}_1^{-1}(t)$ is the same as the monodromy ϕ_2^{-1} associated to a *counter*-clockwise loop around infinity on $\mathsf{w}_2^{-1}(t)$. Note that this implies that this makes use of the assumption that the fibers of w_1 and w_2 are topologically the same Calabi-Yau manifold which we denote Q. This is a stronger assumption than just the fact that they are both mirror to Z. Let r_1 and r_2 be such that $|\lambda| \leq r_1$ for every λ in the critical locus of w_1 and $|\lambda| \leq r_2$ for every λ in the critical locus of w_2. Then we may choose a local trivialization of Y_1 and Y_2 over U_{r_1} and U_{r_2} respectively, where $U_{r_1} = \{z \in \mathbb{C} : |z| > r_1\}$. Let $P_{r_i} = \mathsf{w}_i^{-1}(U_{r_i})$ for $i = 1, 2$. The local trivializations of P_1 and P_2 that we have chosen are topologically equivalent to expressing P_i as a gluing of the ends of $D_i = Q \times [-1, 1] \times (-1, 1)$ to itself along the map

$$\widetilde{\phi}_i : p \times \{-1\} \times (z) \mapsto \phi_i(p) \times \{1\} \times (z).$$

where ϕ_i is the monodromy symplectomorphism, and which identifies $Q \times \{-1\} \times (-1, 1)$ with $Q \times \{1\} \times (-1, 1)$. Therefore, we may glue Y_1 to Y_2. Recall that the condition that $X_1 \cup_Z X_2$ smooths to a Calabi-Yau variety conjecturally implies that $\phi_1 = \phi_2^{-1}$. In this case we can identify D_1 with D_2 by the map

$$\tau : p \times [x] \times (z) \mapsto p \times [-x] \times (-z)$$

Under this identification of D_1 and D_2, it is clear that $\tau \cdot \widetilde{\phi}_1 = \widetilde{\phi}_2$. Thus the identification τ gives an isomorphism between P_1 and P_2, thus it allows us to glue Y_1 to Y_2 along P_1 and P_2 to produce a C^∞ manifold W. This gluing respects the fibrations w_1 and w_2 thus X is equipped with a fibration π over the gluing of $\mathbb{C}$ with $\mathbb{C}$ described above. It is clear that the base of this fibration is just the 2-sphere S^2.

Theorem 6.2.1. *Let Y_1 and Y_2 be Landau-Ginzburg models of d-dimensional quasi-Fano varieties X_1 and X_2 which contain the same anticanonical Calabi-Yau hypersurface Z and that $K_{X_1}|_Z + K_{X_2}|_Z = 0$. Let V be a Calabi-Yau variety obtained from $X_1 \cup_Z X_2$ by smoothing and let W be the variety obtained by gluing Y_1 to Y_2 as above. Then*

$$\chi(V) = (-1)^d \chi(W).$$

Proof. Start by recalling the long exact sequence of the pair $(Y_i, \mathsf{w}_i^{-1}(t))$ for t a regular value of w_i, and the fact that Euler numbers (which we denote $\chi(X)$) are additive in long exact sequences.

$$\cdots \to \mathrm{H}^n(Y_i, \mathbb{C}) \to \mathrm{H}^n(\mathsf{w}_i^{-1}(t), \mathbb{C}) \to \mathrm{H}^{n+1}(Y_i, \mathsf{w}_i^{-1}(t); \mathbb{C}) \to \mathrm{H}^{n+1}(Y_i, \mathbb{C}) \to \dots$$

Thus we have that $\chi(Y_i) = \chi(Y_i, \mathsf{w}_i^{-1}(t)) + \chi(\mathsf{w}_i^{-1}(t))$. Under mirror symmetry, we see that $\chi(Y_i, \mathsf{w}^{-1}(t)) = (-1)^d \chi(X_i)$ (this is a consequence of Hodge number mirror symmetry as described in Chapter 2) where d is the dimension of Y_i. Thus $\chi(Y_i) = (-1)^d \chi(X_i) + \chi(\mathsf{w}_i^{-1}(t)))$. On the other hand, we have the Mayer-Vietoris exact sequence

$$\cdots \to \mathrm{H}^n(W, \mathbb{C}) \to \mathrm{H}^n(Y_1, \mathbb{C}) \oplus \mathrm{H}^n(Y_2, \mathbb{C}) \to \mathrm{H}^n(Y_1 \cap Y_2, \mathbb{C}) \to \dots$$

and so $\chi(W) = \chi(Y_1) + \chi(Y_2) - \chi(Y_1 \cap Y_2)$. Since $Y_1 \cap Y_2$ is a fibration over an annulus, we can compute its cohomology using the Wang sequence [120, Theorem 11.33]

$$\cdots \to \mathrm{H}^n(Y_1 \cap Y_2, \mathbb{C}) \to \mathrm{H}^n(\mathsf{w}^{-1}(t), \mathbb{C}) \xrightarrow{T_n - \mathrm{Id}} \mathrm{H}^n(\mathsf{w}^{-1}(t), \mathbb{C}) \to \dots$$

Where T_n is the action of monodromy on $H^n(\mathsf{w}^{-1}(t), \mathbb{C})$ associated to a small loop around our annulus. Thus $\chi(Y_1 \cap Y_2) = 0$. Since $\mathsf{w}^{-1}(t)$ is topologically mirror to Z by assumption, we have that $\chi(Z) = (-1)^{d-1}\chi(\mathsf{w}^{-1}(t))$ and therefore

$$\chi(W) = (-1)^d(\chi(X_1) + \chi(X_2) - 2\chi(Z)).$$

Let $g : \mathscr{V} \to \Delta$ be a smoothing of $X_1 \cup_Z X_2$, in other words, assume that $g^{-1}(0) = X_1 \cup_Z X_2$ and that $g^{-1}(s) = V_s$ be a smooth Calabi-Yau variety for $s \in \Delta \setminus \{0\}$. Then we can compute the Euler characteristic of V_s by applying [94, Proposition IV.6] which claims that

$$\chi(V_s) = \chi(X_1) + \chi(X_2) - 2\chi(Z).$$

Therefore we have that $\chi(W) = (-1)^d \chi(V)$ as claimed. $\square$

This is precisely the relationship between Euler characteristics of mirror dual Calabi-Yau varieties. Now we will provide more evidence for the fact that W this is the mirror dual of the original Calabi-Yau variety V in the case where V is a Calabi-Yau threefold.

Proposition 6.2.2. *Let W be as above and let S be a fiber of the map π. Assume that* $\dim W = \dim S + 1 = 3$. *Then $b_2(W) = 1 + b_2(Y_1) + b_2(Y_2) - \ell$ where ℓ is rank of the subgroup of* $\mathrm{H}^2(S, \mathbb{C})$ *spanned by the intersection of the images of* $\mathrm{H}^2(Y_1, \mathbb{C})$ *and* $\mathrm{H}^2(Y_2, \mathbb{C})$ *under the natural restriction maps.*

Proof. Let U be the annulus along which D_1 and D_2 are glued, and let $Q = \pi^{-1}(U)$ be its preimage in W. We have that

$$0 \to \mathrm{H}^2(Y_i, S; \mathbb{C}) \to \mathrm{H}^2(Y_i, \mathbb{C}) \to \mathrm{H}^2(S, \mathbb{C}) \to \dots$$

The image of the restriction map $\mathrm{H}^2(Y_i, \mathbb{C}) \to \mathrm{H}^2(S, \mathbb{C})$ is the space of monodromy invariant cycles in the monodromy representation of w_i. Thus $h^2(Y_i)$ is equal to $h^{2,1}(Y_i) + \operatorname{rank} H^2(S, \mathbb{C})^{\rho_i}$, where ρ_i is the monodromy representation of $(Y_i^\vee, \mathsf{w}_i)$. Now we will compute the rank of $\mathrm{H}^2(W, \mathbb{C})$ using the sequence

$$\dots \longrightarrow \mathrm{H}^1(Q, \mathbb{C}) \longrightarrow \mathrm{H}^2(W, \mathbb{C}) \longrightarrow \mathrm{H}^2(Y_1, \mathbb{C}) \oplus \mathrm{H}^2(Y_2, \mathbb{C}) \xrightarrow{r_1^U - r_2^U} \mathrm{H}^2(Q, \mathbb{C}) \longrightarrow \dots.$$

We can use the Wang sequence to compute that $\mathrm{H}^1(Q, \mathbb{C}) = \mathbb{C}$. Using the assumption that $\mathrm{H}^1(Y_1, \mathbb{C}) = \mathrm{H}^1(Y_2, \mathbb{C}) = 0$, we see that $\mathrm{H}^2(W, \mathbb{C})$ is isomorphic to the direct product of $\mathbb{C}$ and the kernel of the restriction map $\mathrm{H}^2(Y_1, \mathbb{C}) \oplus \mathrm{H}^2(Y_2, \mathbb{C}) \xrightarrow{r_1^Q - r_2^Q} \mathrm{H}^2(Q, \mathbb{C})$, where r_i^Q are the natural restriction maps from $\mathrm{H}^2(Y_i, \mathbb{C})$ to $\mathrm{H}^2(Q, \mathbb{C})$. We note that this map fits into a commutative triangle,

$$\begin{tikzcd}
\mathrm{H}^2(Y_1, \mathbb{C}) \oplus \mathrm{H}^2(Y_2, \mathbb{C}) \arrow[r, "r_1^Q - r_2^Q"] \arrow[dr, "r_1^S - r_2^S"'] & \mathrm{H}^2(Q, \mathbb{C}) \arrow[d, "r_Q^S"] \\
& \mathrm{H}^2(S, \mathbb{C})
\end{tikzcd}$$

where the map r_Q^S is injective by the Wang sequence and the vanishing of $h^1(S)$, thus the kernel of $r_1^S - r_2^S$ is the same as the kernel of $r_1^Q - r_2^Q$. It remains to compute the rank of the rank of this kernel. By the assumption that mirror symmetry holds, we have that $h^2(Y_i, S) = h^{1,2}(X_i)$. Therefore, the relative cohomology sequence

$$\cdots \to \mathrm{H}^1(S, \mathbb{C}) = 0 \to \mathrm{H}^2(Y_i, S; \mathbb{C}) \to \mathrm{H}^2(Y_i, \mathbb{C}) \to \mathrm{H}^2(S, \mathbb{C}) \to \dots$$

implies that $h^2(Y_i, S) = \mathrm{rank}\,(\ker(r_i^S))$. Thus the rank of the kernel of $r_1^Q - r_2^Q$ is equal to $h^{1,2}(X_1) + h^{1,2}(X_2) - \mathrm{rank}\,(\mathrm{Im}(r_1^Q) \cap \mathrm{Im}(r_2^Q))$. Thus the proposition follows. $\square$

Therefore, if W admits a complex structure for which it is Calabi-Yau, then we can compute that

$$\begin{aligned}\chi(W) &= 2h^{1,1}(W) - 2h^{1,2}(W)\\ &= 2(1 + h^{1,2}(X_1) + h^{1,2}(X_2) - \ell) - 2h^{1,2}(W)\end{aligned}$$

But we also know that

$$\begin{aligned}\chi(W) &= -\chi(X_1) - \chi(X_2) + 48\\ &= -(4 + 2h^{1,1}(X_1) + 2h^{1,1}(X_2) - 2h^{1,2}(X_1) - 2h^{1,2}(X_2)) + 48\end{aligned}$$

Therefore, we have that $h^{1,2}(W) = \ell - 21 + h^{1,1}(X_1) + h^{1,1}(X_2)$. So in order for W and V to be topologically mirror to one another, we must have

$$\ell - 21 + h^{1,1}(X_1) + h^{1,1}(X_2) = h^{1,1}(X_1) + h^{1,1}(X_2) - k - 1$$

which is equivalent to $\ell + k = 20$. This is true if S and the fiber R of π are Dolgachev-Nikulin dual, given the lattice polarization on S coming from the common image of the restriction maps $\mathrm{H}^2(X_i, \mathbb{Z}) \to \mathrm{H}^2(S, \mathbb{Z})$ and the lattice polarization on a fiber R of π in W given by the restriction $\mathrm{H}^2(W, \mathbb{Z}) \to \mathrm{H}^2(R, \mathbb{Z})$. Thus mirror symmetry for V and W is consistent with mirror symmetry for R and S.

6.3 Batyrev-Borisov duality and K3 fibrations

The results of this section should be thought of as illustrations of the situation considered in Section 6.2. We will show that if V is a Calabi-Yau complete intersection of nef divisors in a d-dimensional toric variety X_Δ, so that there is a nef partition $E_1, \dots, E_k$ determining V, and if we have another nef partition $F_1, \dots, F_{k+1}$ so that $E_k = F_k \cup F_{k+1}$, then there is first of all a Tyurin degeneration associated to this combinatorial data, and also a pencil of quasi-smooth varieties birational to Calabi-Yau $(d-1)$-folds inside of the Batyrev-Borisov dual W. In the case where V is a threefold or surface, we show that this induces a K3 surface fibration on some birational model of W. In the case where V is a hypersurface of dimension 2 or 3, we show that the singular fibers of the fibration on W carry information about the Tyurin degeneration of V. Now let us build Tyurin degenerations out of refinements of nef partitions.

Definition 6.3.1. Let Δ be a reflexive polytope and let $E_1, \dots, E_k$ be a nef partition of the vertices of Δ. Then a refinement of $E_1, \dots, E_k$ is another nef partition $F_1, \dots, F_{k+1}$ so that $E_k = F_k \cup F_{k+1}$.

Let us now take $\mathscr{L}_i$ to be line bundles on X_Δ associated to each E_i. A refinement of our nef partition gives rise to a pair of nef line bundles $\mathscr{L}'_k$ and $\mathscr{L}'_{k+1}$ so that $\mathscr{L}'_k \otimes \mathscr{L}'_{k+1} = \mathscr{L}_k$. Let $s_i \in \mathrm{H}^0(X_\Delta, \mathscr{L}_i)$ be generic sections determining a quasi-smooth Calabi-Yau complete intersection V in X_Δ. If we let s'_k and s'_{k+1} be sections of $\mathscr{L}'_k$ and $\mathscr{L}'_{k+1}$ respectively, then $s'_k s'_{k+1}$ is a section of $\mathscr{L}_k$. Therefore, we can construct a pencil of complete intersections. First, let

$$U = \cap_{i=1}^{k-1} \{s_i = 0\}$$

and assume that U is connected and quasi-smooth. It is clear that U is quasi-Fano as well. Then take the pencil

$$\mathscr{Q} : \{t s_k - s'_k s'_{k+1} = 0\} \cap U$$

in $\mathbb{A}^1 \times X_\Delta$ with t a parameter on $\mathbb{A}^1$. If we assume that X_Δ is a smooth resolution of $\mathbb{P}_\Delta$, then the only singularities of $\mathscr{Q}$ in a neighbourhood of $0 \in \mathbb{A}^1$ are along $t = s_k = s'_k = s'_{k+1} = 0$, which we call Q. We can blow up $t = s'_k = 0$ inside of $\mathbb{A}^1 \times X_\Delta$ and take the proper transform of $\mathscr{Q}$ to resolve this singularity, assuming that the singular locus itself is smooth. If this can be done, then we obtain a Tyurin degeneration of V so that the fiber over 0 of the degeneration is equal to a quasi-Fano variety X_1, which is just $U \cap \{s'_k = 0\}$ blown up at $U \cap \{s'_k = s_k = s'_{k+1} = 0\}$ and a quasi-Fano variety X_2 given by $U \cap \{s'_{k+1} = 0\}$.

Note that since Q is the intersection of a set of nef divisors in X_Δ, it has no base locus, and its singularities are contained in the singular set of X_Δ. Furthermore, the intersection of Q with any torus invariant subvariety of X_Δ is irreducible, thus Q itself is either irreducible or a union of non-intersecting subvarieties of X_Δ. As a result, if X_Δ is smooth, then so is Q, for general enough choices of sections. In the general situation, we can still perform all of the steps above, but we will have singularities occurring at every step in general. Regardless, such an the resulting degeneration should include data corresponding to the quasi-Fano varieties X_1, X_2 and the blown up locus Q. We note that a version of the results of Kawamata and Namikawa for mildly singular varieties has been explored in the thesis of Lee [94]. Another way to interpret the singular case is that the union of X_1 and X_2 is equipped with a log structure (see e.g. [2] and the references therein) which accounts for the subvariety Q and determines the smoothing of $X_1 \cup_Z X_2$ to V.

Now we will look at how this nef partition is reflected in the mirror. At this point we restrict ourselves to the case where V is a hypersurface and $d = 2$ or 3. We have that the Batyrev dual Calabi-Yau variety W is embedded as an anticanonical hypersurface in X_{Δ°. There is also a dual nef partition of Δ_1 and Δ_2 of $\nabla \subseteq \Delta^\circ$. As usual we denote the dual nef partition as ∇_1, ∇_2. It does not follow that this can be extended to a degeneration of W to quasi-Fano varieties dual to X_1 and X_2. We have that W is cut out by equations

$$f = \left\{ \sum_{\rho \in \Delta_i \cap M} a_\rho \prod_{\sigma \in \partial \Delta^\circ \cap N} z_\sigma^{\langle \sigma, \rho \rangle + b_\sigma} = 0 \right\} \text{ for } 1 \leq i \leq k-1.$$

By scaling, we may assume that the coefficient of $\prod_{\sigma \in \partial \Delta^\circ}$ is equal to 1. We will take a pencil $\mathscr{P}(q), q \in \mathbb{P}^1_{s,t}$ of hypersurfaces in W defined by the equations:

$$s \sum_{\rho \in \Delta_1 \cap M \setminus 0_M} a_\rho \prod_{\sigma \in \partial \Delta^\circ \cap N} z_\sigma^{\langle \sigma, \rho \rangle + 1} = t \prod_{\rho \in \partial \Delta^\circ \cap N} z_\rho$$
$$s \sum_{\rho \in \Delta_2 \cap M \setminus 0_M} a_\rho \prod_{\sigma \in \partial \Delta^\circ \cap N} z_\sigma^{\langle \sigma, \rho \rangle + 1} = (s-t) \prod_{\rho \in \partial \Delta^\circ \cap N} z_\rho.$$

Note that the sum of these last two equations is just sf for some constant s, thus for all $[s,t] \in \mathbb{P}^1_{s,t}$ the variety $\mathscr{P}([s:t])$ is contained in W. We need to prove that this is a blown up K3 surface.

Proposition 6.3.2. *If $d-k=3$ then the general member of the above pencil is smooth blown up K3 surface. If $d-k=2$, then the general member of the above pencil is a smooth elliptic curve.*

Proof. If we let $\hat{S}$ be a generic hypersurface in W determined by the equations above, then $\hat{S} \cap (\mathbb{C}^\times)^4$ is determined by equations

$$\sum_{\rho \in \Delta_1 \cap M} a_\rho z^\rho = 0 \qquad \sum_{\rho \in \Delta_2 \cap M} a_\rho z^\rho = 0$$

which naturally compactifies to the K3 surface or an elliptic curve in X_∇ associated to the nef partition ∇_1, ∇_2 depending whether $d=3$ or 4. Therefore $\hat{S}$ is birational to a K3 surface in the $d=4$ case and $\hat{S}$ is birational to an elliptic curve in the $d=3$ case. If we can show that $\hat{S}$ is smooth, then since minimal models of surfaces are obtained by sequentially contracting (-1) curves, we may conclude that $\hat{S}$ is a blown up K3 surface in the $d=4$ case, and that $\hat{S}$ is an elliptic curve in the $d=3$ case. We will restrict our proof to the $d=4$ case.

To see that $\hat{S}$ is smooth for a generic choice of $\hat{S}$, we note that since $\hat{S}$ is contained in a Calabi-Yau hypersurface, which itself does not generically contain any toric strata of X_{Δ°, $\hat{S}$ does not contain any toric strata. Now we look at $\hat{S}$ and W on charts of X_{Δ°. If σ is a point in $\partial\Delta^\circ$, we know that if σ, u_1, u_2, u_3 is a basis of N, then the chart corresponding to σ is $\mathbb{C} \times (\mathbb{C}^\times)^3$ and W is given in this chart by

$$\sum_{\rho\in\Delta} a_\rho x^{\langle\sigma,\rho\rangle+1} \prod_{i=1,2,3} z_i^{\langle u_i,\rho\rangle} = 0.$$

and that the divisor corresponding to σ is given by $x = 0$ in this chart. Now, explicitly, we are given that S is given by the vanishing of the polynomials

$$f_1 = \sum_{\rho\in\Delta_1} a_\rho x^{\langle\sigma,\rho\rangle+1} \prod_{i=1,2,3} z_i^{\langle u_i,\rho\rangle} \qquad f_2 = \sum_{\rho\in\Delta_2} a_\rho x^{\langle\sigma,\rho\rangle+1} \prod_{i=1,2,3} z_i^{\langle u_i,\rho\rangle}$$

on this chart. Note that these polynomials are in $\mathbb{C}[x, u_1^{\pm1}, u_2^{\pm1}, u_3^{\pm1}]$. We know that, away from $x = 0$, this is smooth. Since both of the polynomials above are generic polynomials with Newton polytopes Δ_1 and Δ_2, they do not have any singularities on $(\mathbb{C}^\times)^4 \subseteq \mathbb{C}\times(\mathbb{C}^\times)^3$. Again, restricting to $x = 0$, we see that that $f_1|_{x=0}$ and $f_2|_{x=0}$ are generic Laurent polynomials with some given Newton polytope, since the coefficients are allowed to vary arbitrarily. Now we check to see whether $f_1 = f_2$ is smooth on $(\mathbb{C}^\times)^3 \times \mathbb{C}$ explicitly. The singular locus is given by the vanishing of the minors of the matrix

$$\begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial u_1 & \partial f_1/\partial u_2 & \partial f_1/\partial u_3 \\ \partial f_2/\partial x & \partial f_2/\partial u_1 & \partial f_2/\partial u_2 & \partial f_2/\partial u_3 \end{pmatrix}.$$

Since $f_1 = f_2 = 0$ is smooth where $x \neq 0$, we may search for singularities when $x = 0$. Ignoring the first column of this matrix and letting $x = 0$, we get

$$\begin{pmatrix} \partial f_1|_{x=0}/\partial u_1 & \partial f_1|_{x=0}/\partial u_2 & \partial f_1|_{x=0}/\partial u_3 \\ \partial f_2|_{x=0}/\partial u_1 & \partial f_2|_{x=0}/\partial u_2 & \partial f_2|_{x=0}/\partial u_3 \end{pmatrix}.$$

which we have already determined is rank 2 on $f_1|_{x=0} = f_2|_{x=0}$, since $f_1|_{x=0} = f_2|_{x=0} = 0$ is smooth. Thus $f_1 = f_2 = 0$ is smooth on $\mathbb{C} \times (\mathbb{C}^\times)^3$. Similar arguments suffice to analyze the intersection of $\hat{S}$ with the codimension 2 and 3 strata of X_{Δ°. Since W does not intersect codimension 4 strata of X_{Δ°, neither does $\hat{S}$ so this is enough to show that $\hat{S}$ is smooth.

In the case where $d = 3$, a similar proof shows that members of the pencil in question are smooth and birational to elliptic curves, thus they correspond to a pencil of elliptic curves on W. □

In the case where W is a threefold or a surface, we will show that there is in fact a K3 surface fibration on a birational model of W whose fibers are birational to the members of the pencil $\mathscr{P}$.

Proposition 6.3.3. *If $Z \subseteq W$ is a smooth Calabi-Yau $(d-1)$-fold in a d-dimensional Calabi-Yau manifold, then $|Z|$ is base-point free and hence there is a map $f : W \to \mathbb{P}^1$ so that Z is a fiber of f.*

Proof. By adjunction, $\mathscr{O}_Z(Z) = N_{Z/W} = \omega_Z = \mathscr{O}_Z$. Thus $h^0(N_{Z/W}) = 1$. Let $\iota : Z \hookrightarrow W$, then we have a short exact sequence of sheaves

$$0 \to \mathscr{O}_W \xrightarrow{s} \mathscr{O}_W(Z) \to \iota^*\mathscr{O}_Z(Z) \cong \iota^*\mathscr{O}_Z \to 0$$

where s is a section of $\mathscr{O}_W(Z)$ whose vanishing locus is Z. We have a long exact sequence in cohomology groups which provides the map

$$0 \to \mathrm{H}^0(W, \mathscr{O}_W) \to \mathrm{H}^0(W, \mathscr{O}_W(Z)) \to \mathrm{H}^0(W, \iota^*\mathscr{O}_Z) \to 0$$

and thus a generic section of $\mathscr{O}_W(Z)$ to S is a nonzero section of $\iota^*\mathscr{O}_Z$ which is *a fortiori* non-vanishing. Thus $|Z|$ is base-point free and determines a map to $\mathbb{P}^1$ from W, since $h^0(\mathscr{O}_W(Z)) = 2$ by the exact sequence above. □

Therefore it follows immediately that in the case where $d = 3$, the pencil $\mathscr{P}$ is an elliptic fibration on W. Next we prove that if W is a Calabi-Yau threefold containing a blown up K3 surface $\hat{S}$, then we can get rid of the (-1) curves in S by performing birational transformations

Proposition 6.3.4. *Let W be a Calabi-Yau threefold and let $\hat{S}$ be a blown up K3 surface in W. If C is a (-1) curve in $\hat{S}$, then $N_{C/W} \cong \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$.*

Proof. We have a short exact sequence of sheaves on C,

$$0 \to \Theta_C \to \Theta_W|_C \to N_{C/W} \to 0.$$

Since $c_1(\Theta_C) = 2$ and $c_1(\Theta_W|_C) = c_1(\Theta_W)|_C = 0$, it follows that $N_{C/W} \cong \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(b)$ for $a + b = -2$ (see e.g. [77, Section 1]). We may embed the normal bundle $N_{C/\hat{S}}$ into $N_{C/W}$ to get a short exact sequence of line bundles

$$0 \to N_{C/\hat{S}} \to N_{C/W} \to \mathscr{L} \to 0$$

for some line bundle $\mathscr{L}$. Since C is a (-1) curve in $\hat{S}$, we know that $N_{C/\hat{S}} \cong \mathscr{O}_{\mathbb{P}^1}(-1)$. Furthermore, we have that $c_1(\mathscr{L}) = -1$ from the fact that $c_1(N_{C/W}) = -2$, thus $\mathscr{L} \cong \mathscr{O}_{\mathbb{P}^1}(-1)$. The long exact sequence in cohomology coming from the above short exact sequence proves that $\mathrm{H}^0(C, N_{C/W}) = 0$ and therefore we must have that $N_{C/W} = \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. □

Therefore, any (-1) curve C in $\hat{S}$ may be blown up to produce a a variety $\widetilde{W}$ with exceptional divisor a copy of $\mathbb{P}^1 \times \mathbb{P}^1$. This copy of $\mathbb{P}^1 \times \mathbb{P}^1$ can be smoothly contracted onto either component. The effect of contracting onto one component is just recovering W, but the contraction onto the other component, which we call W^+ is of interest to us. The preimage of $\hat{S}$ in $\widetilde{W}$ is just $\hat{S}$ itself, but when we contract $\hat{W}$ onto W^+, we contract the curve C in $\hat{S}$.

Repeating this for all (-1) curves in $\hat{S}$ we obtain a Calabi-Yau birational model of W in which $\hat{S}$ has been contracted to its minimal model, which is in fact just a K3 surface [20]. We call the Calabi-Yau threefold resulting from this process $\hat{W}$. We have proved that:

Theorem 6.3.5. *If W is a Calabi-Yau threefold containing a smooth blown-up K3 surface $\hat{S}$, then by performing flops on W repeatedly, we may obtain a birational model*

$\hat{W}$ of W which admits a fibration $f : \hat{W} \to \mathbb{P}^1$ so that the minimal model S of $\hat{S}$ is a general fiber of f.

In the case of toric hypersurface Calabi-Yau threefolds, this shows that:

Corollary 6.3.6. *If there is a nef partition of Δ and V is a general complete intersection hypersurface in X_Δ, then V is constructive, and its Batyrev dual W has birational model which admits a K3 surface fibration by K3 surfaces which are Batyrev-Borisov dual to the nef partition Δ_1, Δ_2.*

Of course, the fibers of this fibration are just the members of the pencil of K3 surfaces on W described above with several rational curves contracted in each. In general, it is not clear if this can be done if $\dim V \geq 4$, however, all of the proofs above generalize in the obvious way to refinements of k-partite nef partitions of $(k+3)$-dimensional polytopes to give K3 surface fibrations on a birational model of the Batyrev-Borisov mirror dual. We have restricted ourselves to the hypersurface case for ease of notation.

Remark 6.3.7. It's a bit surprising to me that the birational contortions that we went through above are necessary! There seems to be an inherent incompatibility between Batyrev (and Batyrev-Borisov) duality and K3 surface fibrations on Calabi-Yau threefolds, since one can show that in very basic examples, there are in fact exceptional curves in $\hat{S}$ that cannot be avoided by simply changing birational model of X_{Δ°. A notable exception occurs when either Δ_1 or Δ_2 is 1-dimensional. In this case, one of the two component quasi-Fano varieties to which V degenerates is itself a toric variety. This is mirrored by the fact that X_{Δ° itself admits a morphism to $\mathbb{P}^1$ which induces the required K3 fibration on W. These seem to be a subset of the "toric fibrations" have been studied extensively in the physics literature by by a number of authors ([27, 10, 64] to name a few).

Remark 6.3.8. One can consider more general refinements of nef partitions, by taking a nef partition $F_1, \dots, F_\ell$ so that for each E_i, there is a subset I_i of $\{1, \dots, \ell\}$ so that $E_i = \cup_{j \in I_i} F_j$. These will give rise to generalized degenerations of the Calabi-Yau determined by $E_1, \dots, E_k$ to unions of quasi-Fano varieties, and families of Calabi-Yau varieties of codimension $\ell - k$ in W. The issure of course is that it is hard to prove that this forms a fibration on a birational model of W. Despite this, these families of Calabi-Yau varieties surely have properties related to the LG models of the appropriate quasi-Fano varieties.

Remark 6.3.9. Generalizing Corollary 6.3.6 to higher dimensions seems to be a challenge, since we have made use of both the minimal model program for surfaces and a characterization of flops in three dimensions. Of course, both of these objects have analogues in higher dimensions, but they are much more oblique and not likely to be useful in such a general situation.

6.4 Comparison with LG models

The results in this section can be extended quite generally, however, we will stick to the situation of threefold hypersurfaces. We will analyze the members of the pencil $\mathscr{P}$, which are birational to the fibers of $\hat{W}$ when such a variety exists. Thus in the threefold case, we can give a very accurate description of the singular fibers of $\hat{W}$, up to some birational transformation. Our first goal is to look at the fiber $\mathscr{P}(\infty)$ and understand its meaning in terms of the Tyurin degeneration described in Section 6.3.

Proposition 6.4.1. *The member of the pencil $\mathscr{P}$ corresponding to $s = 0$ is equal to*

$$\bigcup_{\sigma \in (\Delta^\circ \setminus \nabla) \cap N} (D_\sigma \cap W).$$

In other words, the fibration f is associated to the line bundle $\mathscr{O}_W(\sum_{\sigma \in (\Delta^\circ \setminus \nabla) \cap N} D_\sigma|_W)$.

Proof. We may write the pencil $\mathscr{P}$ as the intersection of W with hypersurfaces of the form

$$s \sum_{\rho \in \Delta_1 \cap M \setminus 0_M} a_\rho \prod_{\sigma \in \partial\Delta^\circ \cap N} z_\sigma^{\langle \sigma, \rho \rangle + 1} - t \prod_{\sigma \in \partial \Delta^\circ \cap N} z_\sigma = 0$$

where Δ_1 and Δ_2 is a nef partition of Δ. Thus, at least on an open set of W, we may write this as a rational map from X_Δ to $\mathbb{P}^1$ as a map ϕ given by

$$[z_\sigma] \mapsto \left[\prod_{\sigma \in (\Delta^\circ \setminus \nabla) \cap N} z_\sigma, \left(\sum_{\rho \in \Delta_1 \cap M \setminus 0_M} a_\rho \prod_{\sigma \in \partial \Delta^\circ \cap N} z_\sigma^{\langle \sigma, \rho \rangle + 1} \right) \Big/ \left(\prod_{\sigma \in \partial \nabla \cap N} z_\sigma \right) \right] = [s, t]$$

We want to show that this map is defined on W away from the base locus of $\mathscr{P}$. Note that homogeneity is clear, since, away from perhaps $\prod_{\sigma \in \partial \nabla \cap N} z_\sigma$, both terms are sections of $\mathscr{L} := \mathcal{O}_{X_{\Delta^\circ}}(\sum_{\sigma \in \Delta^\circ \setminus \nabla \cap N} D_\sigma)$. We check that if $\sigma \in \nabla_2$, then the second polynomial in the expression for ϕ has a factor of z_σ, since $\langle \sigma, \rho \rangle \geq 0$ for all $\rho \in \Delta_1$ and $\sigma \in \nabla_2$. Thus this is a section of . Restricted to W, we notice that the second term can also be written as

$$\left(\sum_{\rho \in \Delta_2 \cap M} a_\rho \prod_{\sigma \in \partial \Delta^\circ \cap N} z_\sigma^{\langle \sigma, \rho \rangle + 1} \right) \Big/ \left(\prod_{\sigma \in \partial \nabla \cap N} z_\sigma \right)$$

and thus for the same reason as above, this is a section of $\mathscr{L}$ along D_σ for $\sigma \in \nabla_1$. The fact [17, Proposition 6.3] implies that $\nabla \cap N = (\nabla_1 \cap N) \cup (\nabla_2 \cap N)$, we deduce that this homogeneous function forms an honest global section of $\mathscr{L}|_W$. Thus the map ϕ is well-defined away from the base locus of $\mathscr{L}|_W$ and the fiber of ϕ over ∞ is as required. $\square$

It follows from the proof of Proposition 6.4.1 that the line bundle $\mathscr{O}_W(\hat{S})$ is just $\mathscr{O}_W(\sum_{\sigma \in (\Delta^\circ \setminus \nabla) \cap N} D_\sigma)$. Since W is an anticanonical hypersurface, the intersection of a divisor D_σ with W is empty if and only if σ lives on the relative interior of a facet of Δ°. If σ is on the interior of a codimension 2 face of Δ°, then $D_\rho \cap W$ has $1 + \ell^*(\Gamma(\sigma))\ell^*(\Gamma(\sigma)^\vee)$ irreducible components (see e.g. [130, §3.3]). Here $\Gamma(\sigma)$ is the smallest face of Δ° containing σ and $\Gamma(\sigma)^\vee$ is the face of Δ made up of points ρ satisfying $\langle \sigma, \rho \rangle = -1$.

Proposition 6.4.2. *If σ is in $(\Delta^\circ \setminus \nabla) \cap N$ then $D_\rho \cap W$ has a single irreducible component. Therefore, the fiber over infinity of the K3 surface fibration obtained in*

Section 6.3 has

$$\#(\Delta^\circ \cap \nabla) \cap N$$

irreducible components.

Proof. First, if σ is contained on the relative interior of a facet of Δ°, then $\Gamma(\sigma)^\vee$ is a single vertex η of Δ. Without loss of generality, we can assume that $\eta \in \Delta_1$. Therefore, $\langle \sigma, \rho \rangle \geq -1$ for all points $\rho \in \Delta$ and $\langle \sigma, \rho \rangle = -1$ if and only if $\rho = \eta$, therefore, by definition, σ is in ∇_1. By [17, Proposition 6.3], it follows that all points of ∇ are either in ∇_1 or ∇_2, it follows that no point $\Delta^\circ \setminus \nabla$ is on the interior of a facet of Δ°.

Now if we have σ in $(\Delta^\circ \setminus \nabla) \cap N$ then we must have that there is some $\rho_1 \in \Delta_1$ and $\rho_2 \in \Delta_2$ so that $\langle \rho_1, \sigma \rangle = \langle \rho_2, \sigma \rangle = -1$. Therefore, $\Gamma(\sigma)^\vee$ contains points in both Δ_1 and Δ_2 and is thus a face of neither. It follows from [17, Proposition 6.3] that this implies that $\Gamma(\sigma)^\vee$ does not contain any points on its relative interior. Thus $\ell^*(\Gamma(\sigma)^\vee) = 0$ and hence $D_\sigma \cap W$ has a single irreducible component. □

Remark 6.4.3. In the case where W has arbitrary dimension, the same proposition is true, however, we do not know whether this may be interpreted as a count of components of a singular fiber in a fibration on W.

Next we show that this number has meaning with respect to the mirror Calabi-Yau variety.

Proposition 6.4.4. *If* $\dim \nabla_1 = \dim \nabla_2 = 4$ *then* $V \cap X_1 \cap X_2$ *is an irreducible curve* C *of genus*

$$g(C) = \#(\Delta^\circ \cap \nabla) \cap N - 1.$$

Proof. We have that C is a complete intersection of sections of line bundles $\mathscr{L}_1 = \mathscr{O}_{X_\Delta}(\sum_{\rho \in \Delta_1 \cap N} D_\rho)$ and $\mathscr{L}_1 = \mathscr{O}_{X_\Delta}(\sum_{\rho \in \Delta_2 \cap N} D_\rho)$ and $\omega_{X_\Delta}^{-1}$. We have the Koszul complex resolving $\mathscr{O}_C$ given by

$$\omega_{X_\Delta}^2 \to \mathscr{L}_1^{-1} \otimes \omega_{X_\Delta} \oplus \mathscr{L}_2^{-1} \otimes \omega_{X_\Delta} \oplus \omega_{X_\Delta} \to \mathscr{L}_1^{-1} \oplus \mathscr{L}_2^{-1} \oplus \omega_{X_\Delta} \to \mathscr{O}_{X_\Delta}$$

The corresponding second spectral sequence converges to $\mathrm{H}^i(C, \mathscr{O}_C[3])$, therefore

$$\bigoplus_{p+q=i+3} {''E}_\infty^{p,q} \cong \mathrm{H}^i(C, \mathscr{O}_C).$$

The relevant portion of ${''E}_1^{p,q}$ is given by

$$\begin{array}{ccccccc} \mathrm{H}^4(\omega_{X_\Delta}^2) & \to & \mathrm{H}^4(\mathscr{L}_1^{-1} \otimes \omega_{X_\Delta}) \oplus \mathrm{H}^4(\mathscr{L}_2^{-1} \otimes \omega_{X_\Delta}) \oplus \mathrm{H}^4(\omega_{X_\Delta}) & \to & \mathbb{C} & \to & 0 \\ 0 & \to & 0 & \to & 0 & \to & 0 \\ 0 & \to & 0 & \to & 0 & \to & 0 \\ 0 & \to & 0 & \to & 0 & \to & 0 \\ 0 & \to & 0 & \to & 0 & \to & \mathbb{C} \end{array} \tag{6.1}$$

By [17, Theorem 2.5], we know that $h^4(\omega_{X_\Delta}^2) = \ell^*(2\Delta^\circ)$, $h^4(\omega_{X_\Delta}) = 1, h^4(\mathscr{L}_1^{-1} \otimes \omega_{X_\Delta}) = \ell^*(\nabla_1 + \Delta^\circ)$ and $h^4(\mathscr{L}_2^{-1} \otimes \omega_{X_\Delta}) = \ell^*(\nabla_2 + \Delta^\circ)$. It is not hard to see then that this spectral sequence degenerates at the ${''E}_2$ term and $h^0(\mathscr{O}_C) = 1$, hence C is irreducible. Since $h^i(\mathscr{O}_C) = 0$ for $i > 1$, we have that the top row of Equation 6.1 is exact except in the left-most term. Thus we can compute that

$$g(C) = \ell^*(2\Delta^\circ) - (\ell^*(\Delta^\circ + \nabla_1) + \ell^*(\nabla_2 + \Delta^\circ)).$$

It remains to show that this is precisely the number of points in $(\Delta^\circ \setminus \nabla) \cap N$.

Lemma 6.4.5. *For Q either ∇_i or Δ°, the number $\ell^*(Q + \Delta^\circ)$ is equal to $\ell(Q)$.*

Proof. The polytope Δ° is defined by the inequalities $\langle \rho, \sigma \rangle \geq -1$ for all points $\rho \in \Delta$. Similarly, ∇_1 is defined by the inequalities $\langle \rho, \sigma \rangle \geq -1$ for all points $\rho \in \Delta_1$ and $\langle \sigma, \rho \rangle \geq 0$ for all points $\rho \in \Delta_2$. We shall prove the lemma for $Q = \nabla_1$ and the other cases are similar. Now the polytope $\Delta^\circ + \nabla_1$ is defined by the inequalities $\langle \sigma, \rho \rangle \geq -2$ for $\rho \in \Delta_1$ and $\langle \sigma, \rho \rangle \geq -1$ for $\rho \in \Delta_1$. Therefore a point on the interior of $\Delta^\circ + \nabla_1$ satisfies these inequalities strictly, and thus $(\Delta^\circ + \nabla_1) \cap N$ is the set of points in N so that $\langle \sigma, \rho \rangle \geq 0$ for all $\rho \in \Delta_2$ and $\langle \sigma, \rho \rangle \geq -1$ for all $\rho \in \Delta_1$. Thus this is the set of all integral points in $\Delta^\circ + \nabla_1$. $\square$

Therefore, we have that

$$\ell^*(2\Delta^\circ) - (\ell^*(\Delta^\circ + \nabla_1) + \ell^*(\nabla_2 + \Delta^\circ)) = \ell(\Delta^\circ) - \ell(\nabla_1) - \ell(\nabla_1).$$

By [17, Proposition 6.3], it follows that all points of ∇ are points of ∇_1 or ∇_2, thus this is just

$$\ell(\Delta^\circ) - \ell(\nabla) + 1 = \#(\Delta^\circ \setminus \nabla) + 1$$

Note the (-1) term corresponds to the fact that we have over-counted the origin, as it is the intersection of ∇_1 and ∇_2. □

Remark 6.4.6. A very minor modification of this proof shows that, in the case where W has dimension d, then $h^{d-2,0}(V \cap X_1 \cap X_2) = \#(\Delta^\circ \setminus \nabla) - 1$.

As a corollary, we have that

Theorem 6.4.7. *The member $\mathscr{P}(\infty)$ of the pencil of hypersurfaces in the previous section has exactly $h^{d-2,0}(X_1 \cap X_2 \cap V) + 1$ components.*

Next, we analyze the rest of the fibers in the fibration on $\hat{W}$. Our goal is to show that the fibers over 0 and 1 are essentially the fibers over 0 of the LG models of X_1 and X_2.Thus there is a very real sense in which the fibration on $\hat{W}$ is collecting information regarding the LG models of X_1 and X_2.

Theorem 6.4.8. *The fibers $\mathscr{P}(0)$ and $\mathscr{P}(1)$ are birational to the fibers over 0 of (Y_1, w) and (Y_2, w) respectively. In fact, for any choice of W and $\mathscr{P}(t)$ with $t \neq \infty$, there is a choice of LG model (Y, w) of either X_1 or X_2 so that $\mathscr{P}(t)$ is birational to a fiber of (Y, w).*

Proof. We recall that we have an expression for a birational model of W as a complete intersection in $\mathbb{P}^1 \times X_\nabla$ given by the vanishing of

$$f_1 = s \sum_{\rho \in \Delta_1 \cap M \setminus 0_M} a_\rho \prod_{\sigma \in \partial\Delta^\circ \cap N} z_\sigma^{\langle \sigma, \rho \rangle + 1} - t \prod_{\rho \in \partial\Delta^\circ \cap N} z_\rho$$

$$f_2 = s \sum_{\rho \in \Delta_2 \cap M \setminus 0_M} a_\rho \prod_{\sigma \in \partial\Delta^\circ \cap N} z_\sigma^{\langle \sigma, \rho \rangle + 1} - (s - t) \prod_{\rho \in \partial\Delta^\circ \cap N} z_\rho.$$

Note that f_1 has a factor of $\prod_{\sigma \in \partial \nabla_2 \cap N} z_\rho$ and similarly for f_2, by the definition of ∇_2. If we let $t = 0$, then we have the complete intersection of

$$f_1|_{t=0} = \sum_{\rho \in \Delta_1 \cap M \setminus 0_M} a_\rho \prod_{\sigma \in \partial \Delta^\circ \cap N} z_\sigma^{\langle \sigma, \rho \rangle + 1}$$
$$f_2|_{t=0} = \sum_{\rho \in \Delta_2 \cap M \setminus 0_M} a_\rho \prod_{\sigma \in \partial \Delta^\circ \cap N} z_\sigma^{\langle \sigma, \rho \rangle + 1} - \prod_{\rho \in \partial \Delta^\circ \cap N} z_\rho.$$

We have that $f_1|_{t=0}$ has an additional factor of $\prod_{\sigma \in \nabla_1^{\geq 1}} z_\sigma$. Note that this is precisely the complete intersection deterimining the fiber over 0 of the LG model of X_1, except compactified to X_{Δ° and not X_∇. We define a birational map φ from X_{Δ° to X_∇ which sends z_σ to z_σ if σ is a point in $\partial \nabla \cap N$. If we let $X_{\Delta^\circ}^{[1]}$ and $X_\nabla^{[1]}$ be the complement of all torus invariant loci of codimension ≥ 2 in X_{Δ° and X_∇ respectively, then the restriction of φ simply has the effect of removing tori $(\mathbb{C}^\times)^3$ corresponding to $\sigma \in (\Delta^\circ \setminus \nabla) \cap N$.

Thus φ induces a birational map between the fiber over 0 of (Y_1, w) if no components of $\mathscr{P}(0)$ are contained in torus invariant loci of X_{Δ° of codimension ≥ 2 and no component of $\mathscr{P}(0)$ is contained in a divisor D_σ for $\sigma \in (\Delta^\circ \setminus \nabla) \cap N$. The first claim is trivial. Since each component of $\mathscr{P}(0)$ is of dimension $d-2$ in X_{Δ°, it is contained in a codimension ≥ 2 torus invariant subvariety of X_{Δ° if and only if it is the closure of a the torus invariant subvariety. Since W contains no torus invariant subvarieties of X_{Δ°, this cannot happen.

The second point follows from the fact that if $\sigma \in (\Delta^\circ \setminus \nabla) \cap N$ then $D_\sigma \cap W$ is in $\mathscr{P}(\infty)$. Thus we can have $\mathscr{P}(0)$ intersects D_σ in at most a codimension 2 subvariety of X_{Δ°. An identical argument suffices to show that $\mathscr{P}(1)$ is birational to the fiber over 0 of (Y_2, w), and in fact this shows that for any $t \in \mathbb{P}^1 \setminus \{\infty\}$, a generic choice of W there is a choice of $i = 1$ or 2, and an LG model (Y_i, w) so that $\mathscr{P}(t)$ is birational to a fiber of (Y_i, w). $\square$

As a philosophical remark, this proves that all of the interesting data surrounding the fibration on $\hat{W}$ is related to either the LG models of X_1 and X_2 or the variety $V \cap X_1 \cap X_2$. In the case where both X_1 and X_2 are pullbacks to X_Δ of ample hypersurfaces in $\mathbb{P}_\Delta$ along the natural crepant resolution map, then we can say even more by using Theorem 3.4.9 in Chapter 3.

Corollary 6.4.9. *Let ρ_p be the number of irreducible components in the fiber of f over $p \in \mathbb{P}^1$. If X_1 and X_2 are pullbacks of ample hypersurfaces in $\mathbb{P}_\Delta$, then*

- $\rho_0 = h^{1,2}(X_1) + 1$.
- $\rho_1 = h^{1,2}(X_2) + 1$,
- $\rho_\infty = h^{1,0}(C) + 1$.

Let ℓ be the rank of the image of the restriction map $\mathrm{H}^2(\hat{W}, \mathbb{C}) \to \mathrm{H}^2(W, \mathbb{C})$ for S a smooth fiber of f. Using the same techniques as in the proof of Theorem 3.3.1, one can show that

$$h^{1,1}(W) = \sum_{p \in \mathbb{P}^1} (\rho_p - 1) + \ell + 1$$

By [14], we know that $h^{2,1}(V) = h^{1,1}(\hat{W})$, thus applying Theorem 6.1.1, one sees that in this situation we have

$$\sum_{p \in \mathbb{P}^1 \setminus \{0,1,\infty\}} (\rho_p - 1) + \ell + k = 20,$$

This implies that if Dolgachev-Nikulin mirror symmetry does not hold (in a precise sense) for the K3 surfaces associated to the nef partition Δ_1, Δ_2 and their Batyrev-Borisov duals, then this failure is seen by the rest of the fibers of the fibration f on $\hat{W}$.

Finally, we will illustrate this construction with an example in the case where V is an anticanonical surface in a toric Fano threefold.

Example 6.4.10 (Anticanonical hypersurfaces in $(\mathbb{P}^1)^3$)**.** Let us take, as described, the anticanonical hypersurface in $(\mathbb{P}^1)^3$. This is a K3 surface with Picard lattice of rank 3 and isomorphic to the lattice with Gram matrix

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

There is a Tyurin degeneration of V to a union of two $(1,1,1)$ divisors X_1, X_2 in $(\mathbb{P}^1)^3$. The intersection of $V \cap X_1 \cap X_2$ is just 12 points. On the other side, we see that there is an elliptic fibration on the mirror dual K3 surface has an I_{12} type singular fiber at ∞ and I_2 singular fibers at 0 and 1. The polytope Δ has vertices $\sigma_0, \dots, \sigma_5$ given by the columns of the matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

The appropriate nef partition is $E_1 = \{\sigma_0, \sigma_2, \sigma_4\}$ and $E_2 = \{\sigma_1, \sigma_3, \sigma_5\}$, which has dual nef partition with

$$\nabla_1 = \mathrm{Conv}(\{(1,0,0),(0,1,0),(0,0,1),(1,1,1),(1,1,0),(0,1,1),(0,1,1),(0,0,0)\})$$
$$\nabla_2 = -\nabla_1.$$

We draw the polytopes ∇ and Δ° in the second picture in Figure 6.1. The first picture in Figure 6.1 is just the polytope Δ°, the second denotes ∇_1 and ∇_2 coloured in red and green respectively. The third picture colours the points corresponding to components the fibers over 0 and 1 in red and green respectively. The points of $(\Delta^\circ \setminus \nabla) \cap N$ are drawn in blue. It is clear in the description of the fiber over ∞ that it is actually semi-stable, hence it follows from Kodaira's classification of singular fibers of elliptic fibrations [85] that the resulting fiber is necessarily of type I_{12}. The same cannot be said for the fibers over 0 and 1, since it is not necessarily true that these fibers have normal crossings. Thus Kodaira's classification shows that these fibers are either of type I_2 or of type III.

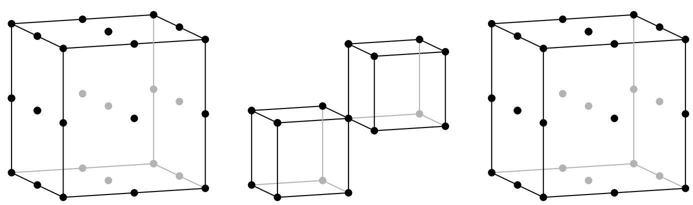

Fig. 6.1 Polytopes related to Example 6.4.10

Chapter 7

Calabi-Yau threefolds fibered by quartic mirror K3 surfaces

7.1 Introduction

In this paper, along with its predecessor [51], we take the first steps towards a systematic study of threefolds fibred by K3 surfaces, with a particular focus on Calabi-Yau threefolds. Our aim in this paper is to gain a complete understanding of a relatively simple case, where the generic fibre in the K3 fibration is a mirror quartic, to demonstrate the utility of our methods and to act as a test-bed for developing a more general theory.

We have chosen mirror quartic K3 surfaces (here "mirror" is used in the sense of Nikulin [109] and Dolgachev [44]) because, from a moduli theoretic perspective, they may be thought of as the simplest non-rigid lattice polarized K3 surfaces. Indeed, mirror quartic K3 surfaces are polarized by the rank 19 lattice

$$M_2 := H \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle,$$

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so move in a 1-dimensional moduli space. By [44, Theorem 7.1], this moduli space is isomorphic to the modular curve $\Gamma_0(2)^+ \setminus \mathbb{H}$, we denote its compactification by $\mathcal{M}_{M_2}$.

Our first main result (Theorem 7.2.2) will show that an M_2-polarized family of K3 surfaces (in the sense of [51, Definition 2.1]) over a quasi-projective base curve U is completely determined by its *generalized functional invariant map* $U \to \mathcal{M}_{M_2}$, which may be thought of as a K3 analogue of the classical functional invariant of an elliptic curve. This also explains why we choose to polarize our K3 surfaces by M_2 instead of $M_1 := H \oplus E_8 \oplus E_8 \oplus \langle -2 \rangle$, which at first would seem like a more obvious choice. Indeed, M_1-polarized K3 surfaces admit an antisymplectic involution that fixes the polarization, which means that the analogue of Theorem 7.2.2 does not hold for them; in analogy with elliptic curves again, the presence of an antisymplectic involution that fixes the polarization means that to uniquely determine an M_1-polarized family of K3 surfaces we would also need a *generalized homological invariant*, to control monodromy around singular fibres, whereas for M_2-polarized familes the lack of such automorphisms means that the generalized functional invariant alone suffices.

A second reason for choosing mirror quartic K3 surfaces is the fact that the mirror quintic Calabi-Yau threefold admits a fibration by mirror quartics [51, Theorem 5.10]. This makes fibrations by mirror quartic K3 surfaces particularly interesting for the study of Calabi-Yau threefolds; the majority of this paper is devoted to this study. Indeed, our second main result (Corollary 7.2.7) provides a complete explicit description of all Calabi-Yau threefolds that admit M_2-polarized K3 fibrations, and we compute Hodge numbers and candidate mirror partners in all cases. Throughout this study we present the mirror quintic as a running example, thereby demonstrating that many of its known properties can be easily recovered from our theory, although we would like to note that our methods apply to a significantly broader class of examples of Calabi-Yau threefolds, many of which are not even known to be toric.

The structure of this paper is as follows. In Section 7.2 we begin by proving Theorem 7.2.2, which shows that any M_2-polarized family of K3 surfaces is uniquely determined by its generalized functional invariant. In particular, this means that any M_2-polarized family of K3 surfaces is isomorphic to the pull-back of a fundamental family of M_2-polarized K3 surfaces, introduced in Section 7.2.1, from the moduli space $\mathcal{M}_{M_2}$. The remainder of Section 7.2 is then devoted to showing how this theory can be used to construct Calabi-Yau threefolds, culminating in Corollary 7.2.7, which gives an explicit description of all Calabi-Yau threefolds that admit M_2-polarized K3 fibrations.

In Section 7.3 we begin our study of the properties of the Calabi-Yau threefolds constructed in Section 7.2, by computing their Hodge numbers. The main results are Proposition 7.3.2, which computes $h^{1,1}$, and Corollary 7.3.7, which computes $h^{2,1}$.

Section 7.4 is devoted to a brief study of the deformation theory of the Calabi-Yau threefolds constructed in Section 7.2. The main result is Proposition 7.4.1, which shows that any small deformation of such a Calabi-Yau threefold is induced by a deformation of the generalized functional invariant map of the K3 fibration on it. In particular, this allows us to relate the moduli spaces of such Calabi-Yau threefolds to Hurwitz spaces describing ramified covers between curves, and gives an easy way to study their degenerations.

Finally, in Section 7.5, we exhibit candidate mirror partners for the Calabi-Yau threefolds constructed in Section 7.2, given as Calabi-Yau smoothings of pairs of Fano threefolds glued along anticanonical K3 surfaces. As evidence for the mirror correspondence between these Calabi-Yau threefolds, in Section 7.5.3 we show that mirror duality of Hodge numbers is satisfied, and exhibit mirror dual filtrations in cohomology. Finally, in Section 7.5.4, we exhibit a relationship between the Landau-Ginzburg models of the pair of Fano threefolds and the M_2-polarized K3 fibration on the original Calabi-Yau threefold. This provides a class of examples illustrating a prediction of Tyurin [141]: that the mirror of a Calabi-Yau smoothing of a pair of Fano threefolds should be expressible in terms of the corresponding Landau-Ginzburg models of those Fanos.

7.2 Construction

We begin by setting up some notation. Let $\mathcal{X}$ be a smooth projective threefold that admits a fibration $\pi\colon \mathcal{X} \to B$ by K3 surfaces over a smooth base curve B. Let $\mathrm{NS}(X_p)$ denote the Néron-Severi group of the fibre of $\mathcal{X}$ over a general point $p \in B$. In what follows, we will assume that $\mathrm{NS}(X_p) \cong M_2$, where M_2 denotes the rank 19 lattice $M_2 := H \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle$.

Denote the open set over which the fibres of $\mathcal{X}$ are smooth K3 surfaces by $U \subset B$ and let $\pi_U\colon \mathcal{X}_U \to U$ denote the restriction of $\mathcal{X}$ to U. We suppose further that $\mathcal{X}_U \to U$ is an M_2-polarized family of K3 surfaces, in the sense of [51, Definition 2.1].

To any such family, we can associate a *generalized functional invariant map* $g\colon U \to \mathcal{M}_{M_2}$, where $\mathcal{M}_{M_2}$ denotes the (compact) moduli space of M_2-polarized K3 surfaces. g is defined to be the map which takes a point $p \in U$ to the point in moduli corresponding to the fibre X_p of $\mathcal{X}$ over p.

[51, Theorem 5.10] gives five examples of Calabi-Yau threefolds admitting such fibrations, arising from the Doran-Morgan classification [52, Table 1]. In each of these cases, the family $\pi_U\colon \mathcal{X}_U \to U$ is the pull-back of a special family of K3 surfaces $\mathcal{X}_2 \to \mathcal{M}_{M_2}$, by the generalized functional invariant map.

Remark 7.2.1. In addition to the five examples from [51, Theorem 5.10], the authors are aware of many more Calabi-Yau threefolds which admit such fibrations. Indeed, the toric geometry functionality of the computer software *Sage* may be used to perform a search for such fibrations on toric Calabi-Yau threefolds with small Hodge number $h^{2,1}$, yielding dozens of additional examples; details will appear in future work.

Our first result will show that this is not a coincidence: in fact, any M_2-polarized family of K3 surfaces $\pi_U\colon \mathcal{X}_U \to U$ is determined up to isomorphism by its generalized functional invariant, so we can obtain any such family by pulling back the special family $\mathcal{X}_2$. We will therefore begin our study of threefolds fibred by M_2-polarized K3 surfaces by studying the family $\mathcal{X}_2$.

Theorem 7.2.2. *Let $\mathcal{X}_U \to U$ denote an M_2-polarized family of K3 surfaces over a quasi-projective curve U, such that the Néron-Severi group of a general fibre of*

$\mathcal{X}_U$ is isometric to M_2. Then $\mathcal{X}_U$ is uniquely determined (up to isomorphism) by its generalized functional invariant map $g\colon U \to \mathcal{M}_{M_2}$.

Proof. Suppose for a contradiction that $\mathcal{X}_U$ and $\mathcal{Y}_U$ are two non-isomorphic M_2-polarized familes of K3 surfaces over U, that satisfy the conditions of the theorem and have the same generalized functional invariant $g\colon U \to \mathcal{M}_{M_2}$.

Let $\{U_i\}$ denote a cover of U by simply connected open subsets and let $\mathcal{X}_{U_i}$ (resp. $\mathcal{Y}_{U_i}$) denote the restriction of $\mathcal{X}_U$ (resp. $\mathcal{Y}_U$) to U_i for each i. On each U_i, Ehresmann's Theorem (see, for example, [143, Section 9.1.1]) shows that we can choose markings compatible with the M_2-polarizations on the families of K3 surfaces $\mathcal{X}_{U_i}$ and $\mathcal{Y}_{U_i}$. Thus, by the Global Torelli Theorem [109, Theorem 2.7'], the families $\mathcal{X}_{U_i}$ and $\mathcal{Y}_{U_i}$ are isomorphic for each i.

Therefore, since we have assumed that $\mathcal{X}_U$ and $\mathcal{Y}_U$ are non-isomorphic, they must differ in how the families $\mathcal{X}_{U_i}$ and $\mathcal{Y}_{U_i}$ glue together over the intersections $U_i \cap U_j$. Let $V \subset U_i \cap U_j$ be a connected component of such an intersection, such that the gluing maps differ over V. As $\mathcal{X}_U$ and $\mathcal{Y}_U$ are isomorphic over V, the gluing maps over V must differ by composition with a nontrivial fibrewise automorphism ψ. Moreover, by the polarization condition, ψ must preserve the M_2-polarizations on the fibres over V.

Now, consider the action of ψ on the fibre X_p of $\mathcal{X}_U$ over a general point $p \in V$. As the Néron-Severi group of X_p is isometric to M_2 (by assumption) and $\psi^* \in \mathrm{O}(H^2(X_p, \mathbb{Z}))$ fixes M_2, we see that ψ must act non-symplectically on X_p.

Thus, by [110, Proposition 1.6.1], ψ^* descends to an element of the subgroup $\mathrm{O}(M_2^\perp)^*$ of $\mathrm{O}(M_2^\perp)$ which induces the trivial automorphism on the discriminant group $A_{M_2^\perp}$. Furthermore, since X_p is general, $M_2^\perp$ supports an irreducible rational Hodge structure, so ψ^* must act irreducibly on $M_2^\perp$. Therefore, by [109, Theorem 3.1], it follows that the order n of ψ^* must satisfy $\varphi(n) | \mathrm{rank}\,(M_2^\perp) = 3$, where $\varphi(n)$ denotes Euler's totient function. Using Vaidya's [142] lower bound for $\varphi(n)$,

$$\varphi(n) \geq \sqrt{n} \text{ for } n > 2,\ n \neq 6$$

we see that $\varphi(n)|3$ implies that $n \leq 9$. A simple check then shows that $n = 2$ or $n = 1$. If $n = 2$ then, by irreducibility, ψ^* would have to act as $-\mathrm{Id}$ on $M_2^\perp$ and as the identity on M_2. However, since the discriminant group of $M_2^\perp$ is $A_{M_2^\perp} \cong \mathbb{Z}/4\mathbb{Z}$, such ψ^* would not descend to the identity on $A_{M_2^\perp}$, so this case cannot occur. Therefore, ψ^* must be of order 1. But then ψ must be the trivial automorphism, which is a contradiction. □

7.2.1 A fundamental family

The family $\mathcal{X}_2 \to \mathcal{M}_{M_2}$ is described in [51, Section 5.4.1]. It is given as the minimal resolution of the family of hypersurfaces in $\mathbb{P}^3$ obtained by varying λ in the following expression

$$\lambda w^4 + xyz(x + y + z - w) = 0. \tag{7.1}$$

This family has been studied extensively by Narumiya and Shiga [108], we will make substantial use of their results in the sequel (note, however, that our λ is not the same as the λ used in [108], instead, our λ is equal to μ^4 or $\frac{u}{256}$ from [108]).

Recall from [44, Theorem 7.1] that $\mathcal{M}_{M_2}$ is the compactification of the modular curve $\Gamma_0(2)^+ \setminus \mathbb{H}$. In [51, Section 5.4.1] it is shown that the orbifold points of orders $(2, 4, \infty)$ in $\mathcal{M}_{M_2}$ occur at $\lambda = (\frac{1}{256}, \infty, 0)$ respectively, and the K3 fibres of $\mathcal{X}_2$ are smooth away from these three points. Let U_{M_2} denote the open set obtained from $\mathcal{M}_{M_2}$ by removing these three points. Then the restriction of $\mathcal{X}_2$ to U_{M_2} is an M_2-polarized family of K3 surfaces (in the sense of [51, Definition 2.1]).

As noted in the previous section, it follows from Theorem 7.2.2 that any M_2-polarized family of K3 surfaces $\mathcal{X}_U \to U$ can be realized as the pull-back of $\mathcal{X}_2$ by the generalized functional invariant map $g\colon U \to \mathcal{M}_{M_2}$.

7.2.2 Constructing Calabi-Yau threefolds

In the remainder of this paper, we will use this theory to construct Calabi-Yau threefolds fibred by M_2-polarized K3 surfaces and study their properties. We note that, in this paper, a *Calabi-Yau threefold* will always be a smooth projective threefold

$\mathcal{X}$ with $\omega_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$ and $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$. We further note that the cohomological condition in this definition implies that any fibration of a Calabi-Yau threefold by K3 surfaces must have base curve $\mathbb{P}^1$, so from this point we restrict our attention to the case where $B \cong \mathbb{P}^1$.

Recall that, by [51, Theorem 5.10], we already know of several Calabi-Yau threefolds with $h^{2,1} = 1$ that admit fibrations by M_2-polarized K3 surfaces. It is noted in [51, Section 5.4] that the generalized functional invariant maps determining these fibrations all have a common form, given by a pair of integers (i, j): the map g is an $(i+j)$-fold cover ramified at two points of orders i and j over $\lambda = \infty$, once to order $(i+j)$ over $\lambda = 0$, and once to order 2 over a point that depends upon the modular parameter of the threefold, where $i, j \in \{1, 2, 4\}$ are given in [51, Table 1].

The aim of this section is to extend this construction of Calabi-Yau threefolds to a more general setting. Let $g\colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$ be an n-fold cover and let $[x_1, \ldots, x_k]$, $[y_1, \ldots, y_l]$ and $[z_1, \ldots, z_m]$ be partitions of n encoding the ramification profiles of g over $\lambda = 0$, $\lambda = \infty$ and $\lambda = \frac{1}{256}$ respectively. Let r denote the degree of ramification of g away from $\lambda \in \{0, \frac{1}{256}, \infty\}$, defined to be

$$r := \sum_{\substack{p \in \mathbb{P}^1 \\ g(p) \notin \{0, \frac{1}{256}, \infty\}}} (e_p - 1),$$

where e_p denotes the ramification index of g at the point $p \in \mathbb{P}^1$.

Let $\pi_2\colon \bar{\mathcal{X}}_2 \to \mathcal{M}_{M_2}$ denote the threefold fibred by (singular) K3 surfaces defined by Equation (7.1); then $\bar{\mathcal{X}}_2$ is birational to $\mathcal{X}_2$. Let $\bar{\pi}_g\colon \bar{\mathcal{X}}_g \to \mathbb{P}^1$ denote the normalization of the pull-back $g^*\bar{\mathcal{X}}_2$.

Proposition 7.2.3. *The threefold $\bar{\mathcal{X}}_g$ has trivial canonical sheaf if and only if $k + l + m - n - r = 2$ and either $l = 2$ with $y_1, y_2 \in \{1, 2, 4\}$, or $l = 1$ with $y_1 = 8$.*

Proof. We begin by noting that a simple adjunction calculation shows that $\bar{\mathcal{X}}_2$ has canonical sheaf $\omega_{\bar{\mathcal{X}}_2} \cong \pi_2^*\mathcal{O}_{\mathcal{M}_{M_2}}(-1)$. We need to study the effects of the map $\bar{\mathcal{X}}_g \to \bar{\mathcal{X}}_2$ on this canonical sheaf.

It is an easy local computation using Equation (7.1) to show that the pull-back $g^*\bar{\mathcal{X}}_2$ is normal away from the fibres over $g^{-1}(\infty)$. To see what happens on the remaining fibres, suppose that $p \in \mathbb{P}^1$ is a point with $g(p) = \infty$ and let y_i denote the order of ramification of g at p. Then the fibre over p is contained in the non-normal locus of $g^*\bar{\mathcal{X}}_2$ if and only if $y_i > 1$. Away from the fibre over p the normalization map $\bar{\mathcal{X}}_g \to g^*\bar{\mathcal{X}}_2$ is an isomorphism, whilst on the fibre over p it is an $\mathrm{hcf}(y_i, 4)$-fold cover.

With this in place, we perform two adjunction calculations. The first is for the map of base curves $g \colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$. As $\mathcal{M}_{M_2} \cong \mathbb{P}^1$, we find that we must have

$$k + l + m - n - r - 2 = 0. \tag{7.2}$$

Next, we compute $\omega_{\bar{\mathcal{X}}_g}$. We find:

$$\omega_{\bar{\mathcal{X}}_g} \cong \bar{\pi}_g^* \mathcal{O}_{\mathbb{P}^1}\left(k + m - n - r - \sum_{i=1}^{l}\left(\frac{y_i}{\mathrm{hcf}(y_i, 4)} - 1\right)\right).$$

Putting these equations together, we see that the condition that $\omega_{\bar{\mathcal{X}}_g}$ is trivial is equivalent to

$$l - 2 + \sum_{i=1}^{l}\left(\frac{y_i}{\mathrm{hcf}(y_i, 4)} - 1\right) = 0.$$

Since $l \geq 1$ and $(\frac{y_i}{\mathrm{hcf}(y_i,4)} - 1)$ is nonnegative for any integer $y_i > 0$, we must therefore have either $l = 2$ and $y_i = \mathrm{hcf}(y_i, 4)$, in which case $y_i | 4$, or $l = 1$ and $y_1 = 2\,\mathrm{hcf}(y_1, 4)$, in which case $y_1 = 8$. Together with Equation (7.2), this proves the proposition. □

Next we will show that we can resolve most of the singularities of $\bar{\mathcal{X}}_g$.

Proposition 7.2.4. *If Proposition 7.2.3 holds, then there exists a projective birational morphism $\mathcal{X}_g \to \bar{\mathcal{X}}_g$, where $\mathcal{X}_g$ is a normal threefold with trivial canonical sheaf and at worst $\mathbb{Q}$-factorial terminal singularities. Furthermore, any singularities of $\mathcal{X}_g$ occur in its fibres over $g^{-1}(\frac{1}{256})$, and $\mathcal{X}_g$ is smooth if g is unramified over $\lambda = \frac{1}{256}$ (which happens if and only if $m = n$).*

Remark 7.2.5. There exist examples of maps $g\colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$, satisfying the conditions of this proposition and ramified over $\lambda = \frac{1}{256}$, for which the corresponding threefolds $\mathcal{X}_g$ are not smooth; see Example 7.4.5.

Proof. We prove this proposition by showing that the singularities of $\bar{\mathcal{X}}_g$ may all be crepantly resolved, with the possible exception of some $\mathbb{Q}$-factorial terminal singularities lying in fibres over $g^{-1}(\frac{1}{256})$.

The threefold $\bar{\mathcal{X}}_2$ has six smooth curves $C_1, \ldots, C_6$ of cA_3 singularities which form sections of the fibration π_2. These lift to the cover $\bar{\mathcal{X}}_g$ so that, away from the fibres over $g^{-1}\{0, \frac{1}{256}, \infty\}$, the threefold $\bar{\mathcal{X}}_g$ also has six smooth curves of cA_3 singularities which form sections of the fibration. These can be crepantly resolved, so $\mathcal{X}_g$ is smooth away from its fibres over $g^{-1}\{0, \frac{1}{256}, \infty\}$.

Now let Δ_0 be a disc in $\mathcal{M}_{M_2}$ around $\lambda = 0$ and let Δ_0' be one of the connected components of its preimage under g. Then $g\colon \Delta_0' \to \Delta_0$ is an x_i-fold cover ramified totally over $\lambda = 0$, for some x_i. The threefold $\bar{\mathcal{X}}_2$ is smooth away from the curves C_i over Δ_0 and the fibre of $\pi_2\colon \bar{\mathcal{X}}_2 \to \mathcal{M}_{M_2}$ over $\lambda = 0$ has four components meeting transversely along six curves $D_1, \ldots, D_6$, with dual graph a tetrahedron.

After pulling back to Δ_0' we see that, after resolving the pull-backs of the six curves C_i of cA_3 singularities, the threefold $\bar{\mathcal{X}}_g$ is left with six curves of cA_{x_i-1} singularities in its fibre over $g^{-1}(0)$, given by the pull-backs of the curves D_j. Friedman [57, Section 1] has shown how to crepantly resolve such a configuration, so $\mathcal{X}_g$ is smooth over Δ_0'.

Next let Δ_∞ be a disc in $\mathcal{M}_{M_2}$ around $\lambda = \infty$ and let Δ_∞' be one of the connected components of its preimage under g. Then $g\colon \Delta_\infty' \to \Delta_\infty$ is a y_i-fold cover ramified totally over $\lambda = \infty$, for some $y_i \in \{1, 2, 4, 8\}$.

The family $\pi_2\colon \bar{\mathcal{X}}_2 \to \mathcal{M}_{M_2}$ has a fibre of multiplicity four over $\lambda = \infty$ and, in addition to the six curves C_i of cA_3 singularities forming sections of the fibration π_2, the threefold $\bar{\mathcal{X}}_2$ also has four curves $E_1, \ldots, E_4$ of cA_3 singularities in its fibre over $\lambda = \infty$. The curves E_j intersect in pairs at six points, which coincide with the six points of intersection of the curves C_i with the fibre $\pi_2^{-1}(\infty)$. Thus, over Δ_∞ the threefold $\bar{\mathcal{X}}_2$ has ten curves of cA_3 singularities, which meet in six triple points.

If $y_i = 1$, then $\bar{\mathcal{X}}_g$ is isomorphic to $\bar{\mathcal{X}}_2$ over Δ_∞. The ten curves of cA_3 singularities may be crepantly resolved by the method in [57, Section 1], so $\mathcal{X}_g$ is smooth over Δ'_∞. This resolution gives three exceptional components over each curve of cA_3 singularities and three exceptional components over each point where three of these curves meet. The resulting singular fibre over $g^{-1}(\infty)$ has 31 components (3 from each of the curves E_j, three from each of the six intersections between these curves, and the strict transform of the original component).

When $y_i = 2$, the four curves E_j lift to four curves of cA_1 singularities in $\bar{\mathcal{X}}_g$. The threefold $\bar{\mathcal{X}}_g$ thus contains ten curves of singularities over Δ'_∞: six curves of cA_3's coming from the pull-backs of the C_i and four curves of cA_1's coming from the pull-backs of the E_j. To crepantly resolve these singularities, first resolve the six curves of cA_3's. The resulting threefold is smooth away from its fibre over $g^{-1}(\infty)$, which contains ten curves of cA_1 singularities (the strict transforms of the pull-backs of the E_j and six further curves coming from the six points where they intersect). These ten curves may be blown up once each to give the threefold $\mathcal{X}_g$, which is smooth over Δ'_∞. The resulting singular fibre over $g^{-1}(\infty)$ has 11 components, arranged in an example of a "flowerpot degeneration" [37] with flowers of type 4α (see [37, Table 3.3]).

Finally, if $4|y_i$, the lifts of the four curves E_j are smooth in $\mathcal{X}_g$. Over Δ'_∞, the threefold $\bar{\mathcal{X}}_g$ thus contains only the six curves of cA_3 singularities coming from the pull-backs of the curves C_i. These may be crepantly resolved without adding any new components to the fibre of $\bar{\mathcal{X}}_g$ over $g^{-1}(\infty)$. Thus we see that, in all cases, the threefold $\mathcal{X}_g$ is smooth over Δ'_∞.

Finally, let $\Delta_{\frac{1}{256}}$ be a disc in $\mathcal{M}_{M_2}$ around $\lambda = \frac{1}{256}$ and let $\Delta'_{\frac{1}{256}}$ be one of the connected components of its preimage under g. Then $g\colon \Delta'_{\frac{1}{256}} \to \Delta_{\frac{1}{256}}$ is an z_i-fold cover ramified totally over $\lambda = \frac{1}{256}$, for some z_i.

The threefold $\bar{\mathcal{X}}_2$ is smooth over $\Delta_{\frac{1}{256}}$ away from the six curves C_i of cA_3's forming sections of the fibration, but its fibre over $\lambda = \frac{1}{256}$ has an additional isolated A_1 singularity. Upon proceeding to the z_i-fold cover $\Delta'_{\frac{1}{256}} \to \Delta_{\frac{1}{256}}$, this becomes an isolated terminal singularity of type cA_{z_i-1} in $\bar{\mathcal{X}}_g$.

Thus $\mathcal{X}_g$ is smooth away from its fibres over $g^{-1}(\frac{1}{256})$, where it can have isolated terminal singularities. By [87, Theorem 6.25], we may further assume that $\mathcal{X}_g$ is $\mathbb{Q}$-factorial. To complete the proof, we note that if g is a local isomorphism over $\Delta_{\frac{1}{256}}$, then $\mathcal{X}_g$ is also smooth over $g^{-1}(\frac{1}{256})$ and thus smooth everywhere. □

Let $\pi_g \colon \mathcal{X}_g \to \mathbb{P}^1$ denote the fibration induced on $\mathcal{X}_g$ by the map $\bar{\pi}_g \colon \bar{\mathcal{X}}_g \to \mathbb{P}^1$. By construction, the restriction of π_g to the smooth fibres is an M_2-polarized family of K3 surfaces, in the sense of [51, Definition 2.1]. Moreover, we have:

Proposition 7.2.6. *Let $\mathcal{X}_g$ be a threefold as in Proposition 7.2.4. Then the cohomology $H^1(\mathcal{X}_g, \mathcal{O}_{\mathcal{X}_g}) = 0$.*

Proof. Since $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$, the proposition will follow immediately from the Leray spectral sequence if we can prove that $R^1(\pi_g)_*\mathcal{O}_{\mathcal{X}_g} = 0$.

To show this, we note that $\mathcal{X}_g$ is a normal projective threefold with at worst terminal singularities and the canonical sheaf $\omega_{\mathcal{X}_g} \cong \mathcal{O}_{\mathcal{X}_g}$ is locally free. So we may apply the torsion-freeness theorem of Kollár [86, Theorem 10.19] to see that $R^1(\pi_g)_*\mathcal{O}_{\mathcal{X}_g}$ is a torsion-free sheaf on $\mathbb{P}^1$. Moreover, since $H^1(X, \mathcal{O}_X) = 0$ for a generic fibre X of $\pi_g \colon \mathcal{X}_g \to \mathbb{P}^1$, the sheaf $R^1(\pi_g)_*\mathcal{O}_{\mathcal{X}_g}$ also has trivial generic fibre. We must therefore have $R^1(\pi_g)_*\mathcal{O}_{\mathcal{X}_g} = 0$. □

Corollary 7.2.7. *Let $\mathcal{X}_g$ be a threefold as in Proposition 7.2.4. If $\mathcal{X}_g$ is smooth, then $\mathcal{X}_g$ is a Calabi-Yau threefold.*

Conversely, let $\mathcal{X} \to \mathbb{P}^1$ be a Calabi-Yau threefold fibred by K3 surfaces, let $U \subset \mathbb{P}^1$ denote the open set over which the fibres of $\mathcal{X}$ are smooth and let $\mathcal{X}_U$ denote the restriction of $\mathcal{X}$ to U. Suppose that $\mathcal{X}_U$ is an M_2-polarized family of K3 surfaces (in the sense of [51, Definition 2.1]) and that the Néron-Severi group of a general fibre of $\mathcal{X}_U$ is isometric to M_2. Then $\mathcal{X}$ is birational to a threefold $\mathcal{X}_g$ as in Proposition 7.2.4.

Proof. To prove the first statement note that, by Proposition 7.2.4, $\mathcal{X}_g$ has trivial canonical bundle. Moreover, $H^1(\mathcal{X}_g, \mathcal{O}_{\mathcal{X}_g}) = 0$ by Proposition 7.2.6. So $\mathcal{X}_g$ is Calabi-Yau.

The converse statement follows from the fact that, if $g\colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$ denotes the generalized functional invariant of $\mathcal{X}$, then Theorem 7.2.2 shows that $\mathcal{X}$ and $\mathcal{X}_g$ are isomorphic over the open set U. $\square$

Example 7.2.8. *We now explain how the five Calabi-Yau threefolds fibred by M_2-polarized K3 surfaces from [51, Theorem 5.10] fit into this picture. As noted in [51, Section 5.4], in each of these cases the generalized functional invariant $g\colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$ has the special form*

$$g\colon (s:t) \mapsto \lambda = \frac{As^{i+j}}{t^i(s-t)^j},$$

where $(s:t)$ are coordinates on $\mathbb{P}^1$, $A \in \mathbb{C}$ is a modular parameter for the threefold, and $i, j \in \{1,2,4\}$ are as listed in [51, Table 1].

In our notation from above, this map g has $(k,l,m,n,r) = (1,2,i+j,i+j,1)$, $[x_1] = [i+j]$, $[y_1,y_2] = [i,j]$, and $[z_1,\ldots,z_{i+j}] = [1,\ldots,1]$. It follows immediately from Proposition 7.2.4 and Corollary 7.2.7 that $\mathcal{X}_g$ is a smooth Calabi-Yau threefold for each choice of i, j, as we should expect. The reason for the special form of the generalized functional invariants g appearing in these cases will be discussed later in Remark 7.3.8.

In particular, we see that the M_2-polarized K3 fibration on the quintic mirror threefold appears as a special case of this construction, with $(i,j) = (1,4)$. Its generalized functional invariant g therefore has $(k,l,m,n,r) = (1,2,5,5,1)$, $[x_1] = [5]$, $[y_1,y_2] = [1,4]$, and $[z_1,\ldots,z_5] = [1,1,1,1,1]$.

7.3 Hodge Numbers

This enables us to construct a large class of Calabi-Yau threefolds $\mathcal{X}_g$ admitting fibrations by M_2-polarized K3 surfaces. We next study the Hodge numbers of these Calabi-Yau threefolds.

Remark 7.3.1. It is easy to see here that the case $l = 1$, $y_1 = 8$ is a smooth degeneration of the case $l = 2$, $(y_1,y_2) = (4,4)$, corresponding to restriction to a sublocus in moduli. Therefore, when discussing the Hodge numbers of the Calabi-Yau threefolds $\mathcal{X}_g$, to avoid pathological cases we will restrict to the case $l = 2$. In this case we have $y_1, y_2 \in \{1, 2, 4\}$ and $k + m - n - r = 0$.

7.3.1 The Hodge number $h^{1,1}$

The Hodge number $h^{1,1}$ is relatively easy to compute. We find:

Proposition 7.3.2. *Let $\mathcal{X}_g$ be a Calabi-Yau threefold as in Corollary 7.2.7 and suppose that $g^{-1}(\infty)$ consists of two points (so that $l = 2$). Then*

$$h^{1,1}(\mathcal{X}_g) = 20 + \sum_{i=1}^{k}(2x_i^2 + 1) + c_1 + c_2,$$

where $[x_1, \ldots, x_k]$ is the partition of n encoding the ramification profile of g over $\lambda = 0$ and c_1, c_2 are given in terms of the partition $[y_1, y_2]$ of n encoding the ramification profile of g over $\lambda = \infty$ by $c_j = 30$ (resp. 10, 0) if and only if $y_j = 1$ (resp. 2, 4).

Proof. Note that $h^{1,1}(\mathcal{X}_g)$ is equal to the rank of $\mathrm{Pic}(\mathcal{X}_g)$, so it suffices to find this rank.

Recall first that, with respect to the fibration π_g on $\mathcal{X}_g$, any irreducible effective divisor D on $\mathcal{X}_g$ is either *horizontal* (i.e. the restriction of $\pi_g \colon \mathcal{X}_g \to \mathbb{P}^1$ to D is surjective) or *vertical* (i.e. $\pi_g(D)$ is a point, so D is a component of a fibre). Moreover, $\mathrm{Pic}(\mathcal{X}_g) \cong \mathrm{Pic}^h(\mathcal{X}_g) \oplus \mathrm{Pic}^v(\mathcal{X}_g)$, where $\mathrm{Pic}^h(\mathcal{X}_g)$ and $\mathrm{Pic}^v(\mathcal{X}_g)$ denote the subspaces of $\mathrm{Pic}(\mathcal{X}_g)$ spanned by the classes of horizontal and vertical divisors respectively. We analyze each of these subspaces in turn.

The subspace of horizontal divisors is easy to access. Let X_p denote the fibre of $\pi_g \colon \mathcal{X}_g \to \mathbb{P}^1$ over a general point $p \in \mathbb{P}^1$. Then any horizontal divisor restricts to a divisor on X_p and this restriction respects linear equivalence, so we have an embedding $\mathrm{Pic}^h(\mathcal{X}_g) \hookrightarrow \mathrm{Pic}(X_p)$. Furthermore, as the restriction of π_g to the smooth fibres of $\mathcal{X}_g$ defines an M_2-polarized family of K3 surfaces (in the sense of [51, Definition 2.1]), monodromy around singular fibres acts trivially on divisor classes in X_p. Thus, every irreducible effective divisor in X_p sweeps out a unique irreducible effective divisor in $\mathcal{X}_g$. This shows that $\mathrm{Pic}^h(\mathcal{X}_g) \cong \mathrm{Pic}(X_p)$, so $\mathrm{rank}\,(\mathrm{Pic}^h(\mathcal{X}_g)) = \rho(X_p) = 19$.

The vertical divisors are more difficult. Firstly, we note that any two fibres are linearly equivalent. This contributes a single divisor class to $\mathrm{Pic}^v(\mathcal{X}_g)$. The remaining vertical divisor classes arise from components of singular fibres of $\mathcal{X}_g$.

The fibres of $\mathcal{X}_g$ are irreducible over all points $p \in \mathbb{P}^1$ with $g(p) = \lambda \notin \{0, \infty\}$, so we only need to consider the points p outside this set. First suppose that p is a point with $g(p) = 0$ and let x denote the order of ramification of g at p. The fibre of $\mathcal{X}_2$ over $\lambda = 0$ is semistable, with four components arranged as a tetrahedron. So the fibre of $\mathcal{X}_g$ over p is isomorphic to the resolution of the pull-back of such a fibre by a base change $t \mapsto t^x$, where t is a local coordinate around $p \in \mathbb{P}^1$.

Pull-backs of semistable fibres of this kind were computed by Friedman [57, Section 1]. By [57, Proposition 1.2], we see that the fibre of $\mathcal{X}_g$ over p has

- 4 components that are strict transforms of the original 4,
- $6(x-1)$ new components arising from the blow-ups of the six curves of cA_{x-1} singularities that occur along the pull-backs of the six edges of the tetrahedron, and
- $2(x-1)(x-2)$ new components arising from the blow-ups of the pull-backs of the four corners of the tetrahedron.

Summing, we obtain $(2x^2+2)$ components in the fibre over p. This gives $(2x^2+1)$ new classes in $\mathrm{Pic}^v(\mathcal{X}_g)$ (as the sum of all $(2x^2+2)$ components is linearly equivalent to the class of a fibre, which we have already counted).

Finally, we consider a fibre of $\mathcal{X}_g$ over a point p with $g(p) = \infty$. Let y denote the order of ramification of g at p; by Proposition 7.2.3 we have $y \in \{1, 2, 4\}$. In each case, the fibre of $\mathcal{X}_g$ over p was computed explicitly in the proof of Proposition 7.2.4. In particular, we found that it has 31 (resp. 11, 1) components when $y = 1$ (resp. 2, 4). Thus, the fibre of $\mathcal{X}_g$ over p contributes 30 (resp. 10, 0) new classes to $\mathrm{Pic}^v(\mathcal{X}_g)$ when $y = 1$ (resp. 2, 4).

Summing over all singular fibres in $\mathcal{X}_g$, we find that

$$\operatorname{rank}(\operatorname{Pic}^v(\mathcal{X}_g)) = 1 + \sum_{i=1}^{k}(2x_i^2+1) + c_1 + c_2,$$

where x_i and c_j are as in the statement of the proposition. Adding in the 19 horizontal divisor classes, we obtain the result. □

7.3.2 Leray filtrations and local systems

The Hodge number $h^{2,1}$ is somewhat more difficult to compute. To find it, we begin by developing some general theoretical results that apply to any K3-fibred threefolds, then specialize these to the case that interests us. Our main tools in this endeavour will be the Leray spectral sequence and Poincaré's formula for the ranks of cohomology groups of local systems.

So let $\pi\colon \mathcal{X} \to B$ be a smooth projective threefold fibred by K3 surfaces over a smooth complete base curve B. Denote the fibre of $\mathcal{X}$ over $p \in B$ by X_p. Zucker [149, Corollary 15.15] has shown that, under these assumptions, the Leray spectral sequence for $\pi\colon \mathcal{X} \to B$ degenerates, so we have a Leray filtration $L^\bullet$ on $H^3(\mathcal{X},\mathbb{Q})$ with $\mathrm{Gr}_L^i = H^{3-i}(B, R^i\pi_*\mathbb{Q})$.

Since B is an algebraic curve, we must have $H^3(B,\pi_*\mathbb{Q}) = 0$. Moreover, $R^1\pi_*\mathbb{Q}$ is a skyscraper sheaf since the generic fibre of π is a K3 surface, giving $H^2(B, R^1\pi_*\mathbb{Q}) = 0$ also. The Leray filtration thus gives rise to the following exact sequence

$$0 \longrightarrow H^1(B, R^2\pi_*\mathbb{Q}) \longrightarrow H^3(\mathcal{X},\mathbb{Q}) \longrightarrow H^0(B, R^3\pi_*\mathbb{Q}) \longrightarrow 0, \tag{7.3}$$

which may be used to study $H^3(\mathcal{X},\mathbb{Q})$.

Now, if $\pi_U\colon \mathcal{X}_U \to U$ is the restriction of π to the locus of smooth fibres and $j\colon U \hookrightarrow B$ is the natural injection, it follows from the local invariant cycle theorem that

$$H^1(B, R^2\pi_*\mathbb{Q}) \cong H^1(B, j_*(R^2(\pi_U)_*\mathbb{Q}))$$

(see [149, (15.1)] for details). This cohomology group can be computed in explicit examples, as we shall see in Section 7.3.3.

To study $H^3(\mathcal{X},\mathbb{Q})$ we must also calculate $H^0(B,R^3\pi_*\mathbb{Q})$. Zucker [149, Section 15] shows that $H^0(B,R^3\pi_*\mathbb{Q})$ admits a pure Hodge structure of weight 3 and that there is a short exact sequence

$$0 \longrightarrow A \longrightarrow H^0(B,R^3\pi_*\mathbb{Q}) \longrightarrow H^0(B,j_*R^3(\pi_U)_*\mathbb{Q}) \longrightarrow 0; \tag{7.4}$$

here $A \subset \bigoplus_{s\in\Sigma} H^3(X_s,\mathbb{Q})$, where $\Sigma \subset B$ denotes the locus of singular fibres in $\mathcal{X}$ and X_s denotes the fibre of $\mathcal{X}$ over $s \in \Sigma$. Specializing this result to our setting, where π is a K3 surface fibration, we see that $H^0(B,j_*R^3(\pi_U)_*\mathbb{Q}) = 0$, so the exact sequence above gives $H^0(B,R^3\pi_*\mathbb{Q}) \cong A$.

A may be described explicitly as follows. Suppose that $\widetilde{\mathcal{X}}$ is a resolution of $\mathcal{X}$, chosen so that each singular fibre $\widetilde{X}_s$ in $\widetilde{\mathcal{X}}$ is a normal crossings union of smooth surfaces, and let $\widetilde{\pi}\colon \widetilde{\mathcal{X}} \to B$ be the induced K3 fibration on $\widetilde{\mathcal{X}}$. Let $\Delta_s \subset B$ be a small closed disc around each point $s \in \Sigma$, let $\widetilde{\mathcal{X}}_{\Delta_s} = \widetilde{\pi}^{-1}(\Delta_s)$ and let $\partial\widetilde{\mathcal{X}}_{\Delta_s}$ be the boundary of $\widetilde{\mathcal{X}}_{\Delta_s}$. Then Zucker [149, Section 15] shows that A may be defined as a sum of images of morphisms of mixed Hodge structures,

$$A = \bigoplus_{s\in\Sigma} \mathrm{im}\left(\phi_s\colon H^3(\widetilde{\mathcal{X}}_{\Delta_s},\partial\widetilde{\mathcal{X}}_{\Delta_s}) \to H^3(\widetilde{X}_s)\right),$$

from which it follows that A admits a pure Hodge structure of weight 3. The exact definitions of the maps ϕ_s appearing here will not concern us, as we only need to use the fact that they are morphisms of mixed Hodge structures; the interested reader may refer to [149, Section 15] for more details. We compute A in the following proposition.

Proposition 7.3.3. *Let $\pi\colon \mathcal{X} \to B$ be a smooth projective threefold fibred by K3 surfaces. Let $X_p = \pi^{-1}(p)$ for p a point in B. If every X_p is either:*

1. *a K3 surface with at worst ADE singularities, or*

2. *a normal crossings union of smooth surfaces S_i, with $H^3(S_i,\mathbb{Q}) = 0$,*

then $A = 0$. In particular, this implies that $H^3(\mathcal{X},\mathbb{Q}) \cong H^1(B,j_*R^2(\pi_U)_*\mathbb{Q})$.

Proof. To show that $A \subset \bigoplus_{s\in\Sigma} H^3(X_s)$ is trivial, it suffices to show that its restriction to $H^3(X_s)$ is trivial for every $s \in \Sigma$.

Begin by letting X_s be any fibre satisfying case (1) of the proposition. Then X_s is still a K3 surface, so $H^3(X_s) = 0$. Thus the restriction of A to $H^3(X_s)$ must be trivial in this case.

Now consider a fibre X_s satisfying case (2). As X_s is a normal crossings union of smooth surfaces, we may choose a resolution $\widetilde{\mathcal{X}}$ so that $\widetilde{\mathcal{X}} \cong \mathcal{X}$ in a neighbourhood Δ_s of s. Therefore, by the definition of A above, to show that the restriction of A to $H^3(X_s)$ is trivial, it suffices to show that the image

$$\mathrm{im}\left(\phi_s \colon H^3(\mathcal{X}_{\Delta_s}, \partial\mathcal{X}_{\Delta_s}) \to H^3(X_s)\right)$$

is trivial.

To do this, we use the Mayer-Vietoris spectral sequence as described by Griffiths and Schmid in [66, Section 4]. Let X_s have irreducible components $\{S_i\}$, then define $(X_s)_{i_1,\ldots,i_p} = \bigcap_{i_0\ldots,i_p} S_{i_j}$ for a disjoint set of indices $i_0,\ldots,i_p$ and let $X_s^{[p]} = \coprod_{i_0<\cdots<i_p}(X_s)_{i_0,\ldots,i_p}$. The E_1 term of the Mayer-Vietoris spectral sequence is then given by $E_1^{p,q} = H^q(X_s^{[p]}, \mathbb{Q})$.

This spectral sequence degenerates at the E_2 level and converges to $H^{p+q}(X_s, \mathbb{Q})$. Its graded pieces Gr_i^W are the weight-graded pieces of the functorial mixed Hodge structure on X_s. Thus only $H^3(X_s^{[0]}, \mathbb{Q})$, $H^2(X_s^{[1]}, \mathbb{Q})$ and $H^1(X_s^{[2]}, \mathbb{Q})$ may contribute to $H^3(X_s, \mathbb{Q})$. Moreover, by the condition that S_i is a smooth rational surface for all values of i, we have $H^3(S_i, \mathbb{Q}) = 0$ and hence $\mathrm{Gr}_3^W H^3(X_s, \mathbb{Q}) = H^3(X_s^{[0]}, \mathbb{Q}) = 0$.

Zucker [149, Section 15] notes that the weight filtration on $H^3(\mathcal{X}_{\Delta_s}, \partial\mathcal{X}_{\Delta_s})$ has $W_i = 0$ for $i \leq 2$. By strictness, we thus see that

$$\mathrm{im}(\phi_s) \cap W_2(H^3(X_s, \mathbb{Q})) = 0$$

and

$$\phi_s(W_i H^3(\mathcal{X}_{\Delta_s}, \partial\mathcal{X}_{\Delta_s})) = \mathrm{im}(\phi_s) \cap W_i(H^3(X_s)) = \mathrm{im}(\phi_s) \cap W_3(H^3(X_s))$$

for all $i \geq 3$. So in particular

$$\operatorname{im}(\phi_s) = \phi_s(W_3 H^3(\mathcal{X}_{\Delta_s}, \partial \mathcal{X}_{\Delta_s})) \subset W_3(H^3(X_s))$$

and the map

$$W_3(H^3(X_s)) \longrightarrow \mathrm{Gr}_3^W(H^3(X_s))$$

is injective on the image of ϕ_s. Thus the image of the induced map

$$W_3 H^3(\mathcal{X}_{\Delta_s}, \partial \mathcal{X}_{\Delta_s}) \xrightarrow{\phi_s} W_3(H^3(X_s)) \longrightarrow \mathrm{Gr}_3^W(H^3(X_s)) = 0$$

is equal to the image of ϕ_s, so the restriction of A to $H^3(X_s)$ is trivial in this case.

Therefore, we have $A = 0$ under the conditions of the proposition. It thereby follows from the exact sequence (7.4) that there is an isomorphism

$$H^0(B, R^3\pi_*\mathbb{Q}) \cong H^0(B, j_* R^3(\pi_U)_*\mathbb{Q}) = 0$$

and substitution into the exact sequence (7.3) gives $H^3(\mathcal{Y}, \mathbb{Q}) \cong H^1(B, j_* R^2(\pi_U)_*\mathbb{Q})$.
$\square$

7.3.3 The Hodge number $h^{2,1}$

Now let $\mathbb{V}$ be an irreducible $\mathbb{Q}$-local system on a quasi-projective curve U and let $j\colon U \hookrightarrow B$ be the canonical injection of U into its smooth closure. Associated to $\mathbb{V}$ and a base-point $p \in U$, we have a representation $\rho\colon \pi_1(U, p) \to \mathrm{GL}(\mathbb{V}_p)$, where $\mathbb{V}_p$ is the fibre of $\mathbb{V}$ at p.

Denote the points in $B - U$ by $\{q_1, \ldots, q_s\}$. Via this representation, to each q_i we may associate a local monodromy matrix γ_i, coming from a counterclockwise loop about q_i. This allows us to associate an integer

$$R(q_i) := \operatorname{rank} \mathbb{V}_p - \dim(\mathbb{V}_p^{\gamma_i})$$

to each q_i, where $\mathbb{V}_p^{\gamma_i}$ is the subspace of elements of $\mathbb{V}_p$ that are fixed under the action of γ_i

With this in place, we may compute $h^1(B, j_*\mathbb{V})$ using the following variation on Poincaré's formula in classical topology, due to del Angel, Müller-Stach, van Straten and Zuo [40, Proposition 3.6]:

$$h^1(B, j_*\mathbb{V}) = \sum_{i=1}^{n} R(q_i) + 2(g(B) - 1)\mathrm{rank}\,(\mathbb{V}). \tag{7.5}$$

As a result of this formula and Proposition 7.3.3, if the singular fibres of π satisfy the assumptions of Proposition 7.3.3 and we know the local monodromy matrices, then we can easily deduce the Betti number $b_3(\mathcal{X})$ of a K3 surface fibred threefold $\mathcal{X}$. These conditions are satisfied by the examples discussed in Section 7.2.

Example 7.3.4. *Let $\mathcal{X}_2 \to \mathcal{M}_{M_2}$ be the K3-fibred threefold discussed in Section 7.2.1. Recall that $\mathcal{X}_2$ has three singular fibres, over the points $(q_1, q_2, q_3) = (0, \frac{1}{256}, \infty)$, and that the family of K3 surfaces over $U_{M_2} := \mathcal{M}_{M_2} - \{q_1, q_2, q_3\}$ is an M_2-polarized family of K3 surfaces (in the sense of [51, Definition 2.1]).*

If π_U denotes the restriction of the fibration $\mathcal{X}_2 \to \mathcal{M}_{M_2}$ to U_{M_2}, then $R^2(\pi_U)_*\mathbb{Q}$ is a $\mathbb{Q}$-local system on U_{M_2}. It is easy to see from the explicit description in the proof of Proposition 7.2.4 that the singular fibres of $\mathcal{X}_2$ are either nodal K3 surfaces or normal crossings unions of smooth rational surfaces. Thus we may apply Proposition 7.3.3 to deduce that $b_3(\mathcal{X}_2) = h^1(\mathcal{M}_{M_2}, j_*(R^2(\pi_U)_*\mathbb{Q}))$, where $j\colon U_{M_2} \to \mathcal{M}_{M_2}$ denotes the inclusion.

It therefore remains to compute this cohomology. The discussion of [51, Section 2.1] gives a splitting of $R^2(\pi_U)_*\mathbb{Q}$ as a direct sum of two irreducible $\mathbb{Q}$-local systems

$$R^2(\pi_U)_*\mathbb{Q} = \mathcal{NS}(\mathcal{X}_2) \oplus \mathcal{T}(\mathcal{X}_2),$$

where $\mathcal{NS}(\mathcal{X}_2)$ consists of those classes which are in $\mathrm{NS}(X_t) \otimes \mathbb{Q}$ for every smooth fibre X_p of $\mathcal{X}_2$, and $\mathcal{T}(\mathcal{X}_2)$ is the orthogonal complement of $\mathcal{NS}(\mathcal{X}_2)$. In our situation,

$\mathcal{NS}(\mathcal{X}_2)$ is a trivial rank 19 local system, and $\mathcal{T}(\mathcal{X}_2)$ is an irreducible local system of rank 3.

We therefore have

$$H^1(\mathcal{M}_{M_2}, j_*(R^2(\pi_U)_*\mathbb{Q})) = H^1(\mathcal{M}_{M_2}, j_*\mathcal{T}(\mathcal{X}_2)) \oplus H^1(\mathcal{M}_{M_2}, j_*\mathcal{NS}(\mathcal{X}_2)),$$

and triviality of $\mathcal{NS}(\mathcal{X}_2)$ allows us to reduce this expression to

$$H^1(\mathcal{M}_{M_2}, j_*(R^2(\pi_U)_*\mathbb{Q})) = H^1(\mathcal{M}_{M_2}, j_*\mathcal{T}(\mathcal{X}_2)).$$

We will compute this last cohomology group using Equation (7.5). According to [108, Section 5], the Picard-Fuchs equation of the family of K3 surfaces $\mathcal{X}_2$ is hypergeometric of type ${}_3F_2(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}; 1, 1; 256\lambda)$. From this, we may use a theorem of Levelt ([96, Theorem 1.1], see also [22, Theorem 3.5]) to compute that the global monodromy representation of $\mathcal{T}(\mathcal{X}_2) \otimes \mathbb{R}$ is given by the monodromy matrices

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 3 & 1 & 0 \\ -3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A^{-1}B = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

around $\lambda = \infty$, $\lambda = 0$ and $\lambda = \frac{1}{256}$ respectively (here the names of the matrices have been chosen to agree with [22]). Thus the local system $\mathcal{T}(\mathcal{X}_2) \otimes \mathbb{C}$ has local monodromy matrices given by

$$\gamma_\infty = \begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & -\sqrt{-1} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma_{\frac{1}{256}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using Equation (7.5), we can therefore compute that

$$b_3(\mathcal{X}_2) = h^1(\mathcal{M}_{M_2}, j_*(R^2(\pi_U)_*\mathbb{Q})) = h^1(\mathcal{M}_{M_2}, j_*\mathcal{T}(\mathcal{X}_2)) = 3 + 1 + 2 - 2 \cdot 3 = 0$$

Local systems $\mathbb{V}$ satisfying $h^1(\mathbb{P}^1, j_*\mathbb{V}) = 0$ are called extremal local systems *in [32].*

Remark 7.3.5. Note that we choose not to use the explicit monodromy matrices computed in [108, Section 4] for this calculation. This is because the method used to compute monodromy matrices in [108] contains a sign ambiguity, corresponding to the choice of primitive fourth root of unity in the transformation [108, (4.1)]. Making the opposite choice has the effect of applying an antisymplectic involution on the fibres, which multiplies the monodromy matrices γ_∞ and γ_0 by a factor of -1. As this sign is crucial in the computation of $R(0)$ and $R(\infty)$, we choose to avoid ambiguity and instead compute the monodromy matrices directly from the Picard-Fuchs equation.

Next we consider the general case. Let $\pi_g \colon \mathcal{X}_g \to \mathbb{P}^1$ be a K3 surface fibred Calabi-Yau threefold as in Corollary 7.2.7 and suppose that $g^{-1}(\infty)$ consists of two points (so that $l = 2$). Recall that $\mathcal{X}_g$ is defined by a degree n cover $g \colon \mathbb{P}^1 \to \mathcal{M}_2$ with ramification profiles $[x_1, \ldots, x_k]$, $[y_1, \ldots, y_l]$ and $[z_1, \ldots, z_m]$ over $\lambda = 0$, $\lambda = \infty$ and $\lambda = \frac{1}{256}$ respectively, and ramification degree r away from these three points. Let $U \subset \mathbb{P}^1$ be the preimage $g^{-1}(U_{M_2})$ and let $j \colon U \to \mathbb{P}^1$ denote the inclusion.

Now, by the proof of Proposition 7.2.4, the singular fibres of $\mathcal{X}_g$ are all either nodal K3 surfaces or normal crossings unions of smooth rational surfaces, so the argument of Example 7.3.4 gives $b_3(\mathcal{X}_g) = h^1(\mathbb{P}^1, j_*\mathcal{T}(\mathcal{X}_g))$. But, by construction, the local system on U given by $\mathcal{T}(\mathcal{X}_g)$ is equal to $g^*\mathbb{V}$, where $\mathbb{V}$ is the local system over U_{M_2} given by $\mathcal{T}(\mathcal{X}_2)$. The cohomology of this local system is computed by:

Proposition 7.3.6. *Let $\mathbb{V}$ be the local system over U_{M_2} given by $\mathcal{T}(\mathcal{X}_2)$. We have*

$$h^1(\mathbb{P}^1, j_*g^*\mathbb{V}) = 2 + 2k + (m_{\text{odd}} - n),$$

where m_{odd} denotes the number of $z_1, \ldots, z_m$ which are odd.

In particular, if g is unramified over $\lambda = \frac{1}{256}$, then $h^1(\mathbb{P}^1, j_*g^*\mathbb{V}) = 2 + 2k$.

Proof. If g ramifies to order a at a point q in $\mathbb{P}^1 - U$, then the monodromy of the pulled-back local system $g^*\mathbb{V}$ about q is given by $\gamma_{g(q)}^a$ where $\gamma_{g(q)}$ is the monodromy matrix of $\mathcal{T}(\mathcal{X}_2)$ around $g(q)$. Therefore we can compute, using the explicit expressions for local monodromy found in Example 7.3.4, that

- if g ramifies to order y at a preimage of ∞, then $R(q) = 4 - \mathrm{hcf}(y, 4)$,
- if g ramifies to order z at a preimage of $\frac{1}{256}$, then $R(q) = 2 - \mathrm{hcf}(z, 2)$, and
- at any preimage of 0, we have $R(q) = 2$.

Thus we calculate

$$h^1(\mathbb{P}^1, j_*g^*\mathbb{V}) = \sum_{i=1}^{l}(4 - \mathrm{hcf}(y_i, 4)) + \sum_{j=1}^{m}(2 - \mathrm{hcf}(z_j, 2)) + 2k - 6. \tag{7.6}$$

Now we impose the conditions of Proposition 7.2.3. By assumption we have $l = 2$, and $y_i = \mathrm{hcf}(y_i, 4)$ for both y_1 and y_2. Equation (7.6) thus gives

$$\begin{aligned} h^1(\mathbb{P}^1, j_*g^*\mathbb{V}) &= (4 - y_1) + (4 - y_2) + \sum_{j=1}^{m}(2 - \mathrm{hcf}(z_j, 2)) + 2k - 6 \\ &= 2 + 2k + (m_{\mathrm{odd}} - n)\,. \end{aligned}$$

Note that $(m_{\mathrm{odd}} - n) \leq 0$ is an even number. □

Since $\mathcal{X}_g$ is a Calabi-Yau threefold, we therefore have:

Corollary 7.3.7. *Let $\mathcal{X}_g$ be a Calabi-Yau threefold as in Corollary 7.2.7 and suppose that $g^{-1}(\infty)$ consists of two points (so that $l = 2$). Then*

$$h^{2,1}(\mathcal{X}_g) = k + \left(\frac{m_{\mathrm{odd}} - n}{2}\right),$$

where k denotes the number of ramification points of g over $\lambda = 0$, m_{odd} denotes the number of ramification points of odd order of g over $\lambda = \frac{1}{256}$, and n is the degree of g.

Moreover, if g is unramified over $\lambda = \frac{1}{256}$, then $h^{2,1}(\mathcal{X}_g) = k = r$, the degree of ramification of g away from $\lambda \in \{0, \frac{1}{256}, \infty\}$.

Remark 7.3.8. We can now explain the general form for the generalized functional invariant maps g of the Calabi-Yau threefolds fibred by M_2-polarized K3 surfaces listed in [51, Theorem 5.10] (see Example 7.2.8). Indeed, in these cases $h^{2,1}(\mathcal{X}_g) = 1$ by assumption so, if we assume that the map g is unramified over $\lambda = \frac{1}{256}$ (which guarantees smoothness of $\mathcal{X}_g$, by Proposition 7.2.4), then $k = r = 1$ by Corollary 7.3.7. From this, we see that g is totally ramified over $\lambda = 0$ and has a single ramification of degree 2 away from $\lambda \in \{0, \frac{1}{256}, \infty\}$. Moreover, if we write $[y_1, y_2] = [i, j]$, for some $i, j \in \{1, 2, 4\}$, then we must have $\deg(g) = n = i + j$.

This shows that, if g is unramified over $\lambda = \frac{1}{256}$ and $h^{2,1}(\mathcal{X}_g) = 1$, then the generalized functional invariant map $g\colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$ must have the form given in Example 7.2.8.

To conclude this section, we demonstrate the application of this theory by calculating the Hodge numbers in our running example of the quintic mirror threefold:

Example 7.3.9. *Recall from Example 7.2.8 that the fibration of the quintic mirror threefold by M_2-polarized K3 surfaces has generalized functional invariant g with $(k, l, m, n, r) = (1, 2, 5, 5, 1)$, $[x_1] = [5]$, $[y_1, y_2] = [1, 4]$, and $[z_1, \ldots, z_5] = [1, 1, 1, 1, 1]$. The Hodge numbers of this threefold are well known; here we illustrate how to recover them from the results above.*

Firstly, we have $h^{2,1}(\mathcal{X}_g) = k = 1$, by Corollary 7.3.7. Moreover, by Proposition 7.3.2, we have

$$h^{1,1}(\mathcal{X}_g) = 20 + (2x_1^2 + 1) + c_1 + c_2 = 20 + 51 + 30 + 0 = 101$$

as expected.

7.4 Deformations and Moduli Spaces

Now consider the setting where g is unramified over the point $\lambda = \frac{1}{256}$ and has only simple ramification points away from $\lambda \in \{0, \frac{1}{256}, \infty\}$. In this case Corollary 7.3.7 raises an obvious question. It is easy to see that, in this setting, r is equal to the number of simple ramification points of g away from $\lambda \in \{0, \frac{1}{256}, \infty\}$. Moreover, for the corresponding threefolds $\mathcal{X}_g$, we also have $h^{2,1}(\mathcal{X}_g) = r$. So to what extent are small deformations of $\mathcal{X}_g$ determined by the locations of these simple ramification points?

In more generality, we may ask to what extent small deformations of the threefold $\mathcal{X}_g$ are determined by deformations of the map g. In fact, we find:

Proposition 7.4.1. *Let $\mathcal{X}_g$ be a Calabi-Yau threefold as in Corollary 7.2.7 and suppose that $g^{-1}(\infty)$ consists of two points (so that $l = 2$). Moreover, suppose that g is unramified over $\lambda = \frac{1}{256}$. Then any small deformation of $\mathcal{X}_g$ is obtained by deforming the map g in a way that preserves the ramification profiles over $\lambda \in \{0, \infty\}$.*

Proof. Suppose that $\pi_g \colon \mathcal{X}_g \to \mathbb{P}^1$ and $\pi_{g'} \colon \mathcal{X}_{g'} \to \mathbb{P}^1$ are two Calabi-Yau threefolds defined by maps $g, g' \colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$ satisfying the assumptions of the proposition. We say that an isomorphism $\varphi \colon \mathcal{X}_g \to \mathcal{X}_{g'}$ is an isomorphism of fibrations between $(\mathcal{X}_g, \pi_g)$ and $(\mathcal{X}_{g'}, \pi_{g'})$ if there is a commutative diagram

$$\begin{tikzcd} \mathcal{X}_g \arrow[rr, "\varphi"] \arrow[dr, "\pi_g"'] & & \mathcal{X}_{g'} \arrow[dl, "\pi_{g'}"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

Such an isomorphism exists if and only if then there is an automorphism $\phi \colon \mathbb{P}^1 \to \mathbb{P}^1$ so that

$$\begin{tikzcd} \mathbb{P}^1 \arrow[rr, "\phi"] \arrow[dr, "g"'] & & \mathbb{P}^1 \arrow[dl, "g'"] \\ & \mathcal{M}_{\mathrm{M}_2} & \end{tikzcd}$$

Assume that g has ramification profile $[y_1, y_2]$ over $\lambda = \infty$ and $[x_1, \ldots, x_k]$ over $\lambda = 0$. By applying an automorphism ϕ of $\mathbb{P}^1$ as above, we may assume that the ramification points over $\lambda = \infty$ are $(1 : 0)$ and $(0 : 1)$, and that $(1 : 1)$ is a ramification point over $\lambda = 0$ with ramification index x_1. Then g may be written as

$$g \colon (s : t) \longmapsto \frac{a_1 (s - t)^{x_1} \prod_{i=2}^{k} (s - a_i t)^{x_i}}{s^{y_1} t^{y_2}},$$

with parameters $a_1, \dots, a_k \in (\mathbb{C}^\times)^k - \Delta$ where Δ is the union of the big diagonals in $(\mathbb{C}^\times)^k$.

Thus there is an k-dimensional space of maps $g\colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$ with the property that g has ramification profile $[y_1, y_2]$ over $\lambda = \infty$ and $[x_1, \dots, x_k]$ over $\lambda = 0$. By the discussion above, this means that the space of local deformations of the fibration $(\mathcal{X}_g, \pi_g)$ is also k-dimensional.

Now, by a result of Oguiso [114, Example 2.3], K3 fibrations on $\mathcal{X}_g$ correspond to certain rational rays in the nef cone of $\mathcal{X}_g$ so, in particular, there are at most countably many K3 fibrations on $\mathcal{X}_g$. This means that we cannot continuously vary the K3 fibration on $\mathcal{X}_g$ without deforming $\mathcal{X}_g$ itself. Since the K3 fibration $(\mathcal{X}_g, \pi_g)$ may be deformed in k different directions and the deformation space of $\mathcal{X}_g$ is k-dimensional (by Corollary 7.3.7), the claim follows. $\square$

Remark 7.4.2. From the proof of this proposition the reader may note that, under the assumptions that $g^{-1}(\infty)$ consists of two points and g is unramified over $\lambda = \frac{1}{256}$, an open subset of the moduli space of K3-fibred Calabi-Yau threefolds $(\mathcal{X}_g, \pi_g)$ is identified with a moduli space of maps g with fixed ramification profiles over $\{0, \frac{1}{256}, \infty\}$. Moduli spaces of such maps are called *Hurwitz spaces* and have been studied extensively in the literature.

Next, we will show that any Calabi-Yau threefold $\mathcal{X}_g$ is deformation equivalent to a Calabi-Yau threefold $\mathcal{X}_{g'}$ defined by a map $g'\colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$ with only simple ramification away from $\lambda \in \{0, \frac{1}{256}, \infty\}$. In particular this shows that, if we are only interested in generic members of deformation classes, we can safely ignore the type of ramification away from $\lambda \in \{0, \frac{1}{256}, \infty\}$.

Proposition 7.4.3. *Let $g\colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$ be a morphism. Then there exists a deformation $g'\colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$ of g that has only simple ramification away from $\lambda \in \{0, \frac{1}{256}, \infty\}$.*

Thus, if $\mathcal{X}_g$ is a Calabi-Yau threefold as in Corollary 7.2.7, then $\mathcal{X}_g$ is deformation equivalent to a Calabi-Yau threefold $\mathcal{X}_{g'}$ defined by a map $g'\colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$ that is simply ramified away from $\lambda \in \{0, \frac{1}{256}, \infty\}$.

Remark 7.4.4. We note that this result is not unexpected: neither our computation of $h^{1,1}(\mathcal{X}_g)$ nor our computation of $h^{2,1}(\mathcal{X}_g)$ made any reference to the type of ramification away from $\lambda \in \{0, \frac{1}{256}, \infty\}$, so we should not expect such ramification to affect the deformation type of $\mathcal{X}_g$.

Proof. Assume that g has degree n and let $\Sigma = \{p_1, \ldots, p_s\}$ be the set of branch points of g in $\mathcal{M}_{M_2}$. Choose a set of discs Δ_i around each $p_i \in \Sigma$, small enough that no pair of discs intersects, and choose non-intersecting paths β_i from a base-point $p \in \mathbb{P}^1$ to the boundary of each Δ_i. For each Δ_i, let γ_i be the path obtained by following the path β_i from p to the boundary of Δ_i, going around $\partial\Delta_i$ once counterclockwise, then traversing β_i backwards to p.

The classes of γ_i generate $\pi_1(p\,, \mathcal{M}_{M_2} - \Sigma)$ and the concatenation $\gamma_1 \cdots \gamma_s$ is a contractible loop. Label the points above p by the integers $\{1, \ldots, n\}$. Then to each point $p_i \in \Sigma$, we may associate an element $\sigma_i \in S_n$ which describes the action of monodromy around γ_i on the points over p. This monodromy representation determines g up to reordering of the points over p. Since $\mathbb{P}^1$ is connected, the subgroup of S_n generated by $\{\sigma_1, \ldots, \sigma_s\}$ is transitive.

If g is not simply ramified away from $\lambda \in \{0, \frac{1}{256}, \infty\}$, then there exists a $p_i \notin \{0, \frac{1}{256}, \infty\}$ so that the corresponding σ_i is not a transposition. Let $\sigma_i = \tau_1 \cdots \tau_{s'}$ be a minimal decomposition of such a σ_i into transpositions. Then we claim that the set $P' = \{\sigma_1, \ldots, \sigma_{i-1}, \tau_1, \ldots, \tau_{s'}, \sigma_{i+1}, \ldots, \sigma_s\}$ can be used to define a new cover $g' \colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$, so that the points q_j over which g' ramifies with cycle structure τ_j have only simple ramification.

To define this cover, let t be a complex coordinate on the disc Δ_i, chosen so that $\Delta_i = \{t \in \mathbb{C} \mid |t| < 1\}$ and p_i lies at $t = 0$. Let q denote the point where the path β_i meets the boundary of Δ_i. Take points $q_1, \ldots, q_{s'} \in \Delta_i$ and define non-intersecting loops δ_j from q around each q_j. Then let γ'_j denote the path obtained by following the path β_i from p to q, going around δ_j once counterclockwise, then traversing β_i backwards to p. After relabelling if necessary, we may assume that $\gamma_i = \gamma'_1 \cdots \gamma'_{s'}$ in $\pi_1(p\,, \mathcal{M}_{M_2} - \Sigma')$, where $\Sigma' := \{p_1, \ldots, p_{i-1}, q_1, \ldots, q_{s'}, p_{i+1}, \ldots p_s\}$.

By construction, the set $\{\gamma_1, \ldots, \gamma_{i-1}, \gamma'_1, \ldots, \gamma'_{s'}, \gamma_{i+1}, \ldots, \gamma_s\}$ forms a basis for $\pi_1(p\,, \mathcal{M}_{M_2} - \Sigma')$. Assign the set P' of elements of S_n to these loops by associating σ_i to γ_i and τ_j to γ'_j. This defines a representation $\rho\colon \pi_1(p\,, \mathcal{M}_{M_2} - \Sigma') \to S_n$ whose image is transitive by construction so, by the Riemann existence theorem, there is a unique connected curve C and morphism $g'\colon C \to \mathcal{M}_{M_2}$, such that g' is branched over Σ' and ρ is the monodromy representation of the associated Galois cover of $\mathcal{M}_{M_2} - \Sigma'$. Using the Riemann-Hurwitz formula, it is easy to check that $C \cong \mathbb{P}^1$

Now take a deformation g'_t of g', obtained by multiplying each point $q_1, \ldots, q_{s'}$ by the local coordinate t, and an appropriate deformation of the loops γ'_j. At $t = 0$, the points q_j all go to p_i and the map g degenerates to a map g'_0 whose monodromy about p_i is $\tau_1 \cdots \tau_{s'} = \sigma_i$. By the uniqueness part of the Riemann existence theorem, the map g'_0 is exactly g.

We may now repeat this procedure for each $p_i \neq \{0, \frac{1}{256}, \infty\}$ over which the corresponding ramification of g is not simple, to obtain a map $g'\colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$ that deforms to g and has simple ramification away from $\{0, \frac{1}{256}, \infty\}$.

Given this, the statement about the threefolds $\mathcal{X}_g$ and $\mathcal{X}_{g'}$ follows from the fact that the deformation $g \rightsquigarrow g'$ induces a deformation $\mathcal{X}_g \rightsquigarrow \mathcal{X}_{g'}$. As this deformation does not affect a neighbourhood of the fibres above $\lambda \in \{0, \frac{1}{256}, \infty\}$, we see that $\mathcal{X}_{g'}$ must also be Calabi-Yau. $\square$

We conclude this section by asking what happens to the threefolds $\mathcal{X}_g$ when the map g degenerates. Such degenerations occur when a ramification point away from $\lambda \in \{0, \frac{1}{256}, \infty\}$ moves to one of these points. In this situation it is easy to see what occurs: the ramification profile defining $\mathcal{X}_g$ changes and the threefold becomes singular. If the new ramification profile defines a smooth Calabi-Yau (according to Corollary 7.2.7), then this singular threefold admits a Calabi-Yau resolution, with new Hodge numbers given by Proposition 7.3.2 and Corollary 7.3.7. Geometrically, the Calabi-Yau threefold $\mathcal{X}_g$ undergoes a *geometric transition* to a new Calabi-Yau threefold with different Hodge numbers.

Example 7.4.5. *In our running example of the quintic mirror threefold, the generalized functional invariant g has one simple ramification away from $\lambda \in \{0, \frac{1}{256}, \infty\}$. As noted above, moving this ramification point corresponds to deforming the quintic mirror in its (1-dimensional) complex moduli space. This can also be seen from the explicit form of the generalized functional invariant given in Example 7.2.8, where varying the modular parameter A changes the location of this simple ramification point, whilst keeping the other ramification points fixed.*

It is easy to see what happens when this simple ramification point moves to $\lambda \in \{0, \frac{1}{256}, \infty\}$. At $\lambda = \frac{1}{256}$ (corresponding to $A = \frac{1}{5^5}$), the proof of Proposition 7.2.4 shows that $\mathcal{X}_g$ acquires a single isolated cA_1 (node) singularity. Moreover, by [106, Lemma 3.5] and [105, Theorem 2.5], the resulting singular threefold is $\mathbb{Q}$-factorial, so does not admit a crepant resolution. In particular, this degeneration provides an example of a map g that satisfies the conditions of Proposition 7.2.4 but does not *give rise to a smooth Calabi-Yau threefold.*

When the simple ramification point moves to $\lambda = \infty$ (corresponding to $A = \infty$), the map g becomes totally ramified over $\lambda \in \{0, \infty\}$. The degenerate threefold acquires an additional $\mathbb{Z}/5\mathbb{Z}$ action, which acts to permute the sheets of this cover.

Finally, when the simple ramification point moves to $\lambda = 0$ (corresponding to $A = 0$), we obtain a degeneration with maximally unipotent monodromy *(see Section 7.5.1).*

As we can see, we have obtained the three well-known boundary points in the complex moduli space of the quintic mirror threefold.

7.5 Mirror Symmetry

We conclude this paper with an exploration of the mirror dual varieties associated to the K3-fibred Calabi-Yau threefolds that are constructed above.

We begin by assuming that $\mathcal{X}_g$ is a Calabi-Yau threefold as in Corollary 7.2.7. Suppose further that $g^{-1}(\infty)$ consists of two points and that g is unramified over

$\lambda = \frac{1}{256}$ (so $l = 2$ and $m = n$). In this case, by the discussion in the last section, we see that the deformation class of $\mathcal{X}_g$ is determined by two pieces of data:

1. a choice of $i, j \in \{1, 2, 4\}$ (these determine both the ramification profile $[y_1, y_2] = [i, j]$ over $\lambda = \infty$ and the degree $\deg(g) = n = i + j$); and

2. a partition $\mu = [x_1, \ldots, x_k]$ of $n = i + j$ (this defines the ramification profile over $\lambda = 0$).

Moreover, it follows from Proposition 7.2.4 and Corollary 7.2.7 that any threefold $\mathcal{X}_g$ determined by such data must be Calabi-Yau.

In the remainder of this section, we will denote a general Calabi-Yau in the deformation class determined by a choice of such data by $X^{\mu}_{i,j}$. Our aim is to exhibit a candidate mirror partner for $X^{\mu}_{i,j}$.

7.5.1 Monodromy operators and the Kähler cone

Mirror symmetry predicts that if Y and $Y^{\vee}$ are mirror dual Calabi-Yau threefolds, then there is a relation between monodromy operators acting on $H^3(Y, \mathbb{Q})$ and divisors in the closure of the Kähler cone of $Y^{\vee}$. We will briefly sketch some of the details of this relationship here, the interested reader may find more details in [35, Chapter 5].

Suppose that $\mathcal{Y} \to (\Delta^*)^n$ is a family of Calabi-Yau threefolds over the punctured polydisc, with fibre $\mathcal{Y}_t = Y$ above some $t \in (\Delta^*)^n$. For each $i \in \{1, \ldots, n\}$, let T_i be the unipotent monodromy operator acting on $H^d(Y, \mathbb{Q})$ coming from the loop $(t_1, \ldots, t_{i-1}, e^{2\pi it}, t_{i+1}, \ldots, t_n)$, where $(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ are fixed constants, and let $N_i = \log(T_i)$. The family $\mathcal{Y}$ is said to have *maximally unipotent monodromy* at $(0, \ldots, 0)$ if

1. for any n-tuple $(a_1, \ldots, a_n)$ of positive integers, the weight filtration on $H^d(Y, \mathbb{Q})$ induced by $\sum_{i=1}^{n} a_i N_i$ has $\dim W_0 = \dim W_1 = 1$ and $\dim W_2 = n + 1$, and

2. if $g_0, \ldots, g_n$ is a basis of W_2 chosen so that g_0 spans W_0, and m_{ij} are defined by $N_i g_j = m_{ij} g_0$, then the matrix (m_{ij}) is invertible.

If $\mathcal{Y}$ has maximally unipotent monodromy, then mirror symmetry should produce a map which assigns to each N_i a divisor D_i in the closure of the Kähler cone of $Y^\vee$. Moreover, there should be an identification under mirror symmetry which gives an isomorphism $H^{3-i,i}(Y) \cong H^{i,i}(Y^\vee)$, and hence an isomorphism $H^3(Y,\mathbb{C}) \cong \bigoplus_{i=0}^3 H^{i,i}(Y^\vee)$, so that the action of N_i on $H^3(Y,\mathbb{C})$ agrees with the action of the cup product operator $L_i(-) = (-) \cup [-D_i]$ under this correspondence. Thus, for any n-tuple $(a_1,\ldots,a_n)$ of non-negative integers, the weight filtration on $H^3(Y,\mathbb{C})$ induced by $N := \sum_{i=1}^n a_i N_i$ should be mirrored by the filtration on $\bigoplus_{i=0}^3 H^{i,i}(Y^\vee)$ induced by $L := \sum_{i=1}^n a_i L_i$, and the limit Hodge decomposition should correspond to the decomposition $\bigoplus_{i=0}^3 H^{i,i}(Y^\vee)$.

Now we specialize this discussion to the case of a degeneration $\mathcal{Y} \to \Delta$ of Calabi-Yau threefolds over the unit disc $\Delta \subset \mathbb{C}$. Assume that that total space $\mathcal{Y}$ is smooth and the central fibre of $\mathcal{Y}$ is a union of threefolds Y_1 and Y_2 that meet normally along a smooth K3 surface S. Assume further that $K_{Y_i} \sim -S$ and $h^{0,1}(Y_i) = 0$ for $i \in \{1,2\}$, such Y_i are called *quasi-Fano threefolds*. Degenerations of this form have been studied by Lee [94], who calls them *Tyurin degenerations*.

Let Y be a general fibre in $\mathcal{Y}$ and let T be the monodromy operator acting on $H^3(Y,\mathbb{Q})$ associated to a counterclockwise loop around 0. In order to apply the predictions of mirror symmetry, we assume that T may be identified with a loop $\prod_{i=1}^n T_i^{a_i}$ around a point of maximally unipotent monodromy in the complex moduli space of Y, where T_i are as above and a_i are non-negative integers. Define $N := \log(T) = \sum_{i=1}^n a_i N_i$. We will use the Clemens-Schmid exact sequence associated to N to compute the limit mixed Hodge structure on $H^3(Y)$, then see what this allows us to deduce about the mirror threefold $Y^\vee$.

Remark 7.5.1. We note that the Tyurin degeneration $\mathcal{Y}$ cannot have maximally unipotent monodromy, for purely topological reasons (see, for instance, [101, Corollary 2]), so T must correspond to a loop around some positive-dimensional boundary component of the compactified complex moduli space of Y. In particular, this implies that some of the a_i must be zero.

We begin by looking at the mixed Hodge structure on $H^3(\mathcal{Y})$ given by Griffiths and Schmid [66, Section 4]. The weight filtration on $H^3(\mathcal{Y},\mathbb{Q})$ has

$$\mathrm{Gr}_3^W = H^3(Y_1,\mathbb{Q}) \oplus H^3(Y_2,\mathbb{Q})$$

and, if r_1 and r_2 are the restriction maps $r_i\colon H^2(Y_i,\mathbb{Q}) \to H^2(S,\mathbb{Q})$, then

$$\mathrm{Gr}_2^W = H^2(S,\mathbb{Q})/(\mathrm{im}(r_1)+\mathrm{im}(r_2)).$$

These weight graded pieces are then equipped with the appropriate Hodge filtrations. Define integers $u := \mathrm{rank}\,(\mathrm{Gr}_2^W) - 2$ and $v := \frac{1}{2}\mathrm{rank}\,(\mathrm{Gr}_3^W)$. Noting that K_{Y_i} is anti-effective, so that $h^{3,0}(Y_i) = 0$, we see that $v = h^{2,1}(Y_1) + h^{2,1}(Y_2)$.

The Clemens-Schmid exact sequence gives us an exact sequence of mixed Hodge structures

$$\cdots \longrightarrow H_5(\mathcal{Y}) \longrightarrow H^3(\mathcal{Y}) \xrightarrow{i^*} H^3_{\mathrm{lim}}(Y) \xrightarrow{N} H^3_{\mathrm{lim}}(Y) \longrightarrow H_3(\mathcal{Y}) \longrightarrow \cdots$$

where i^* is the pull-back on cohomology induced by the inclusion $i\colon Y \hookrightarrow \mathcal{Y}$.

Lemma 7.5.2. $H_5(\mathcal{Y}) = 0$*, so the map* i^* *is an injection.*

Proof. The Mayer-Vietoris sequence for $Y_1 \cup_S Y_2$ gives

$$\cdots \longrightarrow H_5(Y_1)\oplus H_5(Y_2) \longrightarrow H_5(\mathcal{Y}) \longrightarrow H_4(S) \xrightarrow{\alpha} H_4(Y_1)\oplus H_4(Y_2) \longrightarrow \cdots,$$

where the map α is induced by the inclusions $S \hookrightarrow Y_i$.

Now, $H_5(Y_1) \oplus H_5(Y_2)$ vanishes by Poincaré duality and the assumption that $h^{0,1}(Y_i) = 0$. Moreover, as S is an effective anticanonical divisor in both Y_1 and Y_2, the image of the class $[S] \in H_4(S)$ of S under α is non-trivial. But $[S]$ generates $H_4(S)$, so α must be injective. Thus the sequence above gives $H_5(\mathcal{Y}) = 0$. □

Applying this lemma and some standard results on the Clemens-Schmid exact sequence (see, for instance, [101]) we obtain the following limit mixed Hodge structure on $H^3(Y)$

$$\begin{array}{ccccc} & \mathrm{Gr}_F^3 & \mathrm{Gr}_F^2 & \mathrm{Gr}_F^1 & \mathrm{Gr}_F^0 \\ \mathrm{Gr}_4^M & \mathbb{C} & \mathbb{C}^u & \mathbb{C} & 0 \\ \mathrm{Gr}_3^M & 0 & \mathbb{C}^v & \mathbb{C}^v & 0 \\ \mathrm{Gr}_2^M & 0 & \mathbb{C} & \mathbb{C}^u & \mathbb{C} \end{array}$$

where M is the monodromy weight filtration induced by N and F is the limit Hodge filtration.

Therefore, the divisor $D = \sum_{i=1}^n a_i D_i$ on $Y^\vee$ which corresponds to N under mirror symmetry should have

$$\begin{array}{rcccc} & H^{0,0}(Y^\vee) & H^{1,1}(Y^\vee) & H^{2,2}(Y^\vee) & H^{3,3}(Y^\vee) \\ \mathrm{coimage}(L) & \mathbb{C} & \mathbb{C}^u & \mathbb{C} & 0 \\ \ker(L)/\mathrm{im}(L) & 0 & \mathbb{C}^v & \mathbb{C}^v & 0 \\ \mathrm{im}(L) & 0 & \mathbb{C} & \mathbb{C}^u & \mathbb{C} \end{array}$$

where, as before, $L(-) = (-) \cup [-D]$ denotes the cup-product operator. In particular, we see that $L^2 = 0$. Since D is in the closure of the Kähler cone of $Y^\vee$, results of Oguiso [114, Example 2.3] show that mD is the class of a fibre in a fibration of $Y^\vee$ by K3 or abelian surfaces, for some positive integer m.

Remark 7.5.3. We conjecture that mD will always be the class of fibre in a K3 fibration on $Y^\vee$. Oguiso [114, Example 2.3] gives a simple criterion to test for this: mD defines a K3 fibration on $Y^\vee$ if and only if $c_2(Y^\vee) \cdot D > 0$. We can translate this to a statement on Y as follows. Let $c \in H^{1,2}(Y)$ be the class corresponding to $c_2(Y^\vee) \in H^{2,2}(Y^\vee)$ under the isomorphism between these two groups induced by mirror symmetry. Then $c_2(Y^\vee) \cdot D > 0$ if and only if $c \notin \ker(N) = \mathrm{im}(i^*)$ (or, equivalently, if $c \in \mathrm{Gr}_4^M H^{1,2}(Y)$).

In light of this remark, we will assume throughout the remainder of this section that mD defines a K3 fibration on $Y^\vee$. Then the calculation above also shows that the classes in $\mathrm{Pic}(Y^\vee)$ supported on fibres span a $v+1$ dimensional subspace, where one of these classes is mD itself. Moreover, there is a rank u subspace of $\mathrm{Pic}(Y^\vee)$

with $L(\eta) \neq 0$ for each class $\eta \neq 0$ in this subspace. By the global invariant cycles theorem, classes in this second subspace come from monodromy invariant cycles on fibres of the K3-fibration on $Y^\vee$. Thus the K3 surface fibration on $Y^\vee$ induced by mD is $\Lambda^\vee$-polarized (in the sense of [51, Definition 2.1]), for some lattice $\Lambda^\vee$ of rank u.

Therefore we see that, if Y admits a Tyurin degeneration to a union of threefolds $Y_1 \cup_S Y_2$, and if restriction of divisors from Y_1 and Y_2 induces a lattice polarization of S by a lattice Λ of rank $20-u$, then we expect the mirror $Y^\vee$ to admit an $\Lambda^\vee$-polarized K3 surface fibration, for some lattice $\Lambda^\vee$ of rank u. Moreover, the space of vertical divisors in $Y^\vee$ should have rank $v+1 = h^{2,1}(Y_1) + h^{2,1}(Y_2) + 1$.

This is precisely what we will see in the following section. We will build certain Calabi-Yau threefolds $Y^\mu_{i,j}$ which admit Tyurin degenerations, so that there is a mirror matching between the Hodge numbers of $Y^\mu_{i,j}$ and $X^\mu_{i,j}$, and such that the limit mixed Hodge structure on $H^3(Y^\mu_{i,j})$ matches the filtrations on $\bigoplus_{i=0}^3 H^{i,i}(X^\mu_{i,j})$ as described above. The $Y^\mu_{i,j}$ are therefore candidates for mirror pairs to the $X^\mu_{i,j}$.

7.5.2 Constructing mirror partners from smoothings

We will construct the $Y^\mu_{i,j}$ by smoothing a carefully chosen union of threefolds Y_1 and Y_2 as above. First, however, we recall how to construct a Calabi-Yau threefold from a more general union of threefolds with smooth anticanonical divisor.

Let S be a smooth K3 surface. Fix once and for all a pair of threefolds Y_1 and Y_2, such that S is a member of $|-K_{Y_i}|$ and $h^{1,0}(Y_i) = 0$ for $i \in \{1, 2\}$. Then we may take the union $Y_1 \cup_S Y_2$ of Y_1 and Y_2 glued along the image of S.

Specializing a result of Kawamata and Namikawa [83, Theorem 4.2] to our setting, we see that the union $Y_1 \cup_S Y_2$ is smoothable to a Calabi-Yau threefold Y if and only if the condition

$$-K_{Y_1}|_S - K_{Y_2}|_S \sim 0$$

is satisfied. Moreover, the resulting Calabi-Yau threefold Y is unique up to deformation.

Now we go about constructing the threefolds $Y^\mu_{i,j}$. Let V_1, V_2 and V_4 be, respectively, a quartic hypersurface in $\mathbb{P}^4$, a double cover of $\mathbb{P}^3$ branched along a quartic, and $\mathbb{P}^3$

itself. Let $S \subset V_i$ be a generic anticanonical hypersurface. Then S is a primitively $\langle 4 \rangle$-polarized K3 surface and $-K_{V_i}|_S \sim iH$, where H is the hyperplane section on S induced by the embedding into $\mathbb{P}^3$ defined by the polarization.

If we fix a primitively $\langle 4 \rangle$-polarized K3 surface S along with embeddings $S \hookrightarrow V_i$ and $S \hookrightarrow V_j$ that realize S as an anticanonical divisor, then we can form the gluing $V_i \cup_S V_j$. Unfortunately, this $V_i \cup_S V_j$ cannot be smoothed to a Calabi-Yau threefold, since $-K_{V_1}|_S - K_{V_2}|_S \sim (i+j)H$.

We may rectify this by blowing up V_i and V_j along smooth curves contained in S. In particular, if we have smooth curves $C_1, \dots, C_k \subset S$ so that $C_1 + \cdots + C_k \sim (i+j)H$, we may blow up, say, V_i sequentially in the curves $C_1, \dots, C_k$ to get a threefold $\widetilde{V_i}$ with exceptional divisors $E_1, \dots, E_k$. As a result of the canonical bundle formula for smooth blow-ups,

$$-K_{\widetilde{V_i}}|_S \sim iH - (E_1 + \cdots + E_k)|_S \sim iH - (C_1 + \cdots + C_k) \sim -jH$$

and thus $-K_{\widetilde{V_i}}|_S - K_{V_j}|_S \sim 0$.

Hence, we see that $\widetilde{V_i} \cup_S V_j$ can be smoothed to a Calabi-Yau threefold. In other words, there is an analytic family with fourfold total space $\mathcal{V} \to \Delta$ so that the central fibre $\mathcal{V}_0 = \widetilde{V_i} \cup_S V_j$ and a general fibre is a smooth Calabi-Yau threefold. Moreover, we may perform a series of flips to move the surfaces E_{C_l} from V_i to V_j in $\mathcal{V}$ without affecting the general fibre, so in particular we see that the smooth fibre does not depend upon which threefold we chose to blow up the curves C_l in.

Thus, starting from threefolds V_1, V_2 and V_4, we may construct families of Calabi-Yau threefolds based on the following data:

1. a choice of integers $i, j \in \{1, 2, 4\}$, corresponding to the choice of V_i and V_j; and

2. a partition $\mu = [x_1, \dots, x_k]$ of $i + j$, so that $C_s \sim x_s H$ for $s \in \{1, \dots, k\}$.

Let $Y_{i,j}^{\mu}$ denote a general member of the family of Calabi-Yau threefolds obtained from this construction.

7.5.3 Hodge numbers and filtrations

We expect that the threefolds $Y^{\mu}_{i,j}$ are mirror partners to the threefolds $X^{\mu}_{i,j}$. To justify this, we begin by computing the Hodge numbers of the $Y^{\mu}_{i,j}$. This is done using results of Lee [95], who shows that one can use the Clemens-Schmid exact sequence to compute the Hodge numbers of Calabi-Yau threefolds built from Fano threefolds in the way discussed above.

Proposition 7.5.4. *Let $i, j \in \{1, 2, 4\}$ and let $\mu = [x_1, \ldots, x_k]$ be a partition of $i+j$. Then the Hodge numbers of the threefold $Y^{\mu}_{i,j}$ are given by*

$$\begin{aligned} h^{1,1}(Y^{\lambda}_{i,j}) &= k, \\ h^{2,1}(Y^{\lambda}_{i,j}) &= 20 + \sum_{s=1}^{k}(2x_s^2 + 1) + h^{2,1}(V_i) + h^{2,1}(V_j), \end{aligned}$$

where $h^{2,1}(V_i) = 30$ (resp. 10, 0) for $i = 1$ (resp. 2, 4).

Proof. By definition, $Y^{\mu}_{i,j}$ is a smoothing of $\widetilde{V}_i \cup_S V_j$. Define

$$q := \operatorname{rank}(\operatorname{im}(H^2(\widetilde{V}_i, \mathbb{Z}) \oplus H^2(V_j, \mathbb{Z}) \to H^2(S, \mathbb{Z}))).$$

Then Lee [95, Corollary 8.2] shows that the Hodge numbers of $Y^{\mu}_{i,j}$ are given by

$$\begin{aligned} h^{1,1}(Y^{\mu}_{i,j}) &= h^2(\widetilde{V}_i) + h^2(V_j) - q - 1, \\ h^{2,1}(Y^{\mu}_{i,j}) &= 21 + h^{2,1}(\widetilde{V}_i) + h^{2,1}(V_j) - q. \end{aligned}$$

Now, since the Néron-Severi group of S is generated by the restriction of a hyperplane section from V_i, we must have $q = 1$. Moreover, since we blew up V_i a total of k times to obtain $\widetilde{V}_i$, we have $h^2(\widetilde{V}_i) = k + 1$ and $h^2(V_j) = 1$. Thus $h^{1,1}(Y^{\lambda}_{i,j}) = k$.

To compute $h^{2,1}(Y^{\mu}_{i,j})$, we begin by noting that a smooth curve which satisfies $C_s \sim x_s H$ has self-intersection $4x_s^2$ in S. So the genus formula for curves on a surface gives $g(C_s) = 2x_s^2 + 1$. Thus, by standard results on the cohomology of a blow-up (see, for instance, [143, Theorem 7.31]), we find

$$h^{2,1}(\widetilde{V}_i) = h^{2,1}(V_i) + \sum_{s=1}^{k}(2x_s^2 + 1),$$

giving the claimed result for $h^{2,1}(Y^{\mu}_{i,j})$. Finally, the values of $h^{2,1}(V_i)$ are easy to compute explicitly. □

Putting this proposition together with the results of Proposition 7.3.2 and Corollary 7.3.7, we obtain:

Corollary 7.5.5. *Let $i, j \in \{1, 2, 4\}$ be a pair of integers and let μ be a partition of $i + j$. Then there is a mirror duality between the Hodge numbers of the Calabi-Yau threefolds $X^{\mu}_{i,j}$ and $Y^{\mu}_{i,j}$.*

We expect that $X^{\mu}_{i,j}$ is actually mirror to $Y^{\mu}_{i,j}$, but of course this is not a proof. As further evidence, however, we can also compare filtrations as in Section 7.5.1. For the threefolds $Y^{\mu}_{i,j}$, we may compute the limit mixed Hodge structure associated to the degeneration to $\widetilde{V}_i \cup V_j$, to obtain

$$\begin{array}{ccccc} & \mathrm{Gr}^3_F & \mathrm{Gr}^2_F & \mathrm{Gr}^1_F & \mathrm{Gr}^0_F \\ \mathrm{Gr}^M_4 & \mathbb{C} & \mathbb{C}^{19} & \mathbb{C} & 0 \\ \mathrm{Gr}^M_3 & 0 & \mathbb{C}^v & \mathbb{C}^v & 0 \\ \mathrm{Gr}^M_2 & 0 & \mathbb{C} & \mathbb{C}^{19} & \mathbb{C} \end{array}$$

for $v = h^{2,1}(V_i) + h^{2,1}(V_j) + \sum_{s=1}^{k}(2x_s^2 + 1)$.

Now, for the threefolds $X^{\mu}_{i,j}$, let L be the cup product operator with the negative of the class of a fibre of the M_2-polarized K3 surface fibration on $X^{\mu}_{i,j}$. Then, by the proof of Proposition 7.3.2 and Proposition 7.5.4, we see that the rank of the space of vertical divisors in $H^{1,1}(X^{\mu}_{i,j})$ is

$$\mathrm{rank}\,(\ker(L)) = 1 + \sum_{s=1}^{k}(2x_s^2 + 1) + h^{2,1}(V_i) + h^{2,1}(V_j) = v + 1.$$

Moreover, the image of L in $H^{1,1}(X^{\mu}_{i,j})$ is the span of the class of a fibre, the image of L in $H^{2,2}(X^{\mu}_{i,j})$ is the space of classes dual to horizontal divisors in $H^{1,1}(X^{\mu}_{i,j})$ (which has rank 19), and the image of L spans $H^{3,3}(X^{\mu}_{i,j})$. Thus, we obtain

	$H^{0,0}(X^{\mu}_{i,j})$	$H^{1,1}(X^{\mu}_{i,j})$	$H^{2,2}(X^{\mu}_{i,j})$	$H^{3,3}(X^{\mu}_{i,j})$
$\mathrm{coimage}(L)$	$\mathbb{C}$	$\mathbb{C}^{19}$	$\mathbb{C}$	0
$\ker(L)/\mathrm{im}(L)$	0	$\mathbb{C}^{v}$	$\mathbb{C}^{v}$	0
$\mathrm{im}(L)$	0	$\mathbb{C}$	$\mathbb{C}^{19}$	$\mathbb{C}$

and the duality of bifiltered vector spaces discussed in Section 7.5.1 is satisfied in this case.

Finally, we note that restriction of divisors from V_i and V_j induces a lattice polarization of S by the lattice $\langle 4 \rangle$, whilst the K3 surface fibration on $X^{\mu}_{i,j}$ is M_2-polarized. As expected from the calculation in Section 7.5.1, the ranks of these two lattices sum to 20. However, in this case more is true: the lattices $\langle 4 \rangle$ and M_2 are in fact mirror to one another, in the sense of [44].

7.5.4 Relationship to Landau-Ginzburg models of Fano threefolds

We conclude this section with some relations between the fibrations on $X^{\mu}_{i,j}$ and the Landau-Ginzburg models of V_i and V_j, which provide further justification for our claim that $X^{\mu}_{i,j}$ and $Y^{\mu}_{i,j}$ are a mirror pair.

Mirror symmetry can be extended to Fano varieties (see [54][6]). In brief, the mirror to a Fano variety Y should be an open log Calabi-Yau variety U, equipped with a function $w\colon U \to \mathbb{A}^1$ whose fibres are Calabi-Yau varieties mirror dual to an anticanonical hypersurface in Y. These mirrors are called *Landau-Ginzburg models*. We will avoid giving a detailed discussion of Landau-Ginzburg models, except to say that much of the important data in a Landau-Ginzburg model is encoded in its singular fibres. We refer the interested reader to [134][117] [69] for more details.

For the Fano threefolds V_1, V_2 and V_4, candidate Landau-Ginzburg models are known. From [124, Table 1] we see that, on a torus chart $(\mathbb{C}^*)^3$ of U, the fibres of the functions w are given by

$$\begin{aligned} f_1 &= \frac{(x+y+z+1)^4}{xyz}, \\ f_2 &= \frac{(x+y+1)^4}{xyz} + z, \\ f_4 &= x+y+z+\frac{1}{xyz}, \end{aligned}$$

where f_i defines the fibre of the toric Landau-Ginzburg model associated to V_i.

The threefold $\mathcal{X}_2$ is easily seen to be a smooth fibrewise compactification of f_1, and the open threefold U_1 given by removing the fibre over $\lambda = \infty$ from $\mathcal{X}_2$ is the Landau-Ginzburg model of a generic quartic in $\mathbb{P}^4$. Similarly, it is also known [47] that the Landau-Ginzburg models of V_2 and V_4 may be obtained by pulling back $\mathcal{X}_2$ by the map $h_i(s,t) = (s^i : t^i)$ for $i = 2, 4$ respectively, resolving singularities as in the proof of Proposition 7.2.4, and removing the fibre over $\lambda = 0$. Call these threefolds U_2 and U_4.

Let w_i be the induced fibration map $w_i \colon U_i \to \mathbb{A}^1$. The singular fibres of this map may be read off from the proof of Proposition 7.2.4. In particular, we find:

Proposition 7.5.6. *Let $w_i \colon U_i \to \mathbb{A}^1$ be the Landau-Ginzburg model of the Fano threefold V_i constructed above. Then*

- *U_1 has a nodal fibre and a singular fibre with 31 components.*
- *U_2 has two nodal fibres and a singular fibre with 11 components.*
- *U_4 has four nodal fibres.*

It is conjectured [125, Conjecture 1.1] that if $w \colon U \to \mathbb{A}^1$ is the Landau-Ginzburg model of some smooth Fano variety V of dimension d and ρ_x is the number of irreducible components in $w^{-1}(x)$ for $x \in \mathbb{A}^1$, then $h^{d-1,1}(V) = \sum_{x \in \mathbb{A}^1}(\rho_x - 1)$. We note that this is true for the Landau-Ginzburg models of V_i presented above.

Looking at the proof of Propositions 7.2.4 and 7.3.2, we see:

Proposition 7.5.7. *The singular fibres of the Calabi-Yau threefold $X^{\mu}_{i,j}$ are precisely*

(*1*) *The singular fibres of* U_i *and* U_j*, and*

(*2*) *One semistable singular fibre of type III lying above* $\lambda = 0$ *for each element* x_s *of the partition* $\mu = [x_1, \ldots, x_k]$*, consisting of* $(2x_s^2 + 2)$ *smooth rational components.*

This data can also be obtained by looking closely at the decomposition of $H^{1,1}(X_{i,j}^{\mu})$ induced by L. As noted in the previous section, it has graded pieces $\mathrm{im}(L)$, $\ker(L)/\mathrm{im}(L)$, and $\mathrm{coimage}(L)$. The group $\ker(L)/\mathrm{im}(L)$, which has rank $h^{2,1}(V_i) + h^{2,1}(V_j) + \sum_{s=1}^{k}(2x_s^2 + 1)$, breaks up naturally into several disjoint components, corresponding to contributions from distinct singular fibres of $X_{i,j}^{\mu}$. These components have ranks $h^{2,1}(V_i)$, $h^{2,1}(V_j)$, and one of rank $(2x_s^2 + 1)$ for each each element x_s of the partition μ. The divisors spanning the components corresponding to $h^{2,1}(V_i)$ and $h^{2,1}(V_j)$ are the central fibres of the Landau-Ginzburg models of V_i and V_j respectively, and the remaining contributions come from fibres over $\lambda = 0$.

The upshot is, suppose that we degenerate $X_{i,j}^{\mu}$ by deforming the map g so that all ramification points away from $\lambda = \infty$ are moved to $\lambda = 0$, then remove the fibre over $\lambda = 0$. The resulting threefold splits into two components, which are isomorphic to the Landau-Ginzburg models U_i and U_j of the Fano threefolds V_i and V_j. Moreover, the data required to smooth the compactification of $U_i \cup U_j$ to $X_{i,j}^{\mu}$ is the same as the data determining the smoothing of $V_i \cup_S V_j$ to $Y_{i,j}^{\mu}$, namely the partition μ. This provides a class of examples illustrating a prediction of Tyurin [141]: that the mirror of any Calabi-Yau threefold admitting a degeneration to a pair of Fano threefolds should be expressible in terms of the corresponding Landau-Ginzburg models of those Fanos.

Example 7.5.8. *We conclude by returning to our running example of the quintic mirror threefold. In the notation of this section, this threefold is denoted by* $X_{1,4}^{[5]}$.

We begin by studying the candidate mirror $Y_{1,4}^{[5]}$*. In this case we consider a quartic hypersurface* V_1 *in* $\mathbb{P}^4$ *glued to* $V_4 \cong \mathbb{P}^3$ *along a generic anticanonical K3 surface* S*. Without loss of generality, we can realize this configuration by the union of a quartic* *hypersurface and a hyperplane in* $\mathbb{P}^4$. *We write*

$$V_1 \cup_S V_4 = \{x_1 f_4(x_1, x_2, x_3, x_4, x_5) = 0\} \subset \mathbb{P}^4,$$

where $(x_1, x_2, x_3, x_4, x_5)$ *are coordinates on* $\mathbb{P}^4$ *and* f_4 *is a generic homogeneous quartic polynomial.*

This degenerate threefold can be embedded into a family

$$\{x_1 f_4(x_1, x_2, x_3, x_4, x_5) + t g_5(x_1, x_2, x_3, x_4, x_5) = 0\} \subset \mathbb{P}^4 \times \Delta,$$

where $\Delta := \{t \in \mathbb{C} : |t| < 1\}$ *denotes the open unit disc in* $\mathbb{C}$ *and* g_5 *is a generic homogeneous quintic. As we might expect, the generic fibre of this family is a quintic hypersurface in* $\mathbb{P}^4$. *However, this is not a smoothing (indeed, by Kawamata's and Namikawa's criterion,* $V_1 \cup_S V_4$ *is not smoothable to a Calabi-Yau threefold), as the total space of this family is not a smooth fourfold: it has a curve of* cA_1 *singularities along the locus* $\{t = g_5 = f_4 = x_1 = 0\}$. *This locus is given by the intersection of the K3 surface* S *with the locus* $\{g_5 = 0\}$*; it is a smooth curve in the linear system* $|5H|$ *on* S. *Blowing up this locus once in* V_1*, we obtain a smoothing of* $\widetilde{V_1} \cup_S V_4$ *to a quintic hypersurface in* $\mathbb{P}^4$. *This shows, as expected, that* $Y_{1,4}^{[5]}$ *is a quintic Calabi-Yau threefold.*

Finally, degenerate $X_{1,4}^{[5]}$ *by moving the simple ramification point of* g *that lies away from* $\lambda \in \{0, \infty\}$ *to* $\lambda = 0$ *(as mentioned in Example 7.4.5, this corresponds to a degeneration to the maximal unipotent monodromy point in the moduli space of the quintic mirror threefold). Under this operation, the* $\mathbb{P}^1$ *base of the K3 fibration on* $X_{1,4}^{[5]}$ *degenerates to a non-normal curve. After normalizing,* g *splits into two covers: an isomorphism* $g_1 \colon \mathbb{P}^1 \xrightarrow{\sim} \mathcal{M}_{M_2}$*, and a 4-fold cover* $g_4 \colon \mathbb{P}^1 \to \mathcal{M}_{M_2}$ *which is totally ramified over* $\lambda \in \{0, \infty\}$. *The threefold* $X_{1,4}^{[5]}$ *therefore degenerates to a pair of threefolds, obtained by pulling back* $\mathcal{X}_2$ *by the maps* g_1 *and* g_4*, glued along their fibres over* $\lambda = 0$. *Removing the fibres over* $\lambda = 0$*, this degenerate threefold splits into the Landau-Ginzburg models* U_1 *and* U_4 *of* V_1 *and* V_4 *respectively.*

Part IV

The structure of K3 fibrations

Chapter 8

Families of lattice polarized K3 surfaces with monodromy

8.1 Introduction

The concept of lattice polarization for K3 surfaces was first introduced by Nikulin [109] and further developed by Dolgachev [44]. Our aim is to extend this theory to families of K3 surfaces over a (not necessarily simply connected) base, in a way that allows control over the action of monodromy on algebraic cycles.

Our interest in this problem arises from the study of Calabi-Yau threefolds with small Hodge numbers. In their paper [52], Doran and Morgan explicitly classify the possible integral variations of Hodge structure that can underlie a family of Calabi-Yau threefolds over the thrice-punctured sphere $\mathbb{P}^1 - \{0, 1, \infty\}$ with $h^{2,1} = 1$. Explicit examples, coming from toric geometry, of families realising all but one of these variations of Hodge structure were known at the time of publication of [52], and a family realising the fourteenth and final case was recently constructed in [30].

One of the main tools used to study the Calabi-Yau threefolds constructed in [30] was the existence of a *torically induced* fibration (i.e. a fibration of the threefold

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induced by a fibration of the toric ambient space by toric subvarieties) of these threefolds by K3 surfaces polarized by the rank 18 lattice

$$M := H \oplus E_8 \oplus E_8.$$

K3 surfaces polarized by this lattice have been studied by Clingher, Doran, Lewis and Whitcher [29][31] and have a rich geometric structure. In particular, the canonical embedding of the lattice $E_8 \oplus E_8$ into M defines a natural Shioda-Inose structure on them, which in turn defines a canonical Nikulin involution [102]. The resolved quotient by this involution is a new K3 surface, which may be seen to be a Kummer surface associated to a product of two elliptic curves; its geometry is closely related to that of the original K3 surface.

In [30], toric geometry was used to show that this Nikulin involution is induced on the M-polarized K3 fibres by a global involution of the Calabi-Yau threefold. The resolved quotient by this involution is another Calabi-Yau threefold, which is fibred by Kummer surfaces and has geometric properties closely related to the first. Examination of this second Calabi-Yau threefold was instrumental in proving that the construction in [30] realised the “missing” fourteenth variation of Hodge structure from the Doran-Morgan list.

Motivated by the discovery of this K3 fibration and the rich geometry that could be derived from it, we decided to search for similar K3 fibrations on the other threefolds from the Doran-Morgan classification. In a large number of cases (summarized by Theorem 8.5.10), we found fibrations by K3 surfaces polarized by the rank 19 lattices

$$M_n := H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle,$$

which contain the lattice M as a sublattice. Many, but not all, of these fibrations are torically induced.

This raises two natural questions: Do the canonical Nikulin involutions on the fibres of these K3 fibrations extend to global symplectic involutions on the Calabi-Yau threefolds? And if they do, what can be said about the geometry of the new Calabi-Yau threefolds obtained as resolved quotients by these involutions?

Both of these questions may be addressed by studying the behaviour of the Néron-Severi lattice of a K3 surface as it varies within a family. Furthermore, in order for this theory to be useful in the study of K3 fibred Calabi-Yau threefolds it should be able to cope with the possibility of monodromy around singular fibres, meaning that we must allow for the case where the base of the family is not simply connected.

To initiate this study, we introduce a new definition of lattice polarization for families of K3 surfaces and develop the basic theory surrounding it. We note that a related notion of lattice polarizability for families of K3 surfaces was introduced by Hosono, Lian, Oguiso and Yau [75], who also proved statements about period maps and moduli for such families. However, our definition is more subtle than theirs, given that our goal is to derive precise data about the monodromy of algebraic cycles. The relationship between the definitions is discussed in greater detail in Remark 8.2.6.

The structure of this paper is as follows. In Section 8.2 we begin with the central definitions of N-polarized (Definition 8.2.1) and (N, G)-polarized (Definition 8.2.4) families of K3 surfaces, where N is a lattice and G is a finite group. The first is a direct extension of the definition of N-polarization for K3 surfaces to families and does not allow for any action of monodromy on the lattice N. The second is more subtle: it allows for a nontrivial action of monodromy, but this monodromy is controlled by the group G.

The remainder of Section 8.2 proves some basic results about N- and (N, G)-polarized families of K3 surfaces and their moduli. Of particular importance are Proposition 8.2.11 and Corollary 8.2.12, which use this theory to give conditions under which symplectic automorphisms can be extended from individual K3 fibres to entire families of K3 surfaces.

Section 8.3 expands upon these results, focussing mainly on the case where the symplectic automorphism is a Nikulin involution. The main result of this section is Theorem 8.3.3, which shows that the resolved quotient of an N-polarized family of K3 surfaces, where N is the Néron-Severi lattice of a general fibre, by a Nikulin involution is an (N', G)-polarized family of K3 surfaces, where N' is the Néron-Severi lattice of a general fibre of the resolved quotient family and G is a finite group.

In Section 8.4 we specialize all of these results to families of M-polarized K3 surfaces with their canonical Nikulin involution, which extends globally over the family by Corollary 8.2.12. The resolved quotient family is an (N', G)-polarized family of K3 surfaces whose general fibre is a Kummer surface. The first major result of this section, Proposition 8.4.2, places bounds on the size of the group G.

To improve upon this result, in Section 8.4.3 we show that, after proceeding to a finite cover of the base, we may realise these families of Kummer surfaces by applying the Kummer construction fibrewise to a family of Abelian surfaces, a process which we call *undoing the Kummer construction*. As a result of this process we obtain Theorem 8.4.11 and Corollary 8.4.13, which enable explicit calculation of the group G.

In Section 8.5 we further specialize this analysis to families of M_n-polarized K3 surfaces, then apply the resulting theory to the study of the Calabi-Yau threefolds from the Doran-Morgan list. The main results here are Theorems 8.5.10 and 8.5.20, which show that twelve of the fourteen cases from that list admit fibrations by M_n-polarized K3 surfaces. In fact, we prove an even stronger result: for $n \geq 2$ these fibrations are in fact pull-backs of special M_n-polarized families on the moduli space of M_n-polarized K3 surfaces, under the *generalized functional invariant* map, and for $n = 1$ they are pull-backs of a special 2-parameter M_1-polarized family by a closely related map.

We compute the generalized functional invariant maps for all of these fibrations in Sections 8.5.4 and 8.5.5. We find that they all have a standard form, defining multiple covers of the moduli spaces of M_n-polarized K3 surfaces with ramification behaviour determined by a pair of integers (i, j).

Finally, in Section 8.6 we use these results to make an interesting observation concerning an open problem related to the Doran-Morgan classification. Recall that each of the threefolds from this classification moves in a one parameter family over the thrice-punctured sphere. Recently there has been a great deal of interest in studying the action of monodromy around the punctures on the third integral cohomology group of the threefolds. This monodromy action defines a Zariski dense subgroup of $\mathrm{Sp}(4, \mathbb{R})$, which may be either arithmetic or non-arithmetic (more commonly called *thin*). Singh and Venkataramana [138][137] have proved that the monodromy is arithmetic in seven of the fourteen cases from the Doran-Morgan list, and Brav and Thomas [25] have proved that it is thin in the remaining seven. It is an open problem to find geometric criteria that distinguish between these two cases.

In Theorem 8.6.1 we provide a potential solution to this problem: the cases may be distinguished by the values of the pair of integers (i, j) arising from the generalized functional invariants of *torically induced* K3 fibrations on them. Specifically, we find that a case has thin monodromy if and only if neither i nor j is equal to two. This suggests that it may be possible to express the integral monodromy matrices for the families of Calabi-Yau threefolds from the Doran-Morgan list in terms of the families of transcendental cycles for their internal K3-fibrations, and that doing so explicitly may be a good route towards an understanding of the geometric origin of the arithmetic/thin dichotomy.

A different criterion to distinguish the arithmetic and thin cases was recently given by Hofmann and van Straten [73, Section 6], using an observation about the integers m and a from [52, Table 1] (which are called d and k in [73]). Furthermore, the discovery of a yet another criterion has been announced in lectures by M. Kontsevich, using a technique involving Lyapunov exponents. Whilst our result does not appear to bear any immediate relation to either of these other results, it is our intention to investigate the links between them in future work.

8.1.1 Acknowledgements

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8.2 Families of K3 surfaces

Begin by assuming that X is a K3 surface. The Néron-Severi group of divisors modulo homological equivalence on X forms a non-degenerate lattice inside of $H^2(X,\mathbb{Z})$, denoted $\mathrm{NS}(X)$, which is even with signature $(1,\rho-1)$. The lattice of cycles orthogonal to $\mathrm{NS}(X)$ is called the lattice of transcendental cycles on X and is denoted $\mathrm{T}(X)$.

The aim of this section is to develop theoretical tools that will enable us to embark upon a study of the action of monodromy on the Néron-Severi group of a fibre in a family of K3 surfaces.

8.2.1 Families of lattice polarized K3 surfaces

We begin with some generalities on families of K3 surfaces. A family of K3 surfaces will be a variety $\mathcal{X}$ and a flat surjective morphism $\pi\colon \mathcal{X} \to U$ onto some smooth, irreducible, quasiprojective variety U such that for each $p \in U$ the fibre X_p above p is a smooth K3 surface. For simplicity the reader may assume that U has dimension 1 but our results are valid in arbitrary dimension. We further assume that there is a line bundle $\mathcal{L}$ whose restriction to each fibre of π is ample and primitive in $\mathrm{Pic}(X_p)$ for each $p \in U$.

In the analytic topology, there is an integral local system on U given by $R^2\pi_*\mathbb{Z}$ whose fibre above u is isomorphic to $H^2(X_p,\mathbb{Z})$. The Gauss-Manin connection ∇_{GM} is a flat connection on $R^2\pi_*\mathbb{Z}\otimes\mathcal{O}_U$.

The cup-product pairing on $H^2(X_p,\mathbb{Z})$ extends to a bilinear pairing of sheaves

$$\langle\cdot,\cdot\rangle_{\mathcal{X}} = R^2\pi_*\mathbb{Z}\times R^2\pi_*\mathbb{Z}\to R^4\pi_*\mathbb{Z}\cong\mathbb{Z}_U \tag{8.1}$$

where $\mathbb{Z}_U$ is the constant sheaf on U with $\mathbb{Z}$ coefficients. This form extends naturally to arbitrary sub-rings of $\mathbb{C}$.

There is a Hodge filtration on $R^2\pi_*\mathbb{Z}\otimes\mathcal{O}_U$. In particular $\mathcal{H}^{2,0}_{\mathcal{X}} = F^2(R^2\pi_*\mathbb{Z}\otimes\mathcal{O}_U)$, and there is a local subsystem of $R^2\pi_*\mathbb{C}$ which gives rise to $\mathcal{H}^{2,0}_{\mathcal{X}}$. Choosing a flat local section of $\mathcal{H}^{2,0}_{\mathcal{X}}$, which we will call $\omega_{\mathcal{X}}$, we take the local subsystem of $R^2\pi_*\mathbb{Z}$ which is orthogonal to $\omega_{\mathcal{X}}$. Since the pairing $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ is $\mathbb{Z}$ linear and $\omega_{\mathcal{X}}$ is flat, $\omega_{\mathcal{X}}^{\perp}$ is defined globally on U. We will call this local subsystem $\mathcal{NS}(\mathcal{X})$. Note that $\mathcal{NS}(\mathcal{X})$ is the Picard sheaf of the flat morphism π.

We let $\mathcal{T}(\mathcal{X})$ be the integral orthogonal complement of $\mathcal{NS}(\mathcal{X})$. We have an orthogonal direct sum decomposition over $\mathbb{Q}$

$$R^2\pi_*\mathbb{Q} = (\mathcal{T}(\mathcal{X}) \oplus \mathcal{NS}(\mathcal{X})) \otimes_{\mathbb{Z}_U} \mathbb{Q}_U$$

Our aim is to use this to study the action of monodromy on the Néron-Severi lattice of a general fibre of $\mathcal{X}$. In order to gain control of this monodromy, we begin by extending the definition of lattice polarization for K3 surfaces to families.

To do this, let $\mathcal{N}$ be a local subsystem of $\mathcal{NS}(\mathcal{X})$ such that for any $p \in U$, the restriction of $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ to the fibre $\mathcal{N}_p$ over p exhibits $\mathcal{N}_p$ as a non-degenerate integral lattice of signature $(1, n-1)$, which is (non-canonically) isomorphic to a lattice N and embedded into $H^2(X_p, \mathbb{Z})$ as a primitive sublattice containing the Chern class of the ample line bundle $\mathcal{L}_p$. This allows us to define a naïve extension of lattice polarization to families.

Definition 8.2.1. The family $\mathcal{X}$ is *N-polarized* if the local system $\mathcal{N}$ is a trivial local system.

Note that any family of K3 surfaces is polarized by the rank one lattice generated by the Chern class of the line bundle $\mathcal{L}$ restricted to each fibre.

Unfortunately, this definition is too rigid for our needs: it is easy to see that for an N-polarized family of K3 surfaces, a choice of isomorphism $N \cong \mathcal{N}_p$ for any point p determines uniquely an isomorphism $N \cong \mathcal{N}_q$ for any other point q by parallel transport, so this definition does not allow for any action of monodromy on $\mathcal{N}_q$. We will improve upon this definition in Section 8.2.3, but in order to do so we first need to develop some general theory.

8.2.2 Monodromy of algebraic cycles on K3 surfaces

In this section we will begin discusing the action of monodromy on the Néron-Severi group of a general fibre of $\mathcal{X}$. Let p be a point in U such that the fibre above p has

$\mathrm{NS}(X_p) \cong \mathcal{NS}(\mathcal{X})_p$. Parallel transport along paths in U starting at the base point p gives a monodromy representation of $\pi_1(U, p)$

$$\rho_{\mathcal{X}} \colon \pi_1(U, p) \to \mathrm{O}(H^2(X_p, \mathbb{Z}))$$

since we have the pairing in Equation (8.1). Furthermore, $\rho_{\mathcal{X}}$ restricts to monodromy representations of both $\mathcal{NS}(\mathcal{X})$ and $\mathcal{T}(\mathcal{X})$, written as

$$\rho_{\mathcal{NS}} \colon \pi_1(U, p) \to \mathrm{O}(\mathrm{NS}(X_p))$$

and

$$\rho_{\mathcal{T}} \colon \pi_1(U, p) \to \mathrm{O}(\mathrm{T}(X_p)).$$

Similarly for any local subsystem $\mathcal{N}$ of $R^2\pi_*\mathbb{Z}$, we will denote the associated monodromy representation $\rho_{\mathcal{N}}$. Note here that if $\mathcal{X}$ is N-polarized, then the image of $\rho_{\mathcal{N}}$ is the trivial subgroup Id.

Now we prove an elementary but useful result concerning the image of $\rho_{\mathcal{NS}}$. Here we let X be a K3 surface. Recall that the lattice $\mathrm{NS}(X)$ is an even lattice of signature $(1, \operatorname{rank} \mathrm{NS}(X) - 1)$. For such a lattice $\mathrm{NS}(X)$, there is a set of roots

$$\Delta_X = \{w \in \mathrm{NS}(X) : \langle w, w \rangle = -2\}.$$

The Weyl group W_X is the group generated by Picard-Lefschetz reflections across roots in Δ_X. It admits an embedding into the orthogonal group $\mathrm{O}(\mathrm{NS}(X))$. Denote the set of roots in Δ_X which are dual to the fundamental classes of rational curves by Δ_X^+. Then a fundamental domain for the action of W_X on $\mathrm{NS}(X)$ is given by the closure of the connected polyhedral cone

$$K(X) = \{w \in \mathrm{NS}(X) \otimes \mathbb{R} : \langle w, w \rangle > 0, \langle w, \delta \rangle > 0 \text{ for all } \delta \in \Delta_X^+\}.$$

$K(X)$ is the Kähler cone of X [11, Corollary VIII.3.9].

If we let $\mathrm{O}_+(\mathrm{NS}(X))$ be the subgroup of $\mathrm{O}(\mathrm{NS}(X))$ which fixes the positive cone in $\mathrm{NS}(X)$ and let D_X be the subgroup of $\mathrm{O}_+(\mathrm{NS}(X))$ which maps $K(X)$ to itself, then we obtain a semidirect product decomposition

$$\mathrm{O}_+(\mathrm{NS}(X)) = D_X \ltimes W_X.$$

Now let L be an ample line bundle on X. Then the Chern class of L is contained in $K(X)$. Define D_X^L to be the stabilizer of this Chern class in D_X.

Proposition 8.2.2. *Let $\mathcal{X}$ be a family of K3 surfaces and let X_p be a generic fibre of $\mathcal{X}$. Let $\mathcal{L}_p$ be the restriction of the bundle $\mathcal{L}$ on $\mathcal{X}$ to X_p. Then the group $D_{X_p}^{\mathcal{L}_p}$ is finite and contains the image of $\rho_{\mathcal{NS}}$.*

Proof. First we show that $D_{X_p}^{\mathcal{L}_p}$ is a finite group. Let γ be in $D_{X_p}^{\mathcal{L}_p}$. Then γ fixes $\mathcal{L}_p$ by definition. Therefore γ acts naturally on $[\mathcal{L}_p]^\perp$ and fixes $[\mathcal{L}_p]^\perp$ if and only if it fixes all of $\mathrm{NS}(X_p)$. Since $\mathcal{L}_p$ is ample, the orthogonal complement of $[\mathcal{L}_p]$ in $\mathrm{NS}(X_p)$ is negative definite by the Hodge index theorem.

We then recall the fact that $\mathrm{O}(N)$ is finite for any definite lattice N, so $D_{\mathcal{X}}^{\mathcal{L}}$ is contained in a finite group and thus is itself finite.

To see that $\rho_{\mathcal{NS}}$ has image contained in $D_{X_p}^{\mathcal{L}_p}$, we recall that $\rho_{\mathcal{NS}}$ fixes $\mathcal{L}_p \in K(X_p)$ and hence, since the closure of $K(X_p)$ is a fundamental domain for W_{X_p} and the action of W_{X_p} is continuous, $\rho_{\mathcal{NS}}$ must have image in D_{X_p}. □

8.2.3 Monodromy and symplectic automorphisms

We are now almost ready to make a central definition which extends Definition 8.2.1 to cope with the possible action of monodromy on N.

Denote by N^* the dual lattice of N. We may embed $N^* \subseteq N \otimes_{\mathbb{Z}} \mathbb{Q}$ as the sublattice of elements u of $N \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\langle u, v\rangle \in \mathbb{Z}$ for all $v \in N$.

Definition 8.2.3. The *discriminant lattice* of N, which we call A_N, is the finite group N^*/N equipped with the bilinear form

$$b_N\colon A_N \times A_N \to \mathbb{Q} \bmod \mathbb{Z}.$$

induced by the bilinear form on $N \otimes_{\mathbb{Z}} \mathbb{Q}$.

For each lattice N we may define a map $\alpha_N \colon \mathrm{O}(N) \to \mathrm{Aut}(A_N)$ where $\mathrm{Aut}(A_N)$ is the group of automorphisms of the finite abelian group A_N which preserve the bilinear form b_N. Denote the kernel of α_N by $\mathrm{O}(N)^*$. Then we make the central definition:

Definition 8.2.4. Fix an even lattice N with signature $(1, n-1)$ and a subgroup G of $\mathrm{Aut}(A_N)$. Let $\mathcal{X}$ be a family of K3 surfaces and let X_p be a generic fibre of $\mathcal{X}$. Assume that there is local sub-system $\mathcal{N} \subseteq \mathcal{NS}(\mathcal{X})$ which has fibres $\mathcal{N}_p$ that are isometric to N and are embedded into $H^2(X_p, \mathbb{Z})$ as primitive sublattices containing the Chern class of the ample line bundle $\mathcal{L}_p$. Then $\mathcal{X}$ is called an *(N,G)-polarized family of K3 surfaces* if the restriction of the map α_N to the image of $\rho_{\mathcal{N}}$ is injective and has image inside of G.

One sees that if Id is the trivial subgroup of $\mathrm{Aut}(A_N)$, then the definition of an N-polarized family of K3 surfaces is identical to the definition of a family of (N, Id)-polarized K3 surfaces. We also note that, if $G \subset G'$, then any (N,G)-polarized family of K3 surfaces will also be (N,G')-polarized. With this in mind, we identify a special class of (N,G)-polarized families where the group G is as small as possible.

Definition 8.2.5. An (N,G)-polarized family of K3 surfaces $\mathcal{X}$ is called *minimally (N,G)-polarized* if the composition $\alpha_N \cdot \rho_{\mathcal{N}}$ is surjective onto G.

Remark 8.2.6. We note that in [75], the authors introduce a similar notion of N-polarizability for a family of K3 surfaces. A K3 surface X is *N-polarizable* in the sense of [75] if there is a sublattice inside of $\mathrm{NS}(X)$ isomorphic to N, but the primitive embedding of N into $\mathrm{NS}(X)$ is only fixed up to automorphism of the K3 lattice Λ_{K3}. A family of K3 surfaces is then called N-polarizable if each fibre is N-polarizable. There is a well-defined period space of N polarizable K3 surfaces $\mathcal{M}_N^{\circ}$, so that to any family of N-polarizable K3 surfaces there is a well-defined period map.

Our definition is more subtle than this, since our goal is to derive precise data about the monodromy of algebraic cycles. Any (N,G)-polarized family of K3 surfaces is N-polarizable, but the converse does not hold. In fact, both of the families constructed in Section 8.2.4 are families of N-polarizable K3 surfaces, but only one of them is (N, G)-polarized.

There is a close relationship between (N, G)-polarizations and symplectic automorphisms. Recall the following definition:

Definition 8.2.7. Let X be a smooth K3 surface and let $\tau: X \to X$ be an automorphism of X. The automorphism τ is called a *symplectic automorphism* if for some (hence any) non-vanishing holomorphic 2-form ω on X, $\tau^*\omega = \omega$. If τ has order 2, it is called a symplectic involution of X or a *Nikulin involution*.

Symplectic automorphisms of finite order on K3 surfaces exhibit behaviour similar to translation by a torsion section on an elliptic curve. The quotient of an elliptic curve by some subgroup of $\mathrm{Pic}(E)_{\mathrm{tors}}$ is an isogenous elliptic curve, i.e. an elliptic curve E' such that there is a Hodge isometry $H^1(E, \mathbb{Q}) \cong H^1(E', \mathbb{Q})$. Analogously there is a sense in which the resolved quotient of a K3 surface X by a finite group of symplectic automorphisms is isogenous to X: there is a real quadratic extension of $\mathbb{Q}$ under which the Hodge structures on their transcendental lattices are isometric. This will be explained in detail by Proposition 8.3.1.

The following is a consequence of the famous Global Torelli Theorem for K3 surfaces [121][109]. More precisely, it may be seen as a corollary of [43, Theorem 4.2.3].

Theorem 8.2.8. *The kernel of the map $\alpha_{\mathrm{NS}(X_p)}: D^{\mathcal{L}}_{\mathcal{X}} \to \mathrm{Aut}(A_{\mathrm{NS}(X_p)})$ is isomorphic to the finite group of symplectic automorphisms of X_p which fix $[\mathcal{L}_p]$.*

From this, using Proposition 8.2.2, we obtain:

Corollary 8.2.9. *Let $\mathcal{X}$ be a family of K3 surfaces with generic Néron-Severi lattice N. The family $\mathcal{X}$ is (N, G)-polarized for some G in $\mathrm{Aut}(A_N)$ if and only if there is no $\gamma \in \pi_1(U, p)$ such that $\rho_{\mathcal{NS}}(\gamma) = \sigma|_{\mathrm{NS}(X_p)}$ for some symplectic automorphism σ of X_p.*

Therefore, a measure of how far a family of K3 surfaces with generic Néron-Severi lattice N can be from being (N, G)-polarized is given by the size of the group of symplectic automorphisms of a generic N-polarized K3 surface. The number of possible finite groups of symplectic automorphisms of a K3 surface is relatively small. Mukai [104, Theorem 0.3] has shown that such groups are all contained as special subgroups of the Mathieu group M_{23}, and in particular Nikulin [109, Proposition 7.1] has shown that an algebraic K3 surface with symplectic automorphism must have Néron-Severi rank at least 9. This gives:

Corollary 8.2.10. *Any family of K3 surfaces with generic Néron-Severi group N having* $\mathrm{rank}\,(N) < 9$ *is (N, G)-polarized for some $G \subset \mathrm{Aut}(A_N)$.*

We end this subsection with a proposition which determines when a symplectic automorphism on a single K3 surface extends to an automorphism on an entire family of K3 surfaces. This will be useful in Section 8.3, when we will further discuss symplectic automorphisms in families.

Proposition 8.2.11. *Let X_p be a fibre in $\mathcal{X}$ which satisfies $\mathcal{NS}(\mathcal{X})_p \cong \mathrm{NS}(X_p)$, and let τ be a symplectic automorphism of X_p. Then τ extends to an automorphism of $\mathcal{X}$ if and only if its action on $\mathrm{NS}(X_p)$ commutes with the image of $\rho_{\mathcal{X}}$.*

Proof. Since $\mathcal{X}$ is a proper family of smooth manifolds, Ehresmann's theorem (see, for example, [143, Section 9.1.1]) implies that there is a local analytic open subset, called U_0, about $p \in U$, so that there is a marking on the family of K3 surfaces $\mathcal{X}_{U_0}$ on U_0. Therefore [109, Lemma 4.2] and the Global Torelli Theorem [109, Theorem 2.7'] shows that τ extends uniquely to an automorphism on $\mathcal{X}_{U_0}$.

Let $\gamma \in \pi_1(U, p)$, let $\gamma^*\tau$ be the analytic continuation of τ along γ, and let $w \in H^2(X_0, \mathbb{Z})$. Then it is easy to see that

$$\gamma^*\tau(w) = \rho_{\mathcal{X}}(\gamma) \cdot \tau \cdot (\rho_{\mathcal{X}}(\gamma))^{-1}(w).$$

Therefore, the action of τ on $\mathrm{NS}(X_p)$ commutes with the image of $\rho_{\mathcal{X}}$ if and only if the action of $\gamma^*\tau$ on $\mathrm{NS}(X_p)$ agrees with the action of τ. By the Global Torelli Theorem, this happens if and only if the automorphisms τ and $\gamma^*\tau$ are the same. $\square$

Corollary 8.2.12. *Let $\mathcal{X} \to U$ be an N-polarized family of K3 surfaces and suppose $N \cong \mathrm{NS}(X_p)$ for some fibre X_p. If X_p admits a symplectic automorphism τ, then τ extends to an automorphism of $\mathcal{X}$.*

8.2.4 A non-polarizable example

As we have seen, algebraic monodromy of families of K3 surfaces is intimately related to the existence of symplectic automorphisms. In this section, we will give a simple example which will show how the existence of symplectic automorphisms produces non-polarized families of K3 surfaces.

Let us take the pencil of K3 surfaces mirror (in the sense of [44]) to the Fermat pencil of quartics in $\mathbb{P}^3$. We may write these surfaces as a family $\mathcal{X}$ of ADE singular hypersurfaces in $\mathbb{P}^3$:

$$(x + y + z + w)^4 + t^2 xyzw = 0.$$

As a non-compact threefold, we may express these as a singular subvariety of

$$[x : y : z : w] \times t \in \mathbb{P}^3 \times \mathbb{C}^\times.$$

This is an $(E_8^2 \oplus H \oplus \langle -4 \rangle, \mathrm{Id})$-polarized family of K3 surfaces. Each fibre admits A_4 as a group of symplectic automorphisms acting via even permutations on the coordinates x, y, z, w. In particular we have a symplectic involution on each fibre induced by

$$\sigma \colon [x : y : z : w] \mapsto [y : x : w : z],$$

which extends to $\mathcal{X}$ by Corollary 8.2.12. We also have an involution on the base, acting via

$$\eta \colon t \mapsto -t.$$

Therefore, the fibrewise resolutions of the quotient families $\mathcal{Y}_1 = \widetilde{\mathcal{X}/(\mathrm{Id} \times \eta)}$ and $\mathcal{Y}_2 = \widetilde{\mathcal{X}/(\sigma \times \eta)}$ are fibrewise biregular, but are not biregular as total spaces. More importantly both families have the same holomorphic periods, but the monodromy of $\mathcal{NS}(\mathcal{Y}_1)$ is trivial and the monodromy of $\mathcal{NS}(\mathcal{Y}_2)$ is non-trivial around 0.

Thus we see that the family $\mathcal{Y}_1$ is N-polarized. However, by Corollary 8.2.9, the family $\mathcal{Y}_2$ is not (N, G)-polarized for any G since, by construction, monodromy around 0 acts as a Nikulin involution on $\mathcal{NS}(\mathcal{Y}_2)$.

Remark 8.2.13. Of course this examples and examples like it reflect directly the general principle that there does not exist a fine moduli scheme of objects which admit automorphisms, and in particular this example itself proves that the period space of K3 surfaces is not a fine moduli space. If one considers instead the moduli stack of polarized K3 surfaces (see [129]), then such families are distinguished.

8.2.5 Moduli spaces and period maps

In the last subsection of this section, we will study the moduli of (N, G)-polarized families. We begin by establishing some definitions regarding the period spaces of K3 surfaces; much of this material may be found in greater detail in [44].

Define the K3 lattice to be the lattice $\Lambda_{\mathrm{K3}} = E_8^2 \oplus H^3$. The space of marked pseudo-ample K3 surfaces is the type IV symmetric domain

$$\mathcal{P}_{\mathrm{K3}} = \{z \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C}) : \langle z, z\rangle = 0, \langle z, \overline{z}\rangle > 0\}.$$

There is a natural action on $\mathcal{P}_{\mathrm{K3}}$ by the group $\mathrm{O}(\Lambda_{\mathrm{K3}})$. Using terminology of [44], the orbifold quotient

$$\mathcal{M}_{\mathrm{K3}} := \mathrm{O}(\Lambda_{\mathrm{K3}}) \setminus \mathcal{P}_{\mathrm{K3}}$$

is called the *period space of Kähler K3 surfaces.*

For any even lattice N of rank n and signature $(1, n-1)$ equipped with a primitive embedding $N \hookrightarrow \Lambda_{K3}$, one may construct a period space of pseudo-ample marked K3 surfaces with N-polarization. Let

$$\mathcal{P}_N = \{z \in \mathbb{P}(N^{\perp} \otimes \mathbb{C}) : \langle z, z\rangle = 0, \langle z, \overline{z}\rangle > 0\}.$$

There is a natural embedding

$$\varphi_N \colon \mathcal{P}_N \hookrightarrow \mathcal{P}_{\mathrm{K3}}$$

where we suppress the dependence upon choice of embedding of N into Λ_{K3}. Let

$$\mathrm{O}(N^{\perp}) = \{\gamma|_{N^{\perp}} : \gamma \in \mathrm{O}(\Lambda_{\mathrm{K3}}), \gamma(N) \subseteq N\}.$$

The map φ_N descends to an embedding

$$\overline{\varphi_N} \colon \mathrm{O}(N^{\perp}) \setminus \mathcal{P}_N \hookrightarrow \mathrm{O}(\Lambda_{\mathrm{K3}}) \setminus \mathcal{P}_{\mathrm{K3}}.$$

For each group $G_{N^{\perp}}$ in $\mathrm{Aut}(A_{N^{\perp}})$, we may construct a finite index subgroup of $\mathrm{O}(N^{\perp})$,

$$\mathrm{O}(N^{\perp}, G_{N^{\perp}}) = \{\gamma|_{N^{\perp}} \in \mathrm{O}(N^{\perp}) : \alpha_{N^{\perp}}(\gamma|_{N^{\perp}}) \in G_{N^{\perp}}\}.$$

This subgroup is related to (N, G_N)-polarized K3 surfaces in the following way. Recall the following standard lattice theoretic fact from [110].

Proposition 8.2.14. [110, Proposition 1.6.1] *Let N be a primitive sublattice of an even unimodular lattice K, and let $N^{\perp}$ be the orthogonal complement of N in K. Then*

(1) There is a canonical isomorphism ϕ^N between the underlying groups A_N and $A_{N^{\perp}}$ which satisfies

$$b_N(a,b) = -b_{N^{\perp}}(\phi^N(a), \phi^N(b)).$$

(2) If g is an automorphism of N and g' is an automorphism of $N^{\perp}$, then $g \oplus g'$ is an automorphism of $N \oplus N^{\perp}$ which extends to an automorphism of K if and only if the induced actions of g on A_N and of g' on $A_{N^{\perp}}$ are the same under the identification ϕ^N.

Therefore, if a family of K3 surfaces $\mathcal{X}$ is (N, G_N)-polarized, then Proposition 8.2.14 shows that the transcendental monodromy of $\mathcal{X}$ is in $\mathrm{O}(N^{\perp}, G_{N^{\perp}})$ where $G_{N^{\perp}}$ is the subgroup of $A_{N^{\perp}}$ identified with G_N by ϕ^N.

As a particular example, if Id is the trivial subgroup of G_N then the family $\mathcal{X}$ is N-polarized and the group $\mathrm{O}(N^{\perp}, \mathrm{Id})$ corresponds to the group $\mathrm{O}(N^{\perp})^*$. By [44, Proposition 3.3], we have

$$\mathrm{O}(N^{\perp}, \mathrm{Id}) = \mathrm{O}(N^{\perp})^{*} \cong \{\gamma|_{N^{\perp}} : \gamma \in \mathrm{O}(\Lambda_{\mathrm{K3}}), \gamma(w) = w \text{ for all } w \in N\}.$$

In the case where our family is N-polarized we will use the notation and language of [44], but adopt the notation introduced above when the group G_N becomes relevant.

In [44], the space

$$\mathcal{M}_N = \mathrm{O}(N^{\perp})^{*} \setminus \mathcal{P}_N$$

is called the *period space of pseudo-ample N-polarized K3 surfaces.* Dolgachev [44, Remark 3.4] shows that for any N-polarized family of K3 surfaces $\pi\colon \mathcal{X} \to U$, there is a period morphism

$$\Phi_{\mathcal{X}}\colon U \to \mathcal{M}_N.$$

In light of this, define

$$\mathcal{M}_{(N,G_N)} := \mathrm{O}(N^{\perp}, G_{N^{\perp}}) \setminus \mathcal{P}_N.$$

Note that for $G_N \subseteq G'_N$, there is a natural inclusion $\mathrm{O}(N^{\perp}, G_{N^{\perp}}) \subseteq \mathrm{O}(N^{\perp}, G'_{N^{\perp}})$ and therefore there are natural surjective morphisms

$$\mathcal{M}_{(N,G_N)} \to \mathcal{M}_{(N,G'_N)}$$

of degree $[G_N : G'_N]$.

We now take some time to prove the existence of period morphisms associated to the spaces $\mathcal{M}_{(N,G_N)}$.

Theorem 8.2.15. *Let $\mathcal{X} \to U$ be a family of K3 surfaces. If there is some local subsystem $\mathcal{N} \subseteq \mathcal{NS}(\mathcal{X})$, where $\mathcal{N}$ is fibrewise isomorphic to a lattice N of signature $(1, n-1)$ and $\alpha_N \cdot \rho_{\mathcal{NS}}$ is contained inside of a subgroup G_N of $\mathrm{Aut}(A_N)$, then there a period morphism*

$$\Phi_{(N,G_N)}\colon U \to \mathcal{M}_{(N,G_N)}.$$

Proof. Let $\widetilde{U}$ be the simply connected universal covering space of U and $g\colon \widetilde{U} \to U$ be the canonically associated covering map. Then, since $g^*\mathcal{X}$ is marked, pseudo-ample and N-polarized, we have the following diagram

$$\begin{tikzcd} \widetilde{U} \arrow[r] \arrow[d, "g"'] & \mathcal{P}_N \\ U & \end{tikzcd}$$

Now we apply Proposition 8.2.14. Since the image of $\alpha_N \cdot \rho_{\mathcal{N}}$ is in G_N, the image of $\alpha_{N^\perp} \cdot \rho_{\mathcal{N}^\perp}$ is contained in $G_{N^\perp}$ under the identification induced by ϕ^N. Thus $\rho_{N^\perp}$ is contained in $\mathrm{O}(N^\perp, G_{N^\perp})$.

This allows us to canonically complete the diagram above to a commutative square

$$\begin{tikzcd} \widetilde{U} \arrow[r] \arrow[d, "g"'] & \mathcal{P}_N \arrow[d] \\ U \arrow[r, "\Phi_{(N,G_N)}"] & \mathcal{M}_{(N,G_N)} \end{tikzcd}$$

as required. ☐

We note that the assumptions in this proposition are weaker than the assumption that $\mathcal{X} \to U$ is (N, G_N)-polarized, as we do not assume here that the map α_N is injective on the image of $\rho_{\mathcal{NS}}$. What distinguishes (N, G_N)-polarized families of K3 surfaces from the rest is the following observation.

Remark 8.2.16. Let $\mathcal{X} \to D^*$ be an (N, G_N)-polarized family of K3 surfaces over the punctured disc D^*, and let γ be a generator of $\pi_1(D^*, p)$ and $u \in N \subseteq \mathrm{NS}(X_p)$ with $\overline{u}$ its image in A_N. Then under the identification ϕ^N defined in the proof of Theorem 8.2.15,

$$\alpha_{N^\perp}(\rho_{\mathcal{N}^\perp}(\gamma))(\phi^N(\overline{u})) = \phi^N(\alpha_N(\rho_{\mathcal{N}}(\gamma))(\overline{u})).$$

Since α_N is an injection and ϕ^N is an isomorphism, we see that, for an (N, G_N)-polarized family, all data about algebraic monodromy of $\mathcal{N}$ is captured by the monodromy of $\mathcal{N}^\perp$.

This remark will be essential for the calculations that we will do in Section 8.4.

8.3 Symplectic automorphisms in families

In this section, we expand upon Proposition 8.2.11 in the case where τ is a Nikulin involution. The main result is Theorem 8.3.3, which will be used in Section 8.4 to study lattice polarized families of K3 surfaces with Shioda-Inose structure, in an attempt to understand the relationship between such families and their associated families of abelian surfaces.

8.3.1 Symplectic automorphisms and Nikulin involutions

We begin with some background on symplectic automorphisms of K3 surfaces. Let X be a K3 surface and let ω be a non-vanishing holomorphic 2-form on X. For any group Σ of symplectic automorphisms of X, there are two lattices in $H^2(X,\mathbb{Z})$ which may be canonically associated to Σ. The first is the fixed lattice $H^2(X,\mathbb{Z})^\Sigma$. To derive the second, note that, by assumption, Σ fixes ω and hence, since Σ acts as Hodge isometries on $H^2(X,\mathbb{Z})$, we see that Σ must preserve the transcendental Hodge structure on X. This implies that $\mathrm{T}(X) \subseteq H^2(X,\mathbb{Z})^\Sigma$. So we may define a second lattice

$$S_{\Sigma,X} := (H^2(X,\mathbb{Z})^\Sigma)^\perp.$$

When the K3 surface X is understood, we will abbreviate this notation to simply S_Σ. This is appropriate because Nikulin [109, Theorem 4.7] proves that, as an abstract lattice, S_Σ depends only upon Σ. It follows from the fact that $\mathrm{T}(X)$ is fixed by Σ that S_Σ is contained in $\mathrm{NS}(X)$. In [109, Lemma 4.2] it is also shown that S_Σ is a negative definite lattice and contains no elements of square (-2).

In [109, Proposition 7.1], Nikulin determines the lattice S_Σ for any abelian group of symplectic automorphisms Σ. Therefore, since any group contains at least one abelian subgroup, if X admits any nontrivial group Σ of symplectic automorphisms, then S_Σ contains one of the lattices in [109, Proposition 7.1]. The smallest lattice listed therein is $S_{\mathbb{Z}/2\mathbb{Z}}$, which has rank 8.

In general, symplectic automorphisms have fixed point sets of dimension 0. The local behaviour of Σ about the fixed points determines a quotient singularity in X/Σ. It is easy to see from the classification of minimal surfaces that the minimal resolution $Y := \widetilde{X/\Sigma}$ of X/Σ is again a K3 surface: $\sigma^*\omega = \omega$ implies that ω descends to a non-vanishing holomorphic 2-form on the quotient surface and the resulting quotient singularities are crepant.

There is a diagram of surfaces

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "c"'] \arrow[dr, "q"] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & X/\Sigma & \end{tikzcd}$$

where $\tilde{X}$ is the minimal blow up of X on which Σ acts equivariantly with the map c and whose quotient $\widetilde{X}/\Sigma$ is Y.

In $\mathrm{NS}(Y)$ there is a lattice K spanned by exceptional classes. The minimal primitive sublattice of $\mathrm{NS}(Y)$ containing K will be called K_0. Nikulin [109, Propositions 7.1 and 10.1] shows that K_0 and S_Σ have the same rank but are, of course, not isomorphic. The map

$$\theta := q^*c_*\colon K_0^\perp \to H^2(X,\mathbb{Z})^\Sigma$$

is an isomorphism over $\mathbb{Q}$ and satisfies

$$\langle\theta(u),\theta(v)\rangle = |\Sigma|\langle u,v\rangle$$

for any $u, v \in K_0^{\perp}$. Therefore there is a linear transformation g over $\mathbb{Q}(\sqrt{|\Sigma|})$ which relates the lattices $H^2(X, \mathbb{Z})^{\Sigma}$ and $K_0^{\perp}$; a more precise description of this relationship is given in [147, Theorem 2.1].

Since the group Σ acts symplectically, for a class ω spanning $H^{2,0}(Y)$ we have that $\theta(\omega)$ is in $H^{2,0}(X)$, so we see that $\langle \theta(u), \theta(\omega) \rangle = 0$ if and only if $\langle u, \omega \rangle = 0$. Thus $\theta(\mathrm{NS}(Y) \cap K_0^{\perp}) = \mathrm{NS}(X) \cap H^2(X, \mathbb{Z})^{\Sigma}$. In other words, $\theta(\mathrm{T}(Y)) = \mathrm{T}(X)$.

8.3.2 Symplectic quotients and Hodge bundles

If $\mathcal{X}$ is a family of K3 surfaces for which a group of symplectic automorphisms on the fibres extends to a group of automorphisms on the total space, then base-change allows us to relativize the constructions in Section 8.3.1.

We obtain sheaves of local systems $(R^2\pi_*\mathbb{Z})^{\Sigma}$ and $\mathcal{S}_{\Sigma}$ which agree fibrewise with $H^2(X_p, \mathbb{Z})^{\Sigma}$, and S_{Σ, X_p}. The Hodge filtration on $R^2\pi_*\mathbb{Z} \otimes \mathcal{O}_U$ restricted to these sub-sheaves produces integral weight 2 variations of Hodge structure on U.

We wish to compare the variation of Hodge structure on $(R^2\pi_*^{\mathcal{X}}\mathbb{Z})^{\Sigma}$ and the variation of Hodge structure on the subsystem of $R^2\pi_*^{\mathcal{Y}}\mathbb{Z}$ orthogonal to the lattice spanned by exceptional curves in each fibre. Since we deal only with smooth fibrations, the following statements are equivalent to their counterparts for individual K3 surfaces.

Proposition 8.3.1. *Let $\mathcal{X} \to U$ be a family of K3 surfaces on which a group Σ of symplectic automorphisms acts fibrewise and extends to automorphisms of $\pi^{\mathcal{X}} \colon \mathcal{X} \to U$. Let $\pi^{\mathcal{Y}} \colon \mathcal{Y} \to U$ be the resolved quotient threefold. Then*

(1) The Hodge bundles $F^2(R^2\pi_*^{\mathcal{X}}\mathbb{Z} \otimes \mathcal{O}_U)$ and $F^2(R^2\pi_*^{\mathcal{Y}}\mathbb{Z} \otimes \mathcal{O}_U)$ are isomorphic as complex line bundles on U.

(2) If we extend scalars to $\mathbb{Q}(\sqrt{|\Sigma|})$, the induced VHS on $(R^2\pi_*^{\mathcal{X}}\mathbb{Z})^{\Sigma}$ is isomorphic to a sub-VHS of $R^2\pi_*^{\mathcal{Y}}\mathbb{Z}$.

(3) The transcendental integral variations of Hodge structure $\mathcal{T}(\mathcal{X})$ and $\mathcal{T}(\mathcal{Y})$ are isomorphic over $\mathbb{Q}(\sqrt{|\Sigma|})$.

Proof. These are relative versions of the discussion in Section 8.3.1. We use the fact that statements about the local systems $R^2\pi_*^{\mathcal{X}}\mathbb{Z}$ and $R^2\pi_*^{\mathcal{Y}}\mathbb{Z}$ reduce to statements on each fibre. The same is true for statements about the Hodge filtrations on $R^2\pi_*^{\mathcal{X}}\mathbb{Z}\otimes\mathcal{O}_U$ and $R^2\pi_*^{\mathcal{Y}}\mathbb{Z}\otimes\mathcal{O}_U$. Therefore Proposition 8.3.1 reduces to the statements in Section 8.3.1. □

In particular, we can recover from Proposition 8.3.1(3) a result of Smith [139, Theorem 2.12], that the holomorphic Picard-Fuchs equation of $\mathcal{X}$ agrees with the Picard-Fuchs equation of $\mathcal{Y}$, since Picard-Fuchs equations depend only upon the underlying complex VHS.

A corollary to this is that the transcendental monodromy of $\mathcal{Y}$ can be calculated quite easily from the transcendental monodromy of $\mathcal{X}$. If we let g be the $\mathbb{Q}(\sqrt{|\Sigma|})$-linear map relating the lattices $H^2(X_p,\mathbb{Z})^\Sigma$ and $K_0^\perp$

$$g\colon H^2(X_p,\mathbb{Z})^\Sigma \to K_0^\perp$$

for a given fibre X_p, then

$$\rho_{H^2(X_p,\mathbb{Z})^\Sigma}(w) = g^{-1}\rho_{K_0^\perp}(g\cdot w). \tag{8.2}$$

In particular, we have:

Corollary 8.3.2. *Let $\mathcal{X}$ be an N-polarized family of K3 surfaces and suppose $N \cong \mathrm{NS}(X_p)$ for some fibre X_p. Assume that $\mathcal{X}$ admits a group of fibrewise symplectic automorphisms Σ and let $\mathcal{Y}$ be the fibrewise resolution of the quotient $\mathcal{X}/\Sigma$. If $K_0^\perp$ is the sublattice generated by classes orthogonal to exceptional curves on Y_p, then the monodromy representation fixes $K_0^\perp \cap \mathrm{NS}(Y_p)$.*

Proof. By construction, we have that $\mathrm{NS}(X_p)^\Sigma$ is fixed under monodromy. Therefore, the relation in Equation (8.2) implies that its image in $K_0^\perp$ under the $\mathbb{Q}(\sqrt{|\Sigma|})$ isometry g is also fixed. Since g sends the transcendental lattice of X_p to the transcendental lattice Y_p, the image of $\mathrm{NS}(X_p)^\Sigma$ under g is $K_0^\perp \cap \mathrm{NS}(Y_p)$. Thus $K_0^\perp \cap \mathrm{NS}(Y_p)$ is fixed by monodromy of the family $\mathcal{Y}$. □

8.3.3 Nikulin involutions in families

We will now tie our results together. We begin with a family $\mathcal{X}$ of K3 surfaces which admits a fibrewise Nikulin involution and is lattice polarized by a lattice N which is isomorphic to the generic Néron-Severi lattice of the fibres of $\mathcal{X}$. Our goal is to understand how lattice polarization behaves under Nikulin involutions in families. We begin with some generalities on Nikulin involutions.

A Nikulin involution fixes precisely 8 points on X. The resulting quotient X/β has 8 ordinary double points which are then resolved by blowing up to give a new K3 surface Y. We can also resolve these singularities indirectly by blowing up X at the 8 fixed points of β, calling the resulting exceptional divisors $\{E_i\}_{i=1}^{8}$. We see that the blown up K3 surface $\tilde{X}$ also admits an involution $\tilde{\beta}$ whose fixed locus is the exceptional divisor

$$D = \sum_{i=1}^{8} E_i.$$

Let $F_i = q_* E_i$, where $q \colon \tilde{X} \to \tilde{X}/\tilde{\beta} \cong Y$ is the quotient map. The branch divisor in Y is then the sum $f_* D = \sum_{i=1}^{8} F_i$. Since there is a double cover ramified over $f_* D$, there must be some divisor

$$B = \frac{1}{2} f_* D.$$

We call the lattice generated by B and $\{F_i\}_{i=1}^{8}$ the Nikulin lattice, which we denote K_{Nik}.

According to [109, Section 6], K_{Nik} is a primitive sublattice of $\mathrm{NS}(\tilde{X}/\tilde{\beta})$ and, in the case where Σ is a group of order 2, the lattice K_0 discussed in Section 8.3.1 is equal to K_{Nik}. The following theorem is a technical tool, useful for calculations in Section 8.4.

Theorem 8.3.3. *Let $\mathcal{X} \to U$ be an N-polarized family of K3 surfaces and suppose $N \cong \mathrm{NS}(X_p)$ for some fibre X_p. Suppose further that X_p admits a Nikulin involution β; by Corollary 8.2.12 this extends to an involution on $\mathcal{X}$. Let $\mathcal{Y} \to U$ be the resolved quotient family of K3 surfaces and let N' be the Néron-Severi lattice of a generic fibre* *of $\mathcal{Y}$. Then there is a subgroup G of* $\mathrm{Aut}(A_{N'})$ *for which $\mathcal{Y}$ is an (N', G)-polarized family of K3 surfaces.*

Proof. To see that the resulting family $\mathcal{Y}$ is (N', G)-polarized for some G, it is enough to see that monodromy of $\mathcal{Y}$ cannot act trivially on $\mathrm{Aut}(A_{N'})$.

First we note that monodromy of $\mathcal{Y}$ must fix $K_{\mathrm{Nik}}^{\perp} \cap \mathrm{NS}(Y_p)$ by Corollary 8.3.2, where K_{Nik} denotes the Nikulin lattice. Thus the only non-trivial action of monodromy can be upon K_{Nik}.

Suppose for a contradiction that the image of $\rho_{\mathcal{NS}(\mathcal{Y})}$ contains a non-identity element g that lies in the kernel of $\alpha_{N'}$. Recall from Theorem 8.2.8 that such a g must act on $\mathrm{NS}(Y_p)$ in the same way as a non-trivial symplectic automorphism τ. Thus the orthogonal complement of the fixed lattice $\mathrm{NS}(Y_p)^g$ must have rank at least 8. Since K_{Nik} has rank 8 and $K_{\mathrm{Nik}}^{\perp} \cap \mathrm{NS}(Y_p)$ is fixed under monodromy, the orthogonal complement of $\mathrm{NS}(Y_p)^g$ must be contained in K_{Nik}. For reasons of rank this containment cannot be strict, so we must have equality. However, K_{Nik} is generated by elements of square (-2), thus, by [109, Lemma 4.2], it cannot be the lattice S_τ of any automorphism τ of X. This is a contradiction. □

Note that the proof given above does not extend to quotients by arbitrary symplectic automorphisms.

As a result of this theorem, Remark 8.2.16 and Equation (8.2) we may calculate G.

Corollary 8.3.4. *If g is the linear transformation which relates $\mathrm{T}(X_p)$ to $\mathrm{T}(Y_p)$ for some $p \in U$ and $\Gamma_{\mathcal{X}}$ (resp. $\Gamma_{\mathcal{Y}}$) is the image of the monodromy group of $\mathcal{T}(\mathcal{X})$ in $\mathrm{O}(\mathrm{T}(X_p))$ (resp. $\mathcal{T}(\mathcal{Y})$ in $\mathrm{O}(\mathrm{T}(Y_p))$), then $\Gamma_{\mathcal{Y}} = g^{-1}\Gamma_{\mathcal{X}} g$ and the image $\alpha_{\mathrm{T}(\mathcal{Y})}(\Gamma_{\mathcal{Y}})$ is the group G such that $\mathcal{Y}$ is minimally (N', G)-polarized.*

This allows us to control the algebraic monodromy of the family $\mathcal{Y}$ of K3 surfaces. In the following section, we concern ourselves with a geometric situation where it will be important to know exactly what our algebraic monodromy looks like.

8.4 Undoing the Kummer construction.

One of the major motivations for this work is the idea of undoing the Kummer construction globally in families. As we shall see, this has applications to the study of Calabi-Yau threefolds.

8.4.1 The general case.

Begin by assuming that $\mathcal{X}$ is a family of K3 surfaces which admit Shioda-Inose structure. Concretely, a *Shioda-Inose structure* on a K3 surface X is an embedding of the lattice $E_8 \oplus E_8$ into $\mathrm{NS}(X)$. By [102, Section 6], a Shioda-Inose structure defines a canonical Nikulin involution β and the minimal resolution of the quotient X/β is a Kummer surface. Furthermore, if X has transcendental lattice $\mathrm{T}(X)$, then the resolved quotient $Y = \widetilde{X/\beta}$ has transcendental lattice $\mathrm{T}(Y) \cong \mathrm{T}(X)(2)$.

Assume that $\mathcal{X}$ is a lattice polarized family of Shioda-Inose K3 surfaces. Then by Corollary 8.2.12, the Nikulin involution extends to the entire family of K3 surfaces to produce a resolved quotient family $\mathcal{Y}$ of Kummer surfaces.

We would like to find conditions under which one may undo the Kummer construction *in families* starting from the polarized family $\mathcal{X}$ of K3 surfaces with Shioda-Inose structure. In other words, we would like to find conditions under which a family of abelian surfaces $\mathcal{A}$ exists, such that application of the Kummer construction fibrewise to $\mathcal{A}$ yields the family $\mathcal{Y}$ of Kummer surfaces associated to $\mathcal{X}$.

The following proposition provides an easy sufficient condition for undoing the Kummer construction on a family of Kummer surfaces.

Proposition 8.4.1. *Beginning with a family of lattice polarized Shioda-Inose K3 surfaces $\mathcal{X}$ over U, the Kummer construction can be undone on the family of resolved quotient K3 surfaces $\mathcal{Y}$, if $\mathcal{Y}$ itself is lattice polarized.*

In general, however, the family $\mathcal{Y}$ will not be lattice polarized; instead, by Theorem 8.3.3, it will be (N', G)-polarized, for some lattice N' and subgroup G of $\mathrm{Aut}(A_{N'})$. To rectify this, we will have to proceed to a cover $f\colon U' \to U$ to remove the action of the group G, so that the Kummer construction can be undone on the pulled-back family $f^*\mathcal{Y}$.

We begin by finding such a group G. We note, however, that in general $\mathcal{Y}$ will not be *minimally* (N', G)-polarized for this choice of G.

Proposition 8.4.2. *Let $\mathcal{X} \to U$ be a family of N-polarized K3 surfaces with Shioda-Inose structure, where N is isometric to the Néron-Severi lattice of a generic K3 fibre X_p. Then the associated family of Kummer surfaces $\mathcal{Y}$ is an (N', G)-polarized family of K3 surfaces, where N' is the generic Néron-Severi lattice of fibres of $\mathcal{Y}$ and G is the group*

$$\mathrm{O}(N^{\perp})^*/\mathrm{O}(N^{\perp}(2))^*.$$

Furthermore, if $\mathcal{X}$ has transcendental monodromy group $\Gamma_{\mathcal{X}} = \mathrm{O}(N^{\perp})^*$, then $\mathcal{Y}$ is minimally (N', G)-polarized.

Proof. By the results of Section 8.3.2 there is a map

$$g\colon \rho_{\mathcal{T}(\mathcal{X})} \to \rho_{\mathcal{T}(\mathcal{Y})}.$$

Let X_p be a general fibre of $\mathcal{X}$ and let Y_p be the associated fibre of $\mathcal{Y}$. As X_p has Shioda-Inose structure and Y_p is the associated Kummer surface, the transformation g induces the identity map on the level of orthogonal groups,

$$\mathrm{Id}\colon \mathrm{O}(\mathrm{T}(X_p)) \to \mathrm{O}(\mathrm{T}(Y_p))$$

since the lattice $\mathrm{T}(Y_p)$ is just $\mathrm{T}(X_p)$ scaled by 2.

Let $\Gamma_{\mathcal{X}}$ (resp. $\Gamma_{\mathcal{Y}}$) denote the transcendental monodromy group of $\mathcal{X}$ (resp. $\mathcal{Y}$). Then, by Corollary 8.3.4, $\Gamma_{\mathcal{Y}} = g^{-1}\Gamma_{\mathcal{X}}g \cong \Gamma_{\mathcal{X}}$ and $\mathcal{Y}$ is minimally $(N', \alpha_{\mathrm{T}(\mathcal{Y})}(\Gamma_{\mathcal{Y}}))$-polarized. But $\Gamma_{\mathcal{X}} \subset \mathrm{O}(\mathrm{T}(X_p))^* \cong \mathrm{O}(N^{\perp})^*$ (by [44, Proposition 3.3]) and $\alpha_{\mathrm{T}(\mathcal{Y})}$ has kernel $\mathrm{O}(\mathrm{T}(X_p)(2))^* \cong \mathrm{O}(N^{\perp}(2))^*$, so $\alpha_{\mathrm{T}(\mathcal{Y})}(\Gamma_{\mathcal{Y}}) \subset G$, where G is as in the statement of the proposition, with equality if $\Gamma_{\mathcal{X}} = \mathrm{O}(N^{\perp})^*$. □

The group G from this proposition will prove to be very useful in later sections.

8.4.2 M-polarized K3 surfaces.

We will be particularly interested in the case in which our family $\mathcal{X}$ is M-polarized, where M denotes the lattice

$$M := H \oplus E_8 \oplus E_8.$$

Such families admit canonically defined Shioda-Inose structures, so the discussion from Section 8.4.1 holds.

Our interest in such familes stems from the paper [52], in which Doran and Morgan explicitly classify the possible integral variations of Hodge structure that can underlie a family of Calabi-Yau threefolds over $\mathbb{P}^1 - \{0, 1, \infty\}$ with $h^{2,1} = 1$. Their classification is given in [52, Table 1], which divides the possibilities into fourteen cases. Explicit examples, arising from toric geometry, of families of Calabi-Yau threefolds realising thirteen of these cases were known at the time of publication of [52] and are given in the rightmost column of [52, Table 1]. A family of Calabi-Yau threefolds that realised the missing case (hereafter known as the *14th case*) was constructed in [30].

It turns out that many of these threefolds admit fibrations by M-polarized K3 surfaces. The ability to undo the Kummer construction globally on such threefolds therefore provides a new perspective on the geometry of the families in [52, Table 1], which will be explored further in the remainder of this paper.

We begin this discussion with a brief digression into the geometry of M-polarized K3 surfaces, that we will need in the subsequent sections. In this section we will denote an M-polarized K3 surface by (X, i), where X is a K3 surface and i is an embedding $i \colon M \hookrightarrow \mathrm{NS}(X)$.

Clingher, Doran, Lewis and Whitcher [31] have shown that M-polarized K3 surfaces have a coarse moduli space given by the locus $d \neq 0$ in the weighted projective space $\mathbb{WP}(2, 3, 6)$ with weighted coordinates (a, b, d). Thus, by normalizing $d = 1$, we may associate a pair of complex numbers (a, b) to an M-polarized K3 surface (X, i).

Let β denote the Nikulin involution defined by the canonical Shioda-Inose structure on (X, i). Then Clingher and Doran [29, Theorem 3.13] have shown that the resolved quotient $Y = \widetilde{X/\beta}$ is isomorphic to the Kummer surface $\mathrm{Kum}(A)$, where $A \cong E_1 \times E_2$ is an Abelian surface that splits as a product of elliptic curves. By [29, Corollary 4.2] the j-invariants of these elliptic curves are given by the roots of the equation

$$j^2 - \sigma j + \pi = 0,$$

where σ and π are given in terms of the (a, b) values associated to (X, i) by $\sigma = a^3 - b^2 + 1$ and $\pi = a^3$.

There is one final piece of structure on (X, i) that we will need in our discussion. By [29, Proposition 3.10], the K3 surface X admits two uniquely defined elliptic fibrations $\Theta_{1,2} \colon X \to \mathbb{P}^1$, the *standard* and *alternate fibrations*. We will be mainly concerned with the alternate fibration Θ_2. This fibration has two sections, one singular fibre of type I_{12}^* and, if $a^3 \neq (b \pm 1)^2$, six singular fibres of type I_1 [29, Proposition 4.6]. Moreover, Θ_2 is preserved by the Nikulin involution β, so induces a fibration $\Psi \colon Y \to \mathbb{P}^1$ on X. The two sections of Θ_2 are identified to give a section of Ψ, and Ψ has one singular fibre of type I_6^* and, if $a^3 \neq (b \pm 1)^2$, six I_2's [29, Proposition 4.7].

8.4.3 Undoing the Kummer construction for M-polarized families

We will use this background to outline a method by which we can undo the Kummer construction for a family obtained as a resolved quotient of an M-polarized family of K3 surfaces. An illustration of the use of this method to undo the Kummer construction in an explicit example may be found in [30, Section 7.1].

Let N be a lattice that contains a sublattice isomorpic to M. Assume that $\mathcal{X}$ is an N-polarized family of K3 surfaces over U with generic Néron-Severi lattice $N \cong NS(X_p)$, where X_p is the fibre over a general point $p \in U$. Choose an embedding $M \hookrightarrow \mathrm{NS}(X_p)$; this extends uniquely to all other fibres of $\mathcal{X}$ by parallel transport and thus exhibits $\mathcal{X}$ as an M-polarized family of K3's.

This M-polarization induces a Shioda-Inose structure on the fibres of $\mathcal{X}$, which defines a canonical Nikulin involution on these fibres that extends globally by Corollary 8.2.12. Define $\mathcal{Y}$ to be the variety obtained from $\mathcal{X}$ by quotienting by this fibrewise

Nikulin involution and resolving the resulting singularities. Then $\mathcal{Y}$ is fibred over U by Kummer surfaces associated to products of elliptic curves. Let $Y_p \cong \mathrm{Kum}(E_1 \times E_2)$ denote the fibre of $\mathcal{Y}$ over the point $p \in U$, where E_1 and E_2 are elliptic curves.

The aim of this section is to find a cover $\mathcal{Y}'$ of $\mathcal{Y}$ upon which we can undo the Kummer construction. The results of Section 8.4.1 give a way to do this. Let $N' \cong \mathrm{NS}(Y_p)$ denote the generic Néron-Severi lattice of $\mathcal{Y}$. Then Theorem 8.3.3 shows that there is a subgroup G of $\mathrm{Aut}(A_{N'})$ for which $\mathcal{Y}$ is an (N', G)-polarized family of K3 surfaces. We will find a way to compute the action of monodromy around loops in U on N', which will allow us to find the group G such that $\mathcal{Y}$ is a minimally (N', G)-polarized family, along with a cover $\mathcal{Y}'$ of $\mathcal{Y}$ that is an N'-polarized family of K3 surfaces. Then Proposition 8.4.1 shows that we can undo the Kummer construction on $\mathcal{Y}'$.

To simplify this problem we note that, by Corollary 8.3.2, the only non-trivial action of monodromy on N' can be on the Nikulin lattice K_{Nik} contained within it. This lattice is generated by the eight exceptional curves F_i obtained by blowing up the fixed points of the Nikulin involution. Moreover, as β extends to a global involution on $\mathcal{X}$, the set $\{F_1, \ldots, F_8\}$ is preserved under monodromy (although the curves themselves may be permuted). Thus, we can compute the action of monodromy on N' by studying its action on the curves F_i.

To find these curves, we begin by studying the configuration of divisors on a general fibre Y_p. Recall that Y_p is isomorphic to $\mathrm{Kum}(E_1 \times E_2)$, where E_1 and E_2 are elliptic curves. There is a special configuration of twenty-four (-2)-curves on $\mathrm{Kum}(E_1 \times E_2)$ arising from the Kummer construction, that we shall now describe (here we note that we use the same notation as [29, Definition 3.18], but with the roles of G_i and H_j reversed).

Let $\{x_0, x_1, x_2, x_3\}$ and $\{y_0, y_1, y_2, y_3\}$ denote the two sets of points of order two on E_1 and E_2 respectively. Denote by G_i and H_j $(0 \leq i, j \leq 3)$ the (-2)-curves on $\mathrm{Kum}(E_1 \times E_2)$ obtained as the proper transforms of $E_1 \times \{y_i\}$ and $\{x_j\} \times E_2$ respectively. Let E_{ij} be the exceptional (-2)-curve on $\mathrm{Kum}(E_1 \times E_2)$ associated to the point (x_j, y_i) of $E_1 \times E_2$. This gives 24 curves, which have the following intersection numbers:

$$G_i.H_j = 0,$$
$$G_k.E_{ij} = \delta_{ik},$$
$$H_k.E_{ij} = \delta_{jk}.$$

Definition 8.4.3. The configuration of twenty-four (-2)-curves

$$\{G_i, H_j, E_{ij} \mid 0 \leq i, j \leq 3\}$$

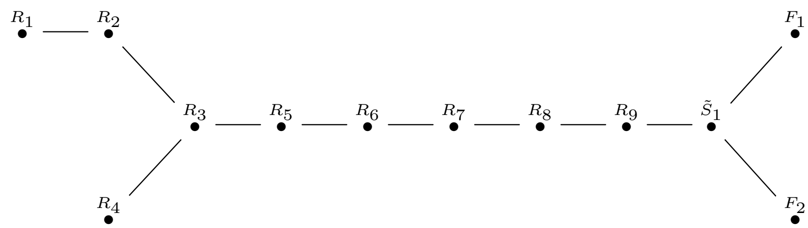
is called a *double Kummer pencil* on $\mathrm{Kum}(E_1 \times E_2)$.

Remark 8.4.4. Note that there may be many distinct double Kummer pencils on $\mathrm{Kum}(E_1 \times E_2)$. However, if E_1 and E_2 are non-isogenous, Oguiso [113, Lemma 1] shows that any two double Kummer pencils are related by a symplectic automorphism on $\mathrm{Kum}(E_1 \times E_2)$.

Clingher and Doran [29, Section 3.4] identify such a pencil on the resolved quotient of an M-polarised K3 surface. We will study this pencil on a fibre of $\mathcal{Y}$ and, by studying the action of monodromy on it, derive the action of monodromy on the curves F_i.

By the discussion in Section 8.4.2, the M-polarization structure on X_p defines an elliptic fibration Θ_2 on it, which is compatible with the Nikulin involution. Furthermore, as $\mathcal{X}$ is an M-polarized family, this elliptic fibration extends to all fibres of $\mathcal{X}$ and is compatible with the fibrewise Nikulin involution. Therefore Θ_2 induces an elliptic fibration Ψ on Y_p which extends uniquely to all fibres of $\mathcal{Y}$, so Ψ must be preserved under the action of monodromy around loops in U.

Using the same notation as in [29, Diagram (26)], we may label some of the (-2)-curves in the fibration Ψ as follows:



Here R_1 is the section of Ψ given uniquely as the image of the two sections of Θ_2 and the remaining curves form the I_6^* fibre. Note that the R_i and $\tilde{S}_1$ are uniquely determined by the structure of Ψ, so must be invariant under the action of monodromy around loops in U. By the discussion in [29, Section 3.5] the curves F_1 and F_2 are two of the eight exceptional curves that we seek, but are determined only up to permutation.

By the discussion in [29, Section 4.6], we may identify these curves with (-2)-curves in a double Kummer pencil as follows: $R_1 = G_2$, $R_2 = E_{20}$, $R_3 = H_0$, $R_4 = E_{30}$, $R_5 = E_{10}$, $R_6 = G_1$, $R_7 = E_{11}$, $R_8 = H_1$, $R_9 = E_{01}$, $\tilde{S}_1 = G_0$, $F_1 = E_{02}$ and $F_2 = E_{03}$. This gives:

Lemma 8.4.5. *In the double Kummer pencil on Y_p defined above, the action of monodromy around loops in U must fix the* 10 *curves G_0, G_1, G_2, H_0, H_1, E_{01}, E_{10}, E_{11}, E_{20}, E_{30}.*

We can improve on this result, but in order to do so we will need to make an assumption:

Assumption 8.4.6. The fibration Ψ on Y_p has six singular fibres of type I_2.

Remark 8.4.7. Recall from the discussion in Section 8.4.2 that this assumption is equivalent to the assumption that the (a, b)-parameters of the M-polarized fibre X_p satisfy $a^3 \neq (b \pm 1)^2$.

Using this, we may now identify all eight of the curves F_i. From the discussion above, we already know $F_1 = E_{02}$ and $F_2 = E_{03}$. [29, Section 3.5] shows that, under

Assumption 8.4.6, the remaining six F_i are the components of the six I_2 fibres in Ψ that are disjoint from the section $R_1 = G_2$.

Kuwata and Shioda [91, Section 5.2] explicitly identify these six I_2 fibres in the double Kummer pencil on Y_p. We see that:

- the section G_3 of Ψ is the unique section that intersects all six of $F_3, \ldots, F_8$,
- the section H_2 of Ψ intersects F_1 and precisely three of $F_3, \ldots, F_8$ (say F_3, F_4, F_5), and
- the section H_3 of Ψ intersects F_2 and the other three $F_3, \ldots, F_8$ (say F_6, F_7, F_8).

Combining this with Lemma 8.4.5 and the fact that the structure of Ψ is preserved under monodromy, we obtain

Proposition 8.4.8. *In addition to fixing the ten curves from Lemma 8.4.5, the action of monodromy around a loop in U must also fix G_3 and either*

(1) fix both $F_1 = E_{02}$ and $F_2 = E_{03}$, in which case H_2 and H_3 are also fixed and the sets $\{F_3, F_4, F_5\}$ and $\{F_6, F_7, F_8\}$ are both preserved, or

(2) interchange $F_1 = E_{02}$ and $F_2 = E_{03}$, in which case H_2 and H_3 are also swapped and the sets $\{F_3, F_4, F_5\}$ and $\{F_6, F_7, F_8\}$ are interchanged.

Whether the action of monodromy around a given loop fixes or exchanges $F_1 = E_{02}$ and $F_2 = E_{03}$ may be calculated explicitly. Recall that the curves $\{F_3, \ldots, F_8\}$ appear as components of the I_2 fibres in the alternate fibration on Y_p. Let x be an affine parameter on the base $\mathbb{P}^1_x$ of the alternate fibration on Y_p, chosen so that the I_6^*-fibre occurs at $x = \infty$. Then the locations of the I_2 fibres is given explicitly by [29, Proposition 4.7]: they lie at the roots of the polynomials $(P(x) \pm 1)$, where

$$P(x) := 4x^3 - 3ax - b, \tag{8.3}$$

for a and b the (a, b)-parameters associated to the M-polarized K3 surface X_p.

Without loss of generality, we may say that $\{F_3, F_4, F_5\}$ appear in the I_2 fibres occurring at roots of $(P(x) - 1)$ and $\{F_6, F_7, F_8\}$ appear in the I_2 fibres occurring at roots of $(P(x) + 1)$. We thus have:

Corollary 8.4.9. *Case (1) (resp. (2)) of Proposition 8.4.8 holds for monodromy around a given loop if and only if that monodromy preserves the set of roots of* $(P(x)+1)$ *(resp. switches the sets of roots of the polynomials* $(P(x) + 1)$ *and* $(P(x) - 1)$*).*

If case (2) of Proposition 8.4.8 holds for some loop in U, we note that the Nikulin lattice is *not* fixed under monodromy around that loop. This presents an obstruction to $\mathcal{Y}$ admitting an N'-polarization. To resolve this we may pull-back $\mathcal{Y}$ to a double cover of U, after which case (1) of the lemma will hold around all loops and the curves $F_1 = E_{02}$, $F_2 = E_{03}$, H_2 and H_3 will all be fixed under monodromy.

Given this, we may safely assume that case (1) holds around all loops in U, so F_1 and F_2 are fixed under monodromy and the sets $\{F_3, F_4, F_5\}$ and $\{F_6, F_7, F_8\}$ are both preserved. All that remains is to find whether monodromy acts to permute $F_3, \ldots, F_8$ within these sets.

Proposition 8.4.10. *Assume that the action of monodromy around all loops in* U *fixes both* F_1 *and* F_2 *(i.e. case (1) of Proposition 8.4.8 holds around all loops in* U*). Then the action of monodromy around a loop in* U *permutes* $\{F_3, F_4, F_5\}$ *(resp.* $\{F_6, F_7, F_8\}$*) if and only if it permutes the roots of* $(P(x) - 1)$ *(resp.* $(P(x) + 1)$*).*

Proof. As $\{F_3, F_4, F_5\}$ appear in the I_2 fibres occurring at roots of $(P(x) - 1)$ and $\{F_6, F_7, F_8\}$ appear in the I_2 fibres occurring at roots of $(P(x)+1)$, they are permuted if and only if the corresponding roots of $(P(x) - 1)$ and $(P(x) + 1)$ are permuted. $\square$

Monodromy around a loop thus acts on $\{F_3, F_4, F_5\}$ and $\{F_6, F_7, F_8\}$ as a permutation in $S_3 \times S_3$. Taken together, the permutations corresponding to monodromy around all loops generate a subgroup H of $S_3 \times S_3$.

Therefore, in order to obtain a N'-polarization on $\mathcal{Y}$, we need to pull everything back to a $|H|$-fold cover $f : V \to U$. This cover is constructed as follows: the $|H|$ preimages of the point $p \in U$ are labelled by permutations in H and, if γ is a loop in U, monodromy around $f^{-1}(\gamma)$ acts on these labels as composition with the corresponding permutation. This action extends to an action of H on the whole of V. In fact, we have:

Theorem 8.4.11. *Let $f\colon V \to U$ be the cover constructed above and let $\mathcal{Y}' \to V$ denote the pull-back of $\mathcal{Y} \to U$. Then $\mathcal{Y}'$ is a N'-polarized family, where N' is the generic Néron-Severi lattice of $\mathcal{Y}$, so we can undo the Kummer construction on $\mathcal{Y}'$. Furthermore, the deck transformation group of f is a subgroup G of S_6 given by:*

- *If case (1) of Proposition 8.4.8 holds around all loops in U, then $G = H$.*
- *If case (2) of Proposition 8.4.8 holds around some loop in U, then there is an exact sequence $1 \to H \to G \to C_2 \to 1$.*

Remark 8.4.12. We note that in the second case there does not seem to be any reason to believe that $G \cong H \rtimes C_2$ in general. Whilst we do not know of any explicit examples where this fails, it does not seem to be inconsistent with the theory as presented.

Proof. Let Y_p' denote one of the preimages of Y_p under the pull-back. Then the argument above shows that each of the eight curves F_i extends uniquely to all smooth fibres of $\mathcal{Y}'$. Thus the Nikulin lattice K_{Nik} is preserved under monodromy and so, by Corollary 8.3.2, N' is also. Therefore $\mathcal{Y}'$ is a N'-polarized family and, by Proposition 8.4.1, we may undo the Kummer construction on $\mathcal{Y}'$.

It just remains to verify the statements about the group G. Note that G can be seen as a subgroup of S_6, given by permutations of the divisors $\{F_3, \ldots, F_8\}$, and that H is the subgroup of G given by those permutations that preserve the sets $\{F_3, F_4, F_5\}$ and $\{F_6, F_7, F_8\}$. If case (1) of Proposition 8.4.8 holds around all loops in U, then all permutations in G preserve the sets $\{F_3, F_4, F_5\}$ and $\{F_6, F_7, F_8\}$, so $G = H$. If case (2) of Proposition 8.4.8 holds around some loop in U then H has index 2 in G, so it must be a normal subgroup with quotient $G/H \cong C_2$. $\square$

Corollary 8.4.13. *$\mathcal{Y}$ is a minimally (N', G)-polarized family of K3 surfaces, where G is the group from Theorem 8.4.11.*

Proof. We just need to show that G is minimal. Note that G was constructed explicitly as the permutation group of the divisors $\{F_1, \ldots, F_8\}$ under monodromy. Furthermore, it is clear from the construction that any permutation in G is induced by monodromy around some loop in U. So $\alpha_{N'}$ is surjective and G is minimal. □

Remark 8.4.14. As the group G from Theorem 8.4.11 is minimal, it will be a subgroup of the group $\mathrm{O}(N^{\perp})^*/\mathrm{O}(N^{\perp}(2))^*$ from Proposition 8.4.2.

8.4.4 The generically M-polarized case.

Suppose now that we are in the case where a general fibre X_p of $\mathcal{X}$ has $\mathrm{NS}(X_p) \cong M$. In this case we have the following version of Proposition 8.4.2.

Proposition 8.4.15. *Suppose that $\mathcal{X}$ is an M-polarized family of K3 surfaces with general fibre X_p satisfying $\mathrm{NS}(X_p) \cong M$. Then the resolved quotient $\mathcal{Y} \cong \widetilde{\mathcal{X}/\beta}$ of $\mathcal{X}$ by the fibrewise Nikulin involution is a (not necessarily minimally) (N', G)-polarized family of K3 surfaces, where $G \cong (S_3 \times S_3) \rtimes C_2$.*

Proof. Recall that $M^{\perp}$ is isomorphic to H^2. The proposition will follow from Proposition 8.4.2 if we can show that

$$\mathrm{O}(H^2)^*/\mathrm{O}(H^2(2))^* \cong (S_3 \times S_3) \rtimes C_2.$$

This quotient is just $\mathrm{Aut}(A_{H^2(2)})$. To see this, note that $\mathrm{O}(H^2)^*$ is isomorphic to $\mathrm{O}(H^2)$, since A_{H^2} is the trivial group, and $\mathrm{O}(H^2)$ is isomorphic to $\mathrm{O}(H^2(2))$, hence

$$\mathrm{O}(H^2(2))/\mathrm{O}(H^2(2))^* \cong \mathrm{O}(H^2)^*/\mathrm{O}(H^2(2))^*.$$

By a standard lattice theoretic fact (see, for example, [110, Theorem 3.6.3]), $\mathrm{O}(H^2(2))$ maps surjectively onto $\mathrm{Aut}(A_{H^2(2)})$. So the group $\mathrm{O}(H^2)^*/\mathrm{O}(H^2(2))^*$ is isomorphic to $\mathrm{Aut}(A_{H^2(2)})$. According to [84, Lemma 3.5] this group is isomorphic to $(S_3 \times S_3) \rtimes C_2$. □

Remark 8.4.16. The results of Section 8.4.3 give an immediate interpretation for this group: the two S_3 factors correspond to permutations of the two sets of divisors $\{F_3, F_4, F_5\}$ and $\{F_6, F_7, F_8\}$, whilst the C_2 corresponds to the action which interchanges these two sets (and also swaps F_1 and F_2).

Example 8.4.17. In [30], the family of threefolds Y_1 that realise the 14th case variation of Hodge structure admit torically induced fibrations by M-polarized K3 surfaces with general fibre X_p satisfying $\mathrm{NS}(X_p) \cong M$. In [30, Section 7.1] we apply the results of the previous section to undo the Kummer construction for the resolved quotient $W \cong \widetilde{Y_1/\beta}$ of Y_1 by the fibrewise Nikulin involution. It is an easy consequence of those calculations that W is *minimally* (N', G)-polarized, for $G \cong (S_3 \times S_3) \rtimes C_2$.

It turns out, however, that the 14th case is the only case from [52, Table 1] that admits a torically induced M-polarized fibration with general fibre X_p satisfying $\mathrm{NS}(X_p) \cong M$. In most other cases (see Theorem 8.5.10) the Néron-Severi lattice of the general fibre is a lattice enhancement of M to a lattice

$$M_n := M \oplus \langle -2n \rangle,$$

with $1 \leq n \leq 4$. In particular, note that M_n-polarized K3 surfaces are also M-polarized, so the analysis of this section still holds. We will examine this case in the next section.

8.5 Threefolds fibred by M_n-polarized K3 surfaces.

In this section we will specialize the analysis of Section 8.4 to the case where we have a family $\mathcal{X}$ of M_n-polarized K3 surfaces. We will then apply this theory to study M_n-polarized families of K3 surfaces arising from threefolds in the Doran-Morgan classification [52, Table 1].

8.5.1 The groups G.

We begin with the analogue of Proposition 8.4.2 in the M_n-polarized case.

Proposition 8.5.1. *Suppose that $\mathcal{X}$ is an M_n-polarized family of K3 surfaces with general fibre X_p satisfying $\mathrm{NS}(X_p) \cong M_n$. Then the resolved quotient $\mathcal{Y} \cong \widetilde{\mathcal{X}/\beta}$ of $\mathcal{X}$ by the fibrewise Nikulin involution is a (not necessarily minimal) (N', G)-polarized family of K3 surfaces, where N' is the generic Néron-Severi latice of $\mathcal{Y}$ and*

- *if $n = 1$ then $G = S_3 \times C_2$,*
- *if $n = 2$ then $G = D_8$, the dihedral group of order 8,*
- *if $n = 3$ then $G = D_{12}$, and*
- *if $n = 4$ then $G = D_8$.*

Proof. This will follow from Proposition 8.4.2 if we can show that

$$\mathrm{O}(M_n^{\perp})^*/\mathrm{O}(M_n^{\perp}(2))^* \cong G,$$

where G is as in each of the four cases in the statement of the proposition. We proceed by obtaining generators for $\mathrm{O}(M_n^{\perp})^* \cong \mathrm{O}(H \oplus \langle 2n \rangle)^*$ and then determining their actions on $A_{H(2)\oplus\langle 4n \rangle}$ to compute the group G.

In the case $n = 1$, the generators of $\mathrm{O}(H \oplus \langle 2 \rangle)^*$ are

$$g_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

whose induced actions on $A_{H(2)\oplus\langle 4 \rangle}$ have orders 2, 3 and 2 respectively. One may check that $g_1 g_2 g_1 = g_2^2$, and hence g_1 and g_2 generate a copy of S_3. It is clear that g_3 commutes with g_1 and g_2, so the subgroup of $\mathrm{Aut}(A_{H(2)\oplus\langle 4 \rangle})$ generated by g_1, g_2 and g_3 is isomorphic to $S_3 \times C_2$.

In the case $n = 2$, the group $\mathrm{O}(H \oplus \langle 4 \rangle)^*$ has a non-minimal set of generators

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ -1 & -1 & -3 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let the automorphism induced on $A_{H(2)\oplus\langle 8\rangle}$ by g_i be denoted h_i. Then $h_1^2 = h_2^2 = h_3^2 = \mathrm{Id}$. We check h_1h_3 has order 4 and it is easy to see that

$$h_1(h_1h_3)h_1 = h_3h_1 = (h_1h_3)^{-1}.$$

Therefore, h_1 and h_1h_3 generate a copy of D_8. Finally, one checks that $(h_1h_3)h_1 = h_2$, so the group of automorphisms $\langle h_1, h_2, h_3\rangle$ is isomorphic to D_8.

In the case when $n = 3$, we may calculate generators of $\mathrm{O}(H \oplus \langle 6\rangle)^*$ to find

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 6 \\ 1 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 3 & 6 \\ 3 & 4 & 12 \\ -1 & -2 & -5 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As before, let the corresponding automorphisms of $H(2) \oplus \langle 12\rangle$ be called h_1, h_2 and h_3. We calculate that

$$h_1^2 = h_2^3 = h_3^2 = (h_1h_3)^6 = \mathrm{Id}.$$

Furthermore, $(h_1h_3)^2 = h_2$ and

$$h_1(h_1h_3)h_1 = h_3h_1 = (h_1h_3)^{-1}.$$

Therefore, the group $\langle h_1, h_2, h_3\rangle$ is isomorphic to D_{12}.

In the case when $n = 4$, we may calculate generators of $\mathrm{O}(H \oplus \langle 8\rangle)^*$ to obtain

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 8 \\ 1 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 9 & 4 & 24 \\ 4 & 1 & 8 \\ -3 & -1 & -7 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Once again, let the corresponding automorphisms of $H(2) \oplus \langle 16\rangle$ be called h_1, h_2 and h_3. We calculate that

$$h_1^2 = h_2^2 = h_3^2 = \mathrm{Id}$$

We check that $h_1h_2 = h_2h_1$ and $h_3h_2 = h_2h_3$. Once again, we also have $(h_1h_3)^2 = h_2$ and

$$h_1(h_1h_3)h_1 = h_3h_1 = (h_1h_3)^{-1}.$$

Therefore the group $\langle h_1, h_2, h_3\rangle$ is isomorphic to D_8. □

8.5.2 Some special families

There are some special families of M_n-polarized K3 surfaces that we can use to vastly reduce the amount of work that we have to do to undo the Kummer construction for the M_n-polarized cases from [52, Table 1].

We begin by noting that the moduli space $\mathcal{M}_{M_n}$ of M_n-polarized K3 surfaces is a 1-dimensional modular curve [44, Theorem 7.1]. Denote by U_{M_n} the open subset of $\mathcal{M}_{M_n}$ obtained by removing the orbifold points.

Definition 8.5.2. $\mathcal{X}_n \to U_{M_n}$ will denote an M_n-polarized family of K3 surfaces over U_{M_n}, with period map $U_{M_n} \to \mathcal{M}_{M_n}$ given by the inclusion and transcendental monodromy group $\Gamma_{\mathcal{X}_n} = \mathrm{O}(M_n^{\perp})^*$.

Remark 8.5.3. Examples of such families for any n are given by the restriction of the special M-polarized family from [31, Theorem 3.1] to the M_n-polarized loci calculated in [31, Section 3.2]. For $n \leq 4$, we will explicitly construct examples of such families in Sections 8.5.4 and 8.5.5.

Let $\mathcal{Y}_n \to U_{M_n}$ be the family of Kummer surfaces associated to $\mathcal{X}_n \to U_{M_n}$ and let K_n be the Néron-Severi lattice of the Kummer surface associated to a K3 surface with Shioda-Inose structure and Néron-Severi lattice M_n.

Suppose now that we can undo the Kummer construction for $\mathcal{Y}_n$, by pulling back to a cover $C_{M_n} \to \mathcal{M}_{M_n}$. Then if we know that an M_n-polarised family of K3 surfaces $\mathcal{X} \to U$ is the pull-back of a family $\mathcal{X}_n \to U_{M_n}$ by the period map $U \to \mathcal{M}_{M_n}$ (which, in the M_n-polarized case, is more commonly known as the *generalized functional invariant*, see [45]), then we can undo the Kummer construction for the associated family of Kummer surfaces $\mathcal{Y} \to U$ by pulling back to the fibre product $U \times_{\mathcal{M}_{M_n}} C_{M_n}$.

Thus the aim of this section is to find covers $C_{M_n} \to \mathcal{M}_{M_n}$ such that the pull-backs of $\mathcal{Y}_n$ to C_{M_n} are K_n-polarized (and so, by Proposition 8.4.1, the Kummer construction can be undone on these pull-backs).

Lemma 8.5.4. *The families $\mathcal{Y}_n$ are minimally (K_n, G)-polarized, where G is the group $G = \mathrm{O}(M_n^\perp)^*/\mathrm{O}(M_n^\perp(2))^*$*

Proof. This follows from Proposition 8.4.2 and the fact that the families $\mathcal{X}_n$ have transcendental monodromy groups $\mathrm{O}(M_n^\perp)^*$. $\square$

As $\mathcal{M}_{M_n} = \mathrm{O}(M_n^\perp)^* \setminus \mathcal{P}_{M_n}$, this lemma suggests that, in order to undo the action of G, we should define C_{M_n} to be the curve $C_{M_n} := \mathrm{O}(M_n^\perp(2))^* \setminus \mathcal{P}_{M_n}$. This curve may be constructed as a modular curve in the following way.

Recall that

$$\Gamma_0(n) := \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod n \right\}$$

and

$$\Gamma(n) := \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n \right\}.$$

By convention, $\Gamma_0(1)$ and $\Gamma(1)$ are just the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We also have

$$\Gamma_0(n)^+ := \Gamma_0(n) \cup \tau_n \Gamma_0(n) \subseteq \mathrm{SL}_2(\mathbb{R})$$

where

$$\tau_n = \begin{pmatrix} 0 & -1/\sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix}$$

is the *Fricke involution.* With this notation, we have $\mathcal{M}_{M_n} \cong \Gamma_0(n)^+ \setminus \mathbb{H}$ [44, Theorem 7.1].

For any lattice N, let $\mathbb{P}\mathrm{O}(N)$ be defined as the cokernel of the obvious injection $\pm\mathrm{Id} \hookrightarrow \mathrm{O}(N)$. Then we have the exact sequence

$$1 \to \{\pm\mathrm{Id}\} \to \mathrm{O}(N) \to \mathbb{P}\mathrm{O}(N) \to 1.$$

If N is a lattice of signature $(1, n-1)$ with a fixed primitive embedding into Λ_{K3} and Γ and Γ' are two subgroups of $\mathrm{O}(N^\perp)$, the quotients $\Gamma \setminus \mathcal{P}_N$ and $\Gamma' \setminus \mathcal{P}_N$ are the same if and only if Γ and Γ' have the same images in $\mathbb{P}\mathrm{O}(N^\perp)$, in which case Γ and Γ' are said to be *projectively equivalent*.

By [44, Theorem 7.1], there is a map R_n, defined in the following proposition, under which $\Gamma_0(n)^+$ is mapped to a subgroup of $\mathrm{SO}(M_n^\perp)$ that is projectively equivalent to $\mathrm{O}(M_n^\perp)^*$.

Lemma 8.5.5. *The group* $\mathrm{O}(M_n^\perp(2))^*$ *is projectively equivalent to the image of* $\Gamma(2) \cap \Gamma_0(2n)$ *under the map*

$$R_n : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}_{\mathbb{R}}(2,1)$$

which is defined as

$$\begin{pmatrix} a & b \\ cn & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & c^2 n & 2acn \\ b^2 n & d^2 & 2bdn \\ ab & cd & bcn + ad \end{pmatrix}$$

(*see the related map in [75, Equation 5.6]*).

Proof. We know that the pre-image of $\mathrm{O}(M_n^\perp)^*$ under R_n is the subgroup $\Gamma_0(n)^+$ and that $\mathrm{O}(M_n^\perp(2))^* \subseteq \mathrm{O}(M_n^\perp)^*$ is the subgroup which fixes the group $A_{M_n^\perp(2)}$. Since R_n maps the Fricke involution to the automorphism

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which is never trivial or $-\mathrm{Id}$ on $A_{M_n^\perp(2)}$, we may automatically restrict to the image of $\Gamma_0(n)$. Automorphisms which fix $A_{M_n^\perp(2)}$ are matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with $a_{12}, a_{21}, a_{31}, a_{32} \equiv 0 \bmod 2$, $a_{13}, a_{23}, \equiv 0 \bmod 2n$, $a_{11}, a_{22} \equiv 1 \bmod 2$ and $a_{33} \equiv 1 \bmod 2n$. Thus $a^2 \equiv d^2 \equiv 1 \bmod 2$ and hence $a, d \equiv 1 \bmod 2$. Using this and the fact that $ab \equiv cd \equiv 0 \bmod 2$, we find that $b \equiv c \equiv 0 \bmod 2$. Therefore the matrices which map to $\mathrm{O}(M_n^\perp(2))^*$ are precisely those which satisfy

$$\begin{pmatrix} a & b \\ cn & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod 2n$$

and

$$\begin{pmatrix} a & b \\ cn & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2.$$

In other words elements of the group $\Gamma_0(2n) \cap \Gamma(2)$. □

We therefore have

$$C_{M_n} \cong (\Gamma_0(2n) \cap \Gamma(2)) \setminus \mathbb{H}.$$

Let $f \colon C_{M_n} \to \mathcal{M}_{M_n}$ be the natural map coming from the modular description of each curve.

Proposition 8.5.6. *If $n \neq 1$, the pullback $f^*\mathcal{Y}_n$ of $\mathcal{Y}_n$ to C_{M_n} is K_n-polarized.*

Proof. The transcendental monodromy of the pullback $f^*\mathcal{X}_n$ is a group Γ contained in $\mathrm{O}(M_n^\perp)^*$ with quotient space $\Gamma \setminus \mathcal{P}_{M_n} \cong (\Gamma_0(2n) \cap \Gamma(2)) \setminus \mathbb{H}$. By Lemma 8.5.5, the group $\mathrm{O}((M_n^\perp)(2))^*$ has this property.

Suppose that there is another subgroup Γ' of $\mathrm{O}(M_n^\perp)^*$ with this property. Let $\gamma \in \Gamma$ be any element and let $g \in \mathbb{P}\mathrm{O}(M_n^\perp)$ be its image. Since Γ and Γ' are projectively equivalent, there is some $\gamma' \in \Gamma'$ which maps to g.

If Γ and Γ' are not the same group, we can find some $g \in \mathbb{P}\mathrm{O}(M_n^{\perp})$ such that there are $\gamma \in \Gamma$ and $\gamma' \in \Gamma$ which map to g yet have $\gamma \neq \gamma'$. Thus $\gamma^{-1}\gamma' \neq \mathrm{Id}$ but $\gamma^{-1}\gamma'$ maps to the identity in $\mathbb{P}\mathrm{O}(M_n^{\perp})$. However, for $n \neq 1$, [75, Lemma 1.15] shows that the kernel of $\mathrm{O}(M_n^{\perp})^* \to \mathbb{P}\mathrm{O}(M_n)$ is trivial. This is a contradiction, hence $\Gamma = \Gamma'$.

Therefore, the monodromy group of the family $f^*\mathcal{X}_n$ is $\mathrm{O}(M_n^{\perp}(2))^* \subseteq \mathrm{O}(M_n^{\perp})^*$. By Corollary 8.3.4, the associated family of Kummer surfaces then has transcendental monodromy $\mathrm{O}(M_n^{\perp}(2))^*$ as well. Since this group is contained in the kernel of $\alpha_{\mathrm{T}(\mathcal{Y}_n)}$, we conclude that $\mathcal{Y}_n$ is K_n-polarized. □

Remark 8.5.7. This discussion may be rephrased in the following way. The quotient-resolution procedure taking $\mathcal{X}_n$ to $\mathcal{Y}_n$ defines an isomorphism $\mathcal{M}_{M_n} \xrightarrow{\sim} \mathcal{M}_{(K_n,G)}$, where G is the group from Lemma 8.5.4. The cover $C_{M_n} \to \mathcal{M}_{M_n}$ is then precisely the cover $\mathcal{M}_{K_n} \to \mathcal{M}_{(K_n,G)}$.

In the case where $n = 1$ this proof fails, as the kernel of the map $\mathrm{O}(M_n^{\perp})^* \to \mathbb{P}\mathrm{O}(M_n)$ is nontrivial. It will therefore be necessary for us to do a little more work in order to find a cover of $\mathcal{M}_{M_1}$ on which the pullback of $\mathcal{Y}_1$ is lattice polarized.

The family $\mathcal{X}_1$ is a family of smooth K3 surfaces over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Let g_1 and g_2 in $\mathrm{O}(H \oplus \langle 2 \rangle)^*$ be as in the $n = 1$ case of the proof of Proposition 8.5.1: then g_1 describes monodromy around 1 and g_2 describes monodromy around ∞, and monodromy around 0 is, as usual, given by $g_1 g_2^{-1}$. Around the point 1, the order of monodromy is 2, around 0, the order of monodromy is 6, and around ∞, the order of monodromy is infinite.

The group $\Gamma_0(2) \cap \Gamma(2)$ is just $\Gamma(2)$, since $\Gamma(2) \subseteq \Gamma_0(2)$, and the map from $C_{M_1} = \Gamma(2) \setminus \mathbb{H}$ to $\mathcal{M}_{M_1} = \Gamma_0(1)^+ \setminus \mathbb{H} \cong \Gamma_0(1) \setminus \mathbb{H}$ is just the j-function of the Legendre family of elliptic curves. This map may be written as a rational function,

$$j(t) = \frac{(t^2 - t + 1)^3}{27t^2(t-1)^2}.$$

The function $j(t)$ has three ramification points of order 2 over 1, three ramification points of order 2 over ∞ and two ramification points points of order 3 over 0. Looking back at the proof of Proposition 8.5.1, we see that the monodromy around the preimages of 1 and ∞ must act as $h_1^2 = \mathrm{Id}$ and $h_2^2 = \mathrm{Id}$ on $A_{H(2)\oplus\langle 4\rangle}$. However, monodromy around the preimages of 0 acts on $A_{H(2)\oplus\langle 4\rangle}$ as $(h_1h_2)^2 = -\mathrm{Id}$. Therefore, in order for monodromy to act trivially on $A_{H(2)\oplus\langle 4\rangle}$, we must take a further double cover of $C_{M_1} = \Gamma(2)\setminus\mathbb{H} = \mathbb{P}^1_t$ ramified along the roots of $t^2 - t + 1 = 0$. We thus have:

Proposition 8.5.8. *If $n = 1$, there is a double cover C'_{M_1} of C_{M_1} on which the pull-back of the family $\mathcal{Y}_n$ is K_1-polarized.*

The maps $f\colon C_{M_n} \to \mathcal{M}_{M_n}$ will be calculated in the next section.

8.5.3 Covers for small n

In this section, we will explicitly compute the maps $f\colon C_{M_n} \to \mathcal{M}_{M_n}$ for $n \leq 4$. To do this, we decompose the map $f = f_1 \circ f_2 \circ f_3$, where

$$\begin{aligned} &f_1\colon \Gamma_0(n)\setminus\mathbb{H} \longrightarrow \Gamma_0(n)^+\setminus\mathbb{H}, \\ &f_2\colon \Gamma_0(2n)\setminus\mathbb{H} \longrightarrow \Gamma_0(n)\setminus\mathbb{H}, \\ &f_3\colon C_{M_n} \cong (\Gamma_0(2n)\cap\Gamma(2))\setminus\mathbb{H} \longrightarrow \Gamma_0(2n)\setminus\mathbb{H}. \end{aligned}$$

8.5.3.1 The case $n = 1$

The rational modular curves $\Gamma_0(1)^+\setminus\mathbb{H}$ and $\Gamma_0(1)\setminus\mathbb{H}$ are isomorphic and have two elliptic points of orders 2 and 3 along with a single cusp. The map f_2 is a triple cover ramified with index 3 over the elliptic point of order 2 and indices $(2,1)$ over the elliptic point of order 2 and the cusp. $\Gamma_0(2)\setminus\mathbb{H}$ is a rational modular curve with an elliptic point of order 2 and two cusps. Finally, f_3 is a double cover ramified over the elliptic point and the cusp that is not a ramification point of f_2 and C_{M_1} is a rational modular curve with three cusps.

We thus see that $f\colon C_{M_1} \to \Gamma_0(1)^+\setminus\mathbb{H}$ is a 6-fold cover ramified with indices 2 and 3 at all points over the elliptic points of order 2 and 3 respectively and index 2 at all points over the cusp. It is easy to see that the deck transformation group of f is S_3.

However, from Proposition 8.5.8, we need to take a further double cover of C_{M_1} before we can undo the Kummer construction. This double cover is ramified over the two preimages under f of the elliptic point of order 3. The composition $C'_{M_1} \to \Gamma_0(1)^+ \setminus \mathbb{H}$ is a 12-fold cover ramified with indices 2 and 6 at all points over the elliptic points of order 2 and 3 respectively and index 2 at all points over the cusp. It is easy to see that the deck transformation group of this composition is $S_3 \times C_2$, as expected from Proposition 8.5.1.

8.5.3.2 The case $n = 2$

The rational modular curve $\Gamma_0(2)^+ \setminus \mathbb{H}$ has two elliptic points of orders 2 and 4 and a single cusp. The map f_1 is a double cover ramified over the two elliptic points and $\Gamma_0(2) \setminus \mathbb{H}$ is a rational modular curve with a single elliptic point of order 2 and two cusps. The map f_2 is then a double cover ramified over the elliptic point and one of the cusps, and $\Gamma_0(4) \setminus \mathbb{H}$ is a rational modular curve with three cusps. Finally, f_3 is a double cover ramified over the two cusps that are not ramification points of f_2 and C_{M_2} is a rational modular curve with four cusps.

We thus see that $f\colon C_{M_2} \to \Gamma_0(2)^+ \setminus \mathbb{H}$ is an 8-fold cover ramified with indices 2 and 4 at all points over the elliptic points of order 2 and 4 respectively and index 2 at all points over the cusp. It is easy to see that the deck transformation group of f is D_8, as expected from Proposition 8.5.1.

8.5.3.3 The case $n = 3$

The rational modular curve $\Gamma_0(3)^+ \setminus \mathbb{H}$ has two elliptic points of orders 2 and 6 and a single cusp. The map f_1 is a double cover ramified over the two elliptic points and $\Gamma_0(3) \setminus \mathbb{H}$ is a rational modular curve with one elliptic point of order 3 and two cusps. The map f_2 is then a triple cover ramified with index 3 over the elliptic point and indices $(2, 1)$ over each of the cusps, and $\Gamma_0(6) \setminus \mathbb{H}$ is a rational modular curve with four cusps. Finally, f_3 is a double cover ramified over the two cusps that are not ramification points of f_2 and C_{M_3} is a rational modular curve with six cusps.

We thus see that the map $f\colon C_{M_3} \to \Gamma_0(3)^+ \setminus \mathbb{H}$ is an 12-fold cover ramified with indices 2 and 6 at all points lying over the elliptic points of orders 2 and 6 respectively and index 2 at all points over the cusp. It is easy to see that the deck transformation group of f is D_{12}, as expected from Proposition 8.5.1.

8.5.3.4 The case $n = 4$

The rational modular curve $\Gamma_0(4)^+ \setminus \mathbb{H}$ has an elliptic point of order 2 and two cusps. The two cusps are distinguished by their widths, which are 1 and 2. The map f_1 is a double cover ramified over the elliptic point and the cusp of width 2. The rational modular curve $\Gamma_0(4) \setminus \mathbb{H}$ has three cusps of widths $(4, 1, 1)$. The map f_2 is then a double cover ramified with index 2 over the cusp of width 4 and one of the cusps of width 1. The rational modular curve $\Gamma_0(8) \setminus \mathbb{H}$ has four cusps of widths $(8, 2, 1, 1)$. Finally, f_3 is a double cover ramified over the two cusps of width 1. The curve C_{M_4} is a rational modular curve with six cusps of widths $(8, 8, 2, 2, 2, 2)$.

We thus see that $f\colon C_{M_4} \to \Gamma_0(4)^+ \setminus \mathbb{H}$ is an 8-fold cover ramified with index 2 at all points lying over the elliptic point and indices 2 and 4 at all points over the cusps of widths 1 and 2 respectively. It is easy to see that the deck transformation group of f is D_8, as expected from Proposition 8.5.1.

Remark 8.5.9. Note that if $n \neq 1$ we may also find a cover of $\mathcal{Y}_n \to U_{M_n}$ that is K_n-polarized using the method of Section 8.4.3 (if $n = 1$ then this method cannot be used, as Assumption 8.4.6 fails; see Section 8.5.5). In the three cases with $n \geq 2$ above it may be seen that this cover agrees with C_{M_n}.

8.5.4 Application to the 14 cases.

We now apply this theory to undo the Kummer construction for families of Kummer surfaces arising from M-polarized fibrations on the fourteen cases in [52, Table 1].

Examining these cases, we find M_n-polarized K3 fibrations with $2 \leq n \leq 4$ on nine of them, listed in the appropriate sections of Table 8.1. In this table, the first column gives the polarization lattice M or M_n, the second gives the mirrors of the threefolds

Lattice	Mirror threefold	Toric?	(i,j)	Arithmetic/thin
M_1	$\mathbb{WP}(1,1,1,1,2)[6]$	Yes	$(1,2)$	Arithmetic
	$\mathbb{WP}(1,1,1,1,4)[8]$	Yes	$(1,3)$	Thin
	$\mathbb{WP}(1,1,1,2,5)[10]$	Yes	$(2,3)$	Arithmetic
	$\mathbb{WP}(1,1,1,1,1,3)[2,6]^*$	Yes	$(1,1)$	Thin
	$\mathbb{WP}(1,1,1,2,2,3)[4,6]^*$	Yes	$(2,2)$	Arithmetic
M_2	$\mathbb{P}^4[5]$	Yes	$(1,4)$	Thin
	$\mathbb{WP}(1,1,1,1,2)[6]$	Yes	$(2,4)$	Arithmetic
	$\mathbb{WP}(1,1,1,1,4)[8]$	Yes	$(4,4)$	Thin
	$\mathbb{P}^5[2,4]$	Yes	$(1,1)$	Thin
	$\mathbb{WP}(1,1,1,1,2,2)[4,4]$	Yes	$(2,2)$	Arithmetic
M_3	$\mathbb{P}^4[5]$	No	$(2,3)$	Thin
	$\mathbb{P}^5[2,4]$	No	$(1,3)$	Thin
	$\mathbb{P}^5[3,3]$	Yes	$(1,2)$	Arithmetic
	$\mathbb{WP}(1,1,1,1,1,2)[3,4]^*$	Yes	$(2,2)$	Arithmetic
	$\mathbb{P}^6[2,2,3]$	Yes	$(1,1)$	Thin
M_4	$\mathbb{P}^5[2,4]$	No	$(2,2)$	Thin
	$\mathbb{P}^6[2,2,3]$	No	$(1,2)$	Thin
	$\mathbb{P}^7[2,2,2,2]$	Yes	$(1,1)$	Thin
M	$\mathbb{WP}(1,1,1,1,4,6)[2,12]$	Yes	$(1,1)$	Thin

Table 8.1 Lattice polarized K3 fibrations on the threefolds from [52, Table 1].

that have M- or M_n-polarized K3 fibrations, and the third states whether or not these fibrations are torically induced (the meanings of the fourth and fifth columns will be discussed later). More precisely, we have:

Theorem 8.5.10. *There exist K3 fibrations with M_n-polarized generic fibre, for $2 \leq n \leq 4$, on nine of the threefolds in [52, Table 1], given by the mirrors of those listed in the appropriate sections of Table 8.1. Furthermore, if $\mathcal{X} \to \mathbb{P}^1$ denotes one of these fibrations and $U \subset \mathbb{P}^1$ is the open set over which the fibres of $\mathcal{X}$ are nonsingular, then the restriction $\mathcal{X}|_U \to U$ agrees with the pull-back of a family $\mathcal{X}_n$ (see Definition 8.5.2) by the generalized functional invariant map $U \to \mathcal{M}_{M_n}$. The family $\mathcal{X}|_U \to U$ is thus an M_n-polarized family of K3 surfaces.*

Remark 8.5.11. The M_1-polarized cases in the first section of Table 8.1 will require some extra work, so they will be discussed separately in Section 8.5.5. The 14th case of [30] has already been discussed in Example 8.4.17, where we recalled that the family of threefolds Y_1 realising the 14th case variation of Hodge structure admit torically induced M-polarized K3 fibrations. By [30, Section 8.2], these threefolds Y_1 can be thought of as mirror to complete intersections $\mathbb{WP}(1,1,1,1,4,6)[2,12]$. This case is included in the final row of Table 8.1.

Remark 8.5.12. To check which of the fibrations listed in Table 8.1 are torically induced, one may use the computer software *Sage* to find all fibrations of the toric ambient spaces by toric subvarieties that induce fibrations of the Calabi-Yau threefold by M-polarized K3 surfaces. The resulting list may be compared to the list of fibrations in Table 8.1, giving the third column of this table. This also proves that Table 8.1 contains *all* torically induced fibrations of the Calabi-Yau threefolds from [52, Table 1] by M-polarized K3 surfaces.

We will prove Theorem 8.5.10 by explicit calculation: we find families $\mathcal{X}_n$ satisfying Definition 8.5.2 and show that they pull back to give the families $\mathcal{X}|_U$ under the generalized functional invariant maps.

In each case, we will see that the generalized functional invariant map is completely determined by the pair of integers (i,j) from the fourth column of Table 8.1. In fact, we find that it is an $(i+j)$-fold cover of $\mathcal{M}_{M_n} \cong \Gamma_0(n)^+ \setminus \mathbb{H}$ having exactly four ramification points: one of order $(i+j)$ over the cusp (or, in the M_4-polarized case, the cusp of width 1), two of orders i and j over the elliptic point of order $\neq 2$ (or, in the M_4-polarized case, the cusp of width 2), and one of order 2 which varies with the value of the Calabi-Yau deformation parameter.

We thus have everything we need to undo the Kummer construction in the families arising as the resolved quotients of the families $\mathcal{X}|_U$ from Theorem 8.5.10. By the discussion in Section 8.5.2, in order to undo the Kummer construction we just need to pull back to the cover $C_{M_n} \times_{\mathcal{M}_{M_n}} U$, where the map $C_{M_n} \to \mathcal{M}_{M_n}$ is as calculated in Section 8.5.3 and $U \to \mathcal{M}_{M_n}$ is the generalized functional invariant map, described above.

8.5.4.1 M_2-polarized families

We begin the proof of Theorem 8.5.10 with the M_2 case. Note first that an M_2-polarized K3 surface is mirror (in the sense of [44]) to a $\langle 4\rangle$-polarized K3 surface, which is generically a hypersurface of degree 4 in $\mathbb{P}^3$.

By the Batyrev-Borisov mirror construction [17], the mirror of a degree 4 hypersurface in $\mathbb{P}^3$ is a hypersurface in the toric variety polar dual to $\mathbb{P}^3$. The intersection of this hypersurface with the maximal torus is isomorphic to the locus in $(\mathbb{C}^\times)^3$ defined by the rational polynomial

$$x_1 + x_2 + x_3 + \frac{\lambda}{x_1x_2x_3} = 1, \tag{8.4}$$

where $\lambda \in \mathbb{C}$ is a constant. This is easily compactified to a singular hypersurface of degree 4 in $\mathbb{P}^3$, given by the equation

$$\lambda w^4 + xyz(x + y + z - w) = 0,$$

where (w, x, y, z) are coordinates on $\mathbb{P}^3$.

Consider the family of surfaces over $\mathbb{C}$ obtained by varying λ. By resolving the singularities of the generic fibre and removing any singular fibres that remain, we obtain a family of K3 surfaces $\mathcal{X}_2 \to U_2 \subset \mathbb{C}$. Dolgachev [44, Example (8.2)] exhibited elliptic fibrations on the K3 fibres of $\mathcal{X}_2$ and used them to give a set of divisors generating the lattice M_2. It can be seen from the structure of these elliptic fibrations that these divisors are invariant under monodromy, so there can be no action of monodromy on M_2. We thus see that $\mathcal{X}_2$ is an M_2-polarized family of K3 surfaces.

The action of transcendental monodromy on $\mathcal{X}_2$ was calculated by Narumiya and Shiga [108] (note that our parameter λ is different from theirs: our λ is equal to μ^4 or $\frac{u}{256}$ from their paper). In [108, Section 4] they find that the fibre X_λ of $\mathcal{X}_2$ is smooth away from $\lambda \in \{0, \frac{1}{256}\}$ and the monodromy action has order 2 around $\lambda = \frac{1}{256}$, order 4 around $\lambda = \infty$, and infinite order around $\lambda = 0$. Furthermore, they show [108, Remark 6.1] that the monodromy of $\mathcal{X}_2$ generates the $(2, 4, \infty)$ triangle group (which is isomorphic to $\Gamma^0(2)^+ \cong \mathrm{O}(M_2^{\perp})^*)$, so the period map of $\mathcal{X}_2 \to U_2$ must be injective. Thus the family $\mathcal{X}_2 \to U_2$ satisfies Definition 8.5.2.

We can use the local form (8.4) of the family $\mathcal{X}_2$ to find M_2-polarized families of K3 surfaces on the threefolds from [52, Table 1]. For example:

Example 8.5.13. The first M_2-polarized case from Table 8.1 is the mirror to the quintic threefold. By the Batyrev-Borisov construction, on the maximal torus we may write this mirror as the locus in $(\mathbb{C}^\times)^4$ defined by the rational polynomial

$$x_1 + x_2 + x_3 + x_4 + \frac{A}{x_1x_2x_3x_4} = 1,$$

where $A \in \mathbb{C}$ is the Calabi-Yau deformation parameter. Consider the fibration induced by projection onto the x_4 coordinate; for clarity, we make the substitution $x_4 = t$. If we further substitute $x_i \mapsto x_i(1-t)$ for $1 \leq i \leq 3$ and rearrange, we obtain

$$x_1 + x_2 + x_3 + \frac{A}{x_1x_2x_3t(1-t)^4} = 1.$$

But, from the local form (8.4), it is clear that this describes an M_2-polarized family of K3 surfaces with

$$\lambda = \frac{A}{t(1-t)^4}.$$

This is the generalized functional invariant map of the fibration. Note that it is ramified to orders 1 and 4 over the order 4 elliptic point $\lambda = \infty$, order 5 over the cusp $\lambda = 0$, and order 2 over the variable point $\lambda = \frac{5^5A}{2^8}$, giving $(i,j) = (1,4)$.

Similar calculations may be performed in the other M_2-polarized cases from Table 8.1. We find that the generalized functional invariants are given by

$$\lambda = \frac{Au^{i+j}}{t^i(u-t)^j},$$

where (t,u) are homogeneous coordinates on the base $U \subset \mathbb{P}^1$ of the K3 fibration, (i,j) are as in Table 8.1, and A is the Calabi-Yau deformation parameter.

8.5.4.2 M_3-polarized families

Here we follow a similar method to the M_2-polarized case. An M_3-polarized K3 surface is mirror to a $\langle 6\rangle$-polarized K3 surface, which may be realised as a complete intersection of type $(2,3)$ in $\mathbb{P}^4$.

By the Batyrev-Borisov construction, on the maximal torus we may express the mirror of a $(2,3)$ complete intersection in $\mathbb{P}^4$ as the locus in $(\mathbb{C}^\times)^3$ defined by the rational polynomial

$$x_1 + \frac{\lambda}{x_1x_2x_3(1-x_2-x_3)} = 1, \tag{8.5}$$

where $\lambda \in \mathbb{C}$ is a constant. This is easily compactified to a singular hypersurface of bidegree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, given by the equation

$$\lambda s^2z^3 + r(r-s)xy(z-x-y) = 0,$$

where (r,s) are coordinates on $\mathbb{P}^1$ and (x,y,z) are coordinates on $\mathbb{P}^2$.

Consider the family of surfaces over $\mathbb{C}$ obtained by varying λ. By resolving the singularities of the generic fibre and removing any singular fibres that remain, we obtain a family of K3 surfaces $\mathcal{X}_3 \to U_3 \subset \mathbb{C}$. We now show that $\mathcal{X}_3$ is an M_3-polarized family that satisfies Definition 8.5.2.

There is a natural elliptic fibration on the fibres of $\mathcal{X}_3$, obtained by projecting onto the $\mathbb{P}^1$ factor. This elliptic fibration has two singular fibers of Kodaira type IV^* at $r=0$ and $r=s$, a fibre of type I_6 at $s=0$, two fibres of type I_1 and a section. In fact, one sees easily that the hypersurface obtained by intersecting with $z=0$ splits into three lines, which project with degree 1 onto $\mathbb{P}^1$ and hence are all sections. If we choose one of these sections as a zero section, the other two are 3-torsion sections and generate a subgroup of the Mordell-Weil group of order 3.

One can check that the lattice spanned by components of reducible fibers and these torsion sections is a copy of the lattice M_3 inside of $\mathrm{NS}(X_\lambda)$, for each fiber X_λ of $\mathcal{X}_3 \to U_3$. Since the 3-torsion sections are individually fixed under monodromy, there can be no monodromy action on this copy of M_3 in $\mathrm{NS}(X_\lambda)$. We thus see that $\mathcal{X}_3$ is an M_3-polarized family of K3 surfaces.

Next we calculate the transcendental monodromy of this family to show that it satisfies Definition 8.5.2.

Lemma 8.5.14. *U_3 is the open subset given by removing the points $\lambda \in \{0, \frac{1}{108}\}$ from $\mathbb{C}$. Transcendental monodromy of the family $\mathcal{X}_3 \to U_3$ has order 2 around $\lambda = \frac{1}{108}$, order dividing 6 around $\lambda = \infty$ and infinite order around $\lambda = 0$.*

Proof. The discriminant of the elliptic fibration on a fibre X_λ of $\mathcal{X}_3$ vanishes for $\lambda \in \{0, \frac{1}{108}, \infty\}$, giving the locations of the singular K3 surfaces that are removed from the family $\mathcal{X}_3$. At $\lambda = \frac{1}{108}$ the two singular fibres of type I_1 collide so that the K3 surface $X_{\lambda=\frac{1}{108}}$ has a single node. Thus there is a vanishing class of square (-2) associated to the fibre $X_{\lambda=\frac{1}{108}}$ and monodromy around this fibre is a reflection across this class. Therefore monodromy around $\lambda = \frac{1}{108}$ has order 2.

We will use this to indirectly calculate the monodromies around other points. After base change $\lambda = \mu^3$ and a change in variables, one finds that the $\lambda = \infty$ fiber can be replaced with an elliptically fibered K3 surface with three singular fibers of type IV^*. Since a generic member of the family $\mathcal{X}_3$ has Néron-Severi rank 19, this fiber can only have a single node, so again the monodromy transformation around it must be of order at most 2. Hence monodromy around $\lambda = \infty$ has order dividing 6.

To determine monodromy around the final point, it is enough to note that the moduli space of M_n-polarized K3 surfaces has a cusp, and the preimage of this cusp under the period map must also have monodromy of infinite order. Since the points $\lambda \in \{\frac{1}{108}, \infty\}$ are of finite order and every other fiber is smooth, $\lambda = 0$ must map to the cusp under the period map and therefore has infinite order monodromy. $\square$

As a result we find:

Proposition 8.5.15. *The period map of $\mathcal{X}_3 \to U_3$ is injective and the subgroup of $\mathrm{O}(M_3^\perp)^*$ generated by monodromy transformations is $\mathrm{O}(M_3^\perp)^*$ itself. The family $\mathcal{X}_3$ thus satisfies Definition 8.5.2.*

Proof. Notice first that, by Lemma 8.5.14, the monodromy group of $\mathcal{X}_3$ is isomorphic to a triangle group of type $(2, d, \infty)$ for $d = 2, 3$ or 6 and contained in $\mathrm{O}(M_3^{\perp})^*$. It is well known that $\mathrm{O}(M_3^{\perp})^* \cong \Gamma_0(3)^+$ is a $(2, 6, \infty)$ triangle group, and since the period map is of finite degree, the monodromy group of $\mathcal{X}_3$ is of finite index in $\Gamma_0(3)^+$. Thus we need to show that the only finite index embedding of a $(2, d, \infty)$ triangle group into the $(2, 6, \infty)$ triangle group is the identity map from the $(2, 6, \infty)$ triangle group to itself. But this is calculated in [140]. $\square$

As before, we can use the local form (8.5) of the family $\mathcal{X}_3$ to find M_3-polarized families of K3 surfaces on the threefolds from [52, Table 1]. We find that the generalized functional invariants are given by

$$\lambda = \frac{Au^{i+j}}{t^i(u-t)^j},$$

where (t, u) are homogeneous coordinates on the base $U \subset \mathbb{P}^1$ of the K3 fibration, (i, j) are as in Table 8.1, and A is the Calabi-Yau deformation parameter.

8.5.4.3 M_4-polarized families

We conclude the proof of Theorem 8.5.10 with the M_4-polarized case. An M_4-polarized K3 surface is mirror to an $\langle 8\rangle$-polarized K3 surface, given generically as a complete intersection of type $(2, 2, 2)$ in $\mathbb{P}^5$.

By the Batyrev-Borisov construction, on the maximal torus we may express the mirror of a complete intersection of type $(2, 2, 2)$ in $\mathbb{P}^5$ as the locus in $(\mathbb{C}^\times)^3$ defined by the rational polynomial

$$x_1 + \frac{\lambda}{x_2(1-x_2)x_3(1-x_3)x_1} = 1. \tag{8.6}$$

This may be easily compactified to a singular hypersurface of multidegree $(2, 2, 2)$ in $(\mathbb{P}^1)^3$ given by

$$\lambda s_1^2 s_2^2 s_3^2 - r_1(s_1 - r_1)r_2(s_2 - r_2)r_3(s_3 - r_3) = 0,$$

where (r_i, s_i) are coordinates on the ith copy of $\mathbb{P}^1$.

As above, we consider the family of surfaces over $\mathbb{C}$ obtained by varying λ. By resolving the singularities of the generic fibre and removing any singular fibres that remain, we obtain a family of K3 surfaces $\mathcal{X}_4 \to U_4 \subset \mathbb{C}$. We now show that $\mathcal{X}_4$ is an M_4-polarized family that satisfies Definition 8.5.2.

Begin by noting that there is an S_3 symmetry on $\mathcal{X}_4$ obtained by permuting copies of $\mathbb{P}^1$. Furthermore, projection of $(\mathbb{P}^1)^3$ onto any one of the three copies of $\mathbb{P}^1$ produces an elliptic fibration on the K3 hypersurfaces. This elliptic fibration has a description very similar to that of the elliptic fibration on $\mathcal{X}_3$. Generically it has two fibres of type I_1^* at $r_i = 0$ and $r_i = s_i$, a fibre of type I_8 at $s_i = 0$, and two fibres of type I_1.

This elliptic fibration has a 4-torsion section. Using standard facts relating the Néron-Severi group of an elliptic fibration to its singular fiber types and Mordell-Weil group (see [99, Lecture VII]), we see that each fiber of $\mathcal{X}_4$ is polarized by a rank 19 lattice with discriminant 8. A little lattice theory shows that this must be the lattice M_4. The embedding of M_4 into the Néron-Severi group must be primitive, otherwise we would find full 2-torsion structure, which is not the case. As in the case of $\mathcal{X}_3$, this embedding of M_4 is monodromy invariant, so $\mathcal{X}_4$ is an M_4-polarized family of K3 surfaces.

Proposition 8.5.16. *U_4 is the open subset given by removing the points $\lambda = \{0, \frac{1}{64}\}$ from $\mathbb{C}$. Transcendental monodromy of the family $\mathcal{X}_4 \to U_4$ has order 2 around $\lambda = \frac{1}{64}$ and infinite order around $\lambda \in \{0, \infty\}$.*

Furthermore, the period map of $\mathcal{X}_4 \to U_4$ is injective and the subgroup of $\mathrm{O}(M_4^{\perp})^*$ generated by monodromy transformations is $\mathrm{O}(M_4^{\perp})^*$ itself. The family $\mathcal{X}_4$ thus satisfies Definition 8.5.2.

Proof. As in the proof of Lemma 8.5.14, to see that fibers of $\mathcal{X}_4$ degenerate only when $\lambda \in \{0, \frac{1}{64}, \infty\}$, it is enough to do a simple discriminant computation. The elliptic fibration described above is well-defined away from $\lambda \in \{0, \infty\}$ and the two I_1 singular fibers collide when $\lambda = \frac{1}{64}$. As before, this shows that monodromy has order 2 around $\lambda = \frac{1}{64}$.

To see that monodromies around $\lambda \in \{0,\infty\}$ have infinite order, we argue as follows. We have a period map from $\mathbb{P}^1_\lambda$ to $\overline{\mathcal{M}_{M_4}}$, the Baily-Borel compactification of the period space of M_4-polarized K3 surfaces. The monodromy of $\mathcal{X}_4$ is a $(2,k,l)$ triangle group for some choice of k,l, and lies inside of $\mathrm{O}(M_4^\perp)^* \cong \Gamma_0(4)^+$ (which is a $(2,\infty,\infty)$ triangle group) as a finite index subgroup, since the period map is dominant. However, by [140], the only $(2,k,l)$ triangle group of finite index inside of the $(2,\infty,\infty)$ triangle group is the $(2,\infty,\infty)$ triangle group itself (equipped with the identity embedding). Therefore the period map is the identity and monodromy around $\lambda \in \{0,\infty\}$ is of infinite order. □

As in the previous cases, we can use the local form (8.6) of the family $\mathcal{X}_4$ to find M_4-polarized families of K3 surfaces on the threefolds from [52, Table 1]. We find that the generalized functional invariants are given by

$$\lambda = \frac{Au^{i+j}}{t^i(u-t)^j},$$

where (t,u) are homogeneous coordinates on the base $U \subset \mathbb{P}^1$ of the K3 fibration, (i,j) are as in Table 8.1, and A is the Calabi-Yau deformation parameter. This completes the proof of Theorem 8.5.10.

8.5.5 The case $n = 1$

It remains to address the case of threefolds from [52, Table 1] that are fibred by M_1-polarized K3 surfaces. Unfortunately many of the results that we have proved so far do not apply in this case: Assumption 8.4.6 does not hold (this follows easily from Remark 8.4.7 and the expressions for the (a,b,d)-parameters of M_1-polarized K3 surfaces, below), so the methods of Section 8.4.3 do not apply, and the torically induced fibrations of these threefolds by M_1-polarized K3 surfaces (computed with *Sage*) cannot all be seen as pull-backs of special M_1-polarized families $\mathcal{X}_1$ from the moduli space $\mathcal{M}_{M_1}$, so we cannot directly use the results of Section 8.5.2 either.

Instead, we will construct a special 2-parameter M_1-polarized family of K3 surfaces $\mathcal{X}_1^2 \to U_1^2$, which is very closely related to a family $\mathcal{X}_1$ satisfying Definition 8.5.2 (this relationship will be made precise in Proposition 8.5.18 and Remark 8.5.19), and show that the M_1-polarized fibrations $\mathcal{X} \to U$ on our threefolds are pull-backs of this family by maps $U \to U_1^2$.

Now let $\mathcal{Y}_1^2 \to U_1^2$ denote the family of Kummer surfaces associated to $\mathcal{X}_1^2 \to U_1^2$ and suppose that we can construct a cover $V \to U_1^2$ that undoes the Kummer construction for $\mathcal{Y}_1^2$. Then, as before, we may undo the Kummer construction for the family of Kummer surfaces associated to $\mathcal{X} \to U$ by pulling back to the fibre product $U \times_{U_1^2} V$.

To construct the 2-parameter family $\mathcal{X}_1^2 \to U_1^2$, we begin by noting that an M_1-polarized K3 surface is mirror to a $\langle 2 \rangle$-polarized K3 surface, which can generically be expressed as a hypersurface of degree 6 in $\mathbb{WP}(1,1,1,3)$. By the Batyrev-Borisov construction, an M_1-polarized K3 surface can be realised torically as an anticanonical hypersurface in the polar dual of $\mathbb{WP}(1,1,1,3)$. The defining polynomial of a generic such anticanonical hypersurface is

$$a_0x_0^6 + a_1x_1^6 + a_2x_2^6 + a_3x_3^2 + a_4x_0x_1x_2x_3 + a_5x_0^2x_1^2x_2^2, \tag{8.7}$$

where x_0, x_1, x_2 are variables of weight 1 and x_3 is a variable of weight 3.

On the maximal torus, the family defined by this equation is isomorphic to the vanishing locus in $(\mathbb{C}^\times)^3$ of the rational polynomial

$$y + z + \frac{\alpha}{x^3yz} + x + 1 + \frac{\beta}{x} = 0, \tag{8.8}$$

where $\alpha = \frac{a_0a_1a_2a_3^3}{a_4^6}$ and $\beta = \frac{a_3a_5}{a_4^2}$. Consider the family of K3 surfaces over $\mathbb{C}^2$ obtained by varying α and β. By resolving the singularities of the generic fibre and removing any singular fibres that remain, we obtain the 2-parameter family of K3 surfaces $\mathcal{X}_1^2 \to U_1^2 \subset \mathbb{C}^2$.

We can express the (a, b, d)-parameters (see Section 8.4.2) of a fibre of $\mathcal{X}_1^2$ in terms of α and β as

$$a = 1, \qquad b = \frac{2^6 3^3 \alpha}{(4\beta - 1)^3} + 1, \qquad d = \left(\frac{2^6 3^3 \alpha}{(4\beta - 1)^3} \right)^2,$$

where this parameter matching was computed using the elliptic fibrations on M-polarized K3 surfaces in Weierstrass normal form.

Introducing a new parameter

$$\gamma := \frac{2^6 3^3 \alpha}{(4\beta - 1)^3},$$

we see from the expressions for (a, b, d) above that γ parametrizes the moduli space $\mathcal{M}_{M_1}$, so the generalized functional invariant of the family $\mathcal{X}_1^2$ is given by γ. Then we find:

Lemma 8.5.17. *U_1^2 is the open set $U_1^2 := \{(\alpha, \beta) \in \mathbb{C}^2 \mid \gamma \notin \{0, -1, \infty\}\}$. Furthermore, $\mathcal{X}_1^2 \to U_1^2$ is an M_1-polarized family of K3 surfaces.*

Proof. Using the computer software *Sage*, it is possible to explicitly compute a toric resolution of a generic K3 surface defined in the polar dual of $\mathbb{WP}(1, 1, 1, 3)$ by Equation (8.7). From this, we find that the singular fibres of this family occur precisely over $\gamma \in \{0, -1, \infty\}$.

To see that $\mathcal{X}_1^2 \to U_1^2$ is an M_1-polarized family, we note that $\mathcal{X}_1^2$ is a family of hypersurfaces in the polar dual to $\mathbb{WP}(1, 1, 1, 3)$. By [130], there is a toric resolution Y of the ambient space such that the fibres X of $\mathcal{X}_1^2$ become smooth K3 surfaces in Y and the restriction map

$$\mathrm{res} : \mathrm{NS}(Y) \to \mathrm{NS}(X)$$

is surjective. Furthermore the image of res is the lattice M_1. This defines a lattice polarization on each fiber and, since this polarization is induced from the ambient threefold, it is unaffected by monodromy. Thus $\mathcal{X}_1^2$ is a family of M_1-polarized K3 surfaces. $\square$

Changing variables in (8.8) and completing the square in x, the family $\mathcal{X}_1^2$ may be written on $(\mathbb{C}^\times)^3$ as the vanishing locus of

$$\frac{x^2}{4\beta - 1} + y + z + \frac{\gamma}{yz} + 1 = 0.$$

Furthermore, we note that points $(\alpha, \beta) \in U_1^2$ correspond bijectively with points (β, γ) in $\{(\beta, \gamma) \in \mathbb{C}^2 \mid \beta \neq \frac{1}{4},\ \gamma \notin \{0, -1\}\}$. Using this we can reparametrize U_1^2 by β and γ, and thus think of $\mathcal{X}_1^2 \to U_1^2$ as the 2-parameter family parametrized by β and γ given on the maximal torus by the expression above.

After performing this reparametrization, the generalized functional invariant map of the family $\mathcal{X}_1^2$ is given simply by projection onto γ. The fibres of this map are 1-parameter families of K3 surfaces with the same period, parametrized by $\beta \in \mathbb{C} - \{\frac{1}{4}\}$, which are therefore isotrivial. It is tempting to expect that these isotrivial families are in fact trivial, but this is not the case. Instead, we find:

Proposition 8.5.18. *Monodromy around the line $\beta = \frac{1}{4}$ fixes the Néron-Severi lattice of a generic fibre of $\mathcal{X}_1^2$ and acts on the transcendental lattice as multiplication by $-\mathrm{Id}$.*

Furthermore, the family $\hat{\mathcal{X}}_1^2$ obtained by pulling back $\mathcal{X}_1^2$ to the double cover of U_1^2 ramified over the line $\beta = \frac{1}{4}$ is isomorphic to a direct product $\mathcal{X}_1 \times \mathbb{C}^\times$, where $\mathcal{X}_1$ is an M_1-polarized family of K3 surfaces satisfying Definition 8.5.2.

Proof. The double cover of U_1^2 ramified over the line $\beta = \frac{1}{4}$ is given by the map $\mathbb{C}^\times \times (\mathbb{C} - \{0, -1\}) \to U_1^2$ taking $(\mu, \gamma) \to (\beta, \gamma) = (\mu^2 + \frac{1}{4}, \gamma)$. After a change of variables $x \mapsto x\mu$, the family $\hat{\mathcal{X}}_1^2$ may be written on the maximal torus $(\mathbb{C}^\times)^3$ as the vanishing locus of the rational polynomial

$$x^2 + y + z + \frac{\gamma}{3^3 yz} + 1 = 0. \tag{8.9}$$

This family does not depend upon μ, so $\hat{\mathcal{X}}_1^2$ is isomorphic to a direct product $\mathcal{X}_1 \times \mathbb{C}^\times$, for some family $\mathcal{X}_1 \to (\mathbb{C} - \{0, -1\})$ parametrized by γ, and its monodromy around $\mu = 0$ is trivial. Furthermore, for two K3 surfaces X_1 and X_2 in $\hat{\mathcal{X}}_1^2$ lying above a fiber X in $\mathcal{X}_1^2$ there are natural isomorphisms

$$\phi_1 : X_1 \to X, \qquad \phi_2 : X_2 \to X.$$

The automorphism $\phi_1^{-1} \cdot \phi_2$ is the non-symplectic involution given on the maximal torus by $(x, y, z) \mapsto (-x, y, z)$, which fixes the lattice $M_1 = \mathrm{NS}(X)$.

Therefore monodromy around $\beta = 1/4$ has order 2 and acts on T_X in the same way as a non-symplectic involution ι with fixed lattice $M_1 = \mathrm{NS}(X)$. Thus, $T_X = (\mathrm{NS}(X)^\iota)^\perp$ and so ι acts irreducibly on T_X with order 2. It must therefore act as $-\mathrm{Id}$.

It remains to prove that the 1-parameter family $\mathcal{X}_1 \to (\mathbb{C} - \{0, -1\})$ given on the maximal torus by varying γ in (8.9) satisfies Definition 8.5.2. We have already noted that the generalized functional invariant map $(\mathbb{C} - \{0, -1\}) \to \mathcal{M}_{M_1}$ defined by γ is injective. Furthermore, using the expressions for a, b and d calculated earlier we see that $\gamma = -1$ at the elliptic point of order 2, $\gamma = \infty$ at the elliptic point of order 3, and $\gamma = 0$ at the cusp. All that remains is to check that the monodromy of the family $\mathcal{X}_1 \to (\mathbb{C} - \{0, -1\})$ has the appropriate orders around each of these points.

This family $\mathcal{X}_1$ has been studied by Smith [139, Example 2.15], where it appears as family $\mathcal{D}$ in Table 2.2 (and we note that Smith's parameter μ is equal to $-\frac{1}{\gamma}$ in our notation). Its monodromy around the points $\gamma \in \{0, -1, \infty\}$ is given by the symmetric squares of the matrices calculated in [139, Example 3.9]; in particular we find that this monodromy has the required orders. $\square$

Remark 8.5.19. We note that the complicating factor in the M_1-polarized case is the fact that a generic M_1-polarized K3 surface X admits a non-symplectic involution which fixes $M_1 \subseteq \mathrm{NS}(X)$. It is this which prevents some of the torically induced fibrations of the threefolds in [52, Table 1] by M_1-polarized K3 surfaces from being expressible as pull-backs of an M_1-polarized family $\mathcal{X}_1$ from the moduli space $\mathcal{M}_{M_1}$. However, from Proposition 8.5.18, we find that we *can* express these fibrations as pull-backs of $\mathcal{X}_1$ if we proceed to a double cover of the base which kills this involution.

Given this result, it is easy to undo the Kummer construction for the family $\mathcal{Y}_1^2 \to U_1^2$ of Kummer surfaces associated to the family $\mathcal{X}_1^2$. First, pull back $\mathcal{Y}_1^2$

Mirror Threefold	α	β	γ
$\mathbb{WP}(1,1,1,1,2)[6]$	$\frac{A(t+u)^3}{tu^2}$	0	$-\frac{2^6 3^3 A(t+u)^3}{tu^2}$
$\mathbb{WP}(1,1,1,1,4)[8]$	$\frac{Au}{t}$	$\frac{t}{u}$	$\frac{2^6 3^3 Au^4}{t(4t-u)^3}$
$\mathbb{WP}(1,1,1,2,5)[10]$	$\frac{Au^2}{t^2}$	$\frac{t}{u}$	$\frac{2^6 3^3 Au^5}{t^2(4t-u)^3}$
$\mathbb{WP}(1,1,1,1,1,3)[2,6]^*$	$-\frac{Au^2}{t(t+u)}$	k	$-\frac{2^6 3^3 Au^2}{(4k-1)^3 t(t+u)}$
$\mathbb{WP}(1,1,1,2,2,3)[4,6]^*$	$\frac{Au^4}{t^2(t+u)^2}$	k	$\frac{2^6 3^3 Au^4}{(4k-1)^3 t^2(t+u)^2}$

Table 8.2 Values of α and β for threefolds admitting M_1-polarized fibrations.

to the double cover $(\mathbb{C} - \{0,-1\}) \times \mathbb{C}^\times \cong U_{M_1} \times \mathbb{C}^\times$ of U_1^2 ramified over the line $\beta = \frac{1}{4}$ (where U_{M_1} is defined as in Section 8.5.2). The result is the family of Kummer surfaces associated to the family $\hat{\mathcal{X}}_1^2 \cong \mathcal{X}_1 \times \mathbb{C}^\times$. This is exactly the family $\mathcal{Y}_1 \times \mathbb{C}^\times$, where $\mathcal{Y}_1 \to U_{M_1}$ is the family of Kummer surfaces associated to $\mathcal{X}_1$. The Kummer construction can then be undone for this family by pulling back to the cover $V = C_{M_1} \times \mathbb{C}^\times$ of $U_{M_1} \times \mathbb{C}^\times$, where the cover $C_{M_1} \to U_{M_1}$ is as calculated in Section 8.5.3.

Thus, given a family $\mathcal{X} \to U$ of M_1-polarized K3 surfaces that can be expressed as the pull-back of the family $\mathcal{X}_1^2$ by a map $U \to U_1^2$, we may undo the Kummer construction for the associated family of Kummer surfaces $\mathcal{Y} \to U$ by pulling back to the cover $V \times_{U_1^2} U$.

We conclude by applying this to the cases from [52, Table 1]. We find:

Theorem 8.5.20. *There exist K3 fibrations with M_1-polarized generic fibre on five of the threefolds in [52, Table 1], given by the mirrors of those listed in Table 8.2.*

Furthermore, if $\mathcal{X} \to \mathbb{P}^1_{t,u}$ denotes one of these fibrations and $U \subset \mathbb{P}^1_{t,u}$ is the open set over which the fibres of $\mathcal{X}$ are nonsingular, then the restriction $\mathcal{X}|_U \to U$ agrees with the pull-back of the family $\mathcal{X}_1^2$ by the map $U \to U_1^2$ defined by α and β in Table 8.2 (in this table (t,u) are coordinates on the base $U \subset \mathbb{P}^1_{t,u}$ of the fibration, A is the Calabi-Yau deformation parameter and $k \in \mathbb{C} - \{0, \frac{1}{4}\}$ is a constant). The family $\mathcal{X}|_U \to U$ is thus an M_1-polarized family of K3 surfaces.

Proof. This is proved in the same way as Theorem 8.5.10, by comparing the forms of the maximal tori in the threefolds from [52, Table 1] to the local form of the family $\mathcal{X}_1^2$ given by Equation (8.8). $\square$

Finally, we note that the generalized functional invariants in these cases are given by γ in Table 8.2. We see that, as in Section 8.5.4, they are all $(i+j)$-fold covers of $\mathcal{M}_{M_1} \cong \Gamma_0(1) \setminus \mathbb{H}$ (where (i,j) are as in Table 8.1) having exactly four ramification points: one of order $(i+j)$ over the cusp, two of orders i and j over the elliptic point of order 3, and one of order 2 which varies with the value of the Calabi-Yau deformation parameter A.

Remark 8.5.21. There is precisely one case from [52, Table 1] that has not been discussed: the mirror of the complete intersection $\mathbb{WP}(1,1,2,2,3,3)[6,6]$. However, it can be seen that this threefold does not admit any torically induced M-polarized K3 fibrations, and our methods have not yielded any that are not torically induced either.

8.6 Application to the arithmetic/thin dichotomy

Recall that each of the threefolds X from [52, Table 1] moves in a one parameter family over the thrice-punctured sphere $\mathbb{P}^1 - \{0,1,\infty\}$. Recently there has been a great deal of interest in studying the action of monodromy around the punctures on the third cohomology $H^3(X,\mathbb{Z})$. This monodromy action defines a Zariski dense subgroup of $\mathrm{Sp}(4,\mathbb{R})$, which may be either arithmetic or non-arithmetic (more commonly called *thin*). Singh and Venkataramana [138][137] have proved that the monodromy is arithmetic in seven of the fourteen cases from [52, Table 1], and Brav and Thomas [25] have proved that it is thin in the remaining seven. The arithmetic/thin status of each of the threefolds from Theorems 8.5.10 and 8.5.20 is given in the fifth column of Table 8.1.

It is an open problem to explain this behaviour geometrically. To this end, we are able to make an interesting observation concerning the arithmetic/thin dichotomy for the M_n-polarized families with Theorems 8.5.10 and 8.5.20. Specifically, from Table 8.1 we observe that a threefold admitting a torically induced fibration by M_n-polarized

K3 surfaces has thin monodromy if and only if neither of the values (i, j) associated to this fibration are equal to 2.

This observation may also be extended to the 14th case [30]. In this case, recall that the threefold Y_1, which moves in a one-parameter family realising the 14th case variation of Hodge structure, admits a torically induced fibration by M-polarized K3's rather than M_n-polarized K3's. Thus the generalized functional invariant map from Y_1 has image in the 2-dimensional moduli space of M-polarized K3 surfaces, rather than one of the modular curves $\mathcal{M}_{M_n}$. However, from [30, Section 5.1 and Equation (4.5)], we see that the image of the generalized functional invariant map from Y_1 is contained in the special curve in the M-polarized moduli space defined by the equation $\sigma = 1$ (where σ and π are the rational functions from Section 8.4.2).

By the results of [31, Section 3.1], the moduli space of M-polarized K3 surfaces may be identified with the Hilbert modular surface

$$(\mathrm{PSL}(2, \mathbb{Z}) \times \mathrm{PSL}(2, \mathbb{Z})) \rtimes \mathbb{Z}/2\mathbb{Z} \setminus \mathbb{H} \times \mathbb{H},$$

with natural coordinates given by σ and π. The $\sigma = 1$ locus is thus parametrized by π and has an orbifold structure induced from the Hilbert modular surface. This orbifold structure has an elliptic point of order six at $\pi = 0$, an elliptic point of order two at $\pi = \frac{1}{4}$, and a cusp at $\pi = \infty$.

The generalized functional invariant map for the K3 fibration on Y_1 is given by the rational function π, which is calculated explicitly in [30, Equation (4.4)]. It is a double cover of the $\sigma = 1$ locus ramified over the cusp and a second point that varies with the value of the Calabi-Yau deformation parameter. This agrees perfectly with the description of the generalized functional invariants for the M_n-polarized cases from Section 8.5.4, with $(i, j) = (1, 1)$, thereby giving the final row of Table 8.1. From this table, we observe:

Theorem 8.6.1. *Suppose that $\mathcal{X}$ is a family of Calabi-Yau threefolds from [52, Table 1] that admit a torically induced fibration by M_n-polarized K3 surfaces (resp. M-polarized K3 surfaces with $\sigma = 1$). By our previous discussion, the generalized* *functional invariant of this fibration is a* $(i+j)$*-fold cover of the modular curve* $\mathcal{M}_{M_n} \cong \Gamma_0(n)^+ \setminus \mathbb{H}$ *(resp. the orbifold curve given by the* $\sigma = 1$ *locus in the moduli space of* M*-polarized K3 surfaces), where* i *and* j *are given by Table 8.1, which is totally ramified over the cusp and ramified to orders* i *and* j *over the remaining orbifold point of order* $\neq 2$*. Then* $\mathcal{X}$ *has thin monodromy if and only if neither* i *nor* j *is equal to* 2*.*

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