The most beautiful thing we can experience is the mysterious. It is the source of all true art and science. He to whom this emotion is stranger, who can no longer pause to wonder and stand rapt in awe, is as good as dead: his eyes are closed.

Albert Einstein (1879-1955)

University of Alberta

Higher-Dimensional Gravitational Objects with External Fields

by

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To

My Parents Mansour and Rosa,
My Sisters Shiva and Shahrzad

Abstract

This thesis summarizes a study of higher-dimensional distorted objects such as a distorted 5-dimensional Schwarzschild-Tangherlini black hole. It considers a particular type of distortion corresponding to an external, static distribution of matter and fields around this object. The corresponding spacetime can be presented in the generalized Weyl form which has an $\mathbb{R} \times U(1) \times U(1)$ group of isometries. This is a natural generalization of the 4-dimensional Weyl form which was presented in the paper by Emparan and Reall [1]. In the frame of this generalized Weyl form one can derive an exact analytic solution to the Einstein equations which describes the non-linear interaction of the black hole with external matter and gravitational fields. This research focuses on the effects of such interaction on the event horizon and the interior of the black hole. A similar study was presented in the papers [2] for 4-dimensional neutral black holes, where special duality relations between a neutral black hole horizon and singularity were derived. In relation to this work it is interesting to study which properties of distorted black holes remain present in the 5-dimensional case. This thesis also gives an investigation of the d-dimensional Fisher solution which represents a static, spherically symmetric, asymptotically flat spacetime with a massless scalar field. This solution has a naked singularity. It is shown that the d-dimensional Schwarzschild-Tangherlini solution and the Fisher solution are dual to each other.

[1] R. Emparan and H. S. Reall, Phys. Rev. D, 65, 084025 (2002).

[2] V. P. Frolov and A. A. Shoom, Phys. Rev. D, 76, 064037 (2007).

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List of Symbols and Abbreviations

Here, we present some symbols and notations, and give definitions which are used in the thesis. Other notations and definitions are explained in the text.

Symbols and sign conventions

The following special symbols are used:

$\simeq$	approximately equal to
$\sim$	order of magnitude estimate
$\approx$	asymptotically approximate to
$\equiv$	defined to be equal to
$i = \sqrt{-1}$	imaginary unit (if not a subscripts or superscript)
$\wedge$	wedge product
$\boldsymbol{v} \cdot \boldsymbol{w}$	scalar product of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$
$T^{\alpha\beta} = \mathrm{diag}(T^{00}, ...)$	represents a geometrical object $T^{\alpha\beta}$,
	whose off-diagonal components are zeros
$\Gamma(x)$	the Gamma function
$\delta(x)$	the Dirac delta function

(1)

Units

The fundamental constants are [1]:

the speed of light in vacuum:	$c = 2.99792458 \times 10^8 \ \mathrm{m\,s^{-1}}$
the 4-dimensional gravitational constant:	$G_{(4)} = 6.67428(67) \times 10^{-11} \ \mathrm{kg^{-1}\,m^3\,s^{-2}}$
the Planck constant:	$h = 6.62606896(33) \times 10^{-34} \ \mathrm{kg\,m^2\,s^{-1}}$
the reduced Planck constant:	$\hbar = \dfrac{h}{2\pi} = 1.054571628(53) \times 10^{-34} \ \mathrm{kg\,m^2}\, s^{-1}$
the electron charge magnitude:	$e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$
	$= 4.80320427(12) \times 10^{-10}\,\mathrm{esu}$

The Planck mass is

$$M_{pl} = \sqrt{\frac{\hbar c}{G}} = 2.17644(11) \times 10^{-8}\,\mathrm{kg}.$$

The Planck energy is

$$E_{pl} = M_{pl}c^2 = \sqrt{\frac{\hbar c^5}{G}} \simeq 1.956086{\times}10^9 \ \mathrm{J} \simeq 1.22089{\times}10^{28} \ \mathrm{eV} = 1.22089{\times}10^{16} \ \mathrm{TeV}.$$

The Planck length (distance) is

$$l_{Pl} = \sqrt{\hbar G/c^3} = 1.616253 \times 10^{-35} \ \mathrm{m}.$$

The Planck time is

$$\tau_{Pl} = \sqrt{\hbar G/c^5} = 5.391241 \times 10^{-44} \ \mathrm{s}.$$

In natural units: $\hbar = c = 1$:

$$\begin{aligned} E_{pl} &= M_{pl} = G^{-\frac{1}{2}} \simeq 1.22089 \times 10^{16} \ \mathrm{TeV} \\ l_{pl} &= G^{\frac{1}{2}} = 1.616253 \times 10^{-35} \ m \simeq 8.190 \times 10^{-17} \ (\mathrm{TeV})^{-1} \\ (1 \ \mathrm{TeV})^{-1} &\simeq 1.973 \times 10^{-19} \ m = 1.973 \times 10^{-17} \ \mathrm{cm} \end{aligned}$$

Basic definitions

For details on the mathematical preliminaries refer to Chapter 2.

The main conventions for geometrical objects are in terminology of [20]. The signature of the spacetime metric $g_{\alpha\beta}$ is: $(- + + \cdots +)$, i.e., it is $+(d-2)$. The time coordinate index is 0: $t \equiv x^0$. Balance of indices and the Einstein summation convention are assumed.

Components of geometrical objects are defined with respect to a coordinate basis, e.g., $T_{\alpha_1\alpha_2\cdots}{}^{\beta_1\beta_2\cdots}$. In a local orthonormal frame these components are $T_{\hat{\alpha}_1\hat{\alpha}_2\cdots}{}^{\hat{\beta}_1\hat{\beta}_2\cdots}$. The partial derivative with respect to coordinate x^α is defined by comma in front of subscript α as follows:

$$f_{,\alpha} \equiv \frac{\partial f}{\partial x^\alpha}\,, \;\; f_{,\alpha\beta} \equiv \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta}\,, \;\; \text{etc.} \tag{2}$$

The covariant derivative of a tensor $T_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_m}$ is defined by the symbol nabla,

$$\begin{aligned} \nabla_\gamma T_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_m} &= T_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_m}{}_{,\gamma} + \Gamma^{\beta_1}{}_{\delta\gamma} T_{\alpha_1\cdots\alpha_n}{}^{\delta\cdots\beta_m} + \cdots + \Gamma^{\beta_m}{}_{\delta\gamma} T_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\delta} \\ &- \Gamma^{\delta}{}_{\alpha_1\gamma} T_{\delta\cdots\alpha_n}{}^{\beta_1\cdots\beta_m} - \cdots - \Gamma^{\delta}{}_{\alpha_n\gamma} T_{\alpha_1\cdots\delta}{}^{\beta_1\cdots\beta_m}\,. \end{aligned} \tag{3}$$

The Riemann tensor is given by

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta} - \Gamma^\alpha{}_{\sigma\delta}\Gamma^\sigma{}_{\beta\gamma}\,.$$

where $\Gamma^\alpha{}_{\beta\gamma}$ are the Christoffel symbols given by

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(g_{\delta\beta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta})\,.$$

The Ricci tensor is defined by $R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}$.

The Ricci scalar is $R = g^{\alpha\beta}R_{\alpha\beta} = R^\alpha{}_\alpha\,.$

The Einstein equations are ($G_{(d)} = 1$)

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi\, T_{\alpha\beta}\,,$$

where $T_{\alpha\beta}$ is the the energy-momentum tensor defined by

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \left(\frac{\delta(\sqrt{-g}\Lambda_m)}{\delta g^{\alpha\beta}} - \left[\frac{\delta(\sqrt{-g}\Lambda_m)}{\delta (g^{\alpha\beta}{}_{,\gamma})} \right]_{,\gamma} \right).$$

Here, $g = det(g_{\alpha\beta})$, and Λ_m is the Lagrangian density of matter.

The Weyl tensor is given by

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{2}{d-2}\left(g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha}\right) + \frac{2}{(d-1)(d-2)}Rg_{\alpha[\gamma}g_{\delta]\beta}\,,$$

where square brackets [] denote anti-symmetrization, and d is the number of dimensions. The Kretschmann scalar is

$$\mathcal{K} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + \frac{4}{d-2}R_{\alpha\beta}R^{\alpha\beta} - \frac{2}{(d-1)(d-2)}R^2\,.$$

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Chapter 1

Introduction

A star dies when the nuclear fuel to provide enough pressure to hold it against its own gravitational force is used up. Such a star undergoes gravitational collapse and possibly a supernova explosion which leaves behind a compact object: a white dwarf, a neutron star, or a black hole. In 1931 Subrahmanyan Chandrasekhar (1910-1995) has found that no white dwarfs can be heavier than $\sim 1.4M_{\odot}$ [1]. This limit is the maximum non-rotating mass which can be sustained by the electron degeneracy pressure against the gravitational collapse. Tolman, Oppenhimer, and Volkoff then calculated the maximum mass limit for a neutron star to be $\sim 0.7M_{\odot}$ [2], [3]. Later the limit was recalculated to be $\sim (1.5 - 3.0)M_{\odot}$ [4] (the uncertainty in the value comes from the fact that the equation of state for the neutron stars is not well understood).

For the stars above this mass limit, gravity overcomes all the other forces, and the gravitational collapse continues with nothing to stop it. Once the star passes through its Schwarzschild radius, it continues to collapse inevitably to zero circumference, creating there a singularity. The resulting object is a black hole. The black hole itself is the region inside the Schwarzschild radius, i.e., the region between the black hole event horizon and the singularity.

Black holes can be well explained by the Einstein theory of general relativity (at least up to the vicinity of the singularity). For example, the static, vacuum, spherically symmetric, asymptotically flat 4-dimensional Schwarzschild black hole is the simplest nontrivial solution of the vacuum Einstein equations.

We say that a space-time has a singularity if it is timelike and null geodesically incomplete [5]. The Hawking-Penrose singularity theorem [5] states that in a space-time where the strong energy condition is satisfied and there are no closed timelike curves, a singularity forms inevitably if a closed trapped surface occurs.

Namely, the laws of physics assure that there exist a singularity inside any black hole resulting from the gravitational collapse of a star. In 1960's Penrose has tried to prove mathematically that all the singularities formed by a reasonable matter and fields are hidden behind a horizon. Failing to prove it in general, he has proposed the cosmic censorship conjecture, stating that whenever a singularity forms, it must be hidden behind a horizon [6], [7]. Namely, no naked singularities can form. The black hole horizon hides the singularity from an external observer. No observer can ever come back or transmit his observations to the external observer.

To test the cosmic censorship conjecture, many models of gravitational collapse were studied both analytically and numerically. It was found that in certain conditions naked singularities may form [8], [9].

Back in the 1950s and early 1960s, John Wheeler had a hope that singularities can provide a laboratory to see the laws of quantum gravity at work with real physical observations and experiments. Cosmic censorship has destroyed this hope. However, even if naked singularities do not exist in nature, they are still mathematical laboratories both for the classical general relativity and the future theory of quantum gravity. Namely, studying a space-time singularity, whether it is hidden behind a black hole horizon or naked, provides us with useful information about how a space-time behaves in its extreme.

In this thesis we consider two different higher dimensional gravitational objects, i.e., black holes and their "siblings", naked singularities. In particular, in the third Chapter we consider a 5-dimensional static, $U(1)\times U(1)$ symmetric solution of the vacuum Einstein equations representing a 5-dimensional Schwarzschild-Tangherlini black hole distorted by external matter. In Chapter 4 we present a 5-dimensional static, $U(1) \times U(1)$ symmetric solution of the Einstein-Maxwell equations representing a 5-dimensional Reissner-Nordström black hole distorted by external distribution of matter. In Chapter 5 we consider the d-dimensional Fisher solution which represents a static, spherically symmetric, asymptotically flat space-time with a massless scalar field. This solution has a naked singularity which divides the manifold into two disconnected parts. The part which is asymptotically flat we call the *Fisher space-time*, and the other part we call the *Fisher universe*. We show that the d-dimensional Schwarzschild-Tangherlini black hole and the d-dimensional naked singularity of the Fisher space-time are dual to each other.

Black holes interact nonlinearly with the external matter and fields. Except in some special cases, where the construction of an exact analytic solution to Einstein equations is possible, the description of the black hole interaction with external matter and fields usually demands numerical computations. In 4-dimensions the

Weyl form [10] solution of the vacuum Einstein equations representing a black hole distorted by external static axisymmetric distribution of matter has been constructed in [11]. In Chapter 3 we use the higher-dimensional analogue of the Weyl solution [12] to construct a $U(1) \times U(1)$ symmetric solution of the vacuum Einstein equations representing a black hole distorted by external matter. We compare the properties of such a black hole to the 4-dimensional distorted black hole. In Chapter 4 we use the analogue of Harrison-Ernst (see [13] and [14]) transformations [15] to produce the distorted 5-dimensional electrically charged black hole. In Chapter 5 we consider a space-time geometry and massless scalar field that have $\mathbb{R} \times SO(d-1)$ symmetry. The results and the derived relation between the geometric quantities of the Fisher space-time, Fisher universe, and Schwarzschild-Tangherlini black hole may suggest that the massless scalar field "transforms" the black hole event horizon into the naked radially weak disjoint singularities of the Fisher space-time and Fisher universe which are "dual to the horizon".

Chapter 2 include background material which helps for going through this thesis. Readers familiar with the materials presented in this Chapter may go directly to Chapter 3. Details of our calculations are presented in the appendices.

The results presented in this thesis were obtained during the course of the author's Ph.D. program at the university of Alberta between the years 2006 to 2010. Chapter 4 represents the 5-dimensional generalization of the 4-dimensional static, axisymmetric distorted charged black hole presented in [16], [17], and [18]. The interior structure of this black hole was analyzed by the author, A. A. Shoom, and V. P. Frolov during the course of the author's PhD. program and was published in a peer reviewed journal [19]. Chapter 5 is based on a published paper in a peer reviewed journal [20].

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Chapter 2

Background Material

In this Chapter we present the background material which is needed for a better understanding of this thesis. This Chapter consist of three sections. In the first Section we present some of the mathematical preliminaries, in the second Section we give a brief review of generalized Weyl solutions, and in Section 3 we discuss the distorted 4-dimensional Schwarzschild black hole. Readers familiar with these materials may skip this Chapter and go directly to Chapter three.

2.1 Mathematical preliminaries

For more details on the subject see ([1]-[5]).

2.1.1 Manifold

A d-dimensional C^∞, real manifold $\mathcal{M}^d$ is a set of points together with a collection of subsets $\{O_\alpha\}$ satisfying the following properties:
1) Each point $p \in \mathcal{M}^d$ lies in at least one O_α, that is $\cup_\alpha O_\alpha$, covers $\mathcal{M}^d$.
2) For each α there is a one-to-one, onto, map $\psi_\alpha : O_\alpha \to U_\alpha$, where U_α is an open subset of $\mathbb{R}^d$.
3) If any two sets O_α and O_β overlap, $O_\alpha \cap O_\beta \neq \Phi$, then $\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(O_\alpha \cap O_\beta) \to \psi_\alpha(O_\alpha \cap O_\beta)$ is a C^∞ map of an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^d$.

2.1.2 Curve

A C^k curve $\lambda(s)$ in $\mathcal{M}^d$ is a C^k map of an interval (a, b) of a real line $\mathbb{R}^1$ into $\mathcal{M}^n$, i.e., $\lambda(s) : (a, b) \to \mathcal{M}^d$. On a chart (O, ψ) the curve λ has a coordinate representation:

$$x = \psi \circ \lambda : \mathbb{R}^1 \to \mathbb{R}^d, \quad x \in \mathbb{R}^d. \tag{2.1}$$

2.1.3 Function

A function f on $\mathcal{M}^d$ is a smooth map from $\mathcal{M}^d$ to $\mathbb{R}^1$. The coordinate representation of f is given by

$$f \circ \psi^{-1} : \mathbb{R}^d \to \mathbb{R}^1. \tag{2.2}$$

2.1.4 Tangent vector

Let f be a function of C^k functions from $\mathcal{M}^d$ into $\mathbb{R}^1$. We define a tangent vector (contravariant vector) as a map $\mathbf{V} : f \to \mathbb{R}$ which is linear and obeys Leibnitz law.

$$\mathbf{V}[af + bg] = a\mathbf{V}[f] + b\mathbf{V}[g], \tag{2.3}$$

$$\mathbf{V}(fg) = f(p)\mathbf{V}[g] + g(p)\mathbf{V}[f], \tag{2.4}$$

where $\mathbf{V}$ and $\mathbf{g}$ are vectors; f and g are functions; a and $b \in \mathbb{R}^1$. The vector $(\partial/\partial s)_\lambda|_{s_0}$ tangent to the C^1 curve $\lambda(s)$ at the point $p \equiv \lambda(s_0)$ is an operator which maps a C^1 function f, defined at $\lambda(s_0)$ into a real number $(\partial f/\partial s)_\lambda|_{s_0}$. If $(x^1, ..., x^d)$ are local coordinates in a neighborhood of p,

$$\left(\frac{\partial f}{\partial s}\right)_\lambda |_{s_0} = \sum_{j=1}^{d} \frac{dx^j(\lambda(s))}{ds}|_{s=s_0} \frac{\partial f}{\partial x^j}|_{\lambda(s_0)} = \frac{dx^j(\lambda(s))}{ds} \frac{\partial f}{\partial x^j}|_{\lambda(s_0)}. \tag{2.5}$$

Here and in what follows, we adopt a summation convention whereby a repeated index implies summation over all the values of that index. Thus, every tangent vector at a point p can be expressed as a linear combination of the coordinate derivatives defined at p

$$\mathbf{X}|_p := X^i \frac{\partial}{\partial x^i}|_p. \tag{2.6}$$

In other words df/ds is obtained by applying the differential operator X to f, i.e.,

$$\frac{df}{dx} = \mathbf{X}[f] = X^i \frac{\partial}{\partial x^i}. \tag{2.7}$$

The space of all tangent to $\mathcal{M}^d$ vectors at p denoted by $\mathcal{T}_p(\mathcal{M}^d)$, or simply $\mathcal{T}_p$ is a d-dimensional vector space.

If $\{\mathbf{E}_i\}$ $(i = 1...d)$ are any set of d vectors at p which are linearly independent, then any other contravariant vector can be written as $\mathbf{X} = X^i\mathbf{E}_i$. Here, the numbers $\{X^i\}$ are components of the vector X with respect to the basis $\{\mathbf{E}_i\}$. Here and in what follows whenever we use the word vector we mean a contravariant vector. One can choose $\{\mathbf{E}_i\}$ as the coordinate basis $\{\frac{\partial}{\partial x^i}\}$.

2.1.5 One-form

A dual space to the tangent space $\mathcal{T}_p(\mathcal{M}^d)$ is called the cotangent space at p which is denoted by $\mathcal{T}_p^*(\mathcal{M}^d)$ or simply $\mathcal{T}_p^*$. An element $\mathbf{\Omega} : \mathcal{T}_p \to \mathbb{R}^1$ of $\mathcal{T}_p^*$ is called a cotangent vector (covariant vector) or a one-form.

If $\mathbf{X}$ is a vector at p, the number into which $\mathbf{\Omega}$ maps $\mathbf{X}$ will be written as $< \mathbf{\Omega}, \mathbf{X} >$; linearity implies

$$< \mathbf{\Omega}, a\mathbf{X} + b\mathbf{Y} >= a < \mathbf{\Omega}, \mathbf{X} > +b < \mathbf{\Omega}, \mathbf{Y} >, \tag{2.8}$$

$$< \mathbf{\Omega} + \mathbf{\Lambda}, \mathbf{X} >=< \mathbf{\Omega}, \mathbf{X} > + < \mathbf{\Lambda}, \mathbf{X} > \quad , \tag{2.9}$$

$$< a\mathbf{\Omega}, X >= a < \mathbf{\Omega}, X > \quad , \tag{2.10}$$

where $\mathbf{X}$, $\mathbf{Y} \in \mathcal{T}_p$; $\mathbf{\Omega}$, $\mathbf{\Lambda} \in \mathcal{T}_p^*$, and a, $b \in \mathbb{R}^1$. Given a vector basis $\{\mathbf{E}_i\}$ at p, one can define a unique set of n one-forms $\{\mathbf{E}^i\}$ by the condition: $\mathbf{E}^i$ maps any vector $\mathbf{X}$ to the number X^i. In particular $< \mathbf{E}^i, \mathbf{E}_j >= \delta^i{}_j$. One can regard $\{\mathbf{E}^i\}$ as the basis of one forms. Any one form $\mathbf{\Omega}$ at p can be expressed as $\mathbf{\Omega} = \Omega_i \mathbf{E}^i$ where the number Ω_i are the components of a covariant vector defined by $\Omega_i :=< \mathbf{\Omega}, \mathbf{E}_i >$.

2.1.6 Tensors

From the space $\mathcal{T}_p$ of vectors at p and the space $\mathcal{T}_p^*$ of one-forms at p, we can form the Cartesian product. A tensor of type (q, r) is a multilinear map

$$\Pi_q^r = \mathcal{T}_{p_1}^* \times ...\mathcal{T}_{p_q}^* \times \mathcal{T}_{p_1} \times ...\mathcal{T}_{p_r} \to \mathbb{R}^1, \tag{2.11}$$

i.e. a map from the ordered set of one-forms and vectors $(\mathbf{\Omega}^1, ..., \ \mathbf{\Omega}^q, \ \mathbf{X}^1 ... \mathbf{X}^r)$, where $\mathbf{\Omega}$s and $\mathbf{X}s$ are arbitrary one-forms and vectors to the real numbers. A tensor of type $(q, \ r)$ at p is a function on Π_q^r which is linear in each argument. That is a tensor of type $(q, \ r)$ at p maps q elements of $\mathcal{T}_p^*$ and r elements of $\mathcal{T}_p$'s to a real number. The space of all such tensors is called the tensor product

$$T_r^q(p) = \mathcal{T}_{p_1} \otimes ... \otimes \mathcal{T}_{p_q} \otimes \mathcal{T}_{p_1}^* \otimes ... \otimes \mathcal{T}_{p_r}^*. \tag{2.12}$$

If $\{\mathbf{E}_i\}$ and $\{\mathbf{E}^i\}$ are dual basis of $\mathcal{T}_p$ and $\mathcal{T}_p^*$ respectively, Then

$$\{\mathbf{E}_{i_1} \otimes ... \otimes \mathbf{E}_{i_q} \otimes \mathbf{E}_{j_1} \otimes ...\mathbf{E}_{j_r}\}, \quad (i_a, \ j_b = 1, .., \ d) \tag{2.13}$$

will be a basis for $T_r^q(p)$. An arbitrary tensor $\mathbf{T} \in T_r^q(p)$ can be expressed in terms of this basis as

$$\mathbf{T} = T^{i_1 ... i_q}{}_{j_1 ... j_r} E_{i_1} \otimes ...E_{i_q} \otimes E^{j_1} ... \otimes E^{j_r}, \tag{2.14}$$

where $\{T^{i_1...i_q}{}_{j_1...j_r}\}$ are the components of $\mathbf{T}$ with respect to the dual bases $\{\mathbf{E}_i\}$, $\{\mathbf{E}^i\}$ and are given by

$$T^{i_1...i_q}{}_{j_1...j_r} = \mathbf{T}(\mathbf{E}^{i_1}, ...\mathbf{E}_{i_q},\ \mathbf{E}_{i_1}...\mathbf{E}_{i_r}). \tag{2.15}$$

2.2 Curvature

Newton's gravitational law states that gravity is a force which acts between all massive objects in the universe. The gravitational attraction between two massive objects is proportional to the product of the object masses and inversely proportional to the squared distance between their gravitational centers. Newton's gravitational law combined with the dynamical laws of motion has successfully explained motion of falling rocks, the Moon, and the planets until the 1900s. In the early 1900s, there were known two puzzling discrepancies between theoretical predictions of Newton's gravitational law and astronomical observations; namely, the peculiarities in Mercury and Moon orbits. The peculiarity in the Moon orbit later turned out to be the misinterpretation of astronomical measurements. On the other hand, the peculiarity in Mercury's orbit (perihelion precession of its orbit) was later successfully explained by Einstein theory of general relativity, which replaced the Newtonian gravitational theory. Here, we give a brief history of Einstein's discovery of the general relativity. This would provide us with the concept of a curved space-time manifold.

Einstein's concept of the special theory of relativity can't be used in non-inertial frames such as those necessary to describe gravity. In 1907 Einstein began searching for new generalized principle of relativity, or in other words he was searching for a new law of gravity. Einstein was convinced that Newton's gravitational law would violate the yet unformulated generalized principle of relativity. One day, in November 1907, while thinking of incorporating gravity into relativity Einstein came to a new idea:*"At the moment there came to me the happiest thought of my life... for an observer falling freely from the roof of a house no gravitational field exists during his fall..."* [6].

Imagine yourself in an elevator. If the spring attached to the elevator gets loose then in this perhaps last experiment of your life before you hit the ground, you shall observe the truth of the Einstein thought. All the objects in the elevator fall with you, that is, all the objects in the elevator remain at rest with respect to you (assuming they have been at rest before the spring gets loose and no other force is exerted on them). For another experiment, while standing in a room, throw a ball into the air at arbitrary angle defined between the horizontal and the vertical.

The path of the ball will be curved. It is the familiar projectile motion. Einstein states that the floor of the room forces us away from our natural motion of free fall. If the room were falling from a cliff, the motion of the ball would look straight. Therefore, in a freely falling frame, gravity is completely eliminated, at least locally. Therefore, we can define such a frame as a locally inertial frame. The world "local" is really essential here. There is *almost* no evidence of gravity in a small freely falling room. That is, we get rid of gravity while we are in the state of free fall.

Did we really get rid of gravity? Well, this is correct up to a precision of our measurement. Increase the precision of the measurement until you begin to notice the relative motion of test particles located in the room. Our free float frame can not be too large, or fall too long without some relative motion of the particles being detected. A freely falling frame is defined locally. Consider, for example, a spaceship orbiting the Earth. Look at two nearby test particles in this spaceship. Increase the precision of your measurement, and you will notice the relative motion of the test particles. In Newton's way of thinking, the gravitational pull is different on the above two test particles, therefore, causing their relative motion. These relative forces are called tidal forces. As Kip Thorne writes: "*The tidal forces felt in large frames seemed to Einstein, in 1911, to be a key to the ultimate nature of gravity... It was clear how Newton's gravitational law explains tidal forces. They are produced by a difference in the strength and direction of gravity's pull, from one place to another...Einstein's challenge was to formulate a completely new gravitational law that is simultaneously compatible with the principle of relativity and explains tidal gravity in some new, simple, compelling way. From mid-1911 to mid-1912, Einstein tried tidal gravity by assuming that time is warped, but space is flat.* [7].

By the summer of 1912 Einstein was convinced that tidal gravity is a manifestation of space-time curvature. Therefore, he decided to "incorporate" *curvature* into Minkowski's space-time. In this section we present the basic idea and mathematical formulation of a space-time curvature.

2.2.1 Parallel displacement

In what follows we use Minkowski coordinates in a flat space-time.

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2. \tag{2.16}$$

We use curvilinear coordinates in a curved space-time. We know how to parallel transport a vector in a flat manifold. We need to define parallel transport of a vector in a curved manifold. We can define parallel transport using embedding of a curved manifold into a higher dimensional Euclidean/Minkowski hyperspace. For example, starting from any vector which belongs to the tangent bundle of the curved manifold, we can parallel transport it in a standard way in the flat hyperspace. Once this is done, the vector does not belong to the tangent bundle of the curved manifold anymore. Then, we need to project the vector onto the manifold, i.e. to consider only the components which belong to the manifold. Consider two infinitesimally close points $p(x^j)$ and $q(x^j + \delta x^j)$ on a manifold. Consider the vector $A^i(x^i)$ at point p. Follow the above procedure to parallel transport this vector into the point q. The error in the actual value of the vector A^i at point q due to neglecting of the orthogonal component, and considering only the component which lies on the manifold is of the second order for infinitesimally separated points p and q. In this section only this projected part is what we mean when we discuss parallel transport of a vector.

Under a parallel translation, *parallel displacement*, of a vector its components in Minkowski coordinates do not change. On the other hand, when we use curvilinear coordinates the components of a vector change under a translation. We will show that a notion of the space-time curvature can be defined in terms of failure of a vector to return to its original value when it gets parallel transported around an infinitesimal closed curve. In general in a curved space-time the parallel displacement of a vector from one point to another depends on the path taken. We will show that space-time curvature is defined in terms of amount of the total change of a vector after parallel displacement around any infinitesimal closed curve.

Consider an arbitrary vector, whose components at the point $p(x^j)$ are $A^i(x^j)$. Let us parallel displace this vector to an infinitesimally close point $q(x^j + dx^j)$. Denote a change in components of the vector which results from this parallel displacement by δA^i. The change in the components depend on the value of components A^i. The sum of two vectors must transform according to the same law of each constituent. It follows that this dependence is linear. Therefore, δA^i has the form

$$\delta A^i = -\Gamma^i{}_{kl} A^k dx^l. \tag{2.17}$$

The quantities $\Gamma^i{}_{kl}$ are called the Christoffel symbols of the second kind, and they are certain functions of coordinates. Note that the Christoffel symbols do not form a tensor. In a Minkowski coordinate system $\Gamma^i{}_{kl} = 0$. To find a change in components of a covariant vector under a parallel displacement, consider the scalar product $A_i B^i$. Scalar quantities do not change under a parallel displacement, i.e., $\delta(A_i B^i) = 0$. Therefore,

$$\delta A_i \; B^i = -A_i \; \delta B^i = \Gamma^i{}_{kl} A_i B^k dx^l \; , \tag{2.18}$$

for arbitrary non-zero A^i. Thus, the change in components of any covariant vector A_i under a parallel displacement is given by

$$\delta A_i = \Gamma^k{}_{il} A_k dx^l. \tag{2.19}$$

2.2.2 Covariant differentiation and Christoffel symbols

The next step is to write the Christoffel symbols and the Riemann tensor explicitly, in terms of coordinates. This is achieved by introducing covariant differentiation. In flat space-time, the differentials dA_i of vector components A_i form one-forms, and the derivatives $\partial A_i/\partial x^k$ of the components of a vector with respect to the coordinates form a tensor. In curved space-time this is not true. This is due to the fact that dA_i is the difference between two vectors located at two infinitesimally separated points of manifold. In order to obtain a one form DA^i which behaves like a differential, we need that two vectors to be subtracted from each other be located at the same point in space. In other words, before subtracting the two vectors from each other, we need to parallel transport one of the vectors to the point where the second one is located. The difference DA^i between the two vectors now read

$$DA^i = dA^i - \delta A^i, \tag{2.20}$$

where dA^i is the difference of the components of two infinitesimally separated vectors located at points $p(x^i)$ and $q(x^i + dx^i)$,

$$dA^i = \frac{\partial A^i}{\partial x^l} dx^l, \tag{2.21}$$

and δA^i is the change in the component of the vector due to parallel displacement of the vector. Substituting (2.17) and (2.21) in (2.20) we derive

$$DA^i = \left(\frac{\partial A^i}{\partial x^l} + \Gamma^i{}_{kl} A^k \right) dx^l. \tag{2.22}$$

Similarly, for a covariant vector we have

$$DA_i = dA_i - \delta A_i, \tag{2.23}$$

where

$$dA_i = \frac{\partial A_i}{\partial x^l} dx^l, \tag{2.24}$$

and δA_i is given by (2.19). Therefore, for a covariant vector

$$DA_i = \left(\frac{\partial A_i}{\partial x^l} - \Gamma^k{}_{il} A_k \right) dx^l. \tag{2.25}$$

The expressions in the parentheses in (2.22) and (2.25) are tensor objects, called the *covariant derivatives* of the vector A^i and covariant vector A_i. Respectively, we denote them by $A^i{}_{;l}$ and $A_{i;l}$.The covariant derivative is a tensor which in curved space-time plays the same role as $\partial A_i/\partial x^k$ in flat space-time. Thus

$$\begin{aligned} &DA^i = A^i{}_{;j} dx^j, \qquad DA_i = A_{i;j} dx^j, \\ &DA_i = g_{ij} DA^j. \end{aligned} \tag{2.26}$$

The covariant derivative of an arbitrary tensor $T_{ij..}{}^{kl..}{}_{;m}$ is given by

$$T_{ij...}{}^{kl...}{}_{;m} = \frac{\partial T_{ij..}{}^{kl..}}{\partial x^m} - \Gamma^n{}_{im} T_{nj..}{}^{kl..} - \Gamma^n{}_{jm} T_{in..}{}^{kl..} - ... + \Gamma^k{}_{nm} T_{ij..}{}^{nl..} + \Gamma^l{}_{nm} T_{ij..}{}^{kn..} + ... \ . \tag{2.27}$$

Note that according to the definition of a tensor,

$$DA_i = D(g_{ik} A^k) = g_{ik} DA^k. \tag{2.28}$$

Therefore, it follows that the covariant derivative of the metric tensor is zero,

$$Dg_{ik} = 0, \Leftrightarrow \ \frac{\partial g_{ik}}{\partial x^l} - g_{mk} \Gamma^m_{\ il} - g_{lm} \Gamma^m_{\ kl} = 0. \tag{2.29}$$

From the relation above one can define the Christoffel symbols of the second kind in terms of the metric tensor g_{ik}. Performing the cyclic permutation of the indices in (2.29) and subtracting the corresponding expressions we derive

$$\Gamma^i{}_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right). \tag{2.30}$$

Clearly, the Christoffel symbols of the second kind are symmetric with respect to the covariant indices, i.e., $\Gamma^i{}_{kl} = \Gamma^i{}_{lk}$. The Christoffel symbols of the first kind are defined as follows:

$$\Gamma_{ikl} := g_{ij} \Gamma^j{}_{kl}. \tag{2.31}$$

Geodesics

Consider a curve defined by the parametric equation $x^i = x^i(s)$. The vector $u^i = dx^i/ds$ is a unit vector tangent to the curve. A geodesic is a curve such that its tangent vector is parallel displaced along itself, i.e.

$$Du^i = 0. \tag{2.32}$$

This equation is a generalization of a motion of a free particle in the special theory of relativity, $du^i/ds = 0$, or $du^i = 0$ where u^i is the four-dimensional velocity. The parameterization which yields (2.32) is called the affine parameterization. Substitute from the expression (2.22), and divide (2.32) the expression by ds; the result is the geodesic equation

$$\frac{d^2x^i}{ds^2} + \Gamma^i{}_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{2.33}$$

2.2.3 Curvature tensor

We will show that space-time curvature is defined in terms of the amount of the total change of a vector after parallel displacement around any infinitesimal closed contour.

The change ΔA_i in a covariant vector A_i after parallel transport around any arbitrary infinitesimal closed contour C can be written in the form

$$\Delta A_k = \oint_C \Gamma^i{}_{kl} A_i dx^l. \tag{2.34}$$

Let us apply the Stokes theorem to the integral above. Note that the values of A_i for points inside the contour are not unique. Namely, they depend to the path taken inside the contour. This non uniqueness is related to the terms of the second order in dx. For the first order accuracy the values of the vector at the points inside the infinitesimal contour are uniquely determined by their values on the contour, that is by the following derivatives (see (2.19))

$$\frac{\partial A_i}{\partial x^l} = \Gamma^m_{il} A_m. \tag{2.35}$$

Thus, Stokes theorem gives

$$\Delta A_k = \frac{1}{2}\left[\frac{\partial(\Gamma^i{}_{kn}A_i)}{\partial x^l} - \frac{\partial(\Gamma^i{}_{kl}A_i)}{\partial x^n}\right]\Delta S^{ln}, \tag{2.36}$$

where ΔS^{ln} is an infinitesimal area enclosed by the contour C. Using (2.35) we derive

$$\Delta A_k = \frac{1}{2} R^i{}_{kln} A_i \Delta S^{ln}, \tag{2.37}$$

where we defined

$$R^i{}_{kln} := \frac{\partial \Gamma^i{}_{kn}}{\partial x^l} - \frac{\partial \Gamma^i{}_{kl}}{\partial x^n} + \Gamma^i{}_{ml}\Gamma^m{}_{kn} - \Gamma^i{}_{mn}\Gamma^m{}_{kl}, \tag{2.38}$$

a fourth rank tensor called the *Riemann tensor* or the *curvature tensor*. Therefore, the non-commutivity of the covariant derivatives implies that the space-time is curved. Here, we have assumed that our manifold is torsion free, i.e.

$$R^i_{kln} A_i := A_{k;ln} - A_{k;nl}, \tag{2.39}$$

In flat space-time covariant derivatives commute with one another. One can show that (2.39) agrees with (2.18), (2.36), and (2.37).

Geodesic deviation equation

We have started our discussion with the relative acceleration of particles in the gravitational field. We have stated that curvature in general relativity plays a similar role to that of the tidal forces in Newtonian gravity. Our final task is to show that indeed, we can relate the tendency of geodesics to accelerate towards or away from each other to the curvature of the manifold.

On a flat surface, such as a sheet of paper, two initially parallel straight lines never cross each other. On a curved surface, such as the surface of a ball, two initially parallel straight lines may cross.

Consider a smooth one parameter family of geodesics $\lambda_t(s)$, parameterized by the affine parameter s, that is for each $t \in R$, the curve λ_t is a geodesic. Let Σ denote a two dimensional submanifold spanned by the geodesic curves $\lambda_t(s)$. Therefore, we may choose s and t as coordinates of Σ. That is, the coordinates of a world point are expressed as functions of s and t, $x^i = x^i(s,t)$. The vector field $u^i = \partial x^i/\partial s$ is tangent to the family of geodesics. A vector field $v^i = \partial x^i/\partial t$ represents the displacement between two nearby geodesics, corresponding to parameter values t and $t + dt$. There is a "gauge freedom" in v^i; namely, v^i changes by addition of a multiple of u^i under a change of the affine parametrization of the geodesics $\lambda_t(s)$. Here, we choose v^i to be always orthogonal to u^i. Since v^i and u^i are coordinate vector fields they commute, i.e,

$$u^i_{;k} v^k = v^i_{;k} u^i. \tag{2.40}$$

The relative acceleration of two infinitesimally close geodesics is given by

$$\frac{D^2 v^i}{ds^2} = (v^i{}_{;k} u^k)_{;l} u^l = R^i_{klm} u^k u^l v^m, \tag{2.41}$$

where we have used $u^i{}_{;i} u^l = 0$, and (2.40). Equation (2.41) is called the geodesic deviation equation.

2.2.4 Properties of the Riemann tensor

From (2.38) and (2.30) the totally covariant components of the curvature tensor are given in terms of metric components and Christoffel symbols by

$$\begin{aligned} R_{ijkl} &= g_{in} R^n{}_{jkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right) \\ &+ g_{np}(\Gamma^n{}_{jk}\Gamma^p{}_{il} - \Gamma^n{}_{jl}\Gamma^p{}_{ik}). \end{aligned} \tag{2.42}$$

Curvature components have the following symmetry properties

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}, \qquad R_{ijkl} = R_{klij} \tag{2.43}$$

They also satisfy the cyclic permutation

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0, \tag{2.44}$$

and the Bianchi identity

$$R^m{}_{ijk;l} + R^m{}_{ikl;j} + R^m{}_{ilj;k} = 0. \tag{2.45}$$

The Riemann tensor can be used to define other important tensors. The *Ricci tensor* is obtained from the curvature tensor by the contraction

$$R_{ij} = R^k{}_{ikj}. \tag{2.46}$$

R_{ij} is symmetric in its indices. The total numbers of its independent components is $N_{\mathcal{R}} = d(d+1)/2$ in d dimensions. In four dimensions, it has 10 components. The *Ricci scalar* is defined as the contraction of the Ricci tensor

$$R = g^{ij} R_{ij} = R^k{}_k. \tag{2.47}$$

From the Ricci tensor and the Ricci scalar one defines the *Einstein tensor*

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R\ . \tag{2.48}$$

This tensor obeys the following property:

$$G^{ij}{}_{;j} = 0. \tag{2.49}$$

The metrics $\hat{\mathbf{g}}$ and $\mathbf{g}$ are said to be conformal if

$$\hat{\mathbf{g}} = \Omega^2 \mathbf{g}, \tag{2.50}$$

for some non-zero suitably differentiable function Ω. The metric components are related by

$$\hat{g}_{ab} = \Omega^2 g_{ab}, \quad \hat{g}^{ab} = \Omega^{-2} g^{ab}, \tag{2.51}$$

Then, for any vectors $\mathbf{X}, \mathbf{Y}, \mathbf{V},\ \mathbf{W}$ defined at a point p we have,

$$\frac{g(\mathbf{X},\mathbf{Y})}{g(\mathbf{V},\mathbf{W})} = \frac{\hat{g}(\mathbf{X},\mathbf{Y})}{\hat{g}(\mathbf{V},\mathbf{W})}, \tag{2.52}$$

that is angles between the vectors and ratios of their magnitudes are preserved under conformal transformations. The Christoffel symbols corresponding to $\hat{\mathbf{g}}$ and $\mathbf{g}$ are related by

$$\hat{\Gamma}^i{}_{jk} = \Gamma^i{}_{jk} + \Omega^{-1}\left(\delta^i{}_j \frac{\partial\Omega}{\partial x^k} + \delta^i{}_k \frac{\partial\Omega}{\partial x^j} - g_{jk} g^{il} \frac{\partial\Omega}{\partial x^l}\right). \tag{2.53}$$

The curvature tensor, the Ricci tensor and the Ricci scalar calculated for $\hat{\mathbf{g}}$ are given by

$$\hat{R}^{ij}{}_{kl} = \Omega^{-2} R^{ij}{}_{kl} + \delta^{[i}{}_{[k} \Omega^{j]}{}_{l]}, \tag{2.54}$$

$$\hat{R}^i_j = \Omega^{-2} R^i{}_j + (d-2)\Omega^{-1}(\Omega^{-1})_{;jk} g^{ik} - (d-1)^{-1}\Omega^{-d}(\Omega^{d-2})_{;jk} g^{jk} \delta^i{}_j, \tag{2.55}$$

$$\hat{R} = \Omega^{-2} R - 2(d-1)\Omega^{-3}\Omega_{;ij} g^{ij} - (d-1)(d-4)\Omega_{;k}\Omega_{;l} g^{kl}, \tag{2.56}$$

where

$$\Omega^i_j := 4\Omega^{-1}(\Omega^{-1})_{;jm} g^{jm} - 2(\Omega^{-1})_{;k}(\Omega^{-1})_{;l} g^{kl} \delta^i{}_j, \tag{2.57}$$

and [ ] defines anti-symmetrization, e.g.,

$$T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji}). \tag{2.58}$$

Let us calculate the number of independent components of the curvature tensor in d-dimensional space ($d \geqslant 4$). It is convenient to use the collective indices $A = [ij]$ and $B = [kl]$. Each of the collective indices has $d(d-1)/2$ independent components. Thus there are $K = d(d-1)/2$ components of R_{AB} with identical indices. Because of the symmetry property $R_{AB} = R_{BA}$, the total number of components of this object is $n_1 = K(K+1)/2$. The cyclic symmetry property (2.44) gives us $n_2 = d!/[(d-4)!4!]$ relations. Therefore, the total number of independent components of curvature tensor is

$$N_R = n_1 - n_2 = \frac{1}{12}d^2(d^2-1). \tag{2.59}$$

If $d = 1$, $R_{ijkl} = 0$. If $d = 2$, there is only one independent component of R_{abcd}, which is essentially a function of R,

$$R_{1212} = \frac{R}{2}\left(g_{11}g_{22} - (g_{12})^2\right). \tag{2.60}$$

If $d = 3$, the curvature tensor and the Ricci tensor both have 6 independent components, and the Ricci tensor completely determines the curvature tensor

$$R_{ijkl} = (g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{jk}R_{il}) - \frac{1}{2}(g_{ik}g_{jl} - g_{il}g_{jk})\, R\,. \tag{2.61}$$

In $d = 4$, the Riemann tensor has 20 independent components. In d-dimensions the remaining components of the curvature tensor can be represented by the *Weyl tensor* given below

$$\begin{aligned} C_{ijkl} &= R_{ijkl} - \frac{1}{(d-2)}(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{jk}R_{il}) \\ &+ \frac{1}{(d-1)(d-2)}(g_{ik}g_{jl} - g_{il}g_{jk})\, R\,. \end{aligned} \tag{2.62}$$

2.2.5 Properties of the Weyl tensor

The Weyl tensor has the same symmetries as the Riemann tensor plus the property

$$C^k{}_{ikj} = 0. \tag{2.63}$$

That is, one can think of the Weyl tensor as that part of the curvature tensor such that all contractions vanish. Therefore, the total number of the independent components of the Weyl tensor are given by

$$N_c = N_R - \frac{d(d+1)}{2} = \frac{1}{12}(d-3)d(d+1)(d+2), \qquad d \geqslant 3 \tag{2.64}$$

By the definition in $d \leqslant 3$ we have $C_{ijkl} = 0$. In four dimensions, the Weyl tensor has 10 independent components. Another characterization of the Weyl tensor is that it is conformally invariant, i.e.

$$\hat{C}^{i}{}_{jkl} = C^{i}{}_{jkl} \ . \tag{2.65}$$

A space-time with zero Weyl tensor is said to be conformally flat; such a space-time can be related by a conformal transformation to a flat metric.
The Weyl tensor is expressed in terms of the Riemann tensor, (2.62) which satisfies the Binachi identity. Thus, the Bianchi identity (2.46) can be written as

$$C^{ijkm}{}_{;m} = J^{ijk} \ , \tag{2.66}$$

where

$$J^{ijk} = \frac{2(d-3)}{(d-2)}(R^{k[i;j]} - \frac{1}{2(d-1)} g^{k[i} R^{;j]}) \ . \tag{2.67}$$

This equation is similar to the Maxwell equation in electrodynamics

$$F^{ij}{}_{;j} = J^{i} \ \ , \tag{2.68}$$

where F^{ij} is the electromagnetic field tensor and J^i is the source 4-current.

2.2.6 Energy conditions

The Einstein equations read

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R = 8\pi \, T_{ij} \ \ ,$$

or

$$R_{ij} = 8\pi \left(T_{ij} - \frac{1}{d-2} g_{ij} T \right) \ , \quad T = g^{ij} T_{ij} \ \ .$$

where T_{ij} is the energy-momentum tensor defined by

$$T_{ij} = -\frac{2}{\sqrt{-g}} \left(\frac{\delta(\sqrt{-g}\Lambda_m)}{\delta g^{ij}} - \left[\frac{\delta(\sqrt{-g}\Lambda_m)}{\delta (g^{ij}{}_{,k})} \right]_{,k} \right) .$$

Here, Λ_m is the Lagrangian density of matter. The distribution of mass, momentum and stress due to matter and to any non-gravitational field is described by energy-momentum tensor. For our discussion we assume that in a suitable orthonormal frame, the components of the energy-momentum tensor are given by

$$T^{\hat{i}\hat{j}} = \mathrm{diag}(\varepsilon\,, \ p_a) \,, \ \ a = 1, ..., d-1 \,,$$

where ε is the energy density, and p_a's are the $(d-1)$ principal pressures.

In general relativity, there are at least seven types of energy conditions. They are the null, weak, strong, and the dominant energy conditions, and the averaged null, strong, and weak energy conditions. Here, we review the following energy conditions.

The *null energy condition* is that for any null vector n^α, $T_{ij}n^i n^j \geqslant 0$, i.e., in terms of principal pressures

$$\varepsilon + p_a \geqslant 0\,, \quad \text{for any } a\ .$$

The *weak energy condition* is that for any timelike vector k^α, $T_{\alpha\beta}k^\alpha k^\beta \geqslant 0$, i.e.,

$$\varepsilon \geqslant 0\,, \quad \text{and } \varepsilon + p_a \geqslant 0\,, \quad \text{for any } a\ .$$

If it this is true for any timelike vector, by continuity it implies the null energy condition. The weak energy condition implies that the local energy density as measured by any timelike observer is positive.

The *strong energy condition* is that for any timelike vector k^i, $T_{ij}k^i k^j \geqslant \frac{T}{d-2}k^i k_i$, i.e.,

$$\varepsilon + p_a \geqslant 0\,, \quad \text{and } \varepsilon + \frac{1}{d-3}\sum_{a=1}^{d-1} p_a \geqslant 0\,, \quad \text{for any } a\ .$$

By Einstein equations this condition implies $R_{ij}k^i k^j \geqslant 0$. The strong energy condition implies the null energy condition. It does not, in general, imply the weak energy condition.

Finally, the *dominant energy condition* is that for any timelike vector k^i, $T_{ij}k^i k^j \geqslant 0$, and $T_{ij}k^j$ is not spacelike, i.e., in terms of the principal pressures

$$\varepsilon \geqslant 0\,, \quad \text{and } p_a \in [-\varepsilon, \varepsilon]\,, \quad \text{for any } a\ .$$

By continuity the dominant energy condition implies the weak and the null energy conditions. It does not, in general, imply the strong energy condition (for details see, e.g., [8]).

2.2.7 3-sphere

A 3-sphere is a 3-dimensional analogue of a sphere. A 3-sphere with center at a point O with the Cartesian coordinates (a, b, c, d) and radius r is the set of all

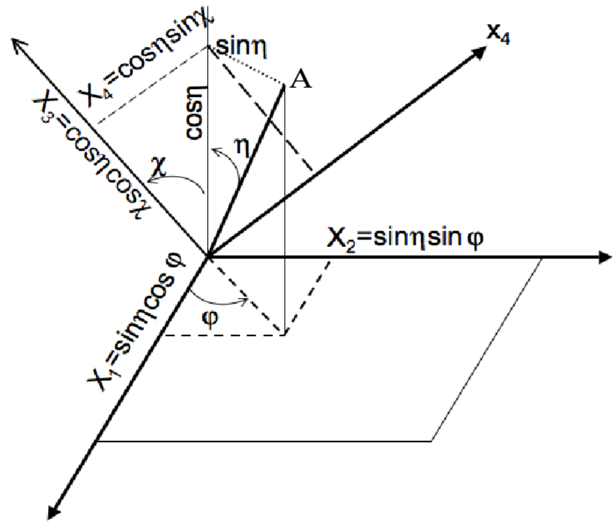

Figure 2.1: The relation of the Cartesian coordinates (x_1, x_2, x_3, x_4) and the Hopf coordinates. A is any point on a unit sphere. The semi-axis $\eta = 0$ corresponds to the orbit of the Killing vector $\frac{\partial}{\partial\phi}$, and the semi-axis $\eta = \pi/2$ corresponds to the orbit of the Killing vector $\frac{\partial}{\partial\chi}$.

points (x_1, x_2, x_3, x_4) in a real 4-dimensional space such that

$$(x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 + (x_4 - d)^2 = r^2 \quad . \tag{2.69}$$

Let us consider a unit sphere centered at the origin $(0, 0, 0, 0)$. We can introduce *hyperspherical coordinates* $(\psi,\ \zeta,\ \theta)$ in the following way

$$x_1 = \cos\psi \ , \tag{2.70}$$
$$x_2 = \cos\zeta \sin\theta \sin\psi \ , \tag{2.71}$$
$$x_3 = \sin\zeta \sin\theta \sin\psi \ , \tag{2.72}$$
$$x_4 = \cos\theta \sin\psi \ . \tag{2.73}$$

where $0 \leqslant \psi \leqslant \pi$, $0 \leqslant \theta \leqslant \pi$, and $0 \leqslant \zeta < 2\pi$. Then, the round metric on a 3-sphere in these coordinates is given by

$$ds^2 = d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta \ d\zeta^2\right) . \tag{2.74}$$

We can introduce *Hopf coordinates* $(\eta,\ \phi,\ \chi)$ in the following way

$$x_1 = \sin\eta\cos\phi\ , \tag{2.75}$$
$$x_2 = \sin\eta\sin\phi\ , \tag{2.76}$$
$$x_3 = \cos\eta\cos\chi\ , \tag{2.77}$$
$$x_4 = \cos\eta\sin\chi\ , \tag{2.78}$$

where $0 \leqslant \eta < \frac{\pi}{2}$, $0 \leqslant \phi < 2\pi$, $0 \leqslant \chi < 2\pi$. The metric on a 3-sphere in Hopf coordinates read

$$ds^2 = d\eta^2 + \sin^2\eta\ d\phi^2 + \cos^2\eta\ d\chi^2\ . \tag{2.79}$$

Using the Hopf coordinates, the metric of the 5-dimensional Minkowski space-time can be written as follows:

$$ds^2 = -dt^2 + dr^2 + r^2(d\eta^2 + \sin^2\eta\ d\phi^2 + \cos^2\eta\ d\chi^2)\ . \tag{2.80}$$

2.3 A review of the generalized Weyl solutions

The study of classical general relativity in more that 4-dimensions has attracted a lot of attention in recent years. Here, we give a brief review of the the history of higher dimensions. For more details, refer to [9], [10], [11].

The idea of extra dimensions goes back to the Finnish physicist Nordström (1881- 1923). In 1914 he has made an attempt to describe gravity and electromagnetism simultaneously by introducing one extra spatial dimension [12]. His theory did not turn out to be correct and was replaced by Einstein's theory. Later, Kaluza's (1885-1954) work which was published in 1921 gave birth to the modern Kaluza-Klein (KK) theories [13], [14]. Kaluza considered a 5-dimensional spacetime with one additional spatial dimension to unify the fundamental forces of gravity and electromagnetism. Initially these theories were rather a mathematical exercise. The formulation of string theory and M-theory in a space-time with a number of dimensions greater than 4 has provided more support for the idea of higher dimensions. The size of these extra dimensions was naturally considered to be of the order of Planck length, $l_{pl} \sim 10^{-33}$ cm. In KK theories, using extra compact dimensions, a tower of 4-dimensional particles with masses proportional to the inverse size of the compact dimension (L^{-1}) are produced. However, the standard model has been successfully tested up to the ~ 100 GeV. This implies that these extra dimensions can not be macroscopic $L < 10^{-17}$ cm. On the other hand, the gravitational force has not been measured beneath the distance of ~ 1 mm [15].

In 1983 Rubakov and Shaposhnikov [16] proposed a novel model in which fermions and bosons are confined to a 4-dimensional subspace of a higher dimensional space-time. Following a similar direction, D-branes have been introduced in string theory [17], where fermions, bosons and gauge fields associated with open strings are confined to propagate only along the brane while gravity associated with closed strings can propagate in the bulk. Finally, in 1998, another idea was proposed by Nima Arkani-Hamed, Savas Dimopoulos and Gia Dvali [18], [19], the so-called ADD model, where the observed weakness of gravity at distances $\geq$ 1mm can be explained. This is the hierarchy problem, where the electrweak scale $m_{EW} \sim 1$ TeV is much smaller that the Planck scale $M_{pl} = G^{-1/2} \sim 10^{16}$ TeV. They have assumed that m_{EW} is the only fundamental short distance of the nature, i.e., our interpretation of M_{pl} as a fundamental energy scale is based on the assumption that gravity is not unmodified over the range where it is actually measured $\sim$ 1 mm down to Planck length. Suppose there are n extra compact dimensions of radius L. The thickness of the brane is at most 10^{-17} cm. The observed Planck scale M_{pl} is then related to the fundamental $(4+n)$ higher dimensional Planck scale $M_{pl(4+n)}$ in the following way:

$$M_{pl}^2 \sim M_{pl(4+n)}^{2+n} L^n. \tag{2.81}$$

Then, according to their philosophy that electroweak scale is the only fundamental short distance of the nature, they set $M_{pl(4+n)}^{2+n} = m_{EW}$. Then the size of extra dimension should be

$$L \sim 10^{\frac{30}{n}-17}\text{cm} \times (\frac{1\text{TeV}}{m_{EW}})^{1+\frac{2}{n}} \quad , \tag{2.82}$$

to produce the observed value of M_{pl}. For $n = 1$, the size of extra dimension given by this model (2.82) is 10^{13} cm. Therefor, this case is empirically excluded. There are many other constraints from collider experiments or some from astrophysical and cosmological experiments [10]. If we include these constraints the ADD model is applicable for $n \geq 4$. The existence of such large extra dimensions ($l_{lp} \ll L \leqslant 1$ mm), by lowering the value of Planck scale, provide the possibility of black hole production in the Large Hadron Collider (LHC).

One may still assume that there are n extra compact dimension of size $L \sim 1$ mm. However, we do not need to assume that $M_{pl(4+n)}^{2+n} = M_{Ew}$. Namely, extra compact spatial dimensions may exist without exactly solving the hierarchy problem. In this case, setting $n = 1$ corresponds to the $M_{pl(5)} \sim 10^5$ TeV, which is not in any way accessible in LHC.

There are other geometrical alternative models attempting to solve the hierarchy problem, such as the Randall-Sundrum (RS) model [20]. All these models have attracted lots of attention to the study of the general relativity in higher dimensional space-time.

Higher dimensional gravity has its own intrinsic features. For example, a variety of black objects exist in higher dimensions which are absent in 4-dimensions. Black objects such as black strings and black p-branes with extended horizons exist only in $d > 4$. To construct the simplest $(d+1)$-dimensional black string solution, add one flat spatial dimension to a d-dimensional solution of the vacuum Einstein equation. This is because the direct product of two Ricci-flat manifold is itself a Ricci flat manifold. The Schwarzschild-Tangherlini metric [21] is a solution of the vacuum d-dimensional Einstein equations given by the following metric:

$$ds^2 = -(1 - \frac{\mu}{r^{d-3}})dt^2 + (1 - \frac{\mu}{r^{d-3}})^{-1}dr^2 + r^2 d\Omega^2_{d-2} \;, \tag{2.83}$$

where $d\Omega^2_{d-2}$ is the metric on a unit $(d-2)$ sphere. The black hole horizon has the topology $S^{(d-2)}$. A 5-dimensional black string solution of the horizon topology $S^{(d-2)} \times \mathbb{R}$ constructed from (2.83) reads

$$ds^2 = -(1 - \frac{\mu}{r^{d-3}})dt^2 + (1 - \frac{\mu}{r^{d-3}})^{-1}dr^2 + r^2 d\Omega^2_{d-2} + dz^2 \;. \tag{2.84}$$

If we identify z periodically, i.e. $z = z + L$, then the above solution represents a black object with horizon topology of $S^{(d-2)} \times \mathbb{T}$. In a similar way one can add p extra flat spatial directions to construct a black p-brane of the horizon topology of $S^{(d-2)} \times \mathbb{R}^p$ or $S^{(d-2)} \times \mathbb{T}^p$. The black p-branes are not globally asymptotically flat and exhibit dynamical instabilities [22]. There exist many other interesting solutions in higher dimensions which we do not discuss here, including rotating and charged black objects (for more details refer to [9]). For example, the 5-dimensional static black ring is the first asymptotically flat solution discovered with a non-spherical horizon topology [23]. Namely, the black ring has a horizon topology of $S^1 \times S^2$ (For construction and detailed analysis of the rotating black ring solution refer to, e.g., [24–27]).

Construction of higher-dimensional solutions even in vacuum is not always as easy as the construction of vacuum black strings. In the search for the exact solutions of Einstein equations, some generating techniques have been invented. One of the earliest of these generating techniques is due to Weyl [28]. In 4-dimensions, any static axisymmetric solution of vacuum Einstein equations can be presented in Weyl form. Remarkably, the Einstein equations can be written in the Weyl form in terms of two metric functions, one of which is a harmonic function satisfying Laplace's equation in a flat three dimensional space, and the other one can be derived from a simple integration involving the first one. Example of the solutions belonging to the 4-dimensional Weyl form are: the Israel-Khan solution representing a set of collinear Schwarzschild black holes [29], a black hole with toroidal horizon [30], and a compactified black hole [31–33].

To extend axial symmetry to higher dimensions and construct the higher dimensional analogue of the Weyl solution, one may look for a solution that admit an isometry group $\mathbb{R}\times O(d-2)$. Here, $\mathbb{R}$ is the time translation. Such a solution is invariant under $(d-2)$ spatial rotations around a given line axis, where the orbits of $O(d-2)$ are $(d-3)$-spheres. However, the construction of such axially symmetric solution has failed (see, e.g., [33–35]). One may instead consider rotation around a line, rotations around spatial codimension-2 hypersurfaces. Assume $d-3$ commuting $U(1)$ symmetries, i.e., a solution with $U(1)^{d-3}$ symmetry in addition to the timelike symmetry $\mathbb{R}$. The vacuum Einstein equations which such symmetry reduce to an integrable two-dimensional $GL(d-2,\ \mathbb{R})$ sigma-model with powerful solution generating techniques. However, such solutions only in $d=4,\ 5$ are globally asymptotically flat.

In higher dimensions a vacuum space-time admitting $d-2$ commuting, orthogonal, non-null Killing vector fields $\xi^{\alpha}_{(i)}=\delta^{\alpha}_{\ i}$ can be written in the following form [23]:

$$ds^2=\sum_{i=1}^{d-2}\epsilon_i e^{2U_i(\rho,z)}(dx^i)^2+e^{2v(\rho,z)}(d\rho^2+dz^2)\ , \tag{2.85}$$

where $\epsilon=\pm1$, depending on whether $\xi_{(i)}$ is spacelike or timelike. Here, we have assumed a special case where the two dimensional surfaces orthogonal to all $\xi_{(i)}$ are spacelike. From the Einstein equations $R_{\mu\nu}=0$, we have

$$U_{i,\rho\rho}+\frac{1}{\rho}U_{i,\rho}+U_{i,zz}=0\ , \tag{2.86}$$

$$v_{,\rho}=\rho\Sigma_{i<j}(U_{i,z}U_{j,z}-U_{i,\rho}U_{j,\rho})\ , \tag{2.87}$$

$$v_{,z}=-\rho\Sigma_{i<j}(U_{i,\rho}U_{j,z}+U_{i,z}U_{j,\rho})\ . \tag{2.88}$$

From (3.10), $U_j(\rho,z)$'s are $d-2$ axisymmetric functions solving the Laplace equation in the flat 3-dimensional space. Also from the Einstein equations follows the following constraint:

$$\sum_{i=1}^{d-2}U_i=\ln\rho+const.\ , \tag{2.89}$$

where the constant term can be adjusted by rescaling the coordinates x^i. The constraint equation (3.11) implies that from the above $d-2$ functions, only $d-3$ functions are independent.

Equation (3.10) is the Laplace equation for $d-2$ harmonic functions in flat 3-dimensional space. Therefore, one can interpret the potentials U_j's as the Newtonian potentials of some axisymmetric sources. $\ln \rho$ is the solution of Laplace's equation that corresponds to the Newtonian potential produced by an infinite rod of zero thickness, with constant mass per unit length $1/2$ lying along the z axis. It turns out that both in $d=4$ and $d>4$, Weyl solutions of physical importance all have sources of the same kind, that is thin rods along the axis of symmetry. It also turns out that one can classify the properties of the solutions in terms of some few characteristics of the rod construction. In this interpretation, the constraint (3.11) implies that the potentials U_j's should add up to give a uniform infinite rod of zero thickness lying along z axis. To cover the whole axis either one of the U_j's has semi-infinite rod sources that extend to $z=\infty$ and $z=-\infty$, or there exists one U_j with a semi-infinte rod source which extends to $z=\infty$ and another U_j has a semi-infinite rod source that extends to $z=-\infty$. There would also exist some finite rod sources for the remaining U_j's. When the sources associated

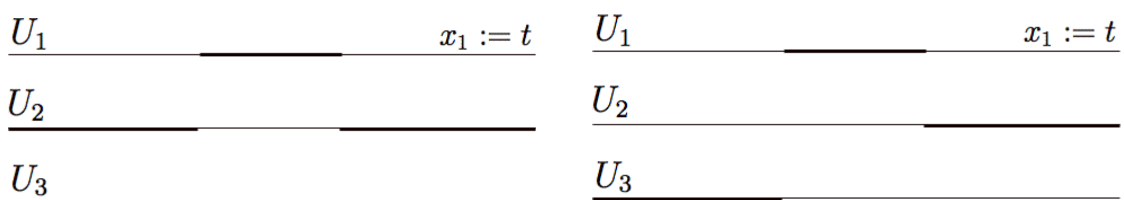

Figure 2.2: Distribution of U_j sources on the z-axis (a) for a uniform black string in 5-dimensions; U_2 has semi-infinite rod sources which extend to $z=\infty$ and $z=-\infty$, (b) for 5-dimensional Schwarzschild black hole; U_2 corresponds to a semi-infinite rod source which extends to $z=\infty$, and U_3 corresponds to a semi-infinite rod source which extends to $z=-\infty$.

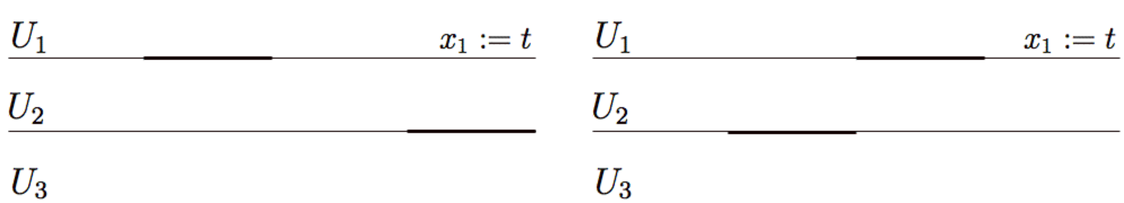

Figure 2.3: Distribution of U_j sources on the z-axis (a) for black ring in 5-dimensions; U_2 corresponds to a semi-infinite rod source which extends to $z=\infty$, and U_3 corresponds to a semi-infinite rod source which extends to $z=-\infty$, and a finite rod source, (b) for black hole plus a KK bubble; U_3 has semi-infinite rod sources which extend to $z=\infty$ and $z=-\infty$.

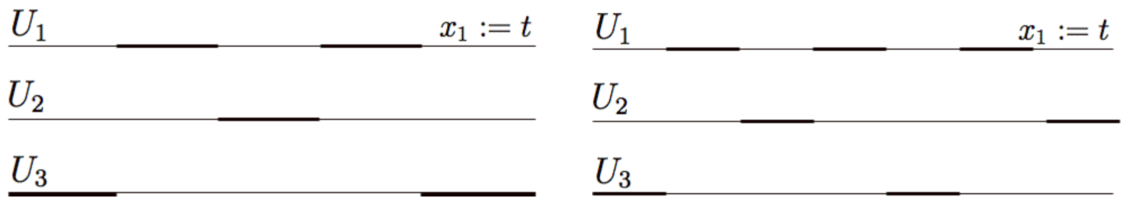

Figure 2.4: Distribution of U_j sources on the z-axis (a) for two black hole configuration, (b) for three black hole configuration.

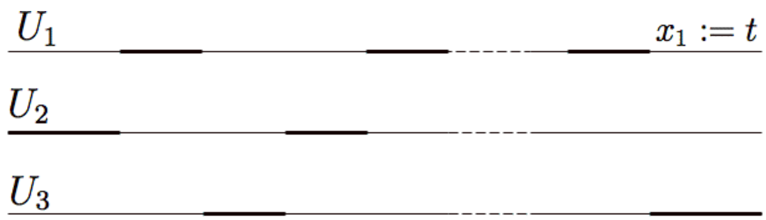

Figure 2.5: Distribution of U_j sources on the z-axis for a finite number of multi black holes (see, [36])

with U_j are bounded (i.e. it involves finite number of finite length rod sources), then U_j approaches a constant far from the sources, i.e., x^j is a flat direction in the asymptotic metric. For example, if one chooses finite rod sources for the time coordinate, then the corresponding solution will have event horizon in space-time. In this case time direction becomes flat in the asymptotic region. Choosing a semi-infinite rod source for time coordinate corresponds to horizon that extends to asymptotic infinity, that is an acceleration horizon. If a spatial coordinate has a finite rod source then it can be interpreted as a Kaluza-Klein (KK) coordinate in the asymptotic region. If the source for the spatial coordinate extends to infinity then the corresponding coordinate is an azimuthal angle.

One can classify the solutions of Einstein equations according to the distribution of the rods on the z axis [23]. A solution will be called of class n if it has n finite rod sources. Then, *class* 0 has no finite rod source; the sources are either an infinite rod or two semi-infinite rods. A flat space-time is the only *class* 0 solution. In *class I* there is a single finite rod so other sources must be a combination of semi infinite rod sources. There are only two ways of doing this (here, we do not distinguish metrics that are related to each other by isometric transformations, also we identify x_1 with the timelike coordinate t). (a) U_2 has a finite rod source and U_1 has two semi-infinite rod sources which extend to $z = \infty$ and $z = -\infty$. Then the other U_i's ($i = 2.., d-2$) are constant. This would represent a 4-dimensional

Schwarzschild solution times some flat directions for $d > 4$ (see Fig. B.1(a) for the 5-dimensional black string). (b) U_2 has a finite rod source and U_1 and U_3 each have a semi-infinite rod source. The other U_i's ($i = 2..d-2$) are constant. This is the five dimensional Schwarzschild solution times $d-5$ flat directions (see Fig. B.1(b) for the 5-dimesnional Schwarzschild-Tangherlini solution). In *class II* there are two finite rods and two semi-infinite rods. There are four ways of distributing these sources. These solutions correspond to: C metric, black ring in 5-dimensions (Fig. B.2(a)), black hole plus KK bubble in 5-dimension (see Fig B.2(b)), and black string and KK bubble depending on the distribution of the sources [23].

The Israel-Khan solution [29] describing an array of collinear 4-dimensional black holes belongs to the 4-dimensional Weyl form. This solution possesses conical singularities. However, an infinite array of black holes is free of conical singularities. This solution represent a black hole localized on the KK circle of $d = 4$ KK theory ([33], [32]). Using the generalized Weyl solutions, one can also construct multi black hole solutions of different configurations (for details see [23], [36]). First, one can construct a non-asymptotically flat two-black-hole solution located at the north and south poles of a KK bubble (see, i.e., Fig. B.4.(a)). No conical singularities is needed to keep the black holes apart. Second, an asymptotically flat three-black-hole solution. In this solution only the central black hole is collinear with the other two black holes along different axes. This solution contains conical singularities (Fig. B.4(b)). Third, one can construction infinite array of black holes which does not represent a black hole localized on a KK circle (see Fig. B.5). Fourth, one can construct an asymptotically flat, static vacuum solution which represents a $U(1) \times U(1)$ compatible collinear multi-black hole solution in 5-dimensions [36]. This notion of collinearity is different from what one needs for a black hole localized on a KK circle of $d = 5$ KK theory; namely, the solution does not possess a $O(d-2)$ symmetry. This solution has conical singularities. The background space-time of this multi-black hole solution, i.e., when all the black holes disappear, is not free of conical singularities, either.

2.4 A distorted 4-dimensional Schwarzschild black hole

The Schwarzschild space-time is given by

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{2.90}$$

where $t \in (-\infty,\ \infty)$, $\rho \in (0,\ \infty)$, $\theta \in [0,\ \pi)$, and $\phi \in [0,\ 2\pi)$ (see [4] for details). It is a static, asymptotically flat, spherically symmetric solution of the vacuum Einstein equations. The black hole horizon is located at $r = 2M$. The parameter $M > 0$ is the gravitational mass of the matter.

In 4-dimensions, any static axisymmetric solution of vacuum Einstein equations can be presented in Weyl form [28]

$$ds^2 = e^{2U}[-\rho^2 dt^2 + e^{2V}(dz^2 + d\rho^2)] + e^{-2U} d\phi^2 \ , \tag{2.91}$$

where $U = U(\rho,\ z)$, $V = V(\rho,\ z)$, and the coordinate ranges are

$$t \in (-\infty,\ \infty), \quad \rho \in (0,\ \infty), \quad z \in (-\infty,\ \infty), \quad \phi \in [0,\ 2\pi) \ . \tag{2.92}$$

Remarkably, the Einstein equations can be written in the Weyl form in terms of two metric functions, one of which is a harmonic function satisfying Laplace's equation in a flat three dimensional space,

$$U_{,\rho\rho} + \frac{1}{\rho} U_{,\rho} + U_{,zz} = 0 \ , \tag{2.93}$$

and the other one can be derived from a simple integration involving the first one, i.e.,

$$V_{,\rho} = \rho(U_{,\rho}^2 - U_{,z}^2) \ , \quad V_{,z} = 2\rho U_{,\rho} U_{,z} \ . \tag{2.94}$$

Equation (2.93) is the integrability condition for the other two Einstein equations (2.94). If the function U is known, V can be determined by simple quadratures.

The metric (B.1) can be written in the Weyl form. Matching the metric (B.1) and (B.2), we derive

$$e^{-2U_S} = r^2 \sin\theta^2, \quad e^{2V_S} = \frac{r^2 \sin^2\theta}{(r^2 - 2mr + m^2 \sin^2\theta)} \ . \tag{2.95}$$

The relation between the Schwarzschild and Weyl coordinate is

$$\rho = \sqrt{r(r - 2M)} \sin\theta, \quad z = (r - M)\cos\theta, \quad r > 2M \ . \tag{2.96}$$

Namely,

$$r = M + \frac{1}{2}(l_+ + l_-), \quad l_\pm = \sqrt{\rho^2 + (z \pm M)^2} \ . \tag{2.97}$$

Using the above transformations we derive

$$e^{-2U_S} = \frac{4\rho^2[M + \frac{1}{2}(l_+ + l_-)]^2}{(l_+ + l_-)^2 - 4M^2} \, , \tag{2.98}$$

$$e^{2V_S} = \frac{16\rho^2[M + \frac{1}{2}(l_+ + l_-)]^2}{[(l_+ + l_-)^2 - 4M^2]^2 - 16M^2\rho^2} \, . \tag{2.99}$$

The inner region of the Schwarzschild black hole $r \in (0,\ 2M)$ corresponds to an imaginary value of the ρ coordinate, and the black hole horizon has a coordinate singularity. It is more convenient to use prolate spherical coordinates $(\eta,\ \theta)$ related to the Weyl coordinates $(\rho,\ z)$ in the following way:

$$\rho = M\sqrt{\eta^2 - 1}\sin\theta \, , \quad z = M\eta\cos\theta \, , \tag{2.100}$$

where $\eta \in (-1,\ \infty)$. The Schwarzschild metric in the prolate spheroidal coordinates reads

$$ds^2 = -\frac{\eta - 1}{\eta + 1}dt^2 + M^2(\eta + 1)^2 \left(\frac{d\eta^2}{\eta^2 - 1} + d\theta^2 + \sin^2\theta d\phi^2 \right) \, , \tag{2.101}$$

where the prolate spheroidal coordinate and the Schwarzschild coordinate are related by

$$r = M(\eta + 1), \quad \theta = \theta, \quad \phi = \phi \, . \tag{2.102}$$

The black hole horizon is at $\eta = 1$, and the singularity is at $\eta = -1$. A Schwarzschild black hole is modified by the presence of external matter. We assume the case where the distribution of matter is static, axisymmetric, and localized outside the black hole horizon, i.e., the space-time in the vicinity of the horizon remains vacuum. The sources are 'moved' to infinity. Therefore, the corresponding space-time is not asymptotically flat. Such a solution is called a local black hole. Here, we present the metric of such a local black hole in Weyl form.

The Schwarzschild black hole distorted by external static, axisymmetric sources is given by

$$U = U_a + \hat{U} \, , \quad V = V_S + \hat{V} \, , \tag{2.103}$$

where $\hat{U}$ and $\hat{V}$ are the distortion fields (for details see [1], [29], [37–42]). The corresponding metric representing the distorted 4-dimensional Schwarzschild black hole reads

$$ds^2 = -\frac{\eta - 1}{\eta + 1}e^{2\hat{U}}dt^2 + M^2(\eta+1)^2 e^{-2\hat{U}} \left[e^{2\hat{V}} \left(\frac{d\eta^2}{\eta^2 - 1} + d\theta^2 \right) + \sin^2\theta d\phi^2 \right] \, , \tag{2.104}$$

where $\hat{U}$ is the solution of the following Laplace equation

$$(\eta^2-1)\widehat{U}_{,\eta\eta}+2\eta\,\widehat{U}_{,\eta}+\widehat{U}_{,\theta\theta}+\cot\theta\,\widehat{U}_{,\theta}=0\ , \tag{2.105}$$

and $\hat{V}$ is given by integration from

$$\widehat{V}_{,\eta} = N\left(\eta[(\eta^2-1)\widehat{U}^2_{,\eta}-\widehat{U}^2_{,\theta}]+2(\eta^2-1)\cot\theta\widehat{U}_{,\eta}\widehat{U}_{,\theta}+2\eta\widehat{U}_{,\eta}+2\cot\theta\widehat{U}_{,\theta}\right)\ , \tag{2.106}$$

$$\widehat{V}_{,\theta} = N(\eta^2-1)\left(\cot\theta[\widehat{U}^2_{,\theta}-(\eta^2-1)\widehat{U}^2_{,\eta}-2\widehat{U}_{,\eta}]+2\eta\widehat{U}_{,\theta}[\widehat{U}_{,\eta}+(\eta^2-1)^{-1}]\right)\ . \tag{2.107}$$

Here, $N=\sin^2\theta(\eta^2-\cos^2\theta)^{-1}$.

The black hole horizon is regular if the space-time invariants are regular on the horizon, and, in addition, there are no conical singularity on the axes of symmetry and thus, on the horizon. It was shown that the Kretschmann invariant calculated on the horizon of a static, asymmetric black hole is regular on the horizon if the its surface gravity is constant and the horizon surface is a regular totally geodesic surface. It follows that the distortion fields $\hat{U}$ and $\hat{V}$ must be regular on the horizon. To derive a solution representing a distorted black hole we start from the Laplace equation (2.105). Here, we do not go through the details of the derivation of the solution. The solution of (2.105) which corresponds to the distortion by external distortion fields is given by

$$\hat{U}(\eta,\theta)=\sum_{n\geq0}a_n\,R^nP_n(\eta\cos\theta/R)\,, \tag{2.108}$$

$$R=(\eta^2-\sin^2\theta)^{1/2}\ , \tag{2.109}$$

We call the coefficients a_n's multiple moments. The solution to Eqs. (2.106), (2.107) for $\widehat{U}$ given by expression (2.108) is given by [43],

$$\widehat{V}=\widehat{V}_1+\widehat{V}_2\ , \tag{2.110}$$

$$\widehat{V}_1=\sum_{n\geqslant1}c_n\sum_{l=0}^{n-1}\left[\cos\theta-\eta-(-1)^{n-l}(\eta+\cos\theta)\right]R^lP_l\left(\eta\cos\theta/R\right)\ , \tag{2.111}$$

$$\begin{aligned}\widehat{V}_2=\sum_{n,k\geqslant1}&\tfrac{nkc_nc_k}{n+k}R^{n+k}\left[P_n\left(\eta\cos\theta/R\right)P_k\left(\eta\cos\theta/R\right)\right.\\&\left.-P_{n-1}\left(\eta\cos\theta/R\right)P_{k-1}\left(\eta\cos\theta/R\right)\right]\ .\end{aligned} \tag{2.112}$$

Here $\widehat{V}_1$ is linear and $\widehat{V}_2$ is quadratic in the a_n's.

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Chapter 3

Distorted 5-dimensional vacuum black hole

3.1 Introduction

String theories, the AdS/CFT correspondence [1, 2], the ADD model [3, 4], and brane-world RS models [5] suggest that higher-dimensional solutions of general relativity may have physical applications. Whether our world has extra dimensions or not should be eventually verified by experiments. One of such experiments is microscopic black hole production, which may be conducted at the LHC. Such a black hole may be created at energies of the order of ~ 10 TeV, if our world has extra spatial dimensions of large size (< 1 mm) or large warping, which become accessible on such energetic scales (see, e.g., [6–13]).

Possible physical applications of higher dimensions have increased interest in higher-dimensional solutions of general relativity. However, the Einstein equations of general relativity, especially higher-dimensional ones, are very complex. To solve them we have to use numerical computations, except for some idealized, highly-symmetrical cases, when construction of analytical solutions becomes possible. For example, one such construction, corresponding to a 4-dimensional, static and axisymmetric vacuum space-time, is due to Weyl [14]. The Weyl solution implies a static and axisymmetric distribution of matter. One of the Einstein equations for the space-time metric represented in the Weyl form reduces to a linear Laplace equation. Therefore, the superposition principle can be applied for the construction of one of the metric functions. Another metric function can be derived by a line integral in terms of the first one. As a result, one can relatively easily con-

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struct many interesting solutions, e.g., the Israel-Khan solution representing a set of collinear Schwarzschild black holes [15], a black hole with a toroidal horizon [16], and a compactified black hole [17–19].

In higher-dimensional space-times we have a very rich variety of black objects classified according to their horizon topology, for example black holes, black strings, and black rings (for a review see, e.g., [20]). However, an exact analytical solution representing a black hole in a space-time with one large, compact extra dimension is not known. The solution representing a black hole in a 5-dimensional space-time with one large, compact extra dimension is not algebraically special [21]. As a result, finding such analytical solution can be a formidable problem. Analytical approximations to the black hole are given in [1, 22, 24–26]. Finding a solution representing a black hole localized on a brane is not a simple problem either. A numerical analysis suggests that in a 5-dimensional, one-brane RS model, only a black hole whose horizon radius is smaller than the bulk curvature can be localized on the brane [27]. Results of a subsequent numerical analysis further suggest that such a black hole may be unstable [28].

Both the sought black hole solutions are axisymmetric, in the sense that they admit an $SO(3)$ isometry group. Orbits of the group are 2-dimensional spheres of nonzero curvature. As was noticed in [19], this nonzero curvature is an essential problem for a construction of such higher-dimensional axisymmetric solutions. However, one can construct algebraically special axisymmetric solutions in d-dimensional space-times [29]. As it was concluded in [30], a d-dimensional, axisymmetric space-time which admits the $SO(d-2)$ isometry group cannot be considered as an appropriate higher-dimensional generalization of the 4-dimensional Weyl form. Instead, it was proposed in [30] to consider a d-dimensional space-time which admits the $\mathbb{R}^1 \times U(1)^{d-3}$ isometry group. Such a generalized Weyl form allows for the construction of many interesting black objects (see, e.g., [20, 30]). However, as it was illustrated in [20], only 4- and 5-dimensional black hole solutions can be presented in the Weyl form. Let us mention that a generalization of the Weyl form to the Einstein-Gauss-Bonnet theory in a 5-dimensional space-time was proposed in [31]. Numerical evidence that a Schwarzschild black hole, a static black ring, and a uniform black string can also be considered within the generalization of the Weyl form was given in [32, 33].

Having the generalized Weyl form, one may try to construct higher-dimensional analogues of 4-dimensional Weyl solutions. For example, a construction of multi-black-hole configurations within the generalized 5-dimensional Weyl form was discussed in the paper [20]. The first configuration discussed there represents a two-black-hole solution which is not asymptotically flat. The second configuration is a three-black-hole solution which is asymptotically flat but suffers from irremovable conical singularities. In addition, the central black hole is collinear with the other two along *different* symmetry axes. The third configuration represents an infinite number of black holes. However, it does not correspond to a 5-dimensional *compactified* black hole ("caged" in the compact dimension). In fact, such a black hole corresponds to a space-time with an infinite number of collinear black holes which admits an $\mathbb{R}^1 \times SO(3)$ isometry group, instead of the $\mathbb{R}^1 \times U(1) \times U(1)$ isometry of the 5-dimensional Weyl form. Asymptotically flat space-times which admit an $\mathbb{R}^1 \times U(1) \times U(1)$ isometry and correspond to 5-dimensional "collinear" black holes were constructed in [34]. The corresponding background space-times have conical singularities and are not flat by the construction. Such space-times have more than one fixed point of the $U(1) \times U(1)$ isometry group, whereas a 5-dimensional Minkowski space-time has only one such point.

Black holes interact with external matter and fields. For example, an accretion disk around a black hole tidally distorts its horizon. An accretion scenario of a black hole which may be produced at the LHC is given in [10, 35]. As it is for any physical objects, properties of black holes are mostly revealed by their interactions. To analyze a black hole interaction is a formidable problem which requires involved numerical computations. However, a study of idealized, highly-symmetrical analytical solutions may provide us with an exact description of black-hole nonlinear interactions with external matter and fields. Among such solutions a black hole distorted by an external, static and axisymmetric distribution of matter deserves particular attention. Such a black hole was analyzed in the papers [36–41, 51].

External matter affects the internal structure of black holes as well. For example, external, asymmetric dynamical distortion of a black hole results in chaotic and oscillatory space-time singularity of the BKL-type, which corresponds to shifts between different Kasner regimes (see, e.g., [43–45]). The interior of a 4-dimensional, distorted, static and axisymmetric, vacuum black hole was studied in [51]. It was shown that in the vicinity of the black hole singularity the space-time has the same Kasner exponents as that of a Schwarzschild black hole. However, the static and axisymmetric distortion does change the geometry of the black hole *stretched singularity* (region near a black hole singularity where the space-time curvature is of the Planckian order, $\sim 10^{66}$ cm^{-2}) and horizon surfaces. The change is such that a certain duality transformation between the geometry of the horizon and the stretched singularity surfaces holds. According to that relation, the geometry of the horizon surface uniquely defines the geometry of the stretched singularity surface. In addition, it was shown that such a distortion noticeably affects the proper time of free fall from the black hole horizon to its singularity. It is interesting to study whether a higher-dimensional distorted black hole has similar properties.

Another motivation to analyze the interior of a higher-dimensional distorted black hole is related to analysis of a topological phase transition between a nonuniform black string, whose horizon wraps the space-time compact dimension, and a compactified black hole (see, e.g., [46, 47]). In such a transition the black string and black hole topological phases meet at the merger point [46, 48–51]. As a result, their near horizon geometry gets distorted. The interior of a nonuniform 6-dimensional black string was studied in [52], where numerical evidence of a space-time singularity approaching the black string horizon at the merger point was presented. What happens to the corresponding compactified black hole approaching the merger point and which way it gets distorted remains an interesting open question.

The main goal of this Chapter is to study a 5-dimensional, distorted, static, vacuum black hole as a distorted Schwarzschild-Tangherlini black hole, which can be presented in the generalized Weyl form, and to compare its properties with those of a 4-dimensional, distorted, static and axisymmetric, vacuum black hole. A 5-dimensional Schwarzschild-Tangherlini black hole is a good approximation to a 5-dimensional compactified black hole if the size of the compact dimension is much larger than the size of the black hole. Thus, the distorted Schwarzschild-Tangherlini black hole may be also considered as a good approximation for such distorted compactified black hole.

This Chapter is organized as follows: In Section II, we construct the 5-dimensional Weyl solution which includes gravitational distortion fields due to remote matter. In Section III, we present the metric of a 5-dimensional, static, vacuum black hole distorted by external gravitational fields and derive the corresponding Einstein equations. A solution to the Einstein equations is derived in Section IV. In Section V, we study the symmetry properties of the distortion fields and present their boundary values on the black hole horizon, singularity, and on its symmetry axes. The space-time near the black hole horizon and singularity is analyzed in Secs. VI and VII, respectively. In Section VIII, we discuss how the black hole distortion affects the maximal proper time of free fall of a test particle moving from the black hole horizon to its singularity. We summarize and discuss our results in Section IX. Details of our calculations are presented in the appendices.

In this Chapter we use the following convention of units: $G_{(5)} = c = 1$, the space-time signature is $+3$, and the sign conventions are that adopted in [20].

3.2 5-dimensional Weyl solution

In this Section we present a 5-dimensional generalization of the Weyl solution in the form suitable for analysis of a distorted 5-dimensional vacuum black hole. To begin with, let us briefly discuss the main properties of the 4-dimensional Weyl solution presented in the following Weyl form:

$$ds^2 = -e^{2U}dt^2 + e^{2(V-U)}(dz^2 + d\rho^2) + \rho^2 e^{-2U} d\phi^2 \,, \tag{3.1}$$

where $t, z \in (-\infty, \infty)$, $\rho \in (0, \infty)$, and $\phi \in [0, 2\pi)$. The metric functions U and V depend on the cylindrical coordinates ρ and z. The Weyl solution represents a general static and axisymmetric metric which solves the corresponding vacuum Einstein equations. One of these equations reduces to the following linear equation for the metric function U:

$$U_{,\rho\rho} + \frac{1}{\rho} U_{,\rho} + U_{,zz} = 0 \,, \tag{3.2}$$

which is defined on the plane (ρ, z). Here and in what follows, $(...)_{,a}$ stands for the partial derivative of the expression $(...)$ with respect to the coordinate x^a. Equation (3.2) can be viewed as a 3-dimensional Laplace equation defined in an auxiliary 3-dimensional Euclidean space. The remaining Einstein equations define the metric function V as follows:

$$V_{,\rho} = \rho \left(U_{,\rho}^2 - U_{,z}^2 \right) \,, \tag{3.3}$$

$$V_{,z} = 2\rho\, U_{,\rho} U_{,z} \,. \tag{3.4}$$

Equation (3.2) is the integrability condition for Eqs. (3.3) and (3.4). If we solve Eq. (3.2) for the metric function U, then the second metric function V can be derived by the following line integral:

$$V(\rho, z) = \int_{(\rho_0, z_0)}^{(\rho, z)} \left[V_{,\rho'}(\rho', z') d\rho' + V_{,z'}(\rho', z') dz' \right] \,, \tag{3.5}$$

where the integral is taken along any path connecting the points (ρ_0, z_0) and (ρ, z). The constant of integration is defined by a point (ρ_0, z_0).

The 4-dimensional Weyl solution admits an $\mathbb{R}^1_t \times SO(2) \cong \mathbb{R}^1_t \times U_\phi(1)$ isometry group. In other words, the Weyl solution is characterized by the two orthogonal, commuting Killing vectors $\xi^\alpha_{(t)} = \delta^\alpha{}_t$ and $\xi^\alpha_{(\phi)} = \delta^\alpha{}_\phi$, which are generators of time translations and 2-dimensional rotations about the symmetry axis z, respectively. Note that the metric function U together with the constant of integration in (3.5) define uniquely the space-time geometry.

The d-dimensional generalization of the Weyl solution which admits $d-2$ commuting, non-null, orthogonal Killing vector fields was presented in the papers [30] and [20]. Here we discuss the 5-dimensional generalized Weyl solution which is characterized by three commuting, non-null, orthogonal Killing vector fields, one of which ($\xi^{\alpha}_{(t)} = \delta^{\alpha}{}_{t}$) is timelike, and other two ($\xi^{\alpha}_{(\chi)} = \delta^{\alpha}{}_{\chi}$ and $\xi^{\alpha}_{(\phi)} = \delta^{\alpha}{}_{\phi}$) are space-like. The Killing vectors are generators of the isometry group $\mathbb{R}^1_t \times U_\chi(1) \times U_\phi(1)$. Thus, the 5-dimensional Weyl solution can be presented as follows:

$$ds^2 = -e^{2U_1}dt^2 + e^{2\nu}(dz^2 + d\rho^2) + e^{2U_2}d\chi^2 + e^{2U_3}d\phi^2 \,, \tag{3.6}$$

where $t, z \in (-\infty, \infty)$, $\rho \in (0, \infty)$, and $\chi, \phi \in [0, 2\pi)$. The metric functions U_i, $i = 1, 2, 3$, and ν depend on the coordinates ρ and z. Each of the functions U_i solves the 3-dimensional Laplace equation (3.2) with the following constraint:

$$U_1 + U_2 + U_3 = \ln \rho \,. \tag{3.7}$$

If the functions U_i are known, the function ν can be derived by the line integral (3.5) using the following expressions:

$$\begin{aligned} \nu_{,\rho} &= -\rho(U_{1,\rho}U_{2,\rho} + U_{1,\rho}U_{3,\rho} + U_{2,\rho}U_{3,\rho} \\ &- U_{1,z}U_{2,z} - U_{1,z}U_{3,z} - U_{2,z}U_{3,z}) \,, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \nu_{,z} &= -\rho(U_{1,\rho}U_{2,z} + U_{1,\rho}U_{3,z} + U_{2,\rho}U_{3,z} \\ &+ U_{1,z}U_{2,\rho} + U_{1,z}U_{3,\rho} + U_{2,z}U_{3,\rho}) \,. \end{aligned} \tag{3.9}$$

The structure of the 5-dimensional Weyl solution can be understood as follows: Given three solutions U_i of the Laplace equation (3.2) which satisfy the constraint (3.7), then norms of the Killing vectors are defined, and with the choice of the integration constant in the line integral for the function ν the space-time geometry is constructed. Because Eq. (3.2) for the metric functions U_i is linear, the superposition principle can be applied for their construction.

Here we shall consider a 5-dimensional Weyl solution representing a background Weyl solution defined by $\widetilde{U}_i$ and $\widetilde{\nu}$, which is distorted by the external, static, axisymmetric fields defined by $\widehat{U}_i$ and $\widehat{\nu}$. The metric functions of the corresponding space-time are

$$U_i = \widetilde{U}_i + \widehat{U}_i \,, \;\; \nu = \widetilde{\nu} + \widehat{\nu} \,, \tag{3.10}$$

where according to the constraint (3.7), we have

$$\widetilde{U}_1 + \widetilde{U}_2 + \widetilde{U}_3 = \ln \rho \,, \;\; \widehat{U}_1 + \widehat{U}_2 + \widehat{U}_3 = 0 \,. \tag{3.11}$$

In what follows, we shall consider static distortion due to the external gravitational fields of remote masses whose configuration obeys the spatial symmetry of $U_\chi(1) \times U_\phi(1)$. Accordingly, we define

$$\widetilde{U}_1 := \widetilde{U} + \widetilde{W} + \ln \rho \,, \;\; \widetilde{U}_2 := -\widetilde{W} \,, \;\; \widetilde{U}_3 := -\widetilde{U} \,, \tag{3.12}$$

$$\widetilde{\nu} := \widetilde{V} + \widetilde{U} + \widetilde{W} \,, \tag{3.13}$$

$$\widehat{U}_1 := \widehat{U} + \widehat{W} \,, \;\; \widehat{U}_2 := -\widehat{W} \,, \;\; \widehat{U}_3 := -\widehat{U} \,, \tag{3.14}$$

$$\widehat{\nu} := \widehat{V} + \widehat{U} + \widehat{W} \,. \tag{3.15}$$

Here the distortion fields $\widehat{U}$ and $\widehat{V}$ define the external gravitational fields, and $\widehat{V}$ defines the interaction between the fields themselves and the background space-time. Then, the metric (3.6) takes the following generalized Weyl form[1] :

$$\begin{aligned} ds^2 \;\; &= \;\; e^{2(\widetilde{U}+\widetilde{W}+\widehat{U}+\widehat{W})}[-\rho^2 dt^2 + e^{2(\widetilde{V}+\widehat{V})}(dz^2 + d\rho^2)] \\ &+ \;\; e^{-2(\widetilde{W}+\widehat{W})}d\chi^2 + e^{-2(\widetilde{U}+\widehat{U})}d\phi^2 \,. \end{aligned} \tag{3.16}$$

The background fields $\widetilde{U}$ and $\widetilde{W}$ satisfy the 3-dimensional Laplace equation (3.2), and the function $\widetilde{V}$ can be derived by the line integral (3.5) using the expressions

$$\widetilde{V}_{,\rho} \;\; = \;\; \rho\,(\widetilde{U}^2_{,\rho} + \widetilde{W}^2_{,\rho} + \widetilde{U}_{,\rho}\widetilde{W}_{,\rho} - \widetilde{U}^2_{,z} - \widetilde{W}^2_{,z} - \widetilde{U}_{,z}\widetilde{W}_{,z}) \,, \tag{3.17}$$

$$\widetilde{V}_{,z} \;\; = \;\; \rho\,(2\widetilde{U}_{,\rho}\widetilde{U}_{,z} + 2\widetilde{W}_{,\rho}\widetilde{W}_{,z} + \widetilde{U}_{,\rho}\widetilde{W}_{,z} + \widetilde{U}_{,z}\widetilde{W}_{,\rho}) \,. \tag{3.18}$$

The distortion fields $\widehat{U}$ and $\widehat{W}$ satisfy the 3-dimensional Laplace equation (3.2),

---

[1] The factor ρ^2 in g_{tt} is a result of the definition of the metric functions. It can be removed by specifying their explicit form. For example, the 5-dimensional flat space-time

$$ds^2 = -dt^2 + dx^2 + dy^2 + x^2 d\phi^2 + y^2 d\chi^2$$

can be derived from the metric (3.16) by taking $\widehat{U} = \widehat{W} = \widehat{V} = 0$ and using the following metric functions:

$$\widetilde{U} = -\ln|x| \,, \;\; \widetilde{W} = -\ln|y| \,, \;\; \widetilde{V} = \ln\left|\frac{xy}{x^2+y^2}\right| \,,$$

where $x^2 = \sqrt{\rho^2 + z^2} - z$ and $y^2 = \sqrt{\rho^2 + z^2} + z$.

and the function $\widehat{V}$ can be derived by the line integral (3.5) using the expressions

$$\begin{aligned}\widehat{V}_{,\rho} &= \rho\,(\widehat{U}_{,\rho}^2 + \widehat{W}_{,\rho}^2 + \widehat{U}_{,\rho}\widehat{W}_{,\rho} - \widehat{U}_{,z}^2 - \widehat{W}_{,z}^2 - \widehat{U}_{,z}\widehat{W}_{,z} \\ &+ \widetilde{U}_{,\rho}\widehat{W}_{,\rho} + \widetilde{W}_{,\rho}\widehat{U}_{,\rho} - \widetilde{U}_{,z}\widehat{W}_{,z} - \widetilde{W}_{,z}\widehat{U}_{,z} \\ &+ 2[\widetilde{U}_{,\rho}\widehat{U}_{,\rho} + \widetilde{W}_{,\rho}\widehat{W}_{,\rho} - \widetilde{U}_{,z}\widehat{U}_{,z} - \widetilde{W}_{,z}\widehat{W}_{,z}])\,, \qquad (3.19)\\ \widehat{V}_{,z} &= \rho\,(2\widehat{U}_{,\rho}\widehat{U}_{,z} + 2\widehat{W}_{,\rho}\widehat{W}_{,z} + \widehat{U}_{,\rho}\widehat{W}_{,z} + \widehat{U}_{,z}\widehat{W}_{,\rho}) \\ &+ \widetilde{U}_{,\rho}\widehat{W}_{,z} + \widetilde{U}_{,z}\widehat{W}_{,\rho} + \widetilde{W}_{,\rho}\widehat{U}_{,z} + \widetilde{W}_{,z}\widehat{U}_{,\rho} \\ &+ 2[\widetilde{U}_{,\rho}\widehat{U}_{,z} + \widetilde{U}_{,z}\widehat{U}_{,\rho} + \widetilde{W}_{,\rho}\widehat{W}_{,z} + \widetilde{W}_{,z}\widehat{W}_{,\rho}])\,. \qquad (3.20)\end{aligned}$$

In the following Sections we construct the metric representing a 5-dimensional distorted Schwarzschild-Tangherlini black hole and study its properties.

3.3 Distorted 5-dimensional vacuum black hole

3.3.1 5-Dimensional Schwarzschild-Tangherlini black hole

A 5-dimensional Schwarzschild-Tangherlini black hole [23] is given by the following metric:

$$ds^2 = -\left(1 - \frac{r_o^2}{r^2}\right)dt^2 + \left(1 - \frac{r_o^2}{r^2}\right)^{-1} dr^2 + r^2 d\omega_{(3)}^2\ , \qquad (3.21)$$

where $t \in (-\infty, +\infty)$, $r \in (0, \infty)$, and $d\omega_{(3)}^3$ is the metric on a 3-dimensional round sphere, which can be presented in the following form[2] :

$$d\omega_{(3)}^2 = d\zeta^2 + \sin^2\zeta\, d\vartheta^2 + \sin^2\zeta \sin^2\vartheta\, d\varphi^2\,, \qquad (3.22)$$

where $\zeta, \vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$ are the hyperspherical coordinates. The black hole event horizon is located at $r = r_o$, and the parameter r_o is related to the black hole mass M as follows:

$$r_o^2 = \frac{8M}{3\pi}\,. \qquad (3.23)$$

The space-time singularity is located at $r = 0$.

To bring the black hole metric (3.21) to the Weyl form (3.16), we use the Hopf coordinates $\lambda \in [0, \pi/2]$ and $\chi, \phi \in [0, 2\pi)$ in which the metric $d\omega_{(3)}^2$ reads

$$d\omega_{(3)}^2 = d\lambda^2 + \cos^2\lambda\, d\chi^2 + \sin^2\lambda\, d\phi^2\,. \qquad (3.24)$$

[2] Note that the metric functions are finite at $|\eta| = |\cos\theta|$

Thus, χ and ϕ are Killing coordinates. The space-time (3.21), (3.24) admits the following orthogonal, commuting Killing vectors:

$$\xi^\alpha_{(t)} = \delta^\alpha_{\ t}\,,\ \ \xi^\alpha_{(\chi)} = \delta^\alpha_{\ \chi}\,,\ \ \xi^\alpha_{(\phi)} = \delta^\alpha_{\ \phi}\,, \tag{3.25}$$

where $\xi^\alpha_{(t)}$ is timelike outside the black hole horizon, and $\xi^\alpha_{(\phi)}$, $\xi^\alpha_{(\chi)}$ are spacelike vectors whose fixed points belong to the orthogonal "axes" $\lambda = 0$ and $\lambda = \pi/2$, respectively. The Hopf coordinates are illustrated in Fig. 3.1.

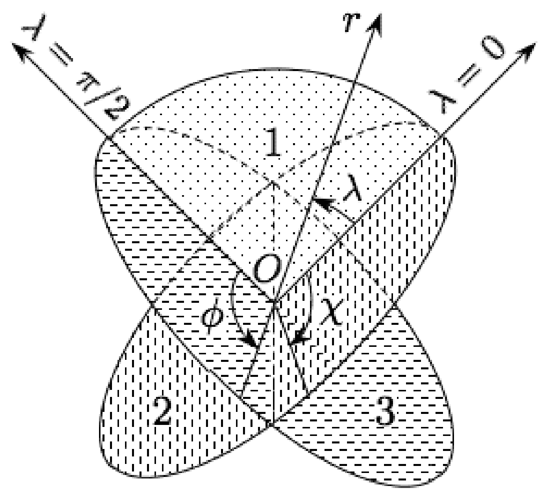

Figure 3.1: The Hopf coordinates (λ, χ, ϕ). The fixed points of the Killing vectors $\xi^\alpha_{(\phi)}$ and $\xi^\alpha_{(\chi)}$ belong to the "axes" defined by $\lambda = 0$ and $\lambda = \pi/2$, respectively. The coordinate origin O is a fixed point of the isometry group $U_\chi(1) \times U_\phi(1)$. Planes 1, 2, and 3, embedded into 4-dimensional space, are orthogonal to each other.

It is convenient to introduce the following coordinate transformations:

$$r = \frac{r_o}{\sqrt{2}}(\eta+1)^{1/2}\,,\ \ \eta \in (-1, \infty), \tag{3.26}$$

$$\lambda = \theta/2\,,\ \ \theta \in [0, \pi]\,. \tag{3.27}$$

In the new coordinates (η, θ) the black hole horizon and singularity are located at $\eta = 1$ and $\eta = -1$, and the black hole interior and exterior regions are defined by $\eta \in (-1, 1)$ and $\eta \in (1, \infty)$, respectively. The metric (3.21), (3.24) takes the following form:

$$\begin{aligned} ds^2 &= -\frac{\eta-1}{\eta+1}dt^2 + \frac{r_o^2}{8}(\eta+1)\left[\frac{d\eta^2}{\eta^2-1} + d\theta^2\right. \\ &+ \left. 2(1+\cos\theta)d\chi^2 + 2(1-\cos\theta)d\phi^2\right]\,. \end{aligned} \tag{3.28}$$

This metric can be written in the Weyl form (3.16) by using the following coordinate transformations:

$$\rho = \frac{r_o^2}{4}\sqrt{\eta^2-1}\sin\theta\,,\quad z=\frac{r_o^2}{4}\eta\cos\theta\,. \tag{3.29}$$

It is more convenient to use (η,θ) coordinates instead of the cylindrical coordinates (ρ,z), which describe the space-time outside the black hole horizon and give additional coordinate singularities in the black hole interior region if analytically continued through the black hole horizon.

The functions $\widetilde{U}$, $\widetilde{W}$, and $\widetilde{V}$ in the coordinates (η,θ) take the following form:

$$e^{2\widetilde{U}} = \frac{4}{r_o^2}(\eta+1)^{-1}(1-\cos\theta)^{-1}\,, \tag{3.30}$$

$$e^{2\widetilde{W}} = \frac{4}{r_o^2}(\eta+1)^{-1}(1+\cos\theta)^{-1}\,, \tag{3.31}$$

$$e^{2\widetilde{V}} = \frac{r_o^2(\eta+1)^3\sin^2\theta}{8(\eta^2-\cos^2\theta)}\,. \tag{3.32}$$

One can check that subject to the transformations (3.29), the functions $\widetilde{U}$ and $\widetilde{W}$ satisfy the Laplace equation (3.2), and the function $\widetilde{V}$ satisfies equations (3.17) and (3.18).

3.3.2 Metric of a 5-dimensional distorted black hole

In the previous subsection we demonstrated that the metric of a 5-dimensional Schwarzschild-Tangherlini black hole can be written in the generalized Weyl form (3.16). Here we present the metric of a 5-dimensional vacuum black hole distorted by external gravitational fields. The fields sources are located at asymptotic infinity and not included into the metric at finite distances, i.e. we consider the vacuum space-time region at finite distances. As a result, the corresponding space-time is not asymptotically flat [3]. We consider the space-time near the black hole regular horizon, far away from the sources. In this case, the solution represents a *local black hole* in analogy with a 4-dimensional distorted vacuum black hole studied in [36]. We focus on the study of the space-time near the black hole horizon and its

[3] Assuming that the external sources are localized at finite distances rather than at infinity, the space-time can be analytically extended to achieve asymptotic flatness in the way described in [36] for a 4-dimensional distorted black hole.

interior region, $\eta \in (-1, 1)$. The corresponding metric is

$$\begin{aligned} ds^2 &= -\frac{\eta - 1}{\eta + 1} e^{2(\widehat{U}+\widehat{W})} dt^2 + \frac{r_o^2}{8}(\eta + 1)\left[e^{2(\widehat{V}+\widehat{U}+\widehat{W})} \left(\frac{d\eta^2}{\eta^2 - 1} + d\theta^2 \right) \right. \\ &+ \left. 2(1+\cos\theta) e^{-2\widehat{W}} d\chi^2 + 2(1-\cos\theta) e^{-2\widehat{U}} d\phi^2 \right] . \end{aligned} \tag{3.33}$$

Here $\widehat{U}$, $\widehat{W}$, and $\widehat{V}$ are function of η and θ. In the absence of distortion fields $\widehat{U}, \widehat{W}$, and $\widehat{V}$, this metric reduces to that of the Schwarzschild-Tangherlini black hole (3.28). The Laplace equation (3.2) and Eqs. (3.19) and (4.14) for the distortion fields $\widehat{U}, \widehat{W}$, and $\widehat{V}$ in the coordinates (η, θ) take the following form:

$$(\eta^2 - 1)\widehat{X}_{,\eta\eta} + 2\eta \widehat{X}_{,\eta} + \widehat{X}_{,\theta\theta} + \cot\theta \widehat{X}_{,\theta} = 0\,, \tag{3.34}$$

where $\widehat{X} := (\widehat{U}, \widehat{W})$, and

$$\begin{aligned} \widehat{V}_{,\eta} &= N\Big(\eta \left[(\eta^2 - 1)(\widehat{U}^2_{,\eta} + \widehat{W}^2_{,\eta} + \widehat{U}_{,\eta}\widehat{W}_{,\eta}) \;-\; \widehat{U}^2_{,\theta} - \widehat{W}^2_{,\theta} - \widehat{U}_{,\theta}\widehat{W}_{,\theta} \right] \\ &+ (\eta^2 - 1)\cot\theta \left[2\widehat{U}_{,\eta}\widehat{U}_{,\theta} + 2\widehat{W}_{,\eta}\widehat{W}_{,\theta} + \widehat{U}_{,\eta}\widehat{W}_{,\theta} + \widehat{U}_{,\theta}\widehat{W}_{,\eta} \right] \\ &+ \frac{3}{2}\eta \left[\widehat{U}_{,\eta} + \widehat{W}_{,\eta} \right] - (\eta^2 - 1)\frac{\cos\theta}{2\sin^2\theta} \left[\widehat{U}_{,\eta} - \widehat{W}_{,\eta} \right] \\ &+ \frac{3}{2}\cot\theta \left[\widehat{U}_{,\theta} + \widehat{W}_{,\theta} \right] + \frac{\eta}{2\sin\theta} \left[\widehat{U}_{,\theta} - \widehat{W}_{,\theta} \right] \Big) - \frac{3}{2}\left[\widehat{U}_{,\eta} + \widehat{W}_{,\eta} \right] , \end{aligned} \tag{3.35}$$

$$\begin{aligned} \widehat{V}_{,\theta} &= -N\Big((\eta^2 - 1)\cot\theta \left[(\eta^2 - 1)(\widehat{U}^2_{,\eta} + \widehat{W}^2_{,\eta} + \widehat{U}_{,\eta}\widehat{W}_{,\eta}) - \widehat{U}^2_{,\theta} - \widehat{W}^2_{,\theta} - \widehat{U}_{,\theta}\widehat{W}_{,\theta} \right] \\ &- \eta(\eta^2 - 1) \left[2\widehat{U}_{,\eta}\widehat{U}_{,\theta} + 2\widehat{W}_{,\eta}\widehat{W}_{,\theta} + \widehat{U}_{,\eta}\widehat{W}_{,\theta} + \widehat{U}_{,\theta}\widehat{W}_{,\eta} \right] - \frac{3}{2}\eta \left[\widehat{U}_{,\theta} + \widehat{W}_{,\theta} \right] \\ &+ (\eta^2 - 1)\frac{\cos\theta}{2\sin^2\theta} \left[\widehat{U}_{,\theta} - \widehat{W}_{,\theta} \right] + \frac{3}{2}(\eta^2 - 1)\cot\theta \left[\widehat{U}_{,\eta} + \widehat{W}_{,\eta} \right] \\ &+ \frac{\eta(\eta^2 - 1)}{2\sin\theta} \left[\widehat{U}_{,\eta} - \widehat{W}_{,\eta} \right] \Big) - \frac{3}{2}\left[\widehat{U}_{,\theta} + \widehat{W}_{,\theta} \right] . \end{aligned} \tag{3.36}$$

Here $N = \sin^2\theta(\eta^2 - \cos^2\theta)^{-1}$ is singular along the lines $\eta = \pm\cos\theta$. However, the function $\widehat{V}$, which is given explicitly in the next Section, is regular along these lines.

If the distortion fields $\widehat{U}$ and $\widehat{W}$ are known, the function $\widehat{V}$ can be derived by the following line integral:

$$\hat{V}(\eta, \theta) = \int_{(\eta_0, \theta_0)}^{(\eta, \theta)} \left[\hat{V}_{,\eta'}(\eta', \theta')d\eta' + \hat{V}_{,\theta'}(\eta', \theta')d\theta' \right] . \tag{3.37}$$

The integral can be taken along any path connecting the points (η_0, θ_0) and (η, θ). Thus, the field $\widehat{V}$ is defined up to arbitrary constant of integration corresponding to the choice of a point (η_0, θ_0). This constant can be chosen to eliminate conical singularities, at least along one connected component of one "axis".

Let us note that the distortion fields $\widehat{U}$ and $\widehat{W}$ define norms of the Killing vectors $\xi^\alpha_{(\phi)}$ and $\xi^\alpha_{(\chi)}$, respectively. Thus, exchange between the "axes" $\theta = 0$ and $\theta = \pi$ is given by the following transformation:

$$\begin{aligned} (\theta, \chi, \phi) \quad &\rightarrow \quad (\pi - \theta, \phi, \chi)\,, \\ \left[\widehat{U}(\eta,\theta), \widehat{W}(\eta,\theta)\right] \quad &\rightarrow \quad \left[\widehat{W}(\eta,\theta), \widehat{U}(\eta,\theta)\right]\,. \end{aligned} \tag{3.38}$$

According to Eqs. (3.35)-(3.37), the distortion field $\widehat{V}$, and hence the metric (3.33), do not change under this transformation, as it has to be.

The distorted black hole horizon is defined by $\eta = 1$. It is regular, if the space-time invariants are finite on the horizon, and there are no conical singularities along the axes of symmetry, and thus, on the horizon. According to the results presented in Section 3.7, the Kretschmann scalar is regular on the black hole horizon if the horizon surface is a regular, totally geodesic surface and its surface gravity is constant. It follows that the distortion fields $\widehat{U}$, $\widehat{W}$, and $\widehat{V}$ must be smooth on a regular horizon. The distortion fields explicitly given in the next Section satisfy this condition.

The metric (3.33) has no conical singularities along the "axes" $\theta = 0$ and $\theta = \pi$, if the space there is locally flat. The no-conical-singularity condition can be formulated as follows: Let us consider a spacelike Killing vector $\xi^\alpha_{(\varphi)} = \delta^\alpha{}_\varphi$, whose orbits are compact near the corresponding symmetry axis defined by $y = y_0$. Let 2π be the period of the Killing coordinate φ, and let

$$dl^2 = A(y)d\varphi^2 + B(y)dy^2\,, \tag{3.39}$$

be a metric of a 2-dimensional surface near the symmetry axis. Then, there is no-conical-singularity corresponding to the symmetry axis if the ratio of the $\xi^\alpha_{(\varphi)}$ orbit circumference at the vicinity of the symmetry axis to the orbit radius, which is defined on the 2-dimensional surface, is equal to 2π, i.e.,

$$\lim_{y\to y_0} \frac{\int_0^{2\pi} A^{1/2}(y)d\varphi}{\int_{y_0}^{y} B^{1/2}(y')dy'} = \lim_{y\to y_0} \frac{2\pi|A_{,y}(y)|}{2\sqrt{A(y)B(y)}} = 2\pi\,. \tag{3.40}$$

If the ratio is less than 2π we have angular deficit, and if it is greater than 2π we have angular excess.

Assuming that the distortion fields $\widehat{U}$ and $\widehat{W}$ are smooth on the "axes", the no-conical-singularity condition for the metric (3.33) and for the "axis" $\theta = 0$, where $(x, y) = (\phi, \theta)$, reads

$$(\widehat{V} + 2\widehat{U} + \widehat{W})|_{\theta=0} = 0\,; \tag{3.41}$$

for the "axis" $\theta = \pi$, where $(x, y) = (\chi, \theta)$, it is given by

$$(\widehat{V} + \widehat{U} + 2\widehat{W})|_{\theta=\pi} = 0\,. \tag{3.42}$$

3.4 Solution

In this Section we derive a solution representing a distorted 5-dimensional vacuum black hole. We start with the Laplace equation (3.34) for the distortion fields $\widehat{U}$ and $\widehat{W}$. In the cylindrical coordinates (ρ, z) (see, (3.29)) the solution is well known and has the following form:

$$\widehat{X}(\rho, z) = \sum_{n\geq 0} \left[A_n\, r^n + B_n\, r^{-(n+1)}\right] P_n(\cos\vartheta)\,, \tag{3.43}$$

where

$$r = (\rho^2 + z^2)^{1/2}\,,\ \ \cos\vartheta = z/r\,, \tag{3.44}$$

and $P_n(\cos\vartheta)$ are the Legendre polynomials of the first kind. The coefficients A_n and B_n in the expansion (3.43) are called the interior and the exterior multipole moments, respectively (see, e.g., [55] and [56]). Distortion fields defined by the exterior multipole moments B_n's alone correspond to asymptotically flat solutions. However, according to the uniqueness theorem formulated in [57], a Schwarzschild-Tangherlini black hole is the only d-dimensional asymptotically flat static vacuum black hole which has non-degenerate regular event horizon. Note that a combination of the distortion fields corresponding to the exterior and the interior multipole moments makes the black hole horizon ($\rho = 0$, $z \in [-r_o^2/4, r_o^2/4]$) singular, because the terms in (3.43) proportional to the A_n's cannot cancel out the divergency at $\rho = z = 0$ due to the terms proportional to the B_n's. Thus, to have a regular horizon we shall consider external sources, whose distortion fields are defined by the interior multipole moments A_n's alone.

Applying the coordinate transformations (3.29) to expressions (3.43) and (3.44) we derive

$$\widehat{U}(\eta,\theta) = \sum_{n\geq 0} a_n\, R^n P_n(\eta\cos\theta/R)\,, \tag{3.45}$$

$$\widehat{W}(\eta,\theta) = \sum_{n\geq 0} b_n\, R^n P_n(\eta\cos\theta/R)\,, \tag{3.46}$$

$$R = (\eta^2 - \sin^2\theta)^{1/2}\,, \tag{3.47}$$

where the coefficients a_n's and b_n's define the distortion fields $\widehat{U}$ and $\widehat{W}$, respectively[4], despite the fact that R is zero for $\eta^2 = \sin^2\theta$ and imaginary for $\eta^2 < \sin^2\theta$. We shall call these coefficients *multipole moments*. In a 4-dimensional space-time, a relation of the multipole moments to their relativistic analogues was discussed in [59]. A general formalism, which includes both the Thorne [60] and the Geroch-Hansen (see, e.g., [61–64]) 4-dimensional relativistic multipole moments is presented in [65]. For a relation between the Thorne [60] and the Geroch-Hansen relativistic multipole moments, see [66, 67].

By analogy with the 4-dimensional case (see, e.g., [68, 69]) the distortion field $\widehat{V}$ can be presented as a sum of terms linear and quadratic in the multipole moments as follows:

$$\widehat{V} = \widehat{V}_1 + \widehat{V}_2\,, \tag{3.48}$$

$$\widehat{V}_1(\eta,\theta) = -\sum_{n\geq 0} 3(a_n/2 + b_n/2)R^n P_n - \sum_{n\geq 1}\Big\{(a_n + b_n/2)\sum_{l=0}^{n-1}(\eta - \cos\theta)R^l P_l + (a_n/2 + b_n)\sum_{l=0}^{n-1}(-1)^{n-l}(\eta + \cos\theta)R^l P_l\Big\}\,, \tag{3.49}$$

$$\widehat{V}_2(\eta,\theta) = \sum_{n,k\geq 1}\frac{nk}{n+k}(a_n a_k + a_n b_k + b_n b_k)R^{n+k}[P_n P_k - P_{n-1}P_{k-1}]\,, \tag{3.50}$$

$$P_n \equiv P_n(\eta\cos\theta/R)\,. \tag{3.51}$$

This form of the distortion field $\widehat{V}$ corresponds to a particular choice of the con-

[4] Using the series expansion of the Legendre polynomials (see, e.g., [58], p. 419)

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}x^{n-2k}\,,$$

where $\lfloor x\rfloor$ is the floor function, one can show that each term $R^n P_n(\eta\cos\theta/R)$ in the expansions (3.45) and (3.46) is real valued and regular even when $\eta^2 \leq \sin^2\theta$.

stant of integration defined by the initial point (η_0, θ_0) in the line integral (3.37). Because we have two "axes", for general a_n and b_n we cannot find such a constant that both the no-conical-singularity conditions (3.41) and (3.42) are satisfied simultaneously. To satisfy these conditions we have to impose an additional constraint on the multipole moments a_n's and b_n's. Using the solution (3.45)–(3.51), the no-conical-singularity conditions (3.41) and (3.42), and the symmetry property of the Legendre polynomials

$$P_n(-x) = (-1)^n P_n(x)\,, \tag{3.52}$$

we derive the following constraint on the multipole moments a_n's and b_n's:

$$\sum_{n\geq 0}(a_{2n} - b_{2n}) + 3\sum_{n\geq 0}(a_{2n+1} + b_{2n+1}) = 0\,. \tag{3.53}$$

In what follows, we shall refer to the constraint (3.53) as the no-conical-singularity condition for the distorted black hole. One can see that the distortion fields $\widehat{U}$, $\widehat{W}$, and $\widehat{V}$ given by expressions (3.45)–(3.51) are smooth on the black hole horizon. Thus, according to the discussion given in the previous Section, the horizon is regular, and this solution represents a local black hole distorted by the external static fields. For this solution the transformation (3.38) takes the following form:

$$\begin{aligned} (\theta, \chi, \phi) \quad &\rightarrow \quad (\pi - \theta, \phi, \chi)\,, \\ [a_n, b_n] \quad &\rightarrow \quad [(-1)^n b_n, (-1)^n a_n]\,. \end{aligned} \tag{3.54}$$

An additional restriction on values of the multipole moments follows from the strong energy condition (SEC) imposed on the external sources of the distortion fields, which follows from the positive mass theorem in a 5-dimensional space-time proven in [70]. If these sources are included, the Einstein equations are not vacuum. In particular, for the metric (3.33) the $\{tt\}$ component of the Einstein equations reads

$$R_{\alpha\beta}\delta^{\alpha}{}_{t}\delta^{\beta}{}_{t} = 8\pi\left(T_{\alpha\beta} - \frac{T^{\gamma}{}_{\gamma}}{3}g_{\alpha\beta}\right)\delta^{\alpha}{}_{t}\delta^{\beta}{}_{t} = \frac{\eta-1}{\eta+1}e^{2(\widehat{U}+\widehat{W})}\left(\triangle\widehat{U} + \triangle\widehat{W}\right)\,, \tag{3.55}$$

where $T_{\alpha\beta}$ is the energy-momentum tensor representing the sources. If the sources satisfy SEC, the right hand side of Eq. (3.55) must be non-negative. The Laplace operator $\triangle$ is a negative operator, hence, SEC implies that

$$\widehat{U} + \widehat{W} \leq 0\,, \tag{3.56}$$

assuming that $\widehat{U}+\widehat{W}=0$ at asymptotically flat infinity. In particular, the condition (3.56) implies that on the black hole horizon, on the "axes" $\theta=0$ and $\theta=\pi$, we have

$$\sum_{n\geq 0}(\pm 1)^n(a_n+b_n)\leq 0\,. \tag{3.57}$$

According to the structure of the 5-dimensional Weyl solution, one has an arbitrary choice to define the distortion fields $\widehat{U}$ and $\widehat{W}$ by specifying the corresponding source functions, which can take any real values (positive or negative), assuming that the SEC (3.57) is satisfied.

To illustrate the effect of the distortion fields on the black hole, we restrict ourselves to the lower order (up to the quadrupole) multipole moments. Values of these moments are subject to the conditions (3.53) and (3.57),

$$a_0-b_0+a_2-b_2+3(a_1+b_1)=0\,, \tag{3.58}$$

$$a_0+b_0\pm(a_1+b_1)+a_2+b_2\leq 0\,. \tag{3.59}$$

The simplest type of distortion is due to a monopole whose values are such that $a_0=b_0\leq 0$. However, this distortion is trivial, for it does not break the spherical symmetry of a 5-dimensional Schwarzschild-Tangherlini black hole. The next, less trivial, distortion is due to a dipole. Taking $\widehat{U}$ as a dipole distortion and $\widehat{W}$ as a monopole distortion and using expression (3.45), we derive the dipole-monopole distortion of the form

$$\begin{aligned}\widehat{U} &= a_0+a_1\,\eta\cos\theta\,,\ \widehat{W}=a_0+3a_1\,,\\ &\quad 2a_0+(3\pm 1)a_1\leq 0\,.\end{aligned} \tag{3.60}$$

According to the transformation (3.54), taking $\widehat{U}$ as a monopole distortion and $\widehat{W}$ as a dipole one corresponds to exchange between the "axes" $\theta=0$ and $\theta=\pi$, and does not give anything new. Note that in next Section we shall bring the metric (3.33) to a dimensionless form such that the monopole does not appear in the solution. Finally, we consider the quadrupole-quadrupole distortion of the form

$$\begin{aligned}\widehat{U} &= \widehat{W}=a_0+\frac{a_2}{2}(1-\eta^2+(3\eta^2-1)\cos^2\theta)\,,\\ &\quad a_0+a_2\leq 0\,.\end{aligned} \tag{3.61}$$

In what follows, to study the distorted black hole we shall consider the dipole-monopole (3.60) and the quadrupole-quadrupole (3.61) distortion fields.

3.5 Symmetries and boundary values of the distortion fields

The space-time (3.33) is symmetric under the continuous group of isometries $\mathbb{R}^1_t \times U_\chi(1) \times U_\phi(1)$. This means that the essential features of the space-time geometry are confined to the (η, θ) plane of orbits, which is invariant under the group of transformations. To study the black hole interior, i.e., the region between the black hole horizon and singularity, it is convenient to introduce instead of η another coordinate ψ as follows:

$$\eta = \cos\psi \,, \quad \psi \in (0, \pi) \,. \tag{3.62}$$

Thus, $\psi = 0$ and $\psi = \pi$ define the black hole horizon and singularity, respectively. The metric on the plane (ψ, θ) corresponding to the black hole interior is

$$d\Sigma^2 = \frac{r_o^2}{8}(1 + \cos\psi)e^{2(\widehat{V}+\widehat{U}+\widehat{W})}\left(-d\psi^2 + d\theta^2\right) \,. \tag{3.63}$$

We see that the coordinate ψ is timelike. The corresponding conformal diagram illustrating the geometry of the black hole interior is presented in Fig. 3.2. In the diagram, the lines $\psi \pm \theta = const$ are null rays propagating within the 2-dimensional plane (ψ, θ). Three of such rays are illustrated in Fig. 3.2 by arrows. One of the rays starts at point A on the horizon, goes through the "axis" $\theta = \pi$, and terminates at the singularity, at point B.

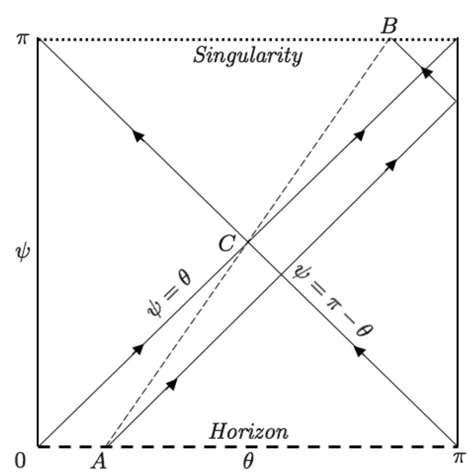

Figure 3.2: Conformal diagram for the (ψ,θ) plane of orbits corresponding to the black hole interior. Arrows illustrate propagation of future directed null rays. Points A and B connected by one of such rays are symmetric with respect to the central point $C(\pi/2, \pi/2)$.

Consider a transformation R_C representing reflection of a point on the (ψ, θ) plane with respect to the central point C

$$R_C: \ (\psi, \theta) \ \rightarrow (\pi - \psi, \pi - \theta) \,. \tag{3.64}$$

This transformation defines a map between functions defined on the plane (ψ, θ), which has the following form:

$$f^* = R_C^*(f): \ \ f^*(\psi, \theta) = f(\pi - \psi, \pi - \theta) \,. \tag{3.65}$$

The coordinates of the points A and B are related by the reflection R_C. Thus, R_C^* is a map between functions defined on the black hole horizon and singularity. Applying this map to the distortion fields $\widehat{U}$, $\widehat{W}$, and $\widehat{V}$, we derive

$$\begin{aligned} \widehat{U}(\pi - \psi, \pi - \theta) &= \widehat{U}(\psi, \theta) \,, & (3.66)\\ \widehat{W}(\pi - \psi, \pi - \theta) &= \widehat{W}(\psi, \theta) \,, & (3.67)\\ \widehat{V}_1(\pi - \psi, \pi - \theta) &= -\widehat{V}_1(\psi, \theta) - 3[\widehat{U}(\psi, \theta) + \widehat{W}(\psi, \theta)] \,, & (3.68)\\ \widehat{V}_2(\pi - \psi, \pi - \theta) &= \widehat{V}_2(\psi, \theta) \,. & (3.69) \end{aligned}$$

We shall use these relations to define values of the distortion fields on the black hole horizon and singularity, as well as on the symmetry "axes".

To begin with let us introduce the following notations:

$$\begin{aligned} u_0 &:= \sum_{n\geq 0} a_{2n} \,, \ \ u_1 := \sum_{n\geq 0} a_{2n+1} \,, & (3.70)\\ w_0 &:= \sum_{n\geq 0} b_{2n} \,, \ \ w_1 := -\sum_{n\geq 0} b_{2n+1} \,, & (3.71) \end{aligned}$$

Then the no-conical-singularity condition (3.53) can be written as

$$u_0 + 3u_1 = w_0 + 3w_1 \,. \tag{3.72}$$

In addition, we define the following functions:

$$\begin{aligned} u_\pm(\sigma) &:= \sum_{n\geq 0} (\pm 1)^n a_n \cos^n(\sigma) - u_0 \,, & (3.73)\\ w_\pm(\sigma) &:= \sum_{n\geq 0} (\pm 1)^n b_n \cos^n(\sigma) - w_0 \,, & (3.74) \end{aligned}$$

where $\sigma := (\psi, \theta)$. Thus, for the dipole-monopole distortion (3.60) we have

$$\begin{aligned} u_{\pm}(\sigma) &= \pm a_1 \cos\sigma\,,\ \ u_0 = a_0\,,\ \ u_1 = a_1\,, \\ w_{\pm}(\sigma) &= 0\,,\ \ w_0 = a_0 + 3a_1\,,\ \ w_1 = 0\,, \end{aligned} \tag{3.75}$$

and for the quadrupole-quadrupole distortion (3.61) we have

$$\begin{aligned} u_{\pm}(\sigma) &= w_{\pm}(\sigma) = -a_2 \sin^2\sigma\,, \\ u_0 &= w_0 = a_0 + a_2\,,\ \ u_1 = w_1 = 0\,. \end{aligned} \tag{3.76}$$

Using the definitions above it is convenient to introduce renormalized distortion fields, which do not depend on the monopole moments a_0 and b_0, as follows:

$$\mathcal{U}(\psi,\theta) := \widehat{U}(\psi,\theta) - u_0 - 3u_1\,, \tag{3.77}$$

$$\mathcal{W}(\psi,\theta) := \widehat{W}(\psi,\theta) - w_0 - 3w_1\,, \tag{3.78}$$

$$\mathcal{V}(\psi,\theta) := \widehat{V}(\psi,\theta) + \frac{3}{2}\left[u_0 + w_0 + 3(u_1 + w_1)\right]. \tag{3.79}$$

With the aid of the expressions above we derive values of the renormalized distortion fields on the black hole horizon

$$\mathcal{U}(0,\theta) := u_+(\theta) - 3u_1\,, \tag{3.80}$$

$$\mathcal{W}(0,\theta) := w_+(\theta) - 3w_1\,, \tag{3.81}$$

$$\mathcal{V}(0,\theta) := 4(u_1 + w_1)\,, \tag{3.82}$$

and the singularity

$$\mathcal{U}(\pi,\theta) = u_-(\theta) - 3u_1\,, \tag{3.83}$$

$$\mathcal{W}(\pi,\theta) = w_-(\theta) - 3w_1\,, \tag{3.84}$$

$$\mathcal{V}(\pi,\theta) = -3(u_-(\theta) + w_-(\theta)) + 5(u_1 + w_1)\,, \tag{3.85}$$

as well as on the "axis" $\theta = 0$

$$\mathcal{U}(\psi,0) = u_+(\psi) - 3u_1\,, \tag{3.86}$$

$$\mathcal{W}(\psi,0) = w_+(\psi) - 3w_1\,, \tag{3.87}$$

$$\mathcal{V}(\psi,0) = -2u_+(\psi) - w_+(\psi) + 3(2u_1 + w_1)\,, \tag{3.88}$$

and on the "axis" $\theta = \pi$

$$\begin{aligned} \mathcal{U}(\psi,\pi) &= u_-(\psi) - 3u_1\,, & (3.89)\\ \mathcal{W}(\psi,\pi) &= w_-(\psi) - 3w_1\,, & (3.90)\\ \mathcal{V}(\psi,\pi) &= -u_-(\psi) - 2w_-(\psi) + 3(u_1 + 2w_1)\,. & (3.91) \end{aligned}$$

In what follows, we consider for convenience the dimensionless form of the metric dS^2, which is related to the metric ds^2 as follows:

$$ds^2 = \Omega^2 dS^2, \tag{3.92}$$

where

$$\Omega^2 = \frac{1}{\kappa^2} e^{2(u_0 - u_1 + w_0 - w_1)}\,, \tag{3.93}$$

is the conformal factor, and κ is the surface gravity of the distorted black hole corresponding to $\xi^\alpha_{(t)} = \delta^\alpha{}_t$,

$$\kappa = \left.\frac{e^{-\widehat{V}}}{r_0}\right|_{\eta=1} = \frac{1}{r_0} e^{(3u_0 + u_1 + 3w_0 + w_1)/2}\,. \tag{3.94}$$

Note that the space-time (3.33) is not asymptotically flat, so the surface gravity (3.94) is defined only up to an arbitrary red-shift factor. The dimensionless metric is given by

$$\begin{aligned} dS^2 &= -\frac{\eta - 1}{\eta + 1} e^{2(\mathcal{U}+\mathcal{W})} dT^2 + \frac{1}{8}(\eta + 1)\left[e^{2(\mathcal{V}+\mathcal{U}+\mathcal{W})} \left(\frac{d\eta^2}{\eta^2 - 1} + d\theta^2 \right) \right.\\ &+ \left. 2(1 + \cos\theta) e^{-2\mathcal{W}} d\chi^2 + 2(1 - \cos\theta) e^{-2\mathcal{U}} d\phi^2 \right], \end{aligned} \tag{3.95}$$

where the dimensionless time T is defined as follows:

$$T = \kappa\, e^{4(u_1 + w_1)} t\,. \tag{3.96}$$

Using the transformation (3.62), one can present the metric (3.95) in $(T, \psi, \theta, \chi, \phi)$ coordinates, which are more convenient for analysis of the black hole interior.

3.6 Space-time near the horizon

3.6.1 Intrinsic curvature of the horizon surface

In this Section we study geometry of the 3-dimensional distorted horizon surface of the space-time (3.95), defined by $T = const$, $\eta = 1$. The metric of the horizon surface reads

$$\begin{aligned} d\Sigma_+^2 &= \frac{1}{4}\Big(e^{2(u_+(\theta)+w_+(\theta)+u_1+w_1)}d\theta^2 + 2(1+\cos\theta)e^{-2(w_+(\theta)-3w_1)}d\chi^2 \\ &+ 2(1-\cos\theta)e^{-2(u_+(\theta)-3u_1)}d\phi^2\Big). \end{aligned} \tag{3.97}$$

Here and in what follows, the '$+$' subscript stands for a quantity defined on the black hole horizon surface. Using this metric one can calculate the dimensionless area of the black hole horizon surface,

$$\mathcal{A}_+ = 2\pi^2 e^{4(u_1+w_1)}. \tag{3.98}$$

The dimensional area is equal to

$$A_+ = \Omega^3 \mathcal{A}_+ = 2\pi^2 r_o^3 e^{-(u_1+w_1+3u_0+3w_0)/2}. \tag{3.99}$$

To study the geometry of a 2-dimensional surface, one can calculate its intrinsic (Gaussian) curvature invariant and illustrate its shape by an isometric embedding of the surface into a 3-dimensional flat space; one can calculate its extrinsic curvature as well. To study the geometry of a 3-dimensional hypersurface is not that simple, for there are generally more than one curvature invariant, and its isometric local embedding generally requires $3(3+1)/2 = 6$-dimensional flat space. However, if the hypersurface admits a group of isometries, one can analyze its geometry by studying the geometry of the sections of the isometry orbits. In our case the 3-dimensional hypersurface defined by the metric (3.97) admits a $U_\chi(1)\times U_\phi(1)$ group of isometries. As a result, we have (θ,χ) and (θ,ϕ) 2-dimensional sections. For completeness, we consider (χ,ϕ) 2-dimensional sections as well. Following an analysis of the horizon surface of a 5-dimensional black hole and black ring presented in [71], we define the Gaussian curvatures of the sections as the corresponding Riemann tensor components of the metric (3.97) calculated in an orthonormal frame,

$$K_{+\phi} := \frac{8(1-\cos\theta)}{\sin^2\theta} e^{-2(u_+(\theta)+u_1+4w_1)} \mathcal{R}_{+\theta\chi\theta\chi}\,, \tag{3.100}$$

$$K_{+\chi} := \frac{8(1+\cos\theta)}{\sin^2\theta} e^{-2(w_+(\theta)+w_1+4u_1)} \mathcal{R}_{+\theta\phi\theta\phi}\,, \tag{3.101}$$

$$K_{+\theta} := \frac{4}{\sin^2\theta} e^{2(u_+(\theta)+w_+(\theta)-3u_1-3w_1)} \mathcal{R}_{+\chi\phi\chi\phi}\,. \tag{3.102}$$

Explicit form of these expressions is presented in Appendix A. For a round 3-dimensional sphere, which represents the horizon surface of a 5-dimensional Schwarzschild-Tangherlini black hole, we have

$$K_{+\phi} = K_{+\chi} = K_{+\theta} = 1. \tag{3.103}$$

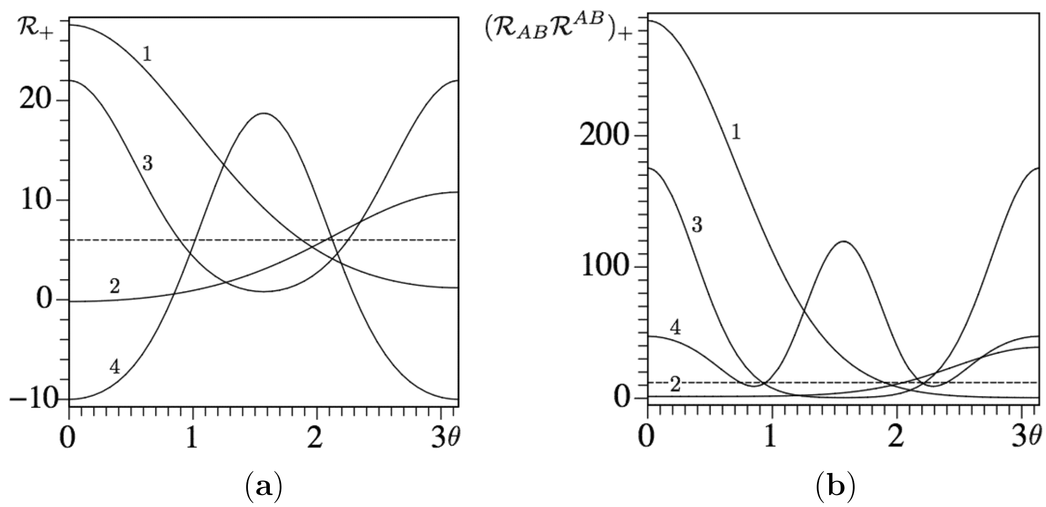

Figure 3.3: Intrinsic curvature invariants of the horizon surface. (**a**) Dimensionless Ricci scalar. (**b**) the trace of the square of the Ricci tensor. Dipole-monopole distortion: $a_1 = -1/5$, $b_1 = 0$ (line 1), $a_1 = 1/5$, $b_1 = 0$ (line 2). Quadrupole-quadrupole distortion: $a_2 = b_2 = -1/7$ (line 3), $a_2 = b_2 = 1/7$ (line 4). The horizontal dashed lines represent the dimensionless Ricci scalar and the trace of the square of the Ricci tensor of a Schwarzschild-Tangherlini black hole.

In the case of the distortion fields $\widehat{U} = 0$, $\widehat{W} \neq 0$ we have $K_{+\chi} = K_{+\theta}$, and in the case of the distortion fields $\widehat{U} \neq 0$, $\widehat{W} = 0$ we have $K_{+\phi} = K_{+\theta}$.

Components of the Ricci tensor corresponding to a 3-dimensional hypersurface are related to the Gaussian curvatures of the sections as follows:

$$\mathcal{R}^{\phi}_{+\phi} = K_{+\theta} + K_{+\chi}\,, \;\; \mathcal{R}^{\chi}_{+\chi} = K_{+\theta} + K_{+\phi}\,,$$
$$\mathcal{R}^{\theta}_{+\theta} = K_{+\chi} + K_{+\phi}\,. \tag{3.104}$$

The corresponding Ricci scalar and the trace of the square of the Ricci tensor are

$$\mathcal{R}_{+} = \mathcal{R}^{\phi}_{+\phi} + \mathcal{R}^{\chi}_{+\chi} + \mathcal{R}^{\theta}_{+\theta}\,, \tag{3.105}$$
$$(\mathcal{R}_{AB}\mathcal{R}^{AB})_{+} = (\mathcal{R}^{\phi}_{+\phi})^2 + (\mathcal{R}^{\chi}_{+\chi})^2 + (\mathcal{R}^{\theta}_{+\theta})^2\,. \tag{3.106}$$

The Ricci scalar $\mathcal{R}_{+}$ and the trace of the square of the Ricci tensor $(\mathcal{R}_{AB}\mathcal{R}^{AB})_{+}$ of the horizon surface are natural invariant measures of its intrinsic curvature. The dimensional Ricci scalar and the trace of the square of the Ricci tensor are equal to $\Omega^{-2}\mathcal{R}_{+}$ and $\Omega^{-4}(\mathcal{R}_{AB}\mathcal{R}^{AB})_{+}$, respectively.

Here we calculate the Gaussian curvatures of the sections for the dipole-monopole distortion (4.63),

$$K_{+\phi} = K_{+\theta} = e^{-2a_1(1+\cos\theta)}\left[1 + 2a_1(1-\cos\theta)\right]\,, \tag{3.107}$$
$$K_{+\chi} = e^{-2a_1(1+\cos\theta)}\left[1 - 2a_1(3+5\cos\theta) - 8a_1^2\sin^2\theta\right]\,, \tag{3.108}$$

and for the quadrupole-quadrupole distortion (3.76),

$$K_{+\phi} = k_{+}\,, \;\; K_{+\chi} = k_{-}\,,$$
$$k_{\pm} = e^{4a_2\sin^2\theta}\left[1 + 8a_2(1 \pm 2\cos\theta - 4\cos^2\theta) - 48a_2^2\cos^2\theta\sin^2\theta\right]\,, \tag{3.109}$$
$$K_{+\theta} = e^{4a_2\sin^2\theta}\left[1 - 8a_2\cos^2\theta - 16a_2^2\cos^2\theta\sin^2\theta\right]\,. \tag{3.110}$$

Using these expressions together with Eqs. (3.104)–(3.106) we can calculate the corresponding dimensionless Ricci scalar and the trace of the square of the Ricci tensor of the horizon surface. For an undistorted black hole the dimensionless Ricci scalar is $\mathcal{R}_{\mathrm{ST}+} = 6$, and the trace of the square of the Ricci tensor is $(\mathcal{R}_{AB}\mathcal{R}^{AB})_{\mathrm{ST}+} = 12$. The Ricci scalar and the trace of the square of the Ricci tensor are shown in Figs. 3.3(**a**) and (**b**), respectively. These figures illustrate that the intrinsic curvature of a distorted horizon surface strongly varies over it.

3.6.2 Shape of the horizon surface

Distortion fields change the shape of the horizon surface. To visualize the effect of the distortion fields on the horizon surface, we consider an isometric embedding of

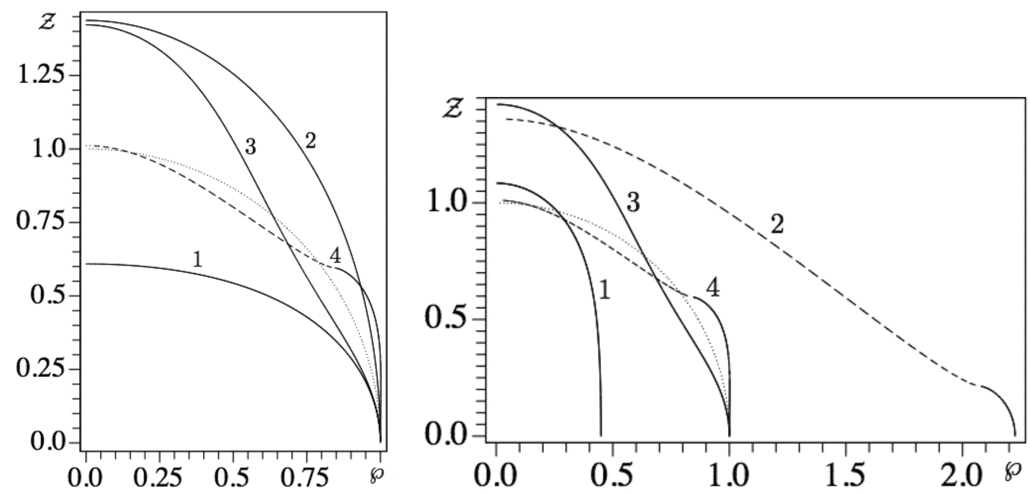

Figure 3.4: Rotational curves of the horizon surface. (**a**): section (θ, χ). (**b**): section (θ, ϕ). Dipole-monopole distortion: $a_1 = -1/5$, $b_1 = 0$ (line 1), $a_1 = 1/5$, $b_1 = 0$ (line 2). Quadrupole-quadrupole distortion: $a_2 = b_2 = -1/7$ (line 3), $a_2 = b_2 = 1/7$ (line 4). Regions of the sections embedded into pseudo-Euclidean space are illustrated by dashed lines. Dotted arcs of unit radius represent the horizon surface of an undistorted black hole.

its 2-dimensional sections into a flat 3-dimensional space with the following metric:

$$dl^2 = \epsilon d\mathcal{Z}^2 + d\wp^2 + \wp^2 d\varphi^2, \tag{3.111}$$

where $\epsilon = +1$ corresponds to Euclidean space, $\epsilon = -1$ corresponds to pseudo-Euclidean space, and $(\mathcal{Z}, \wp, \varphi)$ are the cylindrical coordinates.

The section (χ, ϕ) defined by $\theta = const$ represents a 2-dimensional torus whose radii are defined by the distortion fields. We shall consider the embedding of the (θ, χ) and (θ, ϕ) 2-dimensional sections, which according to the metric (3.97) are parametrized in the cylindrical coordinates as follows:

$$\mathcal{Z} = \mathcal{Z}(\theta)\,, \quad \wp = \wp(\theta)\,. \tag{3.112}$$

The geometry induced on the section (3.112) is given by

$$dl^2 = (\epsilon \mathcal{Z}_{,\theta}^2 + \wp_{,\theta}^2) d\theta^2 + \wp^2 d\varphi^2\,. \tag{3.113}$$

The metric of the section (θ, χ) defined from the metric (3.97) by $\phi = const$ reads

$$d\Sigma^2_{+\phi} = \frac{1}{4}\left(e^{2(u_+(\theta)+w_+(\theta)+u_1+w_1)}d\theta^2 + 2(1+\cos\theta)e^{-2(w_+(\theta)-3w_1)}d\chi^2\right). \quad (3.114)$$

Matching the metrics (3.113) and (3.114), we derive the embedding map

$$\begin{aligned} \varphi = \chi\,, \;\; \wp(\theta) &= \frac{1}{\sqrt{2}}(1+\cos\theta)^{1/2}e^{-w_+(\theta)+3w_1}\,, \\ \mathcal{Z}(\theta) &= \int_0^\theta \mathcal{Z}_{,\theta'}d\theta'\,, \\ \mathcal{Z}_{,\theta} &= \left[\epsilon\left(\frac{1}{4}e^{2(u_+(\theta)+w_+(\theta)+u_1+w_1)} - \wp^2_{,\theta}\right)\right]^{1/2}. \end{aligned} \quad (3.115)$$

The metric of the section (θ, ϕ) defined from the metric (3.97) by $\chi = const$ reads

$$d\Sigma^2_{+\chi} = \frac{1}{4}\left(e^{2(u_+(\theta)+w_+(\theta)+u_1+w_1)}d\theta^2 + 2(1-\cos\theta)e^{-2(u_+(\theta)-3u_1)}d\phi^2\right). \quad (3.116)$$

Matching the metrics (3.113) and (3.116), we derive the embedding map

$$\begin{aligned} \varphi = \phi\,, \;\; \wp(\theta) &= \frac{1}{\sqrt{2}}(1-\cos\theta)^{1/2}e^{-u_+(\theta)+3u_1}\,, \\ \mathcal{Z}(\theta) &= \int_\pi^\theta \mathcal{Z}_{,\theta'}d\theta'\,, \\ \mathcal{Z}_{,\theta} &= -\left[\epsilon\left(\frac{1}{4}e^{2(u_+(\theta)+w_+(\theta)+u_1+w_1)} - \wp^2_{,\theta}\right)\right]^{1/2}. \end{aligned} \quad (3.117)$$

Rotational curves illustrating embeddings of the sections (θ, χ) and (θ, ϕ) for the dipole-monopole (4.63) and the quadrupole-quadrupole (3.76) distortions are shown in Figs. 3.4(**a**) and 3.4(**b**), respectively. These curves belong to plane 1 in Fig. 3.1. To reconstruct the shape of the 3-dimensional horizon surface, we have to rotate these curves in planes 2 and 3 (see Fig. 3.1) around the "axes" $\lambda = \pi/2$ and $\lambda = 0$.

3.6.3 Metric near the horizon

The functions $u_+(\theta)$ and $w_+(\theta)$, which specify the geometry of the horizon surface, uniquely determine the space-time geometry in the vicinity of the black hole horizon. Using the expansion of the distortion fields $\widehat{U}$, $\widehat{W}$, and $\widehat{V}$ in the vicinity of the horizon (see Eqs. (B.8) and (B.11) in Appendix B) and the definition of the renormalized distortion fields (3.77)–(3.79), we derive an approximation for the metric (3.95) near the black hole horizon:

$$dS_+^2 = A_+ dT^2 + B_+(-d\psi^2 + d\theta^2) + C_+ d\chi^2 + D_+ d\phi^2 \,, \tag{3.118}$$

$$\begin{aligned}
A_+ &= \frac{1}{4} e^{2(u_+(\theta)+w_+(\theta)-3u_1-3w_1)} \Bigg[\psi^2 + \frac{1}{2}\Bigg(u_{+,\theta\theta} + w_{+,\theta\theta} \\
&+ \cot\theta (u_{+,\theta} + w_{+,\theta}) + \frac{1}{3} \Bigg) \psi^4 + \mathcal{O}(\psi^6) \Bigg] \,, \\
B_+ &= \frac{1}{4} e^{2(u_+(\theta)+w_+(\theta)+u_1+w_1)} \Bigg[1 + \frac{1}{2}\Bigg(u_{+,\theta\theta} + w_{+,\theta\theta} \\
&+ 2(u_{+,\theta}^2 + u_{+,\theta} w_{+,\theta} + w_{+,\theta}^2) - 2\cot\theta (u_{+,\theta} + w_{+,\theta}) \\
&- \frac{u_{+,\theta} - w_{+,\theta}}{\sin\theta} - \frac{1}{2} \Bigg) \psi^2 + \mathcal{O}(\psi^4) \Bigg] \,, \\
C_+ &= \frac{1}{2}(1 + \cos\theta) e^{-2(w_+(\theta)-3w_1)} \\
&\times \left[1 - \frac{1}{2}\left(w_{+,\theta\theta} + \cot\theta\, w_{+,\theta} + \frac{1}{2} \right) \psi^2 + \mathcal{O}(\psi^4) \right] \,, \\
D_+ &= \frac{1}{2}(1 - \cos\theta) e^{-2(u_+(\theta)-3u_1)} \\
&\times \left[1 - \frac{1}{2}\left(u_{+,\theta\theta} + \cot\theta\, u_{+,\theta} + \frac{1}{2} \right) \psi^2 + \mathcal{O}(\psi^4) \right] \,.
\end{aligned} \tag{3.119}$$

This approximation allows us to calculate the Kretschmann scalar $\mathcal{K} :=^{(5)} R_{\alpha\beta\gamma\delta}\, {}^{(5)}R^{\alpha\beta\gamma\delta}$, which is a space-time curvature invariant, at the horizon surface. In next Section we demonstrate that there is a simple relation between the space-time Kretschmann scalar [5] calculated on the horizon of a 5-dimensional, static, distorted black hole and the trace of the square of the Ricci tensor of its horizon surface, which is

$$\mathcal{K}_+ = 6(\mathcal{R}_{AB}\mathcal{R}^{AB})_+ \,. \tag{3.120}$$

This relation is valid not only for a distorted black hole given by a 5-dimensional Weyl solution, but also for an arbitrary distorted, asymmetric, static, vacuum black hole.

It is interesting to note that the same relation holds for 4D static space-times.

[5] Note that the Chern-Pontryagin scalar is zero in our case.

Namely, if we consider $\mathcal{R}_{AB}$ as the Ricci tensor of the 2D horizon surface of a 4D static asymmetric black hole, then the relation becomes (see, e.g., [51] and [72])

$$\mathcal{K}_+ = 3\mathcal{R}^2 \,. \tag{3.121}$$

Consequently, according to figures 3.3 and 3.4, the space-time curvature at the horizon is greater at the points where the horizon surface is more curved.

3.7 Space-time invariants

In this Section we derive a relation between the Kretschmann scalar $\mathcal{K}$ calculated on the horizon of a 5-dimensional static, asymmetric, distorted vacuum black hole and the trace of the square of the Ricci tensor, $\mathcal{R}_{AB}\mathcal{R}^{AB}$, of the horizon surface. The corresponding space-time admits the Killing vector $\xi^\alpha = \delta^\alpha{}_0$, $(x^0 := t)$, which is timelike in the domain of interest, $\xi^\alpha\xi_\alpha = g_{00} := -k^2 < 0$, and hypersurface orthogonal. The space-time metric $g_{\alpha\beta}$, $(\alpha, \beta, ... = 0, \dots, 4)$ can be presented in the form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -k^2dt^2 + \gamma_{ab}(x^c)dx^a dx^b \,, \tag{3.122}$$

where γ_{ab}, $a, b, c, ... = 1, \dots, 4$, is the metric on a 4-dimensional hypersurface $t = const$. The black hole horizon defined by $k = 0$ is a non-degenerate Killing horizon. One can show that the vacuum Einstein equations ${}^{(5)}R_{\alpha\beta} = 0$ for the metric (3.122) reduce to [6]

$$R_{ab} - k^{-1}\nabla_a\nabla_b k = 0 \,, \tag{3.123}$$

$$\nabla_a\nabla^a k = 0 \,, \tag{3.124}$$

where R_{ab} and ∇_a are the Ricci tensor and the covariant derivative defined with respect to the 4-dimensional metric γ_{ab}. Equation (3.124) implies that k is a harmonic function. Thus, k can be considered in each 4-dimensional hypersurface $t = const$. As a result, the metric (3.122) can be written in the following form:

$$ds^2 \;=\; -k^2dt^2 + \kappa^{-2}(k, x^C)dk^2 + h_{AB}(k, x^C)dx^A dx^B \,, \tag{3.125}$$

[6] One can derive the Einstein equations (3.123) and (3.124) starting from the 5-dimensional vacuum Einstein equations and using Eqs. (3.127) and (3.129) adopted to the metric (3.122) (for details see, e.g., [73]).

where h_{AB}, $A, B, C, ... = 1, 2, 3$, is the metric on an orientable 3-dimensional hypersurface Σ_k. One can show that

$$\kappa^2(k, x^C) = -\frac{1}{2}(\nabla^\alpha \xi^\beta)(\nabla_\alpha \xi_\beta)\,, \tag{3.126}$$

where ∇_α is the covariant derivative defined with respect to the metric (3.122). Thus, $\kappa(k = 0, x^C)$ coincides with the surface gravity of a 5-dimensional vacuum black hole.

To present geometric quantities of the 5-dimensional space-time (3.125) in terms of these corresponding to Σ_k, we use the following relations:

$$R_{ABCD} = \mathcal{R}_{ABCD} + \mathcal{S}_{AD}\mathcal{S}_{BC} - \mathcal{S}_{AC}\mathcal{S}_{BD}\,, \tag{3.127}$$
$$R_{kABC} = \kappa^{-1}(\mathcal{S}_{AB;C} - \mathcal{S}_{AC;B})\,, \tag{3.128}$$
$$R_{AkkB} = \kappa^{-1}(h_{AC}\mathcal{S}_B{}^C{}_{,k} + (\kappa^{-1})_{;AB} + \kappa^{-1}\mathcal{S}_{AC}\mathcal{S}_B{}^C)\,, \tag{3.129}$$

where the first two equations are due to Gauss and Codazzi (see, e.g., [73] and [74]). Here $\mathcal{R}_{ABCD}$ is the intrinsic curvature, and

$$\mathcal{S}_{AB} = \frac{\kappa}{2} h_{AB,k} \tag{3.130}$$

is the extrinsic curvature of a hypersurface Σ_k. The semicolon ; stands for the covariant derivative defined with respect to the metric h_{AB}.

Using expressions (3.125) and (3.130), we derive

$$\nabla_k\nabla_k k = \kappa^{-1}\kappa_{,k}\,,\ \ \nabla_A\nabla_k k = \nabla_k\nabla_A k = \kappa^{-1}\kappa_{,A}\,,$$
$$\nabla_A\nabla_B k = \kappa\mathcal{S}_{AB}\,,\ \ \nabla_a\nabla^a k = \kappa(\kappa_{,k} + \mathcal{S})\,,\ \ \mathcal{S} \equiv \mathcal{S}_A{}^A\,. \tag{3.131}$$

Applying expressions (3.127)-(3.131) to the Einstein equations (3.123) and (3.124), we derive the following set of equations:

$$\kappa_{,k} + \mathcal{S} = 0\,, \tag{3.132}$$
$$\mathcal{R}_A{}^B = \kappa\mathcal{S}_A{}^B{}_{,k} + \kappa(\kappa^{-1})_{;A}{}^{;B} + \mathcal{S}\mathcal{S}_A{}^B + k^{-1}\kappa\mathcal{S}_A{}^B\,, \tag{3.133}$$
$$\kappa_{,A} + k(\mathcal{S}_{,A} - \mathcal{S}_A{}^B{}_{;B}) = 0\,, \tag{3.134}$$
$$\mathcal{R} := h^{AB}\mathcal{R}_{AB} = \mathcal{S}^2 - S_{AB}\mathcal{S}^{AB} + 2k^{-1}\kappa\mathcal{S}\,, \tag{3.135}$$
$$\mathcal{S}_{,k} + (\kappa^{-1})_{;A}{}^{;A} + \kappa^{-1}\mathcal{S}_{AB}\mathcal{S}^{AB} + k^{-1}\kappa_{,k} = 0\,. \tag{3.136}$$

Equations (3.130), (3.132), and (3.133) define a complete system for determining κ, h_{AB}, and $\mathcal{S}_{AB}$ as functions of k. The constraint equations (3.134) and (3.135) together with Eq. (3.136) are satisfied for any value of k.

For the static space-time (3.122), the Riemann tensor components are given by[7]

$$ {}^{(5)}R_{attb} = -k\nabla_a\nabla_b k\,,\ \ {}^{(5)}R_{tabc} = 0\,,\ \ {}^{(5)}R_{abcd} = R_{abcd}\,. \tag{3.137} $$

Thus, we arrive to the following expression for the Kretschmann scalar of the space-time (3.125):

$$ \begin{aligned} \mathcal{K} &\equiv {}^{(5)}R_{\alpha\beta\gamma\delta}\,{}^{(5)}R^{\alpha\beta\gamma\delta} = 4k^{-2}(\nabla_a\nabla_b k)(\nabla^a\nabla^b k) \\ &+ 4R_{AkkB}R^{AkkB} + 4R_{kABC}R^{kABC} + R_{ABCD}R^{ABCD}\,. \end{aligned} \tag{3.138} $$

Let us present this expression in terms of 3-dimensional geometric quantities defined on Σ_k. Using Eq. (3.127) we derive

$$ \begin{aligned} R_{ABCD}R^{ABCD} &= \mathcal{R}_{ABCD}\mathcal{R}^{ABCD} + 2\mathcal{R}_{ABCD}(\mathcal{S}^{AD}\mathcal{S}^{BC} - \mathcal{S}^{AC}\mathcal{S}^{BD}) \\ &+ 2(\mathcal{S}_{AB}\mathcal{S}^{AB})^2 - 2\mathcal{S}_{AC}\mathcal{S}^{BC}\mathcal{S}^{AD}\mathcal{S}_{BD}\,. \end{aligned} \tag{3.139} $$

The 3-dimensional Riemann tensor components $\mathcal{R}_{ABCD}$ corresponding to h_{AB} can be presented as follows (see, e.g., [20], p. 550):

$$ \begin{aligned} \mathcal{R}_{ABCD} &= h_{AC}\mathcal{R}_{BD} + h_{BD}\mathcal{R}_{AC} - h_{AD}\mathcal{R}_{BC} \\ &- h_{BC}\mathcal{R}_{AD} - \frac{1}{2}\mathcal{R}(h_{AC}h_{BD} - h_{AD}h_{BC})\,, \end{aligned} \tag{3.140} $$

where the Ricci scalar $\mathcal{R}$ and the trace of the square of the Ricci tensor $\mathcal{R}_{AB}\mathcal{R}^{AB}$ are defined on Σ_k. This is always true in $d = 3$, where the Weyl tensor vanishes. Expression (3.140) implies

$$ \mathcal{R}_{ABCD}\mathcal{R}^{ABCD} = 4\mathcal{R}_{AB}\mathcal{R}^{AB} - \mathcal{R}^2\,. \tag{3.141} $$

Using expressions (3.128), (3.129), (3.131), (3.138), (3.139), (3.140), and (3.141)

[7] Expressions (3.137) can be derived by changing notations in expressions (3.127)-(3.129) as follows: $k \to t$, $\kappa \to ik^{-1}$, and taking into account that the extrinsic curvature of a 4-dimensional hypersurface $t = const$ vanishes (see Eq. (3.130)).

we derive

$$\begin{aligned}\mathcal{K} &= 4k^{-2}\kappa^2\left(\kappa_{,k}^2+2\kappa^{-2}\kappa_{,A}\kappa^{,A}+2\mathcal{S}_{AB}\mathcal{S}^{AB}\right)-8k^{-1}\kappa\mathcal{S}^{AB}(\mathcal{R}_{AB}-\mathcal{S}\mathcal{S}_{AB}\\ &+ \mathcal{S}_{AC}\mathcal{S}_B{}^C)+8\mathcal{R}_{AB}\mathcal{R}^{AB}-\mathcal{R}^2+2\mathcal{S}^2(\mathcal{R}+2\mathcal{S}_{AB}\mathcal{S}^{AB})\\ &- 2\mathcal{S}_{AB}\mathcal{S}^{AB}(\mathcal{R}-\mathcal{S}_{CD}\mathcal{S}^{CD})-8\mathcal{S}\mathcal{S}^{AB}(2\mathcal{R}_{AB}+\mathcal{S}_{AC}\mathcal{S}_B{}^C)\\ &+ 2\mathcal{S}^{AC}\mathcal{S}^B{}_C(8\mathcal{R}_{AB}+\mathcal{S}_{AD}\mathcal{S}_B{}^D)+8\mathcal{S}^{AB;C}(\mathcal{S}_{AB;C}-\mathcal{S}_{AC;B}).\end{aligned} \tag{3.142}$$

Thus, one can see that the black hole horizon $k=0$ is regular if $\kappa_{,A}=0$ and $\mathcal{S}_{AB}=0$ on the horizon, i.e., the surface gravity is constant on the horizon, and the horizon surface, defined by $k=0$ and $t=const$, is a totally geodesic surface which is regular, i.e., $\mathcal{R}_{AB}\mathcal{R}^{AB}$ and $\mathcal{R}$ are finite on the surface.

To derive a relation between the space-time Kretschmann scalar calculated on the horizon and the 3-dimensional geometric quantities defined on the horizon surface we use the following series expansions:

$$\mathcal{A} = \sum_{n\geqslant 0}\mathcal{A}^{(2n)}k^{2n}\,,\;\; \mathcal{B}=\sum_{n\geqslant 0}\mathcal{B}^{(2n+1)}k^{2n+1}\,, \tag{3.143}$$

where $\mathcal{A}:=\{h_{AB},\kappa,\mathcal{R}_{AB},\mathcal{R}\}$ and $\mathcal{B}:=\{\mathcal{S}_{AB},\mathcal{S}\}$. Here the first term $\mathcal{A}^{(0)}$ corresponds to the value of $\mathcal{A}$ calculated on the horizon. To calculate $\mathcal{K}$ on the horizon it is enough to consider the first order expansion only, i.e., $n=0,1$. Substituting expansions (3.143) into equations (3.130), (3.132)-(3.136), we derive

$$\begin{aligned}&(\kappa^{(0)})_{,A}=0\,,\;\;\kappa^{(2)}=-\frac{\mathcal{R}^{(0)}}{4\kappa^{(0)}}\,,\;\;\mathcal{S}^{(1)}_{AB}=\frac{\mathcal{R}^{(0)}_{AB}}{2\kappa^{(0)}}\,,\\ &\mathcal{S}^{(1)}=h^{AB(0)}\mathcal{S}^{(1)}_{AB}=\frac{\mathcal{R}^{(0)}}{2\kappa^{(0)}}\,,\;\;h^{(2)}_{AB}=\frac{\mathcal{R}^{(0)}_{AB}}{2(\kappa^{(0)})^2}\,.\end{aligned} \tag{3.144}$$

Substituting the corresponding expansions (3.143) for $n=0,1$ with the coefficients (3.144) into Eq. (3.142), we derive the following relation between the space-time Kretschmann scalar $\mathcal{K}$ calculated on the horizon of a 5-dimensional static, asymmetric, distorted vacuum black hole and the trace of the square of the Ricci tensor $\mathcal{R}_{AB}\mathcal{R}^{AB}$ of the horizon surface:

$$\mathcal{K}_+=6(\mathcal{R}_{AB}\mathcal{R}^{AB})_+\,. \tag{3.145}$$

It is interesting to note that the same relation holds for 4-dimensional static space-times. Namely, if we consider $\mathcal{R}_{AB}$ as the Ricci tensor of the 2-dimensional horizon surface of a 4-dimensional static asymmetric black hole, then the relation becomes (see, e.g., [51] and [72]) $\mathcal{K}_+=3\mathcal{R}_+^2$.

3.8 Space-time near the singularity

3.8.1 Metric near the singularity

Using the expansion of the distortion fields $\widehat{U}$, $\widehat{W}$, and $\widehat{V}$ at the vicinity of the black hole singularity (see Eqs. (B.8) and (B.11) in Appendix B) and the definition of the renormalized distortion fields (3.77)–(3.79), we derive an approximation of the metric (3.95) near the black hole singularity $\psi_- = \pi - \psi \to 0$:

$$dS_-^2 = A_- dT^2 + B_-(-d\psi_-^2 + d\theta^2) + C_- d\chi^2 + D_- d\phi^2 \,, \tag{3.146}$$

$$\begin{aligned}
A_- &= 4e^{2(u_-(\theta)+w_-(\theta)-3u_1-3w_1)}\left[\frac{1}{\psi_-^2} + \frac{1}{2}\left(u_{-,\theta\theta} + w_{-,\theta\theta}\right.\right.\\
&+ \left.\left.\cot\theta(u_{-,\theta} + w_{-,\theta}) - \frac{1}{3}\right) + \mathcal{O}(\psi_-^2)\right] \,,\\
B_- &= \frac{1}{16}e^{-4(u_-(\theta)+w_-(\theta)-u_1-w_1)}\left[\psi_-^2 - \left(u_{-,\theta\theta} + w_{-,\theta\theta}\right.\right.\\
&- u_{-,\theta}^2 - u_{-,\theta}w_{-,\theta} - w_{-,\theta}^2 - \frac{1}{2}\cot\theta(u_{-,\theta} + w_{-,\theta})\\
&+ \left.\left.\frac{u_{-,\theta} - w_{-,\theta}}{2\sin\theta} + \frac{1}{12}\right)\psi_-^4 + \mathcal{O}(\psi_-^6)\right] \,,\\
C_- &= \frac{1}{8}(1+\cos\theta)e^{-2(w_-(\theta)-3w_1)}\left[\psi_-^2\right.\\
&- \left.\frac{1}{2}\left(w_{-,\theta\theta} + \cot\theta\, w_{-,\theta} + \frac{1}{6}\right)\psi_-^4 + \mathcal{O}(\psi_-^6)\right] \,,\\
D_- &= \frac{1}{8}(1-\cos\theta)e^{-2(u_-(\theta)-3u_1)}\left[\psi_-^2\right.\\
&- \left.\frac{1}{2}\left(u_{-,\theta\theta} + \cot\theta\, u_{-,\theta} + \frac{1}{6}\right)\psi_-^4 + \mathcal{O}(\psi_-^6)\right] \,.
\end{aligned} \tag{3.147}$$

This approximation allows us to calculate the Kretschmann scalar near the singularity, up to corrections that are second order in ψ_-:

$$\mathcal{K}_- \approx \frac{2^8 \cdot 72}{\psi_-^8} e^{8(u_-(\theta)+w_-(\theta)-u_1-w_1)}\left[1 + \mathcal{K}_-^{(2)}\psi_-^2\right] \,, \tag{3.148}$$

$$\begin{aligned}
\mathcal{K}_-^{(2)} = \frac{2}{3}&\left(u_{-,\theta\theta} + w_{-,\theta\theta} - 4u_{-,\theta}^2 - 6u_{-,\theta}w_{-,\theta} - 4w_{-,\theta}^2\right.\\
&\left. - 2\cot\theta(u_{-,\theta} + w_{-,\theta}) + \frac{u_{-,\theta} - w_{-,\theta}}{\sin\theta} + \frac{1}{2}\right) \,.
\end{aligned} \tag{3.149}$$

Higher order terms can be obtained by using the relations given in Appendix B. In the absence of distortion the space-time Kretschmann scalar is equal to the Kretschmann scalar of the 5-dimensional Schwarzschild-Tangherlini space-time

$$\mathcal{K}_{\mathrm{ST}_-} = \frac{2^8 \cdot 72}{\psi_-^8} \,. \tag{3.150}$$

3.8.2 Stretched singularity

In the absence of distortion the approximation (3.147) gives the Schwarzschild-Tangherlini geometry near the singularity

$$dS_-^2 \approx -\frac{\psi_-^2}{16} d\psi_-^2 + \frac{4}{\psi_-^2} dT^2 + \frac{\psi_-^2}{4} d\omega_{(3)}^2 \,. \tag{3.151}$$

Using the transformation

$$\psi_- = 2\sqrt{2}\tau^{1/2} \tag{3.152}$$

the metric (3.151) can be written in the form

$$dS_-^2 \approx -d\tau^2 + \frac{1}{2\tau}\, dT^2 + 2\tau\, d\omega_{(3)}^2 \,. \tag{3.153}$$

Here τ is the maximal proper time of free fall to the singularity from a point near it along the geodesic defined by $(T, \theta, \chi, \phi) = const$. The proper time τ is positive and equals to 0 at the singularity [8]. The metric (3.153) has the Kasner exponents $(-1/2, 1/2, 1/2, 1/2)$. It represents a metric of a collapsing, anisotropic universe which contracts in the (θ,χ,ϕ)-directions and expands in the T-direction.

The Kretschmann scalar (3.150), expressed through the proper time, has the following form:

$$\mathcal{K}_{\mathrm{ST}_-} = \frac{9}{2\tau^4} \,. \tag{3.154}$$

This expression shows that a surface of constant $\mathcal{K}_{\mathrm{ST}_-}$ is at the same time a surface of constant τ.

A space-time in the region where its curvature is of order of the Planckian curvature requires quantum gravity for its description. For the Schwarzschild-Tangherlini geometry such a region is defined by the surface where $\mathcal{K}_{\mathrm{ST}_-} \sim \ell_{\mathrm{Pl}}^{-4}$, where $\ell_{\mathrm{Pl}} \sim 10^{-33}$cm is the Planckian length, which corresponds to the proper time

[8] The proper time τ defined this way runs backward. One can define another proper time $\tau' := \tau_o - \tau$, where $\tau_o \geq \tau$, which runs forward and is equal to τ_o at the singularity. However, we shall use the former definition, which is more convenient for our calculations.

τ of order of the Planckian time $\tau_{\rm Pl} \sim 10^{-44}$s. Since one cannot rely on the classical description in this region, it is natural to consider its boundary as the *stretched singularity*. The *stretched singularity* of the 5-dimensional Schwarzschild-Tangherlini space-time has the topology $\mathbb{R}^1 \times S^3$. Its metric is a direct sum of the metric of a line and the metric of a round 3-dimensional sphere.

What happens to the stretched singularity when a Schwarzschild-Tangherlini black hole is distorted? To answer this question we use the asymptotic form of the metric near the singularity of the distorted black hole (see Eq. (3.146)). Let us consider a timelike geodesic defined by $(T, \chi, \phi) = const$. For such a geodesic the maximal proper time of free fall to the singularity from a point near it corresponds to $E = L_\chi = L_\phi = L_0 = 0$ (see Appendix C). We shall call the corresponding geodesic "radial". According to the calculations given in Appendix C, the "radial" geodesic is uniquely determined by the limiting value θ_0 of its angular parameter θ at which it "hits" the black hole singularity. Let us denote by τ the proper time measured along the "radial" geodesic backward in time from its end point at the singularity. We can use (τ, θ_0) as new coordinates in the vicinity of the singularity. Using the leading order terms in expressions (B.13) and (B.14), we can relate the coordinates (ψ_-, θ) to the new coordinates as follows:

$$\psi_- = 2\sqrt{2}e^{u_-(\theta)+w_-(\theta)-u_1-w_1}\,\tau^{1/2}\,, \;\; \theta = \theta_0\,. \tag{3.155}$$

In the coordinates $(\tau, \theta_0 = \theta)$ the metric (3.146) takes the following form:

$$dS_-^2 \approx -d\tau^2 + \frac{1}{2\tau}e^{-4(u_1+w_1)}\,dT^2 + 2\tau e^{4(u_1+w_1)}\,d\Sigma_-^2\,, \tag{3.156}$$

where

$$\begin{aligned} d\Sigma_-^2 &= \frac{1}{4}\Big(e^{-2(u_-(\theta)+w_-(\theta)+u_1+w_1)}d\theta^2 + 2(1+\cos\theta)e^{2(u_-(\theta)-3u_1)}d\chi^2 \\ &+ 2(1-\cos\theta)e^{2(w_-(\theta)-3w_1)}d\phi^2\Big)\,. \end{aligned} \tag{3.157}$$

The metric (3.156) has the same Kasner exponents as those of (3.153).

The Kretschmann scalar (3.148) in the $(\tau, \theta_0 = \theta)$ coordinates reads

$$\mathcal{K}_- \approx \frac{9}{2\tau^4}\left[1 + \tilde{\mathcal{K}}_-^{(2)}\tau\right] , \tag{3.158}$$

$$\tilde{\mathcal{K}}_-^{(2)} = \frac{16}{3}e^{2(u_-(\theta)+w_-(\theta)-u_1-w_1)}\Big[u_{-,\theta\theta} + w_{-,\theta\theta} - 4u_{-,\theta}^2 - 6u_{-,\theta}w_{-,\theta}$$
$$-4w_{-,\theta}^2 - 2\cot\theta(u_{-,\theta} + w_{-,\theta}) + \frac{u_{-,\theta} - w_{-,\theta}}{\sin\theta} + \frac{1}{2}\Big] . \tag{3.159}$$

We see that the expansion (3.158) coincides in the leading order with the expansion (3.154). Hence, in the presence of distortion, surfaces where the Kretschmann scalar has a constant value $\mathcal{K}_- = \mathcal{K}_c$ are (in the leading order) surfaces of constant τ. For $\tau \sim \tau_{\rm Pl}$ we can neglect the higher order terms in the expansion (3.158) and present the metric on the stretched singularity defined by $\mathcal{K}_c \sim \ell_{\rm Pl}^{-4}$ as follows:

$$dl_-^2 \approx \left[\frac{\mathcal{K}_c}{72}\right]^{1/4} e^{-4(u_1+w_1)}\, dT^2 + \left[\frac{72}{\mathcal{K}_c}\right]^{1/4} e^{4(u_1+w_1)}\, d\Sigma_-^2 , \tag{3.160}$$

where $d\Sigma_-^2$ is given by expression (3.157). According to the form of this metric, the stretched singularity of a distorted black hole has the same topology as the stretched singularity of a Schwarzschild-Tangherlini black hole.

3.8.3 Geometry of the stretched singularity surface: duality transformation

As we found in the previous subsection, the distortion fields do not change the topology of the *stretched singularity* of a Schwarzschild-Tangherlini black hole; however, they do change its geometry. To study the geometry of the stretched singularity, we consider the geometry of its 3-dimensional hypersurface defined by $T = const$. This surface is the Killing vector $\xi^\alpha_{(T)}$ orbit surface, i.e., it is invariant under $\mathbb{R}^1_T$ transformations. The metric on this surface is defined (up to the conformal factor) by $d\Sigma_-^2$ (see (3.157)). We can calculate the intrinsic curvature of the stretched singularity surface and illustrate its shape by an isometric embedding of its 2-dimensional sections, as we did in Sec. VI for horizon surface of a distorted black hole. However, one can notice that the metric $d\Sigma_-^2$ can be obtained from the horizon surface metric $d\Sigma_+^2$ (see (3.97)) by the following duality transformation:

$$u_+ \to -w_- , \;\; u_1 \to -w_1 , \;\; w_+ \to -u_- , \;\; w_1 \to -u_1 . \tag{3.161}$$

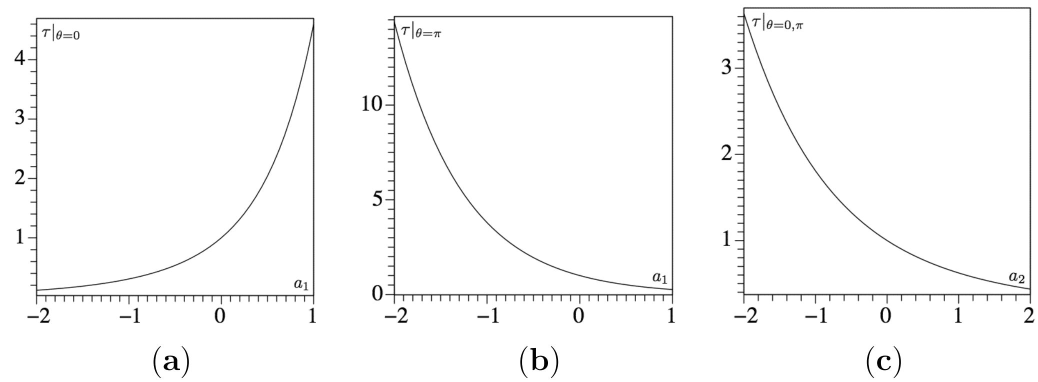

Figure 3.5: The maximal proper time τ in units of R_o. (**a**): The maximal proper time $\tau|_{\theta=0}$ for the dipole-monopole distortion. (**b**): The maximal proper time $\tau|_{\theta=\pi}$ for the dipole-monopole distortion. (**c**): The maximal proper time $\tau|_{\theta=0} = \tau|_{\theta=\pi}$ for the quadrupole-quadrupole distortion.

The duality transformation corresponds to the exchange between the multipole moments (see Eqs. (3.70), (3.71), (3.73), and (3.74))

$$a_{2n+1} \longleftrightarrow b_{2n+1}\,, \quad a_{2n} \longleftrightarrow -b_{2n}\,. \tag{3.162}$$

The no-conical-singularity condition (3.72) remains satisfied under the transformation (3.162). The transformation (3.162) corresponds to the exchange between the "axes" $\theta = 0$ and $\theta = \pi$ and the reversal of the signs of the multipole moments (cf. (3.54)):

$$\begin{aligned} (\theta, \chi, \phi) &\rightarrow (\pi - \theta, \phi, \chi)\,, \\ [a_n, b_n] &\rightarrow [(-1)^{n+1} b_n, (-1)^{n+1} a_n]\,. \end{aligned} \tag{3.163}$$

Because the exchange between the "axes" does not change the space-time of the distorted black hole, the transformation (3.163) reduces to change of signs of the multipole moments. Thus, the stretched singularity intrinsic curvature invariants can be derived from those of the horizon surface illustrated in Fig. 3.3 by exchanging Lines 1, 2, 3, and 4, with Lines 2, 1, 4, and 3, and changing θ to $\pi - \theta$ in each of Figs. 3.3 (**a**) and (**b**). An isometric embedding of the stretched singularity sections can be derived from those of the horizon surface illustrated in Fig. 3.4 by exchanging Lines 1, 2, 3, and 4 in Fig. 3.4 (**a**) with Lines 2, 1, 4, and 3 in Fig. 3.4 (**b**).

According to the duality transformation, given a 5-dimensional distorted black hole, one can find another one whose horizon surface geometry is the same as the geometry of the stretched singularity of the former one.

3.9 Proper time of free fall from the horizon to the singularity

So far we have studied the effect of distortion on the black hole horizon and singularity. However, the distortion fields affect the entire interior region of the black hole. To illustrate this, we consider how the distortion fields change the proper time of free fall of a test particle moving from the horizon to the singularity.

Namely, we study how the proper time depends on the multipole moments of the distortion fields. In our study we consider adiabatic distortion, so that the area A_+ of the distorted black hole horizon surface remains constant, which is equal to the horizon surface area of an undistorted Schwarzschild-Tangherlini black hole. We define the proper time in units of the radius R_o corresponding to the area A_+ (see (3.99)),

$$R_o = \left(\frac{A_+}{2\pi^2}\right)^{1/3} = r_o e^{-(u_1+w_1+3u_0+3w_0)/6} . \tag{3.164}$$

To make our analysis simpler, we consider a test particle moving along a timelike geodesic defined by $(t, \chi, \phi) = const$ and with $L_0 = 0$ (see Appendix C). Such a "radial" motion corresponds to zero angular momenta and energy of the particle . One can show that the proper time is maximal for such a motion. In addition, we consider free fall from the horizon to the singularity along each of the symmetry "axes" $\theta = 0$ and $\theta = \pi$. Using the metric (3.63) we derive

$$\tau|_{\theta=0} = \frac{r_o}{2\sqrt{2}R_o}\int_0^\pi (1+\cos\psi)^{1/2} e^{-u_+(\psi)-u_0} d\psi \,, \tag{3.165}$$

$$\tau|_{\theta=\pi} = \frac{r_o}{2\sqrt{2}R_o}\int_0^\pi (1+\cos\psi)^{1/2} e^{-w_-(\psi)-w_0} d\psi \,. \tag{3.166}$$

For a Schwarzschild-Tangherlini black hole the maximal proper time of free fall along a radial timelike geodesic is equal to 1 in units of $R_o = r_o$.

Let us now calculate the maximal proper time for the dipole-monopole distortion (4.63). In this case the integrals (3.165) and (3.166) can be evaluated exactly,

$$\tau|_{\theta=0} = \frac{\sqrt{\pi}}{2\sqrt{-2a_1}} e^{2a_1/3} \mathrm{erf}(\sqrt{-2a_1}) \,, \tag{3.167}$$

$$\tau|_{\theta=\pi} = e^{-4a_1/3} \,. \tag{3.168}$$

Here erf(x) is the error function (see, e.g., [39], p. 297). The maximal proper time for the quadrupole-quadrupole distortion (3.76) is the same for free fall along both the "axes",

$$\tau|_{\theta=0} = \tau|_{\theta=\pi} = \int_0^1 e^{-4a_2(x^2-x^4)} dx \,. \tag{3.169}$$

The maximal proper time calculated for the black hole distorted by the dipole-monopole and quadrupole-quadrupole distortions is shown in Figs. 3.5(**a**), (**b**), and (**c**). According to these figures, for some values of the multipole moments the maximal proper time is less, equal, or greater than that of a Schwarzschild-Tangherlini black hole of the same horizon area. One can see that due to the external distortion, the singularity of a Schwarzschild-Tangherlini black hole can come close to its horizon.

3.10 Summary of results and discussion

In this Chapter we studied a distorted, 5-dimensional vacuum black hole as a 5-dimensional Schwarzschild-Tangherlini black hole distorted by a static, neutral external distribution of matter. We constructed the corresponding metric which represents such a *local black hole*. In other words, the distortion sources are not included into the metric but are put at infinity. The metric is presented in the 5-dimensional Weyl form which admits the $\mathbb{R}^1 \times U(1) \times U(1)$ isometry group. This metric is a 5-dimensional generalization of the 4-dimensional Weyl form representing the corresponding distorted vacuum black hole studied before (see, e.g., [36–41, 51]).

As a result of our study, we found that distortion fields affect the black hole horizon and singularity. The 5-dimensional distorted black hole has the following properties, which are common with the 4-dimensional one: There is a certain duality transformation between the black hole horizon and the *stretched singularity* surfaces. This transformation implies that distortion of the horizon surface uniquely defines distortion of the stretched singularity surface. Given a 5-dimensional black hole, one can "observe" its distorted stretched singularity surface by observing the horizon surface of the dual distorted black hole. The topology of the stretched singularity is the same as that of a Schwarzschild-Tangherlini black hole. Moreover, the Kasner exponents of the space-time region near the singularities of the black holes are the same as well. One may assume that these properties are the inherent properties of the 4- and 5-dimensional Weyl forms, representing such distorted black holes. Whether all or some of these properties will remain if one changes the $U(1) \times U(1)$ symmetry (for example to $SO(3)$) remains an open question. Thus, we cannot say if a 5-dimensional *compactified* black hole has similar properties. However, a 4-dimensional *compactified* black hole indeed has properties similar to those of a 4-dimensional distorted black hole [18, 51].

The analysis of the maximal proper time of free fall from the distorted black hole horizon to its singularity along the symmetry "axes" shows that the proper time can be less, equal to, or greater than that of a Schwarzschild-Tangherlini black hole of the same horizon area. As a result of external distortion, the black hole stretched singularity can approaches the horizon. In particular, the black hole stretched singularity can approach its horizon. This scenario may suggest that the singularity of a 5-dimensional compactified black hole can approach its horizon during an infinitely slow merger transition between the black hole and the corresponding black string [9]. If so, one cannot rely on a classical description of the transition.

In this Chapter we derived a relation between the Kretschmann scalar calculated on the horizon of a 5-dimensional static, asymmetric, distorted vacuum black hole and the trace of the square of the Ricci tensor of the horizon surface. This relation is a generalization of a similar relation between the Kretschmann scalar calculated on the horizon of a 4-dimensional static, asymmetric, distorted vacuum black hole and the square of the Gaussian curvature of its horizon surface (see [51] and [72]).

Our construction and analysis of a 5-dimensional distorted black hole is based on the 5-dimensional Weyl form (see Sec. II), which is adopted for the construction of 5-dimensional black objects distorted by external gravitational fields. Using this Weyl form one can study other 5-dimensional black objects, e.g., distorted black strings and black rings. One can consider distorted higher(>5)-dimensional black objects as well, by using the corresponding Weyl form.

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[9] When the "north" and the "south" poles of a 4-dimensional compactified black hole come close to each other during an infinitely slow merger transition, its stretched singularity becomes naked at the vicinity of the poles [51].

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Chapter 4

Distorted 5-Dimensional Charged Black hole

In this Chapter first we present the 5-dimensional Reissner-Nordström black hole solution, and explain the parameters present in the metric. Setting the value of the electric charge equal to zero in this solution one has the 5-dimensional Schwarzschild-Tangherlini space-time. We then give a general recipe for transforming any Weyl solution of the vacuum 5-dimensional Einstein equations into the solution of the 5-dimensional Einstein-Maxwell equations. These transformations are the special case and special presentation of a more general transformations presented in [1]. According to [1] one can transform any static d-dimensional solution of the vacuum Einstein equations into a solution of the Einstein-Maxwell-dilaton theory (see also [2]). Setting the dilaton parameter to zero one has the standard solutions of the Einstein-Maxwell theory. Also these transformations can produce both asymptotically flat or non-asymptotically flat solutions. These transformations leave an asymptotically flat solution still asymptotically flat. We use these transformations to produce the distorted 5-dimensional electrically charged black hole. This solution is static and is the $U(1)\times U(1)$ symmetric. This solution is the 5-dimensional generalization of the 4-dimensional static, axisymmetric distorted black hole presented in [3], [4], and [5]. The interior structure of this black hole was analyzed in [6]. In this Chapter we use the following convention of units: $G_{(5)} = c = \hbar = 1$. Space-time signature is $+(d-2)$.

4.0.1 The 5-dimensional Reissner-Nordström solution

The Einstein-Maxwell theory in d-dimensions is described by the action

$$\mathcal{S} = \frac{1}{16\pi}\int d^5x\sqrt{-g}(R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}) \ . \tag{4.1}$$

The Einstein-Maxwell field equations derived from the above action read

$$R_{\alpha\beta} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}T_{\alpha\beta} \ , \tag{4.2}$$

$$T_{\alpha\beta} = F_{\alpha}{}^{\gamma}F_{\beta\gamma} - \frac{1}{4}g_{\alpha\beta}F_{\gamma\nu}F^{\gamma\nu} \ , \tag{4.3}$$

$$\nabla_{\nu}F^{\mu\nu} = 0, \quad \nabla_{[\lambda}F_{\mu\nu]} = 0 \ , \tag{4.4}$$

Here and in what follows ∇_{μ} denotes the covariant derivative defined with respect to the 5d metric. The 5-dimensional Reissner-Nordström solution is the static, spherically symmetric, asymptotically flat solution of the Einstein-Maxwell equations where $F_{\alpha\beta} = \nabla_{\alpha}A_{\beta} - \nabla_{\beta}A_{\alpha}$, and $A_{\alpha} = -\Phi\delta_{\alpha}^{t}$ is electrostatic vector potential.

$$ds^2 = -\bar{f}dt^2 + \bar{f}^{-1}dr^2 + \frac{r^2}{4}d\Omega_3^2 \ , \tag{4.5}$$

$$\bar{f} = 1 - \frac{2m}{r^2} + \frac{Q^2}{r^4} \ , \quad \Phi = \frac{\sqrt{3}Q}{r^2} \ . \tag{4.6}$$

where the metric $d\Omega_3^2$ can be presented in the following form with Hopf coordinates

$$d\Omega_3^2 = \frac{1}{4}\left(d\theta^2 + 2(1+\cos\theta)d\chi^2 + 2(1-\cos\theta)d\phi^2\right) \ , \tag{4.7}$$

where $\theta \in [0,\pi]$, $\chi \in [0, \ 2\pi)$, and $\phi \in [0, \ 2\pi)$. The black hole horizons are at $r^2 = m \pm \sqrt{m^2 - Q^2}$, where the upper sign stands for the event horizon and the lower sign stands for the Cauchy horizon. This space-time has a time-like singularity at $r = 0$. Here Q is the electric charge of the black hole defined by

$$Q = -\frac{1}{8\sqrt{3}\pi^2}\oint_{H} d^3\Sigma_{\alpha\beta}F^{\alpha\beta} \ , \tag{4.8}$$

where we have chosen this definition of Q in order to get Q^2/r^4 in the metric. Komar mass of the black hole is defined by the following expression

$$M = \frac{3}{32\pi G_{(5)}}\oint_{V_3^{\infty}} d^4\Sigma_{\alpha\beta}\nabla^{\alpha}\zeta^{\beta} \ , \tag{4.9}$$

where $\zeta^{\alpha} = \delta_t^{\alpha}$ is a time-like Killing vector normalized to unity at asymptotic infinity, $\zeta_{\alpha}\zeta^{\alpha} = -1$, and

$$d^3\Sigma_{\alpha\beta} = dx^{\gamma} \wedge dx^{\delta} \wedge dx^{\lambda}\frac{\epsilon_{\gamma\delta\lambda\alpha\beta}}{3!\sqrt{-g}} \ , \tag{4.10}$$

where $\epsilon^{01234} = 1$, $\epsilon_{01234} = g$. Here the order for the coordinates is $(x^0, x^1, x^2, x^3, x^4) = (t, r, \theta, \phi, \chi)$. Therefore, parameter m is related to the Komar mass of the black hole by the following expression

$$m = \frac{4M}{3\pi} \ . \tag{4.11}$$

We consider non-extremal black holes with $|Q| < m$.

For our future purposes it is convenient to introduce the following coordinate transformation

$$r = \sqrt{m(1 + p\eta)} \ , \quad p = \frac{\sqrt{m^2 - Q^2}}{m}, \quad \eta \in (-1/p, \infty) \ , \tag{4.12}$$

and rewrite (4.5) in the following form

$$\begin{aligned} ds^2 &= -\frac{p^2(\eta^2 - 1)}{(p\eta + 1)^2} dt^2 + \frac{m}{4}(p\eta + 1) \left[\left(\frac{d\eta^2}{\eta^2 - 1} + d\theta^2 \right) \right. \\ &+ \left. 2(1 + \cos\theta) d\chi^2 + 2(1 - \cos\theta) d\phi^2 \right] \ , \end{aligned} \tag{4.13}$$

$$\Phi = q(p\eta + 1)^{-1}, \quad F_{t\eta} = -F_{\eta t} = pq(p\eta + 1)^{-2} \ , \tag{4.14}$$

$$q = \sqrt{3(1 - p^2)} \ , \tag{4.15}$$

where other components of $F_{\mu\nu}$ are zero. In these coordinates $\eta = 1$ corresponds to the outer horizon $\mathcal{H}_+$ and $\eta = -1$ corresponds to the inner horizon $\mathcal{H}_-$ of the metric (4.13). Here and later, quantities with a subscript '+' are calculated for the outer horizons and quantities with a subscript '-' are calculated for the inner horizon. $\eta = -1/p$ corresponds to the black hole singularity. The horizon areas are

$$\mathcal{A}_\pm = 2\pi^2 \sqrt{m^3(1 \pm p)^3} \ , \tag{4.16}$$

and their surface gravity are

$$\kappa_\pm^2 = -\frac{1}{2} \nabla_\alpha \zeta_\beta \nabla^\alpha \zeta^\beta = \frac{4p^2}{m(1 \pm p)^3}. \tag{4.17}$$

In the case of $Q = 0$ $(p = 1)$, the electrostatic potential Φ vanishes, and the metrics

(4.5) and (4.13) represent the vacuum 5-dimensional Schwarzschild space-time.

$$\begin{aligned} ds^2 &= -\frac{\eta-1}{\eta+1}dt^2 + \frac{m}{4}(\eta+1)\Bigg[\left(\frac{d\eta^2}{\eta^2-1} + d\theta^2\right) \\ &+ 2(1+\cos\theta)d\chi^2 + 2(1-\cos\theta)d\phi^2\Bigg] , \end{aligned} \tag{4.18}$$

4.0.2 Charging vacuum solutions

Here, we present a special case of transformation which when applied to a 5-dimensional Weyl solution produces a 5-dimensional charged Weyl solution satisfying the Einstein-Maxwell equations (4.88) and (4.89). Let us first review the more general case (see [1]). The metric of a 5-dimensional static space-time can be written in the following form

$$ds^2 = -e^{2U}dt^2 + e^{-U}g_{ij}dx^i dx^j , \tag{4.19}$$

where $i, j = (1, 2, 3, 4)$. We assume the electrostatic vector potential of the form $A_\alpha = -\Phi\delta^t_\alpha$. Using (4.19) the Einstein-Maxwell equations read

$$\nabla_i\nabla^j U = \frac{4}{3}e^{-2U}g^{ij}\nabla_i\Phi\nabla_j\Phi, \tag{4.20}$$

$$\mathcal{R}_{ij} = \frac{3}{2}\nabla_i U\nabla_j U - 2e^{-2U}\nabla_i\Phi\nabla_j\Phi , \tag{4.21}$$

$$\nabla_i(e^{-2U}g^{ij}\nabla_j\Phi) = 0 , \tag{4.22}$$

where ∇_i is the covariant derivative and $\mathcal{R}_{ij}$ is Ricci tensor with respect to the 4-dimensional metric g_{ij}. Equations (4.20)- can be derived from the following action

$$\mathcal{S} = \int d^4x\sqrt{det(g_{ij})}\left[\mathcal{R} - \frac{3}{4}g^{ij}Tr(\nabla_i P\nabla_i P^{-1})\right] , \tag{4.23}$$

where $\mathcal{R}$ is the Ricci scalar with respect to the 4-dimensional metric g_{ij} and

$$P = e^{-U}\begin{pmatrix} e^{2U} - \frac{4}{3}\Phi^2 & -\frac{4}{3}\Phi^2 \\ -\frac{4}{3}\Phi^2 & -1 \end{pmatrix} . \tag{4.24}$$

The action (4.23) is invariant under the following symmetry transformation

$$P \to GPG^T , \tag{4.25}$$

where $G \in GL(2, \mathbb{R})$. To have asymptotically flat solutions with

$$U(\infty) = 0, \quad \Phi(\infty) = 0 , \tag{4.26}$$

we should have $G \in SO(1,1)$. Consider a static asymptotically flat solution of the 5-dimensional vacuum Einstein equations encoded the metric g_{ij} and matrix P_0

$$P = e^{-U}\begin{pmatrix} e^{2U} & 0 \\ 0 & -1 \end{pmatrix} . \tag{4.27}$$

The $SO(1,1)$ matrix

$$G = \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} , \tag{4.28}$$

then generates a solution of the 5-dimensional Einstein-Maxwell equations given by

$$ds^2 = -e^{2\bar{U}}dt^2 + e^{-\bar{U}}g_{ij}dx^i dx^j \ , \tag{4.29}$$

$$e^{\bar{U}} = \frac{e^U}{[\cosh^2\gamma - e^{2U}\sinh^2\gamma]} \ , \tag{4.30}$$

$$\bar{\Phi} = \frac{\sqrt{3}}{2}\frac{\tanh\gamma(1-e^{2U})}{[1-e^{2U}\tanh^2\gamma]} \ . \tag{4.31}$$

Here, we present these transformations in a more suitable form. A 5-dimensional Weyl solution is characterized by three commuting, orthogonal Killing vector fields one of which $\xi^\alpha_{(t)} = \delta^\alpha{}_t$ is timelike, and other two $\xi^\alpha_{(\chi)} = \delta^\alpha{}_\chi$, and $\xi^\alpha_{(\phi)} = \delta^\alpha{}_\phi$ are spacelike in the domain of interest. The 5-dimensional Weyl solution can be presented as follows:

$$\begin{aligned} ds^2 &= -e^{2U_1}dt^2 + e^{2\nu}(\eta^2-\cos^2\theta)\left(\frac{d\eta^2}{\eta^2-1}+d\theta^2\right) \\ &+ e^{2U_2}d\chi^2 + e^{2U_3}d\phi^2 \, , \end{aligned} \tag{4.32}$$

where $t \in (-\infty,\infty)$, $\theta \in (0,\pi)$, $\chi,\phi \in [0,2\pi)$, and $\eta \in (-\infty,\infty)$ are Killing coordinates (sometimes η is singular at some point). The metric functions U_i, $i=1,2,3$, and ν depend on the coordinates η and θ. Each of the functions U_i, $i=1,2,3$ solves the following 3-dimensional Laplace equation:

$$(\eta^2-1)U_{i,\eta\eta} + 2\eta U_{i,\eta} + U_{,\theta\theta} + \cot\theta U_{i,\theta} = 0 \, , \tag{4.33}$$

and the following constraint holds:

$$e^{2[U_1+U_2+U_3]} = (\eta^2-1)\sin^2\theta \ . \tag{4.34}$$

Here and in what follows $(...)_{,a}$ stands for the partial derivative of the expression (...) with respect to the coordinate x^a.

If we know a solution of vacuum Einstein equations (4.32) in the Weyl form, using (4.29)-(4.31) we derive the solution of the Einstein-Maxwell equations (4.88)-(4.4)

$$\begin{aligned} ds^2 &= -4p^2 e^{2U_1}\left[1+p-(1-p)e^{2U_1}\right]^{-2} dt^2 + \frac{1}{2}\left[1+p-(1-p)e^{2U_1}\right] \\ &\times \left[e^{2\nu}(\eta^2-\cos^2\theta)\left(\frac{d\eta^2}{\eta^2-1}+d\theta^2\right)+e^{2U_2}d\chi^2+e^{2U_3}d\phi^2\right], \end{aligned} \tag{4.35}$$

(4.36)

where

$$\cosh\gamma = \sqrt{\frac{1+p}{2p}}, \ \ \sinh\gamma = \sqrt{\frac{1-p}{2p}}, \tag{4.37}$$

and the electrostatic potential $\bar{\Phi}$ is given by the following expression

$$\Phi = q(1-e^{2U_1})Y^{-1}, \tag{4.38}$$

From now on we drop bar from Φ. The function ν remains the same under the transformation. These transformations charge the neutral Newtonian sources U_1. γ is the charge angle, i.e., for $\gamma = 0$ or $p = 1$ we have the solution of the vacuum Einstein equations. For example, applying these transformations we can construct the 5-dimensional Reissner-Nordström space-time (4.13) starting from the 5-dimensional vacuum Schwazschild space-time (4.18). where

$$e^{2U_1} = \frac{\eta-1}{\eta+1}, \tag{4.39}$$

$$e^{2U_2} = \frac{m}{2}(\eta+1)(1+\cos\theta), \tag{4.40}$$

$$e^{2U_3} = \frac{m}{2}(\eta+1)(1-\cos\theta), \tag{4.41}$$

$$e^{2\nu} = \frac{m(\eta+1)}{4(\eta^2-\cos^2\theta)}. \tag{4.42}$$

4.0.3 Distorted 5-dimensional charged black hole

We can now apply the transformations introduced in the previous subsection to derive the distorted charged black hole solution. To do this we start from the distorted 5-dimensional Schwarzschild solution and apply the Harrison transformations. The distorted 5-dimensional Schwarzschild space-time presented in Section

(1.3.2) reads

$$\begin{aligned} ds^2 &= -\frac{\eta-1}{\eta+1}e^{2(\widehat{U}+\widehat{W})}dt^2+\frac{m}{4}(\eta+1)[e^{2(\widehat{V}+\widehat{U}+\widehat{W})}\left(\frac{d\eta^2}{\eta^2-1}+d\theta^2\right) \\ &+ 2(1+\cos\theta)e^{-2\widehat{W}}d\chi^2+2(1-\cos\theta)e^{-2\widehat{U}}d\phi^2\Big] , \end{aligned} \tag{4.43}$$

where $\widehat{U}$, $\widehat{W}$, and $\hat{V}$ are given by the following expressions

$$\widehat{U}(\eta,\theta) = \sum_{n\geq 0} a_n\, R^n P_n(\eta\cos\theta/R)\,, \tag{4.44}$$

$$\widehat{W}(\eta,\theta) = \sum_{n\geq 0} b_n\, R^n P_n(\eta\cos\theta/R)\ , \qquad R=(\eta^2-\sin^2\theta)^{1/2}\,, \tag{4.45}$$

$$\widehat{V} = \widehat{V}_1+\widehat{V}_2\,, \tag{4.46}$$

$$\begin{aligned} \widehat{V}_1(\eta,\theta) = -\sum_{n\geq 0}\{&3(a_n/2+b_n/2)R^nP_n\,+\,(a_n+b_n/2)\sum_{l=0}^{n-1}(\eta-\cos\theta)R^lP_l \\ &+(a_n/2+b_n)\sum_{l=0}^{n-1}(-1)^{n-l}(\eta+\cos\theta)R^lP_l\ \}\ , \end{aligned} \tag{4.47}$$

$$\widehat{V}_2(\eta,\theta) = \sum_{n,k\geq 1}\frac{nk}{n+k}(a_na_k+a_nb_k+b_nb_k)R^{n+k}[P_nP_k-P_{n-1}P_{k-1}], \tag{4.48}$$

$$P_n \equiv P_n(\eta\cos\theta/R)\,. \tag{4.49}$$

where the coefficients a_n's and b_n's define the distortion fields $\widehat{U}$ and $\widehat{W}$, respectively. Comparing (4.43) and we read off

$$e^{2U_1} = \frac{\eta-1}{\eta+1}e^{2(\widehat{U}+\widehat{W})}, \tag{4.50}$$

$$e^{2U_2} = \frac{m}{2}(\eta+1)(1+\cos\theta)e^{-2\widehat{W}} \tag{4.51}$$

$$e^{2U_3} = \frac{m}{2}(\eta+1)(1-\cos\theta)e^{-2\widehat{U}} \tag{4.52}$$

$$e^{2\nu} = \frac{m(\eta+1)}{4(\eta^2-\cos^2\theta)}e^{2(\widehat{V}+\widehat{U}+\widehat{W})}. \tag{4.53}$$

Applying (4.38)-(4.0.3) on (4.43) we derive the distorted charged black hole solution

$$\begin{aligned} ds^2 &= -\frac{4p^2(\eta^2-1)}{f^2}e^{2(\widehat{U}+\widehat{W})}dt^2 + \frac{mf}{8}\left[e^{2(\widehat{V}+\widehat{U}+\widehat{W})}\left(\frac{d\eta^2}{\eta^2-1}+d\theta^2\right)\right. \\ &+ \left. 2(1+\cos\theta)e^{-2\widehat{W}}d\chi^2 + 2(1-\cos\theta)e^{-2\widehat{U}}d\phi^2\right] , \quad (4.54) \\ f &= (1+p)(\eta+1)-(1-p)(\eta-1)e^{2(\widehat{U}+\widehat{W})} , \quad (4.55) \\ \Phi &= qf^{-1}[\eta+1-(\eta-1)e^{2(\widehat{U}+\widehat{W})}] . \quad (4.56) \end{aligned}$$

Setting the distortion fields to zero this solution represents the 5-dimensional Reissner-Nordström solution. Setting $p = 1$ this solution represents the distorted 5-dimensional Schwarzschild solution. The only non-zero components of the electromagnetic tensor $F_{\mu\nu}$ are

$$\begin{aligned} F_{t\eta} &= 4pqf^{-2}e^{2(\widehat{U}+\widehat{W})}[1+(\eta^2-1)(\widehat{U}_{,\eta}+\widehat{W}_{,\eta})] \quad (4.57) \\ F_{t\theta} &= 4pqf^{-2}e^{2(\widehat{U}+\widehat{W})}(\eta^2-1)[\widehat{U}_{,\theta}+\widehat{W}_{,\theta}] . \quad (4.58) \end{aligned}$$

The distorted black hole solution (4.54) posses two horizons, the outer (event) horizon $\mathcal{H}_+$ at $\eta = 1$, and the inner (Cauchy) horizon $\mathcal{H}_-$ at $\eta = -1$. In this paper we mainly focus on the study of the horizons $\mathcal{H}_\pm$, and the domain located between the horizons. The no conical singularity condition (3.40) implies that for the "axis" $\theta = 0$ we should have

$$\widehat{V} + 2\widehat{U} + \widehat{W}|_{\theta=0} = 0 , \quad (4.59)$$

and for the "axis" $\theta = \pi$ we should have

$$\widehat{V} + \widehat{U} + 2\widehat{W}|_{\theta=\pi} = 0 . \quad (4.60)$$

Using (4.44)-(4.49) and the symmetry property of the Legendre polynomials

$$P_n(-x) = (-1)^n P_n(x) , \quad (4.61)$$

from the (4.59) and (4.60) we derive the following black hole equilibrium condition

$$\sum_{n\geq 0}(a_{2n} - b_{2n}) + 3\sum_{n\geq 0}(a_{2n+1} + b_{2n+1}) = 0 . \quad (4.62)$$

Let us introduce the following notations which will be useful for our calculations:

$$u_0 := \sum_{n\geq 0} a_{2n}\,, \quad u_1 := \sum_{n\geq 0} a_{2n+1}\,, \tag{4.63}$$

$$w_0 := \sum_{n\geq 0} b_{2n}\,, \quad w_1 := -\sum_{n\geq 0} b_{2n+1}\,, \tag{4.64}$$

$$y_0 = 3(u_0+w_0)+u_1+w_1, \tag{4.65}$$

$$y_1 = 3(u_1+w_1)+u_0+w_0 \tag{4.66}$$

Then the black hole equilibrium condition (4.62) can be written as

$$u_0+3u_1 = w_0+3w_1\,. \tag{4.67}$$

It is convenient to introduce ‘renormalized‘ distortion fields defined as follows:

$$\mathcal{U}(\psi,\theta) := \widehat{U}(\psi,\theta)-u_0-3u_1\,, \tag{4.68}$$

$$\mathcal{W}(\psi,\theta) := \widehat{W}(\psi,\theta)-w_0-3w_1\,, \tag{4.69}$$

$$\mathcal{V}(\psi,\theta) := \widehat{V}(\psi,\theta)+\frac{3}{2}y_1. \tag{4.70}$$

4.0.4 Dimensionless form of the metric

It is convenient to write metric (4.54) in the following dimensionless form adopted to the black hole horizons $\mathcal{H}_\pm$.

$$ds^2 = \Omega_\pm dS^2_\pm \tag{4.71}$$

$$dS^2_\pm = -\frac{(\eta^2-1)}{\Delta_\pm}e^{2(\mathcal{U}+\mathcal{W})}dT^2_\pm+\frac{\Delta_\pm}{8}\left[e^{2(\mathcal{V}+\mathcal{U}+\mathcal{W})}\left(\frac{d\eta^2}{\eta^2-1}+d\theta^2\right)\right.$$
$$\left.+\ 2(1+\cos\theta)e^{-2\mathcal{W}}d\chi^2+2(1-\cos\theta)e^{-2\mathcal{U}}d\phi^2\right], \tag{4.72}$$

$$\Omega_\pm = m(1\pm p)e^{\mp y_1} = m'(1\pm p'), \tag{4.73}$$

$$\Delta_\pm = \left(\frac{\delta^\pm}{\delta}\right)^{\frac{1}{2}}\left[\eta+1-\delta(\eta-1)e^{2(\mathcal{U}+\mathcal{W})}\right], \tag{4.74}$$

$$\delta = \delta_0 e^{2y_1} = \frac{1-p}{1+p}e^{2y_1} = \frac{1-p'}{1+p'}\,. \tag{4.75}$$

For $dS^2_\pm$, $T_\pm = \kappa_\pm e^{\pm 4(u_1+w_1)}t$, and $\kappa_\pm$ is the surface gravity

$$\kappa_\pm = \frac{2p\ e^{\pm\frac{y_0}{2}}}{\sqrt{m(1\pm p)^3}}. \tag{4.76}$$

Together with the original parameters m and p it is convenient to use the related parameters

$$p' = \frac{\sqrt{m'^2 - Q^2}}{m'}, \quad m' = \frac{m}{2}[(1+p)e^{-y_1} + (1-p)e^{y_1}]. \tag{4.77}$$

The curvature and the electromagnetic invariants diverge for $f = 0$, i.e. for

$$\eta = -\frac{1 + p + (1-p)e^{2(\widehat{U}+\widehat{W})}}{1 + p - (1-p)e^{2(\widehat{U}+\widehat{W})}}, \tag{4.78}$$

indicating the space-time singularity. From (4.78) we see that if $(\widehat{U}+\widehat{W}) \leqslant 0$, that is for the distortion fields not satisfying the strong energy conditions, space-time singularities are located always behind the Cauchy horizon.

4.0.5 Duality relations between the inner and outer horizons

To study the region between the outer and inner horizon it is convenient to introduce instead of η another coordinate as follows

$$\eta = \cos\psi\,, \quad \psi \in (0, \pi). \tag{4.79}$$

The surface of the outer horizon is defined by $T_+ = const$ and $\psi = \psi_+ = 0$. The surface of the inner horizon is defined by $T_- = const$ and $\psi = \psi_- = \pi$. The corresponding dimensionless metric derived from (4.72) for the outer horizon reads

$$\begin{aligned} d\Sigma_+^2 &= \frac{1}{4}\Big[e^{2(u_+(\theta)+w_+(\theta)+u_1+w_1)}d\theta^2 + 2(1+\cos\theta)e^{-2(w_+(\theta)-3w_1)}d\chi^2 \\ &+ 2(1-\cos\theta)e^{-2(u_+(\theta)-3u_1)}d\phi^2\Big]. \end{aligned} \tag{4.80}$$

where

$$u_\pm(\sigma) := \sum_{n\geq 0}(\pm 1)^n a_n \cos^n(\sigma) - u_0\,, \tag{4.81}$$

$$w_\pm(\sigma) := \sum_{n\geq 0}(\pm 1)^n b_n \cos^n(\sigma) - w_0\,. \tag{4.82}$$

In our case $\sigma := (\psi, \theta)$. The metric (4.80) coincides with the metric on the distorted Schwarzschild black hole horizon surface. The corresponding dimensionless metric derived from (4.72) for the outer horizon reads

$$d\Sigma_-^2 = \frac{1}{4}\Big[e^{-2(u_-(\theta)+w_-(\theta)+u_1+w_1)}d\theta^2 + 2(1+\cos\theta)e^{2(u_-(\theta)-3u_1)}d\chi^2 + 2(1-\cos\theta)e^{2(w_-(\theta)-3w_1)}d\phi^2\Big]\,. \tag{4.83}$$

The metrics $d\Sigma_+^2$ and $d\Sigma_-^2$ are related to each other by the following transformations

$$w_+ \rightarrow -u_-\,,\;\; u_+ \rightarrow -w_-\,,\;\; u_1 \rightarrow -w_1\,,\;\; w_1 \rightarrow -u_1\;. \tag{4.84}$$

This transformation implies the following transformations between the multipole moments:

$$a_{2n} \longleftrightarrow -b_{2n}\,,\;\; a_{2n+1} \longleftrightarrow b_{2n+1}\,. \tag{4.85}$$

These transformations correspond to exchange between the semi-axes $\theta = 0$ and $\theta = \pi$ and change of signs of the multipole moments $a_n \longleftrightarrow -a_n$, $b_n \longleftrightarrow -b_n$.

4.1 Space-time invariants

In Chapter 3 we derived the following relation between the space-time Kretschmann scalar $\mathcal{K}$ and the trace of the square of the Ricci tensor $\mathcal{R}_{AB}\mathcal{R}^{AB}$ calculated on the horizon surface of a 5-dimensional static asymmetric vacuum black hole:

$$\mathcal{K}|_{\mathcal{H}} = 6\mathcal{R}_{AB}\mathcal{R}^{AB}|_{\mathcal{H}}\,. \tag{4.86}$$

In this Section we generalize this relation to the case of a 5-dimensional, static, asymmetric, electrically charged black hole.

Let us rewrite the Einstein-Maxwell theory in 5-dimensions described by the action with arbitrary coupling constant

$$\mathcal{S} = \frac{1}{16\pi}\int d^5x\sqrt{-g}(R - \frac{1}{2}sF^{\mu\nu}F_{\mu\nu}). \tag{4.87}$$

The Einstein-Maxwell field equations derived from the above action read

$$R_{\alpha\beta} - \frac{1}{2}g_{\mu\nu}R = sT_{\alpha\beta}\,, \tag{4.88}$$

$$T_{\alpha\beta} = F_\alpha{}^\gamma F_{\beta\gamma} - \frac{1}{4}g_{\alpha\beta}F_{\gamma\nu}F^{\gamma\nu}\,, \tag{4.89}$$

$$\nabla_\nu F^{\mu\nu} = 0, \quad \nabla_{[\lambda}F_{\mu\nu]} = 0\,. \tag{4.90}$$

Here and in what follows ∇_μ denotes the covariant derivative defined with respect to the 5d metric. Consider a static, 5-dimensional asymmetric electrically charged black hole. The corresponding space-time admits the Killing vector $\xi^\alpha = \delta^\alpha{}_0$, $(x^0 := t)$, which is timelike in the domain of interest, $\xi^\alpha \xi_\alpha = g_{00} := -k^2 < 0$, and hypersurface orthogonal. The space-time metric $g_{\alpha\beta}$, $(\alpha, \beta, ... = 0, \dots, 4)$ can be presented in the form

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -k^2 dt^2 + \gamma_{ab}(x^c) dx^a dx^b \,, \tag{4.91}$$

where $\gamma_{ab}(x^c)$, $(a, b, c, ... = 1, \dots, 4)$, is the metric on a 4-dimensional hypersurface $t = const$. The black hole horizon defined by $k = 0$ is a non-degenerate Killing horizon. The metric (4.91) can be decomposed further. Assume that $\nabla_\alpha k \nabla^\alpha k$ vanishes nowhere in the domain of the interest. We can consider equi-potential surfaces of constant t and k. Metric (4.91) can be written in the following form:

$$ds^2 = -k^2 dt^2 + \kappa^{-2}(k, x^C) dk^2 + h_{AB}(k, x^C) dx^A dx^B \,, \tag{4.92}$$

where h_{AB}, $(A, B, C, ... = 1, 2, 3)$ is the metric on an orientable 3-dimensional hypersurface Σ_k. One can show that

$$\kappa^2(k, x^C) = -\frac{1}{2}(\nabla^\alpha \xi^\beta)(\nabla_\alpha \xi_\beta) \,, \tag{4.93}$$

where ∇_α is a covariant derivative defined with respect to the metric (4.91). Thus, $\kappa(k = 0, x^C)$ coincides with is the surface gravity of the 5-dimensional charged black hole. For the static space-time (4.91) the Riemann tensor components are given by

$$^{(5)}R_{attb} = -k \nabla_a \nabla_b k \,, \quad ^{(5)}R_{tabc} = 0 \,, \quad ^{(5)}R_{abcd} = R_{abcd} \,. \tag{4.94}$$

Thus, we arrive to the following expression for the Kretschmann scalar of the space-time (4.92):

$$\begin{aligned} \mathcal{K} \;\equiv\; & {}^{(5)}R_{\alpha\beta\gamma\delta}\, {}^{(5)}R^{\alpha\beta\gamma\delta} = 4k^{-2} (\nabla_a \nabla_b k)(\nabla^a \nabla^b k) + 4 R_{AkkB} R^{AkkB} \\ + \;& 4 R_{kABC} R^{kABC} + R_{ABCD} R^{ABCD} \,. \end{aligned} \tag{4.95}$$

To present geometric quantities of the 5-dimensional space-time (4.92) in terms of these corresponding to Σ_k we apply the following relations:

$$R_{ABCD} = \mathcal{R}_{ABCD} + \mathcal{S}_{AD}\mathcal{S}_{BC} - \mathcal{S}_{AC}\mathcal{S}_{BD}\,, \tag{4.96}$$

$$R_{kABC} = \kappa^{-1}(\mathcal{S}_{AB;C} - \mathcal{S}_{AC;B})\,, \tag{4.97}$$

$$R_{AkkB} = \kappa^{-1}(h_{AC}\mathcal{S}_{B}{}^{C}{}_{,k} + (\kappa^{-1})_{;AB} + \kappa^{-1}\mathcal{S}_{AC}\mathcal{S}_{B}{}^{C})\,, \tag{4.98}$$

where the first two equations are due to Gauss and Codazzi (see, e.g., [7] and [8]). Here $\mathcal{R}_{ABCD}$ is the intrinsic and

$$\mathcal{S}_{AB} = \frac{\kappa}{2}h_{AB,k} \tag{4.99}$$

is the extrinsic curvature of a hypersurface Σ_k. The semicolon ; stands for a covariant derivative defined with respect to the metric h_{AB}. Using expressions (4.92) and (4.99) we derive

$$\begin{aligned}&\nabla_k\nabla_k k = \kappa^{-1}\kappa_{,k}\,,\ \ \nabla_A\nabla_k k = \nabla_k\nabla_A k = \kappa^{-1}\kappa_{,A}\,,\\ &\nabla_A\nabla_B k = \kappa\mathcal{S}_{AB}\,,\ \ \nabla_a\nabla^a k = \kappa(\kappa_{,k} + \mathcal{S})\,,\ \ \mathcal{S} \equiv \mathcal{S}_A{}^A\,.\end{aligned} \tag{4.100}$$

Equations (4.99), (4.107), and (4.108) define a complete system for determining κ, h_{AB}, and $\mathcal{S}_{AB}$ as functions of k. The constraint equations (4.109) and (4.110) together with Eq. (3.136) are satisfied for any value of k. The 3-dimensional Riemann tensor components $\mathcal{R}_{ABCD}$ corresponding to h_{AB} can be presented as follows (see, [9], p. 550):

$$\begin{aligned}\mathcal{R}_{ABCD} &= h_{AC}\mathcal{R}_{BD} + h_{BD}\mathcal{R}_{AC} - h_{AD}\mathcal{R}_{BC}\\ &- h_{BC}\mathcal{R}_{AD} - \frac{1}{2}\mathcal{R}(h_{AC}h_{BD} - h_{AD}h_{BC})\,,\end{aligned} \tag{4.101}$$

where the Ricci scalar $\mathcal{R}$ and the trace of the square of the Ricci tensor $(\mathcal{R}_{AB}\mathcal{R}^{AB})$ are defined on Σ_k. Expression (4.101) implies

$$\mathcal{R}_{ABCD}\mathcal{R}^{ABCD} = 4\mathcal{R}_{AB}\mathcal{R}^{AB} - \mathcal{R}^2\,. \tag{4.102}$$

Using expressions (4.97), (4.98), (4.100), (4.95), (4.101), and (4.102) we derive

$$\begin{aligned}\mathcal{K} &= 4k^{-2}\kappa^2\left(\kappa_{,k}^2 + 2\kappa^{-2}\kappa_{,A}\kappa^{,A} + 2\mathcal{S}_{AB}\mathcal{S}^{AB}\right) + 8\mathcal{R}_{AB}\mathcal{R}^{AB} - \mathcal{R}^2\\ &- 8k^{-1}\kappa\mathcal{S}^{AB}(\mathcal{R}_{AB} - \mathcal{S}\mathcal{S}_{AB} + \mathcal{S}_{AC}\mathcal{S}_B{}^C) + 2\mathcal{S}^2(\mathcal{R} + 2\mathcal{S}_{AB}\mathcal{S}^{AB})\\ &- 2\mathcal{S}_{AB}\mathcal{S}^{AB}(\mathcal{R} - \mathcal{S}_{CD}\mathcal{S}^{CD}) - 8\mathcal{S}\mathcal{S}^{AB}(2\mathcal{R}_{AB} + \mathcal{S}_{AC}\mathcal{S}_B{}^C)\\ &+ 2\mathcal{S}^{AC}\mathcal{S}^B{}_C(8\mathcal{R}_{AB} + \mathcal{S}_{AD}\mathcal{S}_B{}^D) + 8\mathcal{S}^{AB;C}(\mathcal{S}_{AB;C} - \mathcal{S}_{AC;B})\,.\end{aligned} \tag{4.103}$$

Thus, one can see that the black hole horizon $k = 0$ is regular if $\kappa_{,A} = 0$ and

$\mathcal{S}_{AB} = 0$ on the horizon, i.e., the surface gravity is constant on the horizon, and the horizon surface

$$\mathcal{H} := (k = 0\,,\ \ t = const) \tag{4.104}$$

is a totally geodesic surface which is regular, i.e., $\mathcal{R}_{AB}\mathcal{R}^{AB}$ and $\mathcal{R}$ are finite on $\mathcal{H}$. One can show that the Einstein equations

$$^{(5)}G_{\alpha\beta} = sT_{\alpha\beta}, \tag{4.105}$$

for the metric (4.91) reduce to

$$^{(5)}G_{tt} = \frac{1}{2}k^2\ ^{(4)}R\,,\ \ ^{(5)}G_{ta} = 0\,,\ \ ^{(5)}G_{ab} =^{(4)} G_{ab} - k^{-1}\nabla_a\nabla_b k + k^{-1}g_{ab}\nabla_c\nabla^c k. \tag{4.106}$$

Here s is the coupling constant. where $^{(4)}R$ and ∇_a are the Ricci scalar and the covariant derivative defined with respect to the 4-dimensional metric γ_{ab}. Applying expressions (4.96)-(4.100) to the Einstein equations (4.106) we derive the following set of equations:

$$k\kappa(\kappa_{,k} + \mathcal{S}) = \frac{2}{3}s(\Psi^2 + \Phi_{,A}\Phi^{,A})\,, \tag{4.107}$$

$$\mathcal{R}_A{}^B = -sk^{-2}\Phi_{,A}\Phi^{,B} + \frac{1}{3}sh^B_A k^{-2}(\Psi^2 + \Phi_{,A}\Phi^{,A}), \tag{4.108}$$

$$\kappa_{,A} + k(\mathcal{S}_{,A} - \mathcal{S}_A{}^B{}_{;B}) = sk^{-1}\Psi\Phi_{,A}\,, \tag{4.109}$$

$$\mathcal{R} := h^{AB}\mathcal{R}_{AB} = \mathcal{S}^2 - S_{AB}\mathcal{S}^{AB} - 2k^{-1}\kappa\kappa_{,k} + \frac{1}{3}sk^{-2}(7\Psi^2 + \Phi_{,A}\Phi^{,A})\,, \tag{4.110}$$

$$\mathcal{S}_{,k} + (\kappa^{-1})_{;A}{}^{;A} + \kappa^{-1}\mathcal{S}_{AB}\mathcal{S}^{AB} + k^{-1}\kappa_{,k} = \frac{1}{3}sk^{-2}\kappa^{-1}(2\Psi^2 - \Phi_{,A}\Phi^{,A})\,. \tag{4.111}$$

where $\Psi = \kappa\Phi_{,k}$. To derive a relation between the Kretschmann scalar and 3-dimensional geometric quantities defined on $\mathcal{H}$ we use the following series expansions:

$$\mathcal{A} \ = \ \sum_{n\geqslant 0}\mathcal{A}^{[2n]}k^{2n}\,,\ \ \mathcal{B} = \sum_{n\geqslant 0}\mathcal{B}^{[2n+1]}k^{2n+1}\,, \tag{4.112}$$

where $\mathcal{A} = \{h_{AB}, \kappa, \mathcal{R}_{AB}, \mathcal{R}, \Phi\}$ and $\mathcal{B} = \{\mathcal{S}_{AB}, \mathcal{S}, \Psi\}$. Here the first term $\mathcal{A}^{[0]}$ corresponds to the value of $\mathcal{A}$ calculated on the horizon. To calculate $\mathcal{K}$ on the horizon it is enough to consider the first order expansion only, i.e., $n = 0, 1$.

Substituting expansions (3.143) into equations (4.99), (4.107)-(4.111) we derive

$$
\begin{aligned}
(\kappa^{[0]})_{,A} &= 0\,, \;\; \kappa^{[2]} = -\frac{\mathcal{R}^{[0]}}{4\kappa^{[0]}} - \frac{7}{24}s\frac{F^{2[0]}}{\kappa^{[0]}}\,, \\
\mathcal{S}^{[1]}_{AB} &= \frac{\mathcal{R}^{[0]}_{AB}}{2\kappa^{[0]}} + \frac{1}{12}s\frac{h^{[0]}_{AB}}{\kappa^{[0]}}F^{2[0]}\,, \;\; \mathcal{S}^{[1]} = \frac{\mathcal{R}^{[0]}}{2\kappa^{[0]}} + \frac{1}{4}s\frac{F^{2[0]}}{\kappa^{[0]}}\,, \\
h^{[2]}_{AB} &= \frac{\mathcal{R}^{[0]}_{AB}}{2(\kappa^{[0]})^2} + \frac{1}{12}s\frac{h^{[0]}_{AB}}{(\kappa^{[0]})^2}F^{2[0]}\,, \\
\Phi^{[0]}{}_{,A} &= 0\,, \;\; \Phi^{[2]} = \frac{\sqrt{-F^{2[0]}}}{2\sqrt{2}\kappa^{[0]}}.
\end{aligned} \tag{4.113}
$$

Using the coefficients (4.113) and substituting the corresponding expansions (4.112) for $n = 0, 1$ into Eq. (4.103) we derive the following relation:

$$
\mathcal{K}|_{\mathcal{H}} = 6\mathcal{R}_{AB}\mathcal{R}^{AB} + \frac{7}{3}s\mathcal{R}F^2 + \frac{55}{36}s^2(F^2)^2|_{\mathcal{H}}\,. \tag{4.114}
$$

This relation represents the Kretschmann invariant $\mathcal{K}$ calculated on the horizon of a 5-dimensional, asymmetric, electrically charged black hole in terms of the electromagnetic field invariant F^2 calculated on the black hole horizon and the geometric invariants of the horizon surface, i.e., the 3-dimensional Ricci scalar $\mathcal{R}$, and the trace of the square of the Ricci tensor $\mathcal{R}_{AB}\mathcal{R}^{AB}$. This relation is a generalization of a similar relation between the Kretschmann scalar calculated on the horizon of a 4-dimensional static asymmetric charged black hole and the electromagnetic field invariant calculated on the horizon and the square of Gussian curvature of the horizon surface obtained in [6].

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Chapter 5

Analysis of the Fisher Solution

5.1 Introduction

In this Chapter we study a solution which was discovered by Fisher [1]. Later the solution was rediscovered by many authors (see, for example, [2–4]) and usually referred to as the Janis-Newman-Winicour solution [5]. Here we study the d-dimensional ($d \geqslant 4$) generalization of this solution which was given in [6] . The solution represents a static, spherically symmetric, asymptotically flat spacetime with a massless scalar field. A massless scalar field is related to a massless particle of zero spin. Such particles are not known, and all known zero spin particles are massive. Thus, such a field may be not realistic (unless a zero spin massless particle is discovered). However, in some cases one may consider such a field as an approximation for a massive scalar field, or regard a massless scalar field as a toy model, which is often useful for its simplicity. There is a more serious reason to consider the Fisher solution as unphysical, for it represents a naked curvature singularity.

The classical description of spacetime breaks down at a curvature singularity. However, spacetime singularities arise in a very large class of solutions of the general theory of relativity, and in fact in very reasonable physical conditions which respect causality and energy conditions [7]. The trouble with naked singularities (except agreeably with the Big Bang one which is in our past) is that they are naked, i.e., one could potentially "see a breakdown of physics" if a naked singularity is present. To avoid formation of a naked singularity in real physical processes, such as gravitational collapse, which are described by classical laws of the general theory of relativity, the cosmic censorship conjecture was formulated, first in weak

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[8] and later in strong form [9]. However, the present issue of its validity is very much open [10].

In attempts to test cosmic censorship, many models of gravitational collapse were studied analytically and numerically (for a popular survey of the subject see [11]). It was found that in certain conditions naked singularities may form. For example, they may form as a result of collapse of collisionless gas spheres [12], or self-similar collapse of a massless, minimally coupled scalar field where the second type phase transition from black hole to naked singularity takes place [13]. However, such examples should be considered with caution, for a rigorous analysis may suggest that the detected naked singularity formation may be ambiguous [14]. A review [15] has many other examples as well as discussion of gravitational radiation and quantum particle creation by naked singularities. There is a recent proposal to search for a naked singularity using Kerr lensing [16]. These examples may imply that we have to study naked singularities rather than disregard them.

Here we study the naked singularity of the Fisher solution which is due to a massless scalar field. The reasons for such a study is to understand deeper how such a field affects spacetime and what type of singularity it "produces." For example, it was shown that a massless scalar field "converts" the Cauchy horizon of a Kerr-Newman black hole into a strong curvature singularity [17]. Another example is a weak instantaneous curvature singularity which appears at the moment of a wormhole formation when a ghost massless scalar field is present [18]. On the other side, it was shown that quantum effects may prevent the formation of a naked singularity due to gravitational collapse of a homogeneous scalar field [19]. This may suggest that a curvature singularity due to massless scalar field may be "smoothed out" by quantum effects.

The main idea of our study is to analyze the naked curvature singularity of the Fisher solution and to show that indeed, a spacetime curvature singularity (at least in our example).

This Chapter is organized as follows. In Sec. 2 we present the d-dimensional Fisher solution and discuss its general properties. In Sec. 3 we study curvature singularities of the Fisher solution. Causal properties of the Fisher solution are discussed in Sec. 4. In Sec. 5 we present an isometric embedding of the Fisher solution. Using results of the previous sections, we return to a discussion of the Fisher solution in Sec. 6. Section 7 contains a summary and discussion of our results. Additional details illustrating our calculations are given in the appendixes. In this Chapter we set $G_{(d)} = c = 1$, where $G_{(d)}$ is the d-dimensional ($d \geqslant 4$) gravitational constant. The spacetime signature is $+(d-2)$. We use the notations and conventions adopted in [20].

5.2 The Fisher solution

5.2.1 Metric

Let us present a d-dimensional generalization of the Fisher solution, which is static, spherically symmetric, asymptotically flat spacetime with a massless, minimally coupled scalar field. The corresponding action has the following form:

$$\mathcal{S}[g_{ab},\, \varphi] = \frac{1}{16\pi}\int d^dx\sqrt{-g}\left(R - \tfrac{d-2}{d-3} g^{ab}\varphi_{,a}\varphi_{,b}\right) , \tag{5.1}$$

where R is the d-dimensional Ricci scalar and φ is the massless, minimally coupled scalar field, rescaled to allow the factor $(d-2)/(d-3)$ that simplifies the equations below. Here and in what follows $(...)_{,a}$ stands for the partial derivative of the expression $(...)$ with respect to the coordinate x^a.

The energy-momentum tensor of the scalar field is

$$T_{ab} = \frac{1}{8\pi}\frac{d-2}{d-3}\left(\varphi_{,a}\varphi_{,b} - \frac{1}{2}g_{ab}\varphi_{,c}\varphi^{,c}\right) . \tag{5.2}$$

Thus, the corresponding Einstein equations are

$$R_{ab} = \tfrac{d-2}{d-3}\varphi_{,a}\varphi_{,b} \,. \tag{5.3}$$

The scalar field solves the massless Klein-Gordon equation

$$\nabla^a\nabla_a\varphi = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{ab}\varphi_{,a}\right)_{,b} = 0 \,. \tag{5.4}$$

Here ∇_a stands for the covariant derivative defined with respect to the d-dimensional metric g_{ab}. An explicit form of Eqs. (5.3) and (5.4) for a static, spherically symmetric spacetime is given in Appendix D . A static, asymptotically flat, spherically symmetric solution to Eqs. (5.3) and (5.4) was derived in [6] in isotropic coordinates, which bring the Einstein equations into a form more suitable for integration. Here we present the solution in different (Schwarzschild-like[1]) coordinates. . The

[1] The radial isotropic coordinate r_i of [6] is related to our coordinate r through the following transformation:

$$r_i^{d-3} = \frac{r^{d-3}}{4}\left(1+\sqrt{F}\right)^2 \;,\;\; F = 1 - \left(\frac{r_o}{r}\right)^{d-3} .$$

duality transformation presented in the next subsection allows one to derive this solution without integration of the Einstein equations. The Fisher metric reads

$$ds^2 = -F^S dt^2 + F^{\frac{1-S}{d-3}-1} dr^2 + r^2 F^{\frac{1-S}{d-3}} d\Omega^2_{(d-2)}\,, \tag{5.5}$$

where $d\Omega^2_{(d-2)}$ is the metric on a unit $(d-2)$-dimensional round sphere. Here

$$F = 1 - \left(\frac{r_o}{r}\right)^{d-3}\,, \tag{5.6}$$

$$r_o^{d-3} = \frac{8\Gamma(\frac{d-1}{2})}{(d-2)\pi^{\frac{d-3}{2}}}(M^2+\Sigma^2)^{\frac{1}{2}}\,, \tag{5.7}$$

and

$$S = \frac{M}{(M^2+\Sigma^2)^{\frac{1}{2}}}\,, \tag{5.8}$$

where $M \geqslant 0$ is the d-dimensional Komar mass [21] measured at asymptotic infinity ($r \to \infty$) and the parameter Σ is defined below.

The scalar field, defined up to an additive constant which is irrelevant to our considerations, reads

$$\varphi = \frac{\Sigma}{2(M^2+\Sigma^2)^{\frac{1}{2}}} \ln|F|\,. \tag{5.9}$$

In the asymptotic region we have

$$\varphi \sim -\frac{4\Gamma(\frac{d-1}{2})}{(d-2)\pi^{\frac{d-3}{2}}}\frac{\Sigma}{r^{d-3}}\,. \tag{5.10}$$

Thus, we define $\Sigma \in (-\infty, \infty)$ as the d-dimensional "scalar charge." Hence, expression (5.8) implies that $S \in [0, 1]$ if we take $M \geqslant 0$.

Calculating the energy-momentum tensor components in a local orthonormal frame, we derive the following energy density ϵ and the principal pressures $p_{\hat{r}}$, $p_{\hat{\alpha}}$:

$$\epsilon = p_{\hat{r}} = -p_{\hat{\alpha}} = \frac{(d-3)\Gamma^2(\frac{d-1}{2})\Sigma^2}{(d-2)\pi^{d-2} r^{2(d-2)} F^{\frac{1-S}{d-3}+1}}\,, \tag{5.11}$$

where the index $\hat{\alpha} = 3, ..., d$ stands for orthonormal components in the compact dimensions of the $(d-2)$-dimensional round sphere. The scalar field obeys the strong and the dominant energy conditions. Thus, by continuity it obeys the weak and the null energy conditions (see, e.g., [2, 22]).

The Fisher solution has the following limiting cases:

The pure scalar charge case: $M = 0$. According to expressions (5.7) and (5.8), this case implies

$$r_o^{d-3}|_{M=0} = r_\Sigma^{d-3} = \frac{8\Gamma(\frac{d-1}{2})|\Sigma|}{(d-2)\pi^{\frac{d-3}{2}}} \ , \ \ S = 0. \tag{5.12}$$

Thus, $\varphi = 1/2 \ln |F|$, and the corresponding metric is

$$ds^2 \ = \ -dt^2 + F_\Sigma^{-\frac{d-4}{d-3}} dr^2 + r^2 F_\Sigma^{\frac{1}{d-3}} d\Omega^2_{(d-2)} \,, \tag{5.13}$$

where

$$F_\Sigma = 1 - \left(\frac{r_\Sigma}{r}\right)^{d-3} . \tag{5.14}$$

We shall call this solution the *massless Fisher* solution.

The pure mass case: $\Sigma = 0$. According to expressions (5.7) and (5.8), this case implies

$$r_o^{d-3}|_{\Sigma=0} = r_M^{d-3} = \frac{8\Gamma(\frac{d-1}{2})M}{(d-2)\pi^{\frac{d-3}{2}}} \ , \ \ S = 1. \tag{5.15}$$

Thus, $\varphi = 0$, and the corresponding metric is known as the d-dimensional Schwarzschild-Tangherlini black hole [23]

$$ds^2 \ = \ -F_M dt^2 + F_M^{-1} dr^2 + r^2 d\Omega^2_{(d-2)} \,, \tag{5.16}$$

where

$$F_M = 1 - \left(\frac{r_M}{r}\right)^{d-3} . \tag{5.17}$$

The uniqueness of the Schwarzschild-Tangherlini solution was proven in [24, 25].

5.2.2 Duality

The Fisher solution presented above possesses a certain duality symmetry. Here we show that the static, spherically symmetric spacetimes (5.5) corresponding to different values of M and Σ are dual to each other. In particular, we show that the Fisher solution is dual to the Schwarzschild-Tangherlini black hole of a particular mass.

Let us present the metric (5.5) in the following form:

$$ds^2 = -k^2 dt^2 + k^{-\frac{2}{d-3}} \bar{g}_{\mu\nu} dx^\mu dx^\nu \ . \tag{5.18}$$

Here, $-k^2$ is the squared norm of the timelike Killing vector $\delta^a{}_t$ and $k^{-2/(d-3)}\bar{g}_{\mu\nu}$ is the $(d-1)$-dimensional spatial metric on a hypersurface orthogonal to $\delta^a{}_t$. We can reduce the d-dimensional action (5.1) for the metric (5.18) to a $(d-1)$-dimensional action for the metric $\bar{g}_{\mu\nu}$. Let us first decompose the Ricci scalar R with respect to a basis defined by the unit timelike vector $k^{-1}\delta^a{}_t$ and $(d-1)$ basis vectors tangential to the hypersurface (see, e.g., [26]),

$$R = \widetilde{R} - 2\widetilde{\nabla}^\mu \widetilde{\nabla}_\mu \ln|k| - 2k^{\frac{2}{d-3}} \bar{g}^{\mu\nu} (\ln|k|)_{,\mu} (\ln|k|)_{,\nu} \ . \tag{5.19}$$

Here the $(d-1)$-dimensional Ricci scalar $\widetilde{R}$ and the covariant derivative $\widetilde{\nabla}_\mu$ are associated with the metric $k^{-2/(d-3)}\bar{g}_{\mu\nu}$. Applying the conformal transformation defined by the conformal factor $k^{-2/(d-3)}$ to the Ricci scalar $\widetilde{R}$ we derive (see, e.g., [2])

$$\widetilde{R} \ = \ k^{\frac{2}{d-3}} \left[\bar{R} + \tfrac{2}{d-3} \bar{\nabla}^\mu \bar{\nabla}_\mu \ln|k| - \tfrac{d-2}{d-3} \bar{g}^{\mu\nu} (\ln|k|)_{,\mu} (\ln|k|)_{,\nu} \right] \ . \tag{5.20}$$

Here the $(d-1)$-dimensional Ricci scalar $\bar{R}$ and the covariant derivative $\bar{\nabla}_\mu$ are associated with the metric $\bar{g}_{\mu\nu}$. Substituting (5.20) into (5.1), eliminating a surface term, and neglecting an integral over the Killing coordinate t we derive the following $(d-1)$-dimensional action for the metric $\bar{g}_{\mu\nu}$:

$$\mathcal{S}[\bar{g}_{\mu\nu}, \, k, \, \varphi] \ = \ \frac{1}{16\pi} \int d^{d-1}x \sqrt{\bar{g}} \left(\bar{R} - \tfrac{d-2}{d-3} \bar{g}^{\mu\nu} \left[\varphi_{,\mu}\varphi_{,\nu} + (\ln|k|)_{,\mu} (\ln|k|)_{,\nu} \right] \right) \ . \tag{5.21}$$

According to the principle of least action, variation of the action (5.21) with respect to the fields $\bar{g}_{\mu\nu}$, k, and φ gives the following equations[2] :

$$\bar{R}_{\mu\nu} = \tfrac{d-2}{d-3} \left[\varphi_{,\mu}\varphi_{,\nu} + (\ln|k|)_{,\mu} (\ln|k|)_{,\nu} \right] \ , \tag{5.22}$$

$$\bar{\nabla}^\mu \bar{\nabla}_\mu (\ln|k|) = 0 \ , \tag{5.23}$$

$$\bar{\nabla}^\mu \bar{\nabla}_\mu \varphi = k^{-\frac{2}{d-3}} \nabla^a \nabla_a \varphi = 0 \ . \tag{5.24}$$

The first equality in Eq. (5.24) holds because the scalar field is static.

[2] Wick rotated solutions of this system correspond to steady Ricci solitons [27].

We see that the action (5.21) and the field equations (5.22)-(5.24) are invariant under the following transformation:

$$\left.\begin{aligned} \ln\,|k'| &= \ln\,|k|\cos\psi + \varphi\sin\psi \\ \varphi' &= -\ln\,|k|\sin\psi + \varphi\cos\psi \end{aligned}\right\}, \tag{5.25}$$

which we shall call a *duality transformation*. Here the primes denote the dual solution and ψ is the duality transformation parameter whose range is defined below. The duality transformation is analogous to the Buscher T-duality transformation [28] (see also [29]). The metric dual to the metric (5.18) is

$$ds^2 = -k'^2 dt^2 + k'^{-\frac{2}{d-3}}\bar{g}_{\mu\nu}dx^\mu dx^\nu\ . \tag{5.26}$$

Thus, we can construct the dual solution (5.26) to the field equations (5.22)-(5.24) if some solution (5.18) is already known. In particular, we can apply the duality transformations (5.25) to generate the Fisher solution (5.5) without integration of the Einstein equations, starting from the Schwarzschild-Tangherlini metric (5.16) with $r_M = r_o$ and taking $\cos\psi = S$. This procedure suggests that we can present the duality transformation (5.25) in different form, in terms of the mass M and the scalar charge Σ. Indeed, starting from the metric (5.5) we have $k^2 = F^S$. Using expressions (5.6)-(5.9) and (5.25) we find that $r_o' = r_o$. Thus, r_o [see, (5.7)] is invariant of the duality transformation (there are other invariants of the duality transformation which we present in Sec. VI). Hence, we can present the duality transformation (5.25) in the following form:

$$\left.\begin{aligned} M' &= M\cos\psi + \Sigma\sin\psi \\ \Sigma' &= -M\sin\psi + \Sigma\cos\psi \end{aligned}\right\}. \tag{5.27}$$

Thus, we have the duality transformation between the mass and the scalar charge acting in the parameter space (M, Σ). To define the range for ψ we consider dual Fisher solutions which have nonnegative mass $M \geqslant 0$. Thus, for a Fisher solution defined by the parameters $(M_o \geqslant 0, \Sigma_o)$ such that

$$\psi_o = \arctan(\Sigma_o/M_o) \in [-\pi/2, \pi/2]\,, \tag{5.28}$$

the corresponding duality transformation parameter is defined by

$$\psi \in [-\pi/2 + \psi_o, \pi/2 + \psi_o]\,. \tag{5.29}$$

In particular, for $\psi = \mp\pi/2 + \psi_o$ we have $M_o' = 0$ and $\Sigma_o' = \pm(M_o^2 + \Sigma_o^2)^{1/2}$, which is a massless Fisher solution (5.13) with $r_\Sigma = r_o$. For $\psi = \psi_o$ we have

$M'_o = (M_o^2 + \Sigma_o^2)^{1/2}$ and $\Sigma'_o = 0$, which is a Schwarzschild-Tangherlini black hole (5.16) with $r_M = r_o$. Here and in what follows, unless stated explicitly, we shall refer to the massless Fisher solution (5.13) and to the Schwarzschild-Tangherlini black hole (5.16) having in mind their *dual to the Fisher solution form*, i.e., for $r_\Sigma = r_o$ and for $r_M = r_o$, respectively. This convention can be expressed in the following way:

$$r_\Sigma = r_o \Longleftrightarrow \Sigma' = (M^2 + \Sigma^2)^{\frac{1}{2}}\ , \ \ M' = 0\,, \tag{5.30}$$

$$r_M = r_o \Longleftrightarrow M' = (M^2 + \Sigma^2)^{\frac{1}{2}}\ , \ \ \Sigma' = 0\,. \tag{5.31}$$

The duality transformation (5.27) is illustrated in Fig. 5.1. From the duality diagram we see that increase (decrease) in the mass M' corresponds to decrease (increase) in the scalar charge Σ'. Thus, the duality transformation can be considered as a change of the mass M and the scalar charge Σ in the original solution to their dual values M' and Σ'. From this point of view, the duality transformation is a mapping between different members of the Fisher family of solutions (M, Σ). In particular, for $\psi_o = 0$, and $\psi = \pi/2$ the Schwarzschild-Tangherlini black hole and the massless Fisher solution are dual to each other (see, [28], p. 216). In general, any Fisher solution is dual to the Schwarzschild-Tangherlini black hole.

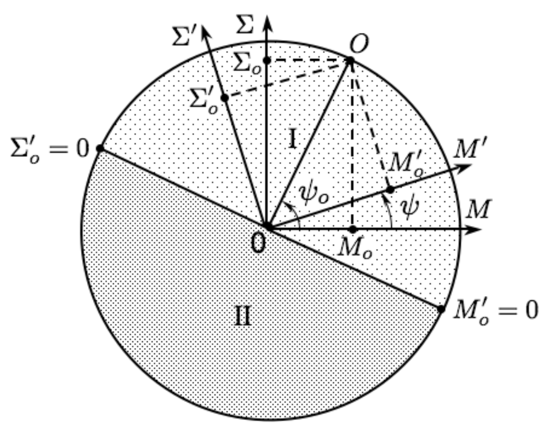

Figure 5.1: Duality diagram. Point O represents the Fisher solution defined by the mass M_o and the scalar charge Σ_o. Sector I represents its dual nonnegative mass solutions ($M'_o \geqslant 0$). One such dual Fisher solution is defined by the mass M'_o and the scalar charge Σ'_o. Sector II represents dual negative mass solutions ($M'_o < 0$) which we do not consider here.

The duality transformation (5.27) is a transformation between different solutions which follow from the same action (5.1). Each of these solutions represents a spacetime of certain properties. That is, all these spacetimes are spherically symmetric, static, and asymptotically flat. However, there is an essential difference between the Schwarzschild-Tangherlini spacetime and the Fisher solution. The Schwarzschild-Tangherlini spacetime represents a black hole of the mass M' whose event horizon is defined by $r = r_o$. The horizon is regular and the spacetime singularity is located behind the horizon at $r = 0$. However, as we shall see in the next section, the Fisher spacetime does not have an event horizon but instead has a naked singularity located at $r = r_o$. In what follows, we shall study the properties of the Fisher solution. We shall see that the spacetime geometry near the naked singularity has interesting properties which may be seen as a manifestation of the duality.

5.2.3 The Fisher universe

As we already mentioned, $r = r_o$ is a naked curvature singularity of the Fisher solution. Thus, we have to cut $r = r_o$ out of the Fisher manifold defined by the coordinates (t, r, x^α), where the index $\alpha = 3, ..., d$ stands for compact coordinates which define the position of a point on a unit $(d-2)$-dimensional round sphere. As we shall see, $r = 0$ is another curvature singularity of the Fisher solution. Thus, the cut divides the Fisher manifold into two disconnected parts defined by $r \in (r_o, \infty)$ and $r \in (0, r_o)$. In what follows, we shall call the region $r \in (r_o, \infty)$ the *Fisher spacetime*, and the region $r \in (0, r_o)$ the *Fisher universe*.

In a traditional approach, one considers that part of a manifold which represents the external field due to some source and which is asymptotically flat, if such exists. Such an approach was taken before in the case of the Fisher solution (see, e.g., [2–4]). Here we shall consider both the parts of the manifold. The reason for such a consideration is motivated by the duality between the Schwarzschild-Tangherlini black hole and the Fisher solution which we discussed above. In particular, the interior of the Schwarzschild-Tangherlini black hole corresponds to $r \in (0, r_o)$. Thus, to consider a dual to the interior part we have to consider the region $r \in (0, r_o)$ of the Fisher solution. However, for $r \in (0, r_o)$ and nonzero scalar charge the metric (5.5) is in general complex valued due to noninteger exponents[3] . One can make the metric real valued by introducing absolute values $|F|$ into the metric functions in an appropriate way. Such a modified metric solves the Einstein equations (C.2)-(C.4) but has the signature $-(d-2)$. As a result, for $r \in (0, r_o)$ the periodic angular coordinate becomes timelike which leads to causality violation, which we would not like to have here. There is another way to make the metric real valued in the region, which is to replace r_o^{d-3} with $r_o^{d-3}\,\mathrm{sign}(r - r_o)$ in the metric functions.

[3] One can show that for an appropriate discrete set of the parameters M and Σ the metric (5.5) can be real valued for $r \in (0, r_o)$. Here we shall not consider such a restrictive choice of the parameters.

However, such a choice implies that the dual Schwarzschild-Tangherlini black hole has negative mass $M' < 0$, which is out of our consideration. However, there is yet another way to get a real valued metric for $r \in (0, r_o)$. Namely, one can apply complex transformations preserving the signature of the metric and keeping mass nonnegative. The following complex transformations bring the metric in the region $r \in (0, r_o)$ to a real valued form:

$$\left.\begin{aligned} t &= (-1)^{\frac{1-S}{2}}\tau \\ r &= (-1)^{\frac{S-1}{2(d-3)}}\rho \end{aligned}\right\} , \quad \left.\begin{aligned} M &= (-1)^{\frac{S-1}{2}}\mu \\ \Sigma &= (-1)^{\frac{S-1}{2}}\sigma \end{aligned}\right\} . \tag{5.32}$$

Note that r_o transforms like r and according to expression (5.8), S is an invariant,

$$S = \frac{M}{(M^2 + \Sigma^2)^{\frac{1}{2}}} = \frac{\mu}{(\mu^2 + \sigma^2)^{\frac{1}{2}}} . \tag{5.33}$$

In the limit $S \to 1$ these transformations become merely a relabeling of the coordinates and parameters and preserve the positive direction of the time and space coordinates. In addition, in the limit $S \to 1$ the two disconnected parts of the Fisher manifold represent the exterior and interior of the Schwarzschild-Tangherlini black hole, and can be analytically extended to a larger manifold which represents the maximal d-dimensional extension of the Schwarzschild-Tangherlini solution. Such an extension was given in the Kruskal coordinates in [30] and in another coordinate system in [31, 32].

Applying the transformations (5.32) to the metric (5.5) we derive

$$ds^2 = \Phi^S d\tau^2 - \Phi^{\frac{1-S}{d-3}-1} d\rho^2 + \rho^2 \Phi^{\frac{1-S}{d-3}} d\Omega^2_{(d-2)}, \tag{5.34}$$

where

$$\Phi = \left(\frac{\rho_o}{\rho}\right)^{d-3} - 1 \ , \ \ \rho_o^{d-3} = \frac{8\Gamma(\frac{d-1}{2})}{(d-2)\pi^{\frac{d-3}{2}}} (\mu^2 + \sigma^2)^{\frac{1}{2}} . \tag{5.35}$$

Here the compact coordinate $\rho \in (0, \rho_o)$ is timelike. The spacetime (5.34) represents an anisotropic universe which we call the Fisher universe. We shall study properties of the Fisher universe in the following sections.

Applying the transformations (5.32) to the scalar field (5.9) we derive

$$\varphi = \frac{\sigma}{2(\mu^2 + \sigma^2)^{\frac{1}{2}}} \ln \Phi . \tag{5.36}$$

Calculating the energy-momentum tensor components of the scalar field in a local orthonormal frame we derive the following energy density ϵ and the principal pressures $p_{\hat{\tau}}$, $p_{\hat{\alpha}}$ [cf. Eq.(5.11)]:

$$p_{\hat{\tau}} = \epsilon = p_{\hat{\alpha}} = \frac{(d-3)\Gamma^2(\frac{d-1}{2})\sigma^2}{(d-2)\pi^{d-2}\rho^{2(d-2)}\Phi^{\frac{1-S}{d-3}+1}} \,. \tag{5.37}$$

Thus, the scalar field represents a stiff fluid. It obeys the strong and the dominant energy conditions. Therefore, by continuity it obeys the weak and the null energy conditions.

In the case of the massless Fisher solution (5.13), the transformation of the t coordinate in (5.32) is the Wick rotation. This case implies

$$\rho_o^{d-3}|_{\mu=0} = \rho_\sigma^{d-3} = \frac{8\Gamma(\frac{d-1}{2})\sigma}{(d-2)\pi^{\frac{d-3}{2}}} \ , \ \ S = 0. \tag{5.38}$$

Thus, $\varphi = 1/2 \ln \Phi$, and the corresponding metric is

$$ds^2 \ \ = \ \ d\tau^2 - \Phi_\sigma^{-\frac{d-4}{d-3}} d\rho^2 + \rho^2 \Phi_\sigma^{\frac{1}{d-3}} d\Omega^2_{(d-2)} \,, \tag{5.39}$$

where

$$\Phi_\sigma = \left(\frac{\rho_\sigma}{\rho}\right)^{d-3} - 1 \,. \tag{5.40}$$

We shall call this solution the *massless Fisher universe*. Analogous to (5.30) the dual to the Fisher universe massless solution corresponds to

$$\rho_\sigma = \rho_o \Longleftrightarrow \sigma' = (\mu^2 + \sigma^2)^{\frac{1}{2}} \ , \ \ \mu' = 0 \,. \tag{5.41}$$

Here and in what follows, unless stated explicitly, we shall refer to the massless Fisher universe (5.39) having in mind the dual to the Fisher universe form (5.41).

In general, the mass transformation in (5.32) for arbitrary $S \in [0, 1]$ has the following form:

$$\mu = M \cos\left(\frac{\pi}{2}[1 - S]\right) + iM \sin\left(\frac{\pi}{2}[1 - S]\right) \,, \tag{5.42}$$

where the first term is the bradyon mass M_B and the second term is the tachyon mass M_T. In these notations, S defines the ratio of the tachyon mass to the bradyon mass as follows:

$$\frac{M_T}{M_B} = i\tan\left(\frac{\pi}{2}[1-S]\right) . \tag{5.43}$$

The scalar charge transformation in (5.32) is analogical to (5.42),

$$\sigma = \Sigma\cos\left(\frac{\pi}{2}[1-S]\right) + i\Sigma\sin\left(\frac{\pi}{2}[1-S]\right) , \tag{5.44}$$

where the first term is a real scalar field charge and the second term is a ghost scalar field charge. However, expressions (5.42) and (5.44) are merely transformations. It is not clear if they have any physical meaning. In the Fisher spacetime and the Fisher universe the mass and the scalar charge are real.

5.3 Curvature singularities

5.3.1 Spacetime invariants

Spacetime curvature singularities, like those located inside of black holes, are associated with infinitely growing spacetime curvature invariants. To determine singularities of the Fisher solution we calculate the Ricci scalar and the Kretschmann invariant. The Ricci scalar is

$$R = \frac{1-S^2}{4}\frac{r_o^{2(d-3)}}{r^{S+d-2}}\frac{(d-2)(d-3)}{\left(r^{d-3}-r_o^{d-3}\right)^{\frac{1-S}{d-3}+1}} . \tag{5.45}$$

We see that the Ricci scalar diverges at $r = r_o$, if $S \neq 1$, and at $r = 0$. According to the transformations (5.32), $r = r_o$ and $r = 0$ correspond to $\rho = \rho_o$ and $\rho = 0$, respectively. The Schwarzschild-Tangherlini black hole solution ($S = 1$) is Ricci flat.

For $S \neq 1$ the Kretschmann invariant presented in Appendix E is proportional to R^2, therefore, it diverges at the same points. For the Schwarzschild-Tangherlini black hole the Kretschmann scalar is

$$\mathcal{K} = \frac{r_o^{2(d-3)}}{r^{2(d-1)}}(d-1)(d-2)^2(d-3) . \tag{5.46}$$

It diverges at $r = 0$. The analysis of the spacetime invariants shows that the Fisher solution is singular at $r = r_o$ ($\rho = \rho_o$) for $S \in [0, 1)$ and at $r = \rho = 0$ for $S = [0, 1]$. Both the singularities are *central*, i.e., the corresponding areal radii vanish at the singularities [see, expressions (5.100) and (5.103)]. We shall study the properties of these singularities.

5.3.2 Strength of the singularities

Spacetime curvature singularities can be characterized according to their strength. A definition of singularity strength based on purely geometric properties of spacetime was proposed in [33]. According to that definition, there are two types of curvature singularities, gravitationally *weak* and *strong*. Namely, if a volume (an area) element defined by linearly independent spacelike vorticity-free Jacobi fields propagating along any timelike (null) geodesic and orthogonal to its tangent vector vanishes at spacetime singularity, the singularity is called strong, otherwise, if the volume (the area) element does not vanish and remains finite, the singularity is called weak. Necessary and sufficient conditions for strong curvature singularities were formulated in [34, 35]. The definition above was subsequently modified in [36], where behavior of each Jacobi field was taken into account. According to the renewed definition, a spacetime singularity is called strong if at least one Jacobi field vanishes or diverges at the singularity. For example, a singularity is called strong if some of the Jacobi fields diverge and others vanish such that the volume element remains finite at the singularity. A *deformationally strong* singularity was defined in [37]. According to that definition, a spacetime singularity is called deformationally strong if the volume element diverges, or at least one Jacobi field diverges, but the volume element remains finite, for other Jacobi fields vanish at the singularity.

Here we shall study the strength of the Fisher spacetime and the Fisher universe singularities. Let us begin with the Fisher spacetime (5.5), $r \in (r_o, \infty)$. We shall study behavior of Jacobi fields defined for radial timelike and null geodesics near the spacetime singularity located at $r = r_o$. Equations for the geodesic motion can be derived from the corresponding Lagrangian $\mathcal{L}$ associated with the metric (5.5),

$$2\mathcal{L} \;=\; -F^S\dot{t}^2 + F^{\frac{1-S}{d-3}-1}\dot{r}^2 = \varepsilon\,, \tag{5.47}$$

where ε is equal to -1 for timelike and 0 for null geodesics. The overdot denotes the differentiation with respect to λ which is the proper time for timelike and the affine parameter for null geodesics. We define λ such that the geodesics approach the singularity located at $r = r_o$ as $\lambda \to -0$. The radial geodesics are defined by the unit tangent vector $k^a = \dot{x}^a$ whose nonzero components in a local orthonormal frame are given by

$$k^{\hat{t}} = F^{\frac{S}{2}}\,\dot{t} = E_\varepsilon\, F^{-\frac{S}{2}}\,, \tag{5.48}$$

$$k^{\hat{r}} = F^{\frac{1-S}{2(d-3)}-\frac{1}{2}}\,\dot{r} = \pm\left[(k^{\hat{t}})^2 + \varepsilon\right]^{\frac{1}{2}}\,, \tag{5.49}$$

where "+" stands for outgoing and "−" stands for ingoing geodesics, and $E_\varepsilon =$ *const* which we define as follows:

$$E_{-1} > 1\,,\;\; E_0 = 1\,. \tag{5.50}$$

We consider ingoing geodesics. One can check that the radial geodesics approach the singularity for finite values of λ. For $S \in (0,1]$ the geodesics approach $r = r_o$ in infinite coordinate time t which measures proper time of an observer which is at rest with respect to the gravitational center (the naked singularity) and located at asymptotic infinity ($r \to \infty$). For $S = 0$ the coordinate time t is finite.

Jacobi fields $Z^{\hat{a}}(\lambda)$ are orthogonal to $k^{\hat{a}}$ and represent the spatial separation of two points of equal values of λ located on neighboring geodesics. They satisfy the Jacobi geodesic deviation equation (see, e.g., [2])

$$\ddot{Z}^{\hat{a}} + R_{\hat{c}\hat{b}\hat{d}}{}^{\hat{a}} Z^{\hat{b}} k^{\hat{c}} k^{\hat{d}} = 0\,, \tag{5.51}$$

where $R_{\hat{c}\hat{b}\hat{d}}{}^{\hat{a}}$ are the Riemann tensor components defined in the local orthonormal frame (see, Appendix E).

For radial timelike geodesics we define two types of the Jacobi fields. The *radial* Jacobi field

$$Z^{\eta}\partial_{\eta} = Z^{\hat{t}}\partial_{\hat{t}} + Z^{\hat{r}}\partial_{\hat{r}}\;\;,\;\; g_{\eta\eta} = 1\,, \tag{5.52}$$

and the $(d-2)$ orthogonal *angular* Jacobi fields

$$Z^{\hat{\alpha}}\partial_{\hat{\alpha}}\;\;,\;\; g_{\hat{\alpha}\hat{\alpha}} = 1\;\;,\;\; \hat{\alpha} = 3, ..., d\,. \tag{5.53}$$

The spacelike vectors $\partial_\eta, \partial_{\hat{\alpha}}$, $\hat{\alpha} = 3, ..., d$ form a $(d-1)$-dimensional orthonormal basis which is parallel propagated along the radial timelike geodesics. As far as we are interested in spatial separations of neighboring geodesics, for radial null geodesics we consider only the angular Jacobi fields (5.53).

The radial Jacobi field satisfies the Jacobi equation

$$\ddot{Z}^{\eta} + R_{\hat{t}\hat{r}\hat{t}}{}^{\hat{r}} Z^{\eta} = 0\,. \tag{5.54}$$

Approximating expressions (5.48), (5.49), and (D.1) near the singularity we derive

$$\ddot{Z}^{\eta} - C|\lambda|^{-2+\frac{2S(d-3)}{d-2+S(d-4)}} Z^{\eta} \approx 0\,, \tag{5.55}$$

where

$$C = \frac{S(1-S)(d-2)(d-3)(2r_o)^{-\frac{2S(d-3)}{d-2+S(d-4)}}}{\left(E_{-1}[d-2+S(d-4)]\right)^{2-\frac{2S(d-3)}{d-2+S(d-4)}}}\,. \tag{5.56}$$

This equation is a particular case of the Emden-Fowler equation (see, Eq. (2.1.2.7), p.132 in [38]). Its solutions are expressed in terms of the modified Bessel functions of the first and second kind. Using asymptotics of the modified Bessel functions for small values of their arguments (see, e.g., Eqs. (9.6.7) and (9.6.9) in [39]) we derive the asymptotic behavior of the radial Jacobi field near the singularity

$$Z^{\eta}(\lambda) \sim c_1 + c_2|\lambda| \sim c_1\,. \tag{5.57}$$

Here and in what follows $c_{1,2} = const \neq 0$. Thus, for $S \in [0,1)$ the radial Jacobi field remains finite at the singularity. Although it is obvious that the Jacobi field is finite in the case of the Schwarzschild-Tangherlini black hole ($S = 1$), for there is no spacetime singularity at $r = r_o$, it is remarkable that the radial Jacobi field is finite at the singularity of the Fisher spacetime. Thus, the singularity at $r = r_o$ is of a special type, which we call *radially weak*.

Now we consider the angular Jacobi fields (5.53). Each of the $(d-2)$ angular Jacobi fields $Z^{\hat{\alpha}}$ satisfies the following equation (no summation over $\hat{\alpha}$):

$$\ddot{Z}^{\hat{\alpha}} + \left[R_{\hat{t}\hat{\alpha}\hat{t}}{}^{\hat{\alpha}}(k^{\hat{t}})^2 + R_{\hat{r}\hat{\alpha}\hat{r}}{}^{\hat{\alpha}}(k^{\hat{r}})^2\right] Z^{\hat{\alpha}} = 0\,. \tag{5.58}$$

This equation is valid for both the radial timelike and null geodesics. Approximating expressions (5.48), (5.49), (D.2), and (D.3) near the singularity and applying the method of Frobenius we derive the asymptotic behavior of the angular Jacobi fields

$$Z^{\hat{\alpha}}(\lambda) \;\sim\; c_1|\lambda|^{\frac{1-S}{d-2+S(d-4)}} + c_2|\lambda|^{\frac{(1+S)(d-3)}{d-2+S(d-4)}} \sim c_1|\lambda|^{\frac{1-S}{d-2+S(d-4)}}\,. \tag{5.59}$$

This expression is valid for the radial timelike and null geodesics for $S \in [0,1]$. There is no singularity for $S = 1$, and the corresponding angular Jacobi fields are finite. For other values of S the angular Jacobi fields vanish.

Let us now study the singularities of the Fisher universe (5.34). We shall study behavior of the Jacobi fields defined for radial timelike and null geodesics approaching the spacetime singularities located at $\rho = \rho_o$ and at $\rho = 0$. Applying the transformations (5.32) to expressions (5.48) and (5.49) we derive the nonzero components of the unit tangent vector

$$k^{\hat{\tau}} = \Phi^{\frac{S}{2}}\,\dot{\tau} = \mathscr{E}_\varepsilon\,\Phi^{-\frac{S}{2}}\,, \tag{5.60}$$

$$k^{\hat{\rho}} = \Phi^{\frac{1-S}{2(d-3)}-\frac{1}{2}}\,\dot{\rho} = \mp\left[(k^{\hat{\tau}})^2 - \varepsilon\right]^{\frac{1}{2}}\,, \tag{5.61}$$

where "$-$" stands for outgoing and "$+$" stands for ingoing geodesics and $\mathscr{E}_\varepsilon = const$ which we define as follows:

$$\mathscr{E}_{-1} \geqslant 0\,, \quad \mathscr{E}_0 = 1\,. \tag{5.62}$$

One can check that the radial geodesics approach the singularities for finite values of λ. For $S \in (0,1]$ and geodesics approaching $\rho = \rho_o$, the finite change of λ corresponds to an infinite change of the spacelike coordinate τ for $\mathscr{E}_{-1} > 0$, whereas for geodesics approaching $\rho = 0$ the change of the spacelike coordinate τ vanishes. For $S = 0$ the change of the coordinate τ is always finite.

The geodesics deviation equations for the radial and angular Jacobi fields (5.52) and (5.53) orthogonal to the tangent vector (5.60) and (5.61) can be constructed by applying the transformations (5.32) to the Riemann tensor components in Eqs. (5.54) and (5.58). Solving the derived equations near the singularity $\rho = \rho_o$ of the Fisher universe, one can see that the behavior of the Jacobi fields is exactly the same as the behavior of the corresponding Jacobi fields (5.57) and (5.59) near the singularity $r = r_o$ of the Fisher spacetime.

Let us examine the singularity at $\rho = 0$. Approximating the Jacobi equation (5.54) near the singularity and applying the method of Frobenius, we derive the asymptotic behavior of the radial Jacobi field,

$$Z^\eta(\lambda) \sim c_1|\lambda|^{-\frac{S(d-3)}{d-2+S}} + c_2|\lambda|^{\frac{(1+S)(d-2)}{d-2+S}} \sim c_1|\lambda|^{-\frac{S(d-3)}{d-2+S}}\,, \tag{5.63}$$

where $S \in (0,1]$. Thus, as in the case of the Schwarzschild-Tangherlini black hole, the radial Jacobi field diverges. However, in the case of the massless Fisher solution ($S = 0$) the radial Jacobi field is finite at the singularity and given by expression (5.57). Thus, this singularity is radially weak as well.

Let us consider the asymptotic behavior of the angular Jacobi fields (5.53) corresponding to the radial timelike and null geodesics approaching the singularity.

For timelike geodesics and for $d = 4$ we have

$$Z^{\hat{\alpha}}(\lambda) \sim c_1|\lambda|^{\frac{1}{2+S}} + c_2|\lambda|^{\frac{1+S}{2+S}} \sim c_1|\lambda|^{\frac{1}{2+S}}\,, \tag{5.64}$$

whereas for $d > 4$ we have

$$Z^{\hat{\alpha}}(\lambda) \;\sim\; c_1|\lambda|^{\frac{1+S}{d-2+S}} + c_2|\lambda|^{\frac{d-3}{d-2+S}} \sim c_1|\lambda|^{\frac{1+S}{d-2+S}}\,. \tag{5.65}$$

For null geodesics we have

$$\begin{aligned} Z^{\hat{\alpha}}(\lambda) \;&\sim\; c_1|\lambda|^{\frac{1+S}{d-2-S(d-4)}} + c_2|\lambda|^{\frac{(1-S)(d-3)}{d-2-S(d-4)}} \\ &\sim\; c_1|\lambda|^{\frac{1+S}{d-2-S(d-4)}}\,,\;\; S \in \left(0, \tfrac{d-4}{d-2}\right]\,, \end{aligned} \tag{5.66}$$

and

$$Z^{\hat{\alpha}}(\lambda) \sim c_2|\lambda|^{\frac{(1-S)(d-3)}{d-2-S(d-4)}}\,,\;\; S \in \left(\tfrac{d-4}{d-2}, 1\right]\,. \tag{5.67}$$

Thus, for $S \in (0, 1]$ and the radial timelike and null geodesics approaching the singularity at $\rho = 0$, the angular Jacobi fields vanish.

To define the strength of the singularities we calculate first the norm of the $(d-1)$-dimensional volume element of a synchronous frame which is defined by 1-forms corresponding to the radial and angular Jacobi fields calculated for the radial timelike geodesics as follows:

$$\|V_{(d-1)}\| = |Z^{\eta}| \prod_{\hat{\alpha}=3}^{d} |Z^{\hat{\alpha}}|\,. \tag{5.68}$$

Near the singularities the norm of the volume element can be approximated according to the behavior of the Jacobi fields [see Eqs. (5.57),(5.59), and (5.63)-(5.65)] as follows:

$$\|V_{(d-1)}\| \sim |\lambda|^{v}\,, \tag{5.69}$$

where the exponent $v = const$ defines how fast the norm of the volume element vanishes or diverges when we approach the singularities ($\lambda \to -0$). Thus, to compare the strength of the singularities of the Fisher spacetime and the Fisher universe we compare the corresponding values of the exponent v. The results are given in Table I.

For null geodesics approaching the singularities we calculate the norm of the $(d-2)$-dimensional area element which is defined by 1-forms corresponding to the

Table 5.1: The values of the exponent v for the radial timelike geodesics approaching the singularities.

d	$r=r_o\,(\rho=\rho_o)$	$\rho=0$
$=4$	$1-S\geqslant 0$	$\frac{2-S}{2+S}\geqslant\frac{1}{3}$
>4	$\frac{(1-S)(d-2)}{d-2+S(d-4)}\geqslant 0$	1

Table 5.2: The values of the exponent a for the radial null geodesics approaching the singularities.

d	$r=r_o\,(\rho=\rho_o)$	$\rho=0$	$\rho=0^b$
$=4$	$1-S\geqslant 0$	1	$1-S\geqslant 0$
>4	$\frac{(1-S)(d-2)}{d-2+S(d-4)}\geqslant 0$	$\frac{(1+S)(d-2)}{d-2-S(d-4)}\geqslant 1$	$\frac{(1-S)(d-2)(d-3)}{d-2-S(d-4)}\geqslant 0$

angular Jacobi fields calculated for the radial null geodesics as follows:

$$\|A_{(d-2)}\| = \prod_{\hat{\alpha}=3}^{d} |Z^{\hat{\alpha}}|. \tag{5.70}$$

Analogous to the norm of the volume element, the norm of the area element can be approximated near the singularities according to the behavior of the angular Jacobi fields [see Eqs. (5.59),(5.66), and (5.67)] as follows:

$$\|A_{(d-1)}\| \sim |\lambda|^{a}\,, \tag{5.71}$$

where the exponent $a = const$ defines how fast the norm of the area element vanishes or diverges when we approach the singularities ($\lambda \to -0$). The values of the exponent a calculated for the radial null geodesics approaching the singularities of the Fisher spacetime and the Fisher universe are given in Table II.

Now we can summarize our results. According to the values of the exponents v and a presented in Tables I and II the volume and the area elements vanish at the singularities, except for the case of $S = 1$ and $r = r_o$, where $v = a = 0$, so the volume element is finite. This case corresponds to the event horizon of the Schwarzschild-Tangherlini black hole. At the black hole singularity ($r = \rho = 0$) the area element is finite as well ($a = 0$). Thus, according to the classifications of spacetime singularities, the singularities of the Fisher spacetime and the Fisher universe are strong. In addition, the strength of the singularity at $\rho = 0$ is greater if the value of S is smaller. However, for the radial timelike geodesics and $d > 4$

the strength does not depend on S. Thus, in general, the scalar field decreases the values of the volume and the area elements. From the tables we see that for $S \in (0,1)$ the singularity at $\rho = 0$ is stronger than the singularity at $r = r_o$, whereas for $S = 0$ these singularities have equal strength.

Let us analyze the behavior of the Jacobi fields. An analysis of the angular Jacobi fields (5.59),(5.64),(5.66), and (5.67) shows that the scalar field contracts the spacetime in the angular directions. However, for the radial timelike geodesics and $d > 4$ [see, (5.65)] it decreases the spacetime contraction in the angular directions caused by the gravitational field. From expressions (5.64) and (5.65) we see that in the case of the Schwarzschild-Tangherlini black hole ($S = 1$) the angular Jacobi fields contract faster for $d = 5, 6$ than for $d = 4$, and for $d = 4$ and $d = 7$ the contraction rates are the same, whereas for $d > 7$ the contraction is less than for $d = 4$. In the presence of the scalar field ($S \neq 1$) for $d > 4$ the contraction is less [see, (5.65)]. An analysis of the radial Jacobi field (5.63) shows that the scalar field decreases its divergence, i.e., the scalar field contracts the Fisher spacetime in the radial direction as well. However, the radial Jacobi fields (5.57) at the singularities at $r = r_o$ and at $\rho = \rho_o$ for $S \in [0, 1)$, as well as at the singularity at $\rho = 0$ for $S = 0$ remain finite. According to our calculations, this is a generic property of the singularities which is valid for any set of initial data. In other words, no fine-tuning is required for such a behavior of the radial Jacobi fields. It implies that a 1-dimensional object, for example, an infinitesimally thin rod, which is moving along a radial timelike geodesic will arrive intact to the singularities without being contracted to zero or stretched to infinity. We call these singularities radially weak.

Finite, nonzero values of the radial Jacobi fields terminating at the radially weak singularities may suggest a C^0 local extension [33] of the 2-dimensional (t, r) and (τ, ρ) spacetime surfaces through the singularities. In Sec. VII we shall discuss such an extension for the singularities of the Fisher solution.

5.4 Causal Properties of the Fisher solution

5.4.1 Closed trapped surfaces

The concept of a *closed trapped surface* introduced by Penrose [40] was crucial for the formulation of the singularity theorems [2]. In a d-dimensional spacetime a closed trapped surface $\mathcal{T}$ is a $(d-2)$-dimensional spacelike compact surface without boundary which is defined according to the following property: future directed outgoing and ingoing null geodesics orthogonal to $\mathcal{T}$ are converging at $\mathcal{T}$. Mathematically, this property is expressed in the following way. Let $\vec{n}^{\pm}$ be future directed null vectors orthogonal to $\mathcal{T}$ and normalized in the following way:

$g(\vec{n}^+, \vec{n}^-) = -1$, where "+" stands for outgoing and "−" stands for ingoing null geodesics. Then, the scale-invariant *trapping scalar* defined on $\mathcal{T}$ is as follows:

$$\Theta_{\mathcal{T}} = \theta^+\theta^- \tag{5.72}$$

must be positive (see, e.g., [2, 41]). Here $\theta^\pm$ are the null expansions of the null geodesics defined on $\mathcal{T}$ and expressed in terms of the null second fundamental forms

$$\chi^\pm_{\alpha\beta} = e^{\ a}_{(\alpha)} e^{\ b}_{(\beta)} \nabla_b n^\pm_a , \tag{5.73}$$

in the following way:

$$\theta^\pm = \gamma^{\alpha\beta}\chi^\pm_{\alpha\beta}\big|_{\mathcal{T}} . \tag{5.74}$$

Here $e^{\ a}_{(\alpha)}$, $\alpha = 3, ..., d$ are the base-vectors tangential to $\mathcal{T}$ and

$$\gamma_{\alpha\beta} = g_{ab}|_{\mathcal{T}}\, e^{\ a}_{(\alpha)} e^{\ b}_{(\beta)} \tag{5.75}$$

is the positive-defined metric induced on $\mathcal{T}$.

Let us examine if closed trapped surfaces are present in the Fisher spacetime and/or the Fisher universe. The Fisher spacetime (5.5), $r \in (r_o, \infty)$ is static and spherically symmetric. Thus, we define $\mathcal{T}$ by $t = const$ and $r = const$. In this case, the trapping scalar (5.72) is

$$\Theta_{\mathcal{T}} = -\left.\frac{g^{rr}}{8}\left(\frac{\partial \ln \gamma}{\partial r}\right)^2\right|_{r=const} , \tag{5.76}$$

where $\gamma = det(\gamma_{\alpha\beta})$ and the indices $\alpha, \beta = 3, ..., d$ stand for angular coordinates. For the Fisher spacetime (3.25), $r \in (r_o, \infty)$ expression (5.76) reads

$$\Theta_{\mathcal{T}} = -\left.\frac{(d-2)^2}{8r^{S+d-2}} \frac{\left(2r^{d-3} - (1+S)r_o^{d-3}\right)^2}{\left(r^{d-3} - r_o^{d-3}\right)^{\frac{1-S}{d-3}+1}}\right|_{r=const.} . \tag{5.77}$$

This expression is negative for $r \in (r_o, \infty)$. Thus, there are no closed trapped surfaces in the Fisher spacetime. For the Fisher universe (3.44) $\mathcal{T}$ is defined by $\tau = const$ and $\rho = const$ and the trapping scalar is

$$\Theta_{\mathcal{T}} = \left.\frac{(d-2)^2}{8\rho^{S+d-2}} \frac{\left((1+S)\rho_o^{d-3} - 2\rho^{d-3}\right)^2}{\left(\rho_o^{d-3} - \rho^{d-3}\right)^{\frac{1-S}{d-3}+1}}\right|_{\rho=const.} . \tag{5.78}$$

Clearly, it is nonnegative for $\rho \in (0, \rho_o)$. The trapping scalar vanishes for

$$\rho = \rho_* = \rho_o \left(\frac{1+S}{2}\right)^{\frac{1}{d-3}} . \tag{5.79}$$

The corresponding spacelike $(d-2)$-dimensional surface is called a *marginally trapped surface*. Note that $R_{\hat{\tau}\hat{\rho}\hat{\tau}}{}^{\hat{\rho}}$ is zero on this surface [see, (D.1) and (5.32)]. For the Schwarzschild-Tangherlini black hole the *marginally trapped surface* coincides with the surface of its event horizon: $\rho_* = \rho_o = r_o$. In the case of the massless Fisher solution we have $\rho_*^{d-3} = \rho_o^{d-3}/2$.

Let us calculate the maximal proper time λ_1 corresponding to the interval $\rho \in (0, \rho_*]$ for the radial timelike geodesics. Using (5.60) and (5.61) and taking $\mathscr{E}_{-1} = 0$ we derive

$$\begin{aligned}\lambda_1 &= \int_{+0}^{\rho_*} d\rho \left[\left(\frac{\rho_o}{\rho}\right)^{d-3} - 1\right]^{\frac{1-S}{2(d-3)}-\frac{1}{2}} \\ &= \frac{\rho_o}{d-3}\mathcal{B}_{\frac{1+S}{2}}\left(\frac{1+S}{2(d-3)}+\frac{1}{2}, \frac{1-S}{2(d-3)}+\frac{1}{2}\right),\end{aligned} \tag{5.80}$$

where $\mathcal{B}_x(a,b)$ is the incomplete beta function (see, e.g., [39], p. 263). The maximal proper time λ_2 corresponding to the interval $\rho \in [\rho_*, \rho_o)$ is

$$\begin{aligned}\lambda_2 &= \int_{\rho_*}^{\rho_o-0} d\rho \left[\left(\frac{\rho_o}{\rho}\right)^{d-3} - 1\right]^{\frac{1-S}{2(d-3)}-\frac{1}{2}} \\ &= \frac{\rho_o}{d-3}\mathcal{B}\left(\frac{1+S}{2(d-3)}+\frac{1}{2}, \frac{1-S}{2(d-3)}+\frac{1}{2}\right) - \lambda_1 ,\end{aligned} \tag{5.81}$$

where $\mathcal{B}(a,b)$ is the beta function (see, e.g., [39], p. 258). According to the symmetry property of the incomplete beta function,

$$\mathcal{B}_x(a,b) = \mathcal{B}(a,b) - \mathcal{B}_{1-x}(b,a) , \tag{5.82}$$

for the massless Fisher solution $(S=0)$ we have

$$\lambda_1 = \lambda_2 = \frac{\rho_o}{2(d-3)}\mathcal{B}\left(\frac{d-2}{2(d-3)}, \frac{d-2}{2(d-3)}\right) . \tag{5.83}$$

For the Schwarzschild-Tangherlini black hole ($S = 1$) we have

$$\lambda_1 = \frac{r_o}{d-3}\mathcal{B}\left(\frac{1}{d-3}+\frac{1}{2},\frac{1}{2}\right)\ ,\ \ \lambda_2 = 0\,. \tag{5.84}$$

In the 4-dimensional case this expression reduces to the well-known result: $\lambda_1 = \pi M$ (see [20], p. 836). Let us see how λ_1 and λ_2 depend on the scalar charge Σ. Figure 5.2(**a**) illustrates the maximal proper time λ_1 and λ_2 as a function of S for the fixed value of the mass $\mu = 1$. Thus, $S = 0$ corresponds to infinite value of the scalar charge σ [see expression (5.33)] and, as a result, $\lambda_1 = \lambda_2 \to \infty$. Note, that for any $d \geqslant 4$ the maximal proper time λ_1 has a local minimum for a certain value of $S \in (0, 1)$.

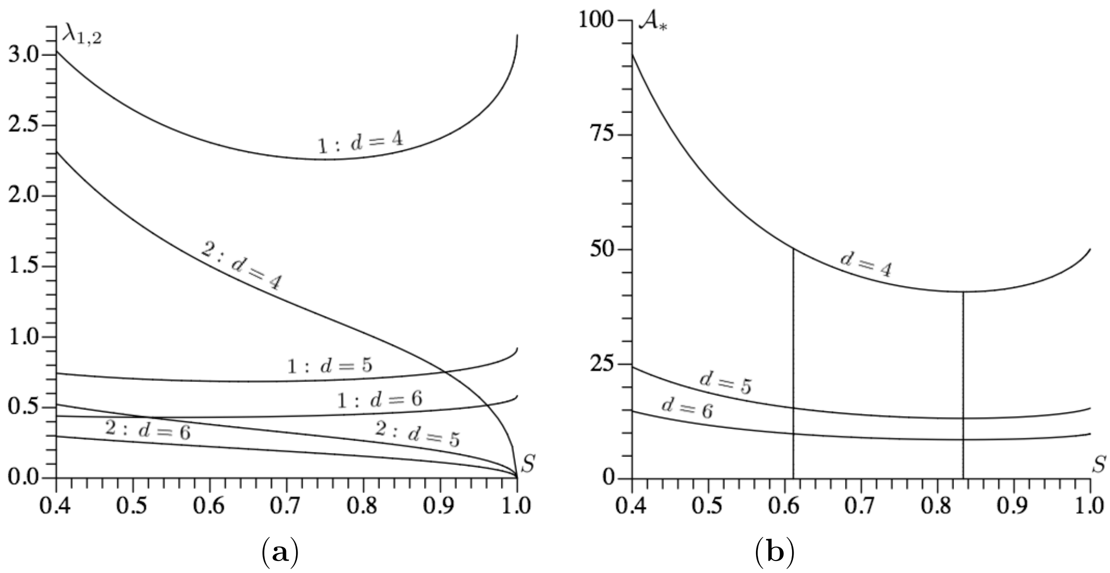

Figure 5.2: (**a**): Maximal proper time $\lambda_{1,2}$ as a function of S for the fixed value of the mass $\mu = 1$ and $d = 4, 5, 6$. The indices 1 and 2 correspond to λ_1 and λ_2, respectively. (**b**): Area $\mathcal{A}_*$ as a function of S for $\mu = 1$ and $d = 4, 5, 6$. In any dimension, the minimal value of $\mathcal{A}_*$ corresponds to $S \approx 0.834$ and for $S \approx 0.611$ the value of $\mathcal{A}_*$ equals to the horizon surface area of the Schwarzschild-Tangherlini black hole of $\mu = M = 1$ and $\Sigma = 0$, ($S = 1$).

Let us calculate the area of the marginally trapped surface defined by (5.79). The areal radius corresponding to ρ_* is

$$\mathscr{R}_* \equiv \mathscr{R}(\rho_*) = 2^{-\frac{1}{d-3}}\rho_o(1-S)^{\frac{1-S}{2(d-3)}}(1+S)^{\frac{1+S}{2(d-3)}}\,. \tag{5.85}$$

For $S = 1$ we have $\mathscr{R}_* = \rho_o = r_o$, which corresponds to the Schwarzschild-Tangherlini black hole, and for $S = 0$ we have $\mathscr{R}_* = 2^{-\frac{1}{d-3}}\rho_o$, which corresponds to the massless Fisher solution. Thus, the area of the $(d-2)$-dimensional marginally trapped surface is

$$\mathcal{A}_* = \frac{2\mathscr{R}_*^{d-2}\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}. \tag{5.86}$$

Figure 5.2(**b**) illustrates how this area depends on the value of S for the fixed value of the mass $\mu = 1$.

5.4.2 Misner-Sharp energy

In a spherically symmetric spacetime the Misner-Sharp energy, which is a spacetime invariant, defines the "local gravitational energy" inside a sphere of the areal radius R (see, e.g.,[20, 42]). It has many interesting properties (see, e.g., [43]). In particular, at spatial infinity in an asymptotically flat spacetime it reduces to the Arnowitt-Deser-Misner energy. For a central singularity, a negative value of the Misner-Sharp energy implies that the singularity is untrapped and timelike. If the dominant energy condition holds on an untrapped sphere, the Misner-Sharp energy is monotonically increasing in outgoing spatial or null directions. As we shall see below, this is exactly the case for the central singularity at $r = r_o$ of the Fisher spacetime. Here we use the following expression for the Misner-Sharp energy generalized to a d-dimensional spacetime:

$$M(r) = \frac{(d-2)\pi^{\frac{d-3}{2}}}{8\Gamma(\frac{d-1}{2})}R(r)^{d-3}\left[1 - g^{rr}(r)\left(\frac{dR(r)}{dr}\right)^2\right]. \tag{5.87}$$

For the Fisher spacetime (5.5), $r \in (r_o, \infty)$ we have

$$R(r) = r\left[1 - \left(\frac{r_o}{r}\right)^{d-3}\right]^{\frac{1-S}{2(d-3)}}, \tag{5.88}$$

and the Misner-Sharp energy is

$$M(r) = \frac{(d-2)\pi^{\frac{d-3}{2}}}{32\Gamma(\frac{d-1}{2})}\frac{r_o^{d-3}\left(4Sr^{d-3} - (1+S)^2 r_o^{d-3}\right)}{r^{(1-S)\frac{d-3}{2}}\left(r^{d-3} - r_o^{d-3}\right)^{\frac{1+S}{2}}}. \tag{5.89}$$

In the limit $r \to \infty$ we have $M(r) \to M$. The Misner-Sharp energy (5.89) vanishes for

$$r = r_e = r_o(4S)^{-\frac{1}{d-3}}(1+S)^{\frac{2}{d-3}}\ , \quad r_e > r_o\,, \tag{5.90}$$

where $S \in [0,1)$, and it is negative for $r \in (r_o, r_e)$. Note that $R_{\hat{\alpha}\hat{\beta}\hat{\alpha}}{}^{\hat{\beta}}$ is zero for $r = r_e$ [see, (D.4)]. For $S = 1$, which corresponds to the Schwarzschild-Tangherlini black hole, we have

$$M(r) = \frac{(d-2)\pi^{\frac{d-3}{2}}}{8\Gamma(\frac{d-1}{2})} r_o^{d-3} = M \geqslant 0\,. \tag{5.91}$$

For a negative mass Schwarzschild-Tangherlini spacetime which has naked singularity, $M(r) = M < 0$ everywhere.

The Misner-Sharp energy (5.87) can be expressed in terms of the trapping scalar $\Theta_{\mathcal{T}}$ [see, (5.76)] as follows:

$$M(r) = \frac{(d-2)\pi^{\frac{d-3}{2}}}{8\Gamma(\frac{d-1}{2})} R(r)^{d-3} \left[1 + \frac{2R(r)^2}{(d-2)^2}\Theta_{\mathcal{T}}\right]\,. \tag{5.92}$$

Thus, it defines a condition when a sphere of the areal radius R is trapped. Another way to define this condition is to introduce the "local (Newtonian) gravitational potential energy" associated with $M(r)$ as follows:

$$U(R) = \frac{4\Gamma(\frac{d-1}{2})M(r)}{(d-2)\pi^{\frac{d-3}{2}} R(r)^{d-3}}\,. \tag{5.93}$$

Then, the trapping condition is the following: if $U(R) > 1/2$ the surface $R = const$ is trapped, if $U(R) = 1/2$ the surface is marginally trapped, and if $U(R) < 1/2$ the surface is untrapped. Figure 5.3 illustrates $M(R)$ and $U(R)$ for $d = 4$. For any $d \geqslant 4$, $M(R)$ is monotonically increasing and $U(R)$ has the maximum $U_m = U(R_m) = S^2/2 \leqslant 1/2$ [4] , where

$$R_m \equiv R(r_m) = r_o(2S)^{-\frac{1}{d-3}}(1-S)^{\frac{1-S}{2(d-3)}}(1+S)^{\frac{1+S}{2(d-3)}}\,, \tag{5.94}$$

[4] Let us note that for a Reisner-Nordström spacetime behavior of the corresponding functions $M(R)$ and $U(R)$ is qualitatively the same. However, the Misner-Sharp energy is negative in the region behind the Cauchy horizon, and maximum of $U(R) \geqslant 1/2$, were the equality sign stands for the extremal Reisner-Nordström spacetime.

and

$$r_m = r_o \left(\frac{1+S}{2S}\right)^{\frac{1}{d-3}} \quad , \quad r_m \geqslant r_e \, . \tag{5.95}$$

For the Schwarzschild-Tangherlini black hole we have $R_m = r_o$. Note that $R_{\hat{r}\hat{\alpha}\hat{r}}{}^{\hat{\alpha}}$ is zero for $r = r_m$ [see, (D.3)].

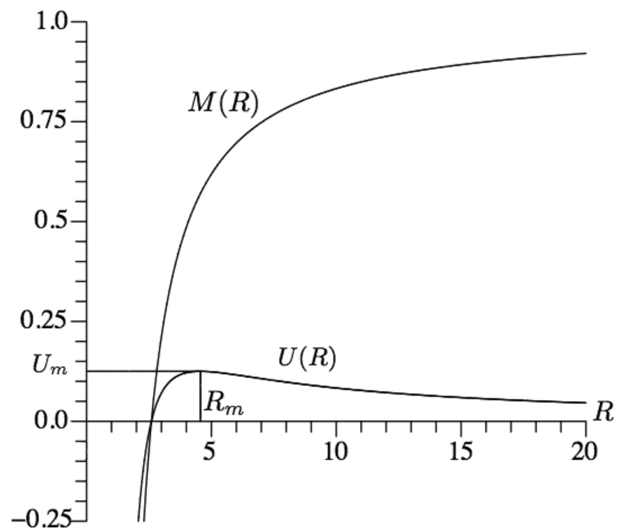

Figure 5.3: Misner-Sharp energy $M(R)$ and the "local (Newtonian) gravitational potential energy" $U(R)$ for $M = 1$, $S = 1/2$, and $d = 4$. The maximum $U_m = U(R_m)$ corresponds to $R_m = 2 \cdot 3^{3/4}$ and is equal to $1/8$.

Let us calculate geometric invariants of the region where $M(r) \leqslant 0$. The proper distance corresponding to nonpositive $M(r)$ is

$$L_e = \int_{r_o+0}^{r_e} dr \left[1 - \left(\frac{r_o}{r}\right)^{d-3}\right]^{\frac{1-S}{2(d-3)} - \frac{1}{2}} \, . \tag{5.96}$$

Figure 5.4(**a**) illustrates the proper distance L_e as a function of S for the fixed value of the mass $M = 1$. According to the figure, the proper distance L_e is a monotonically decreasing function of S. This function diverges for $S \to 0$ corresponding to infinite value of the scalar charge Σ [see expression (5.8)].

Let us calculate the area of the sphere corresponding to zero Misner-Sharp energy. The areal radius corresponding to r_e [see, (5.90)] is

$$R_e \equiv R(r_e) = r_o (4S)^{-\frac{1}{d-3}} (1-S)^{\frac{1-S}{d-3}} (1+S)^{\frac{1+S}{d-3}} \, . \tag{5.97}$$

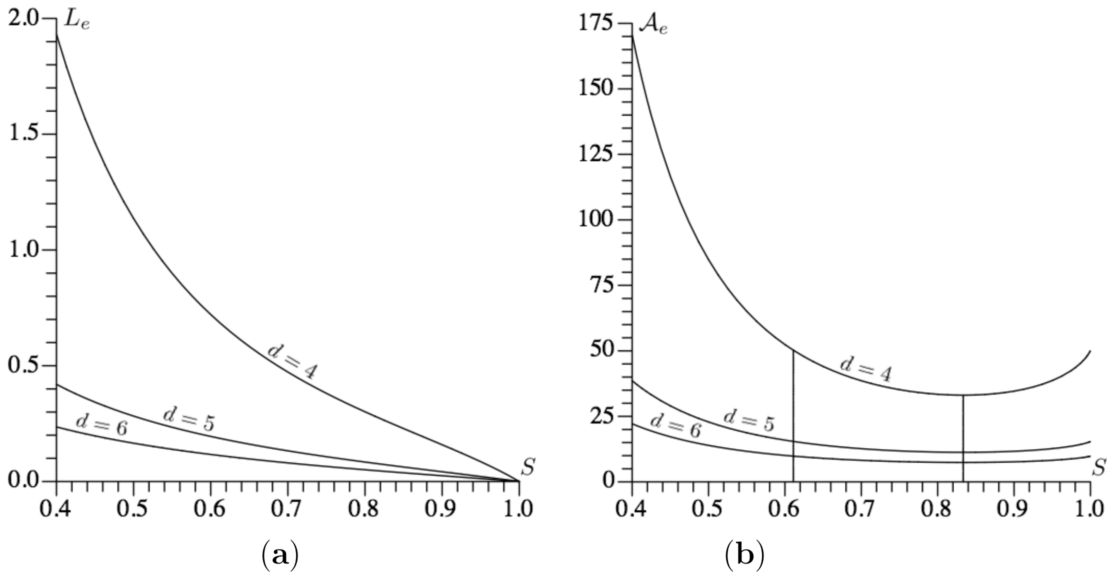

Figure 5.4: (**a**): Proper distance L_e as a function of S for the fixed value of the mass $M = 1$ and $d = 4, 5, 6$. (**b**): Area $\mathcal{A}_e$ as a function of S for $M = 1$. In any dimension, the minimal value of $\mathcal{A}_e$ corresponds to $S \approx 0.834$ and for $S \approx 0.611$ the value of $\mathcal{A}_e$ equals to the horizon surface area of the Schwarzschild-Tangherlini black hole of $M = 1$ and $\Sigma = 0$, ($S = 1$).

For the Schwarzschild-Tangherlini black hole ($S = 1$) we have $R_e = r_o$ and for the massless Fisher solution ($S = 0$) we have $R_e \to +\infty$. The area of the $(d-2)$-dimensional sphere corresponding to zero Misner-Sharp energy is

$$\mathcal{A}_e = \frac{2R_e^{d-2}\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \,. \tag{5.98}$$

Figure 5.4(**b**) illustrates how this area depends on the value of S for the fixed value of the mass $M = 1$. It is remarkable that in any dimension $d \geqslant 4$ both the areas $\mathcal{A}_e$ and $\mathcal{A}_*$ [see Fig. 5.2(**b**)] have minimal values at the same value of $S \approx 0.834$, and for $S \approx 0.611$ they are equal to the horizon surface area of the Schwarzschild-Tangherlini black hole of $M = 1$ and $\Sigma = 0$, ($S = 1$).

5.4.3 Causal structure

To study the causal structure of the Fisher spacetime and the Fisher universe we consider first radial null geodesics. We start from the Fisher spacetime (5.5),

$r \in (r_o, \infty)$ and consider radial null geodesics in the (t, R) plane, where $R = R(r)$ is the areal radius [see, (5.88)], which is a geometric invariant. Using (5.48) and (5.49) we present the solution for the radial null geodesics in the following form:

$$t(r) = \pm \int dr \left[1 - \left(\frac{r_o}{r}\right)^{d-3}\right]^{\frac{1-S}{2(d-3)} - \frac{1+S}{2}}, \quad (5.99)$$

$$R(r) = r \left[1 - \left(\frac{r_o}{r}\right)^{d-3}\right]^{\frac{1-S}{2(d-3)}}, \quad (5.100)$$

where "+" stands for outgoing and "−" stands for ingoing radial null geodesics. The coordinate t is timelike and the areal radius $R(r)$ is spacelike. Local null cones are defined by

$$\frac{dt}{dR} = \pm 2 r^{(1+S)\frac{d-3}{2}} \frac{\left(r^{d-3} - r_o^{d-3}\right)^{\frac{1-S}{2}}}{2r^{d-3} - (1+S) r_o^{d-3}}. \quad (5.101)$$

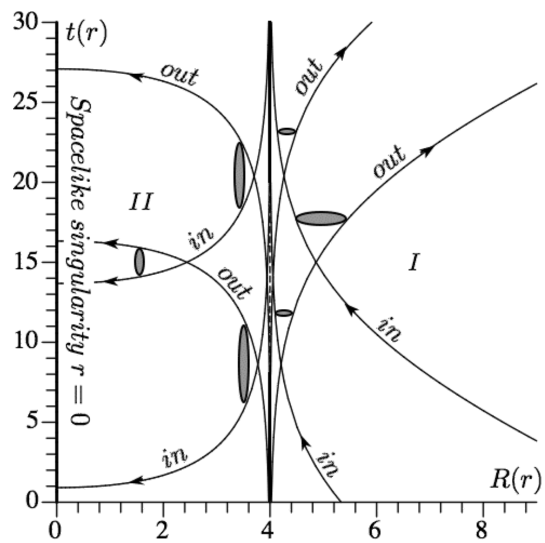

Figure 5.5: Radial null geodesics in the Schwarzschild-Tangherlini spacetime of $M' = 2$, $\Sigma' = 0$, [see, (5.31)] and $d = 4$. The behavior of the geodesics is generic for other values of $d > 4$. The black hole event horizon is located at $R = r_o = 4$. It separates the exterior I and interior II regions. The spacelike singularity is located at $R = r = 0$. The direction of local time is illustrated by the future null cones.

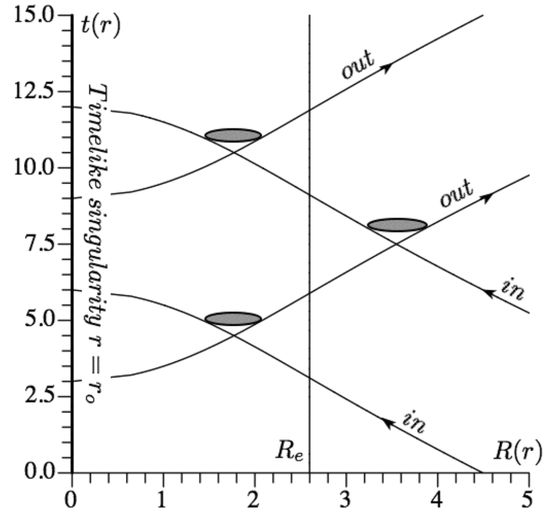

Figure 5.6: Radial null geodesics in the Fisher spacetime of $M = 1$, $S = 1/2$, and $d = 4$. The behavior of the geodesics is generic for other values of $d > 4$ and $S \in [0, 1)$. The areal radius corresponding to zero value of the Misner-Sharp energy is given by $R_e = 3\sqrt{3}/2$. The timelike singularity is located at $R(r_o) = 0$. The direction of local time is illustrated by the future null cones.

The radial null geodesics in the Fisher universe (5.34) can be derived by applying the transformations (5.32) to expressions (5.99) and (5.100), or directly by using (5.60) and (5.61),

$$\tau(\rho) = \mp \int d\rho \left[\left(\frac{\rho_o}{\rho} \right)^{d-3} - 1 \right]^{\frac{1-S}{2(d-3)} - \frac{1+S}{2}}, \tag{5.102}$$

$$\mathscr{R}(\rho) = \rho \left[\left(\frac{\rho_o}{\rho} \right)^{d-3} - 1 \right]^{\frac{1-S}{2(d-3)}}, \tag{5.103}$$

where "$-$" stands for outgoing and "$+$" stands for ingoing radial null geodesics. The coordinate τ is spacelike and the areal radius $\mathscr{R}(\rho)$ is timelike. The local null cones are defined by

$$\frac{d\mathscr{R}}{d\tau} = \mp \frac{1}{2} \rho^{-(1+S)\frac{d-3}{2}} \frac{(1+S)\rho_o^{d-3} - 2\rho^{d-3}}{\left(\rho_o^{d-3} - \rho^{d-3}\right)^{\frac{1-S}{2}}}. \tag{5.104}$$

This expression vanishes at $\rho = \rho_*$ which corresponds to the marginally trapped

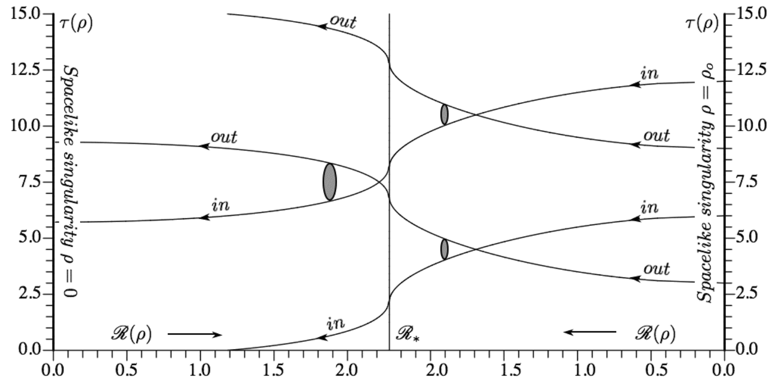

Figure 5.7: Radial null geodesics in the Fisher universe of $\mu = 1$, $S = 1/2$, and $d = 4$. The behavior of the geodesics is generic for other values of $d > 4$ and $S \in [0, 1)$. The marginally trapped surface is located at $\mathscr{R}_* = 3^{3/4}$. The spacelike singularities corresponding to $\rho = \rho_o$ and $\rho = 0$ are located at $\mathscr{R} = 0$. The direction of local time is illustrated by the future null cones.

surface (5.79).

The radial null geodesics corresponding to $S = 1$ are illustrated in Fig. 5.5. To construct a similar picture for the radial null geodesics corresponding to $S \in [0, 1)$ we define the direction of time in the Fisher universe in accordance with the Schwarzschild-Tangherlini black hole interior (see region II in Fig. 5.5). Namely, for $S = 1$ the timelike coordinate $\rho = \rho_o = r_o$ is past and $\rho = r = 0$ is future. We shall keep this convention for other values of $S \in [0, 1)$. The radial null geodesics in the Fisher spacetime and the Fisher universe are illustrated in Figs. 5.6 and 5.7, respectively.

The Fisher universe is an anisotropic universe whose topology is $\mathbb{R}^1_\tau \times \mathbb{R}^1_\rho \times \mathbb{S}^{d-2}$. At the moment of its "Big Bang" ($\rho = \rho_o$) the Fisher universe is a point of zero proper $(d-1)$-dimensional volume. It begins to expand in all spatial directions and at the moment $\rho = \rho_*$ [see, (5.79)] its boundary area along the angular directions reaches the maximal value $\mathcal{A}_*$ [see, (5.86)], and the universe begins to contract in the angular directions and continues to expand in the spatial τ direction. At the moment of its "Big Crunch" ($\rho = 0$) its boundary area along the angular directions vanishes and its expansion along the τ direction diverges.

The causal structure of the Fisher solution can be summarized in the corresponding Penrose diagrams (see Figs. 5.9 and 5.10). For comparison, we present

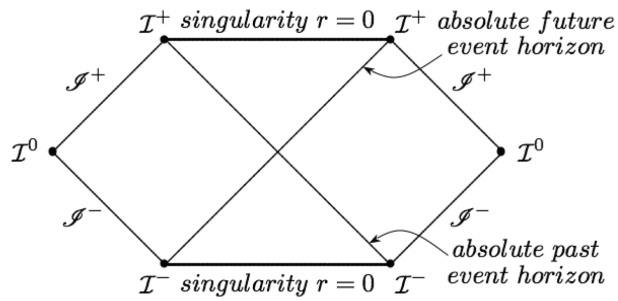

Figure 5.8: Penrose diagram for the maximally extended Schwarzschild-Tangherlini spacetime. Each interior point in the diagram represents a $(d-2)$-dimensional sphere.

the Penrose diagram of the Schwarzschild-Tangherlini spacetime (see Fig. 5.8). The topology of the spacelike singularity located at $r = 0$ is $\mathbb{R}^1_t \times \mathbb{S}^{d-2}$ [5] . Figure 5.9 represents the region conformal to the Fisher spacetime (3.25), $r \in (r_o, \infty)$. It is asymptotically flat and has timelike curvature singularity at $r = r_o$. The topology of the timelike singularity located at $r = r_o$ is $\mathbb{R}^1_t$ for $S \in [0, 1/(d-2))$, and $\mathbb{R}^1_t \times \mathbb{S}^{d-2}$ for $S \in [1/(d-2), 1)$. Figure 5.10 represents the region conformal to the Fisher universe (5.34). The coordinate ρ and the corresponding "tortoise coordinate," which is given by the right-hand side of (5.102), take finite values, whereas $\tau \in (-\infty, \infty)$. There is no conformal transformation which makes the infinite interval $\tau \in (-\infty, \infty)$ finite and does not shrink the finite interval of the tortoise coordinate to a point, thus inducing a coordinate singularity [6] . Here we present spacelike infinities $\tau \to \pm\infty$ by two disjoint points $\mathcal{I}^0$. The spacetime singularities of the Fisher universe located at $\rho = \rho_o$ and $\rho = 0$ are both spacelike. The topology of the spacelike singularities located at $\rho = \rho_o$ and at $\rho = 0$ is $\mathbb{R}^1_\tau \times \mathbb{S}^{d-2}$. According to the time direction convention the singularity at $\rho = \rho_o$ is in the past and the singularity $\rho = 0$ is in future. Thus, any causal curve in the Fisher universe originates at $\rho = \rho_o$ and terminates at $\rho = 0$. As a result, for geodesic families of observers both particle and event horizons exist. The geodesic of one such observer O and the corresponding past and future event horizons are shown in the diagram.

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[5] Here, and in what follows by topology of a spacetime singularity, we mean topology of ideal points of a spacetime which represent the singularity (see, e.g., [2, 44]).

6 A similar problem arises in the construction of the Penrose diagram of the anti-de Sitter spacetime (see, [2], p. 133 and Fig. 20).

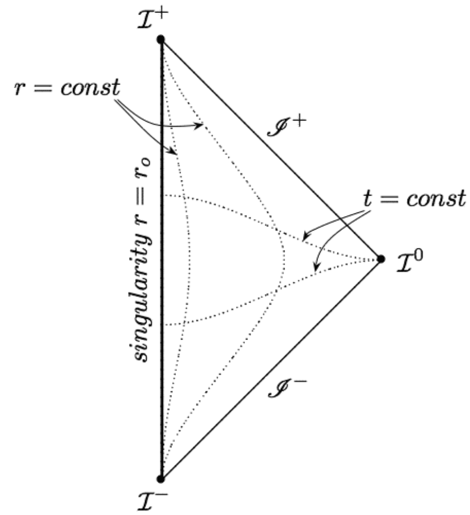


Figure 5.9: Penrose diagram for the Fisher spacetime. Each interior point in the diagram represents a $(d-2)$-dimensional sphere.

5.5 Isometric embedding

One of the ways to study geometry of a d-dimensional (pseudo-)Riemannian space which has an analytic metric of signature $p - q \leqslant d$ is to construct its isometric embedding into a D-dimensional (pseudo-)Euclidean space with the signature $r - s \leqslant D$. A local analytic isometric embedding is always possible if the dimension of the (pseudo-)Euclidean space of the signature $r - s$ is $D = d(d+1)/2$ and $r \geqslant p, s \geqslant q$, [45]. For a global isometric embedding the dimension D generally should be greater [46]. For example, a 4-dimensional Schwarzschild solution

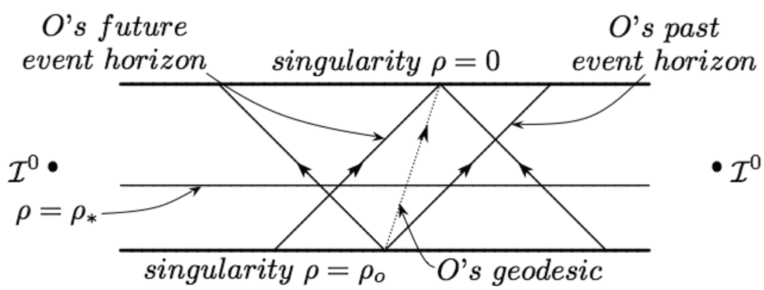


Figure 5.10: Penrose diagram for the Fisher universe. Each interior point in the diagram represents a $(d-2)$-dimensional sphere. The marginally trapped surface of the Fisher universe is schematically illustrated by the infinite line $\rho = \rho_*$.

whose metric has the signature $3 - 1 = 2$ can be isometrically embedded into a 6-dimensional pseudo-Euclidean space of the signature $5 - 1 = 4$ [47]. Examples of isometric local and sometimes global embeddings of some 4-dimensional Lorentzian spacetimes into pseudo-Euclidean spaces of higher dimensions are given in [48]. When dealing with spacetimes of the general theory of relativity one has usually $d \geqslant 4$ and higher values of D. Thus, having an embedding it is impossible to construct the corresponding visual picture illustrating the spacetime geometry. However, if a spacetime has symmetries defined by its Killing vectors, one can study its geometry by considering embeddings of the spacetime (hyper)surfaces orthogonal to the orbits of its Killing vectors. In the case if such a 2-dimensional surface exists, one can construct a 3-dimensional picture illustrating its isometric local embedding.

Here we shall consider local isometric embeddings of 2-dimensional subspaces of the Fisher spacetime and the Fisher universe. Both the spacetimes have a set of Killing vectors which allows us to study their geometry by considering embedding of the corresponding 2-dimensional subspaces. The geometry of the Fisher spacetime (5.5), $r \in (r_o, \infty)$ and the Fisher universe (5.34) is the same for any value of the coordinate t and τ, respectively. In addition, the spacetimes spherical symmetry implies that any 2-dimensional surface defined by $t = const$ ($\tau = const$) and $\theta^\alpha = const$, $\alpha = 3, ..., d-1$, where $\theta^\alpha \in [0, \pi]$ and $\phi \in [0, 2\pi)$ are $d-2$ (hyper)spherical coordinates, has the same geometry. Thus, to visualize the geometry of the spacetimes we present local isometric embeddings of their 2-dimensional subspaces defined by $t = const$ ($\tau = const$) and $\theta^\alpha = \pi/2$, $\alpha = 3, ..., d-1$.

Let us begin with the Fisher spacetime (5.5) whose 2-dimensional subspace metric is given by

$$ds^2 = F^{\frac{1-S}{d-3}-1} dr^2 + r^2 F^{\frac{1-S}{d-3}} d\phi^2, \tag{5.105}$$

where $r \in (r_o, \infty)$ and F is given by (5.6). Let us embed this surface into a 3-dimensional Euclidean space endowed with the following metric:

$$dl^2 = dZ^2 + dR^2 + R^2 d\phi^2, \tag{5.106}$$

where (Z, R, ϕ) are the cylindrical coordinates. To construct the embedding we consider the following parametrization of the surface:

$$Z = Z(r), \, , \;\; R = R(r). \tag{5.107}$$

Thus, the surface metric in the cylindrical coordinates takes the following form:

$$dl^2 = (Z_{,r}^2 + R_{,r}^2) dr^2 + R(r)^2 d\phi^2. \tag{5.108}$$

Matching the metrics (5.105) and (5.108) we derive the following embedding map:

$$R(r) = r\left[1-\left(\frac{r_o}{r}\right)^{d-3}\right]^{\frac{1-S}{2(d-3)}}, \tag{5.109}$$

$$Z(r) = r_o^{\frac{d-3}{2}}\int dr\,\frac{[4Sr^{d-3}-(1+S)^2 r_o^{d-3}]^{\frac{1}{2}}}{2r^{\frac{1-S}{2}}\left(r^{d-3}-r_o^{d-3}\right)^{1-\frac{1-S}{2(d-3)}}}. \tag{5.110}$$

We see that for $r \in (r_o, r_e)$, where r_e is given by (5.90), the coordinate $Z(r)$ is imaginary. Thus, the corresponding region of the surface cannot be isometrically embedded in this way into the 3-dimensional Euclidean space. Note that the Misner-Sharp energy (5.89) and $R_{\hat{\alpha}\hat{\beta}\hat{\alpha}}{}^{\hat{\beta}}$ (D.4) are negative in this region.

Although the region $r \in (r_o, r_e)$ cannot be isometrically embedded in this way into the 3-dimensional Euclidean space, we can embed it isometrically into 3-dimensional pseudo-Euclidean space endowed with the following metric:

$$dl^2 = -d\mathcal{Z}^2 + dR^2 + R^2 d\phi^2, \tag{5.111}$$

where $\mathcal{Z}$ is a timelike coordinate. Repeating the steps above we derive the corresponding embedding map

$$R(r) = r\left[1-\left(\frac{r_o}{r}\right)^{d-3}\right]^{\frac{1-S}{2(d-3)}}, \tag{5.112}$$

$$\mathcal{Z}(r) = r_o^{\frac{d-3}{2}}\int dr\,\frac{[(1+S)^2 r_o^{d-3}-4Sr^{d-3}]^{\frac{1}{2}}}{2r^{\frac{1-S}{2}}\left(r^{d-3}-r_o^{d-3}\right)^{1-\frac{1-S}{2(d-3)}}}. \tag{5.113}$$

Embeddings of the surfaces corresponding to $S=1$ and $S=1/2$ are presented in Figs. 5.11(**a**) and 5.11(**b**), respectively. In the case of the Fisher spacetime, the region between R_e [see, (5.97)] and asymptotic infinity ($R\to\infty$) corresponds to positive Misner-Sharp energy. The region between $R(r_o)$ and R_e corresponds to negative Misner-Sharp energy. At the convolution point R_m [see, (5.94)] we have $Z_{,R} = S/\sqrt{1-S^2}$. For $S=0$ we have $R_e\to\infty$ and the Misner-Sharp energy is negative everywhere.

Let us now consider the Fisher universe (5.34) whose 2-dimensional subspace metric is given by

$$ds^2 = -\Phi^{\frac{1-S}{d-3}-1}d\rho^2 + \rho^2\Phi^{\frac{1-S}{d-3}}d\phi^2, \tag{5.114}$$

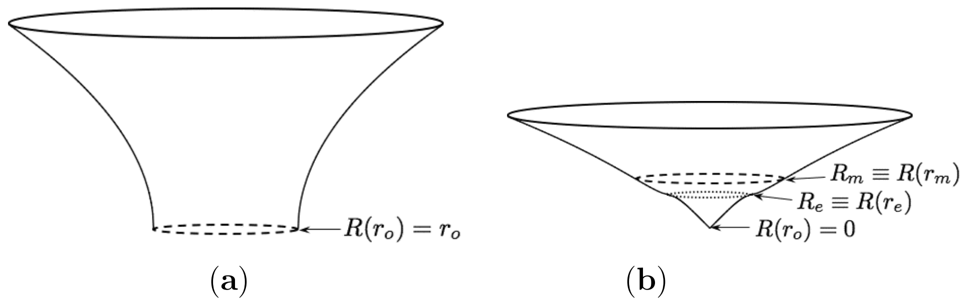


Figure 5.11: Embedding diagrams for $d = 4$. (**a**): Exterior region of the Schwarzschild-Tangherlini black hole of $M' = 2$ and $\Sigma' = 0$, [see, (5.31)]. The dashed circle of the radius $R(r_o) = r_o$ represents its event horizon. (**b**): Fisher spacetime corresponding to $M = 1$ and $S = 1/2$. The point $R(r_o) = 0$ represents the naked timelike singularity. The dotted circle of the radius R_e represents the region where the Misner-Sharp energy is zero. The dashed circle of the radius R_m represents the region where the "local (Newtonian) gravitational potential energy" $U(R)$ is minimal (see Fig. 5.3). The diagrams are qualitatively generic for other values of $d > 4$.

where $\rho \in (0, \rho_o)$ and Φ is given by (5.35). This surface can be isometrically embedded into a 3-dimensional pseudo-Euclidean space endowed with the following metric:

$$dl^2 = -d\mathscr{Z}^2 + d\mathscr{R}^2 + \mathscr{R}^2 d\phi^2, \tag{5.115}$$

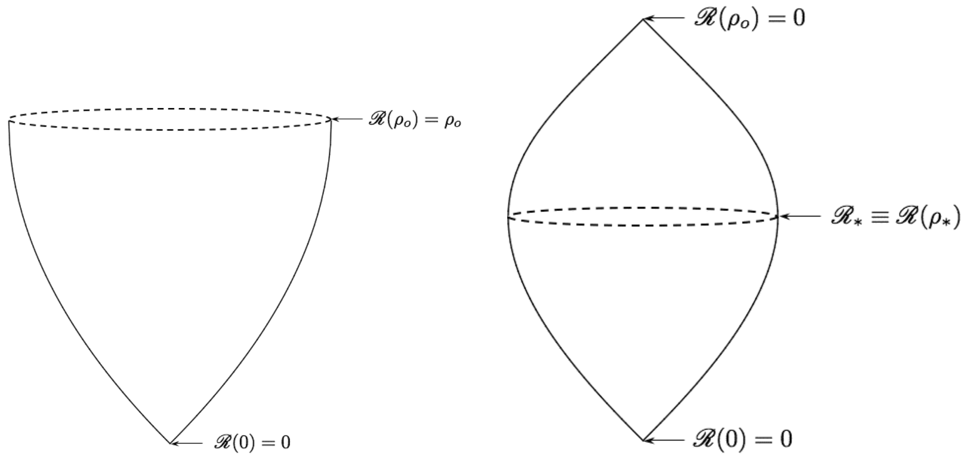
Matching the metrics (5.114) and (5.115) we derive the following embedding map:

$$\mathscr{R}(\rho) = \rho \left[\left(\frac{\rho_o}{\rho} \right)^{d-3} - 1 \right]^{\frac{1-S}{2(d-3)}}, \tag{5.116}$$

$$\mathscr{Z}(\rho) = \rho_o^{\frac{d-3}{2}} \int d\rho \, \frac{[(1+S)^2 \rho_o^{d-3} - 4S\rho^{d-3}]^{\frac{1}{2}}}{2\rho^{\frac{1-S}{2}} (\rho_o^{d-3} - \rho^{d-3})^{1-\frac{1-S}{2(d-3)}}}. \tag{5.117}$$

Embeddings of the surfaces corresponding to $S = 1$ and $S = 1/2$ are presented in Figs. 5.12(**a**) and 5.12(**b**), respectively.

We shall discuss the embedding diagrams in the following section.



(**a**) (**b**)

Figure 5.12: Embedding diagrams for $d = 4$. (**a**): Interior region of the Schwarzschild-Tangherlini black hole of $M' = 2$ and $\Sigma' = 0$, [see, (5.31)]. The dashed circle $\mathscr{R}(\rho_o) = \rho_o = r_o$ represents its event horizon and the point $\mathscr{R}(0) = 0$ represents its spacelike singularity. (**b**): Fisher universe corresponding to $\mu = 1$ and $S = 1/2$. The points $\mathscr{R}(\rho_o) = 0$ and $\mathscr{R}(0) = 0$ represent its spacelike singularities corresponding to the universe's Big Bang and Big Crunch, respectively. The dashed circle of the radius $\mathscr{R}_*$ represents the marginally trapped surface (5.85). For $S = 0$ the diagram is symmetric with respect to the circle. The diagrams are qualitatively generic for other values of $d > 4$.

5.6 The Fisher spacetime and the Fisher universe

So far we were considering the Fisher spacetime and the Fisher universe separately. This approach is based on the fact that the Fisher solution is singular at $r = r_o$ ($\rho = \rho_o$) and the disconnected parts of the Fisher manifold, which represent the Fisher spacetime and the Fisher universe, seem to not be related to each other. However, we can show that there are certain relations between some geometric quantities of the Fisher spacetime and the Fisher universe. Namely, if we consider expressions (5.79), (5.90), and (5.95), we observe that the following relation holds:

$$\frac{\rho_*}{\rho_o} = \frac{r_e}{r_m} = \left(\frac{1+S}{2}\right)^{\frac{1}{d-3}} . \tag{5.118}$$

In the limit $S \to 1$ we have $\rho_* = \rho_o = r_e = r_m$, where $\rho_o = r_o$ defines the event horizon of the Schwarzschild-Tangherlini black hole which is dual to the Fisher solution. There is an analogous relation between surface areas corresponding to ρ_*, r_e, r_m and the area $\mathcal{A}_{EH}$ of the black hole event horizon surface ($r = r_o = \rho_o$) [see Eqs. (5.85),(5.86),(5.94),(5.97), and (5.98)],

$$\frac{\mathcal{A}_*}{\mathcal{A}_{EH}} = \frac{\mathcal{A}_e}{\mathcal{A}_m} = \left[\frac{1}{2}(1-S)^{\frac{1-S}{2}}(1+S)^{\frac{1+S}{2}}\right]^{\frac{d-2}{d-3}} . \tag{5.119}$$

In the limit $S \to 1$ we have $\mathcal{A}_* = \mathcal{A}_{EH} = \mathcal{A}_e = \mathcal{A}_m$. In addition, in Sec. IV we found that in any dimension $d \geqslant 4$ both the areas $\mathcal{A}_e$ and $\mathcal{A}_*$ calculated for the fixed value of the mass $M = \mu = 1$ have minimal values at the same value of $S \approx 0.834$, and for $S \approx 0.611$ they are equal to the horizon surface area of the Schwarzschild-Tangherlini black hole of $M = 1$ and $\Sigma = 0$ [see Figs. 5.2(**b**) and 5.4(**b**)].

An analysis of the Kretschmann invariant (D.5) shows that there is another property which holds for any member of the Fisher family of solutions corresponding to $S \in [0, 1)$. Namely, the ratio of the Kretschmann invariant to the corresponding squared Ricci scalar (5.45) calculated at $\rho = \rho_*$, $r = r_e$, and $r = r_m$ does not depend on S and M (or μ),

$$\left.\frac{\mathcal{K}}{R^2}\right|_{\rho=\rho_*} = \frac{2(2d-5)}{(d-2)(d-3)} , \tag{5.120}$$

$$\left.\frac{\mathcal{K}}{R^2}\right|_{r=r_e} = \frac{d}{d-2} , \tag{5.121}$$

$$\left.\frac{\mathcal{K}}{R^2}\right|_{r=r_m} = \frac{2(2(d-2)^2-1)}{(d-2)(d-3)} , \tag{5.122}$$

where $S \in [0, 1)$. Thus, these ratios, as well as r_o, are invariants of the duality transformation (5.27) corresponding to $S \in [0, 1)$.

The relations (5.118), (5.119) may seem "natural" because both the Fisher spacetime and the Fisher universe originate from the same metric (5.5). However, such relations may have deeper roots. Our analysis of the Fisher solution yields the following results. The Schwarzschild-Tangherlini black hole solution follow from the same action (3.21), and it is dual to the Fisher solution. The duality transformation (5.27) maps the exterior region of the Schwarzschild-Tangherlini black hole $r \in (r_o, \infty)$ into the Fisher spacetime $r \in (r_o, \infty)$ and the interior region of the black hole $r \in (0, r_o)$ into the Fisher universe $\rho \in (0, \rho_o)$. Such a map may be visualized with the help of the embedding diagrams presented in Figs. 5.11 and 5.12 in Sec. V. Namely, according to expressions (5.85),(5.88),(5.94), and (5.97) we have

$$\mathscr{R}_*|_{S\to 1} = R_m|_{S\to 1} = R_e|_{S\to 1} = R(r_o) = r_o\,. \tag{5.123}$$

This expression implies that in the limit, which corresponds to zero value of the scalar charge, the region between the dashed circle of the radius R_m and the point $R(r_o) = 0$ in Fig. 5.11(**b**) maps into the circle of the radius $R(r_o) = r_o$ in Fig. 5.11(**a**), and the region between the dashed circle of the radius $\mathscr{R}_*$ and the point $\mathscr{R}(\rho_o) = 0$ in Fig. 5.12(**b**) maps into the circle of the radius $\mathscr{R}(\rho_o) = \rho_o$ in Fig. 5.12(**a**). Both the circles in Figs. 5.11(**a**) and 5.12(**a**) represent the event horizon of the Schwarzschild-Tangherlini black hole, i.e., $R(r_o) = r_o = \mathscr{R}(\rho_o) = \rho_o$. Thus, the region of the Fisher spacetime between the $(d-2)$-dimensional sphere of the areal radius R_m and the timelike naked singularity at $r = r_o$ and the region of the Fisher universe between the spacelike naked singularity at $\rho = \rho_o$ and the marginally trapped surface at $\rho = \rho_*$ map into the event horizon of the Schwarzschild-Tangherlini black hole. Note that this is not a one-to-one map.

5.7 Summary and discussion

In this Chapter we studied the d-dimensional generalization of the Fisher solution, which has a naked curvature singularity that divides the Fisher manifold into two disconnected parts, the Fisher spacetime and the Fisher universe. The d-dimensional Schwarzschild-Tangherlini solution and the Fisher solution follow from the same action (5.1) and are dual to each other. The duality transformation (5.27) maps the exterior region of the Schwarzschild-Tangherlini black hole into the Fisher spacetime, which has a naked timelike singularity, and the interior region of the black hole into the Fisher universe, which is an anisotropic expanding-contracting universe and which has two spacelike singularities representing its Big Bang and Big Crunch. The Big Bang singularity and the singularity of the Fisher spacetime are radially weak in the sense that a 1-dimensional object moving along a timelike radial geodesic can arrive at the singularities intact. These results and the relations between geometric quantities of the Fisher spacetime, the Fisher universe and the Schwarzschild-Tangherlini black hole presented in Sec. VI may suggest the following scenario. The massless scalar field, which according to the results of Sec. III contracts the spacetime in the angular directions, transforms the event horizon of the Schwarzschild-Tangherlini black hole into the naked radially weak disjoint singularities of the Fisher spacetime and the Fisher universe which are “dual to the horizon.” The properties of the Fisher solution presented above may suggest that one could “join” the Fisher spacetime and the Fisher universe together. If such a “junction” is possible, then a 1-dimensional object traveling along a radial geodesic can pass through the timelike naked singularity of the Fisher spacetime and emerge out of the Big Bang singularity into the Fisher universe.

One may think of how to construct a junction between the Fisher spacetime and the Fisher universe. As it was mentioned at the end of Sec. III, one may suggest a C^0 local extension of the 2-dimensional (t, r) and (τ, ρ) spacetime surfaces through the singularities which could provide a junction between the Fisher spacetime and the Fisher universe. However, this does not solve the problem completely, as far as it may provide a 2-dimensional junction only. Thus, one may try to look for other possibilities. For example, in a domain of Planckian curvatures, $\mathcal{K} \sim l_{Pl}^{-4} = c^6/(\hbar G)^2 \approx 1.47 \times 10^{139} m^{-4}$, quantum effects can be dominant and may "smooth out" curvature singularities. If this is indeed true, then we may expect that the Fisher spacetime and the Fisher universe may be *physically* (in the quantum way) joined together. Another way to smooth out the singularities is to consider the Einstein action with higher curvature interactions which are dominant near a spacetime curvature singularity and may remove it. However, there are arguments based on ground state stability which imply that curvature singularities (eternal and timelike) play a useful role as being unphysical [49]. For example, the timelike singularity of the negative mass Schwarzschild solution, if smoothed out, would give us a negative energy regular solution. As a result, Minkowski spacetime would not be stable. In the case of the Fisher solution, which is a nonvacuum solution, there is a compact region near the singularity (which can be arbitrary small) where the Misner-Sharp energy is negative. However, the energy conditions are not violated. Thus, the singularity of the Fisher spacetime may be "physical."

How generic can the properties of the Fisher solution be? According to a theorem presented in [50] for 4-dimensional spacetime, any static, asymptotically flat solution to Eqs. (5.3) and (5.4) with $\varphi \neq 0$ has a singular, simply connected event horizon defined by $k^2 = 0$, where $-k^2$ is the squared norm of the timelike Killing vector $\delta^a{}_t$ [see, (5.18)]. The event horizon remains singular if a solution to Eqs. (5.3) and (5.4) with $\varphi \neq 0$ is not asymptotically flat. For example, applying the duality transformation (5.25) to a 4-dimensional axisymmetric distorted Schwarzschild black hole discussed in [51], we can construct the corresponding axisymmetric distorted Fisher solution. There are other 4-dimensional singular solutions with a massless scalar field which are generalizations of the Fisher solution. These are the Penney solution, which is a generalization of the Reissner-Nordström solution in the presence of the massless scalar field [52] and the Kerr solution with the addition of the massless scalar field [3]. These solutions indicate that the massless scalar field transforms the event horizon into a naked singularity. Whether the naked singularity in these solutions is radially weak and the solutions have properties similar to the Fisher solution is an open question. We believe that it is likely to be the case.

Finally, one can ask if the Fisher solution is physical indeed. This question can be divided into two parts. The first part is whether such a solution can be considered as a result of a gravitational collapse, disproving cosmic censorship conjecture. Spherical gravitational collapse of a massless scalar field (without scalar charge) was studied, e.g., in [53, 54]. It was found that in some cases naked singularities do appear. However, later it was shown that formation of the naked singularities is an unstable phenomenon [55]. An alternative to gravitational collapse is the existence of primordial singularities (see, e.g., [56]). The second part of the question is concerned with the stability of the Fisher solution. To the best of our knowledge this issue is open. The related problem of stability of the negative mass Schwarzschild solution under linearized gravitational perturbations was discussed in [57]. It was found that for a physically preferred boundary conditions corresponding to the perturbations of finite energy the spacetime is stable. A different conclusion concerning to stability of the negative mass Schwarzschild solution had been reached in [58]. It would be interesting to study the stability of the Fisher spacetime singularity.

We hope that in the future more can be said about the issues discussed here.

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Chapter 6

Conclusion

In this thesis we have considered two different gravitational objects, a 5-dimensional distorted black hole and a d-dimensional naked singularity. In Chapter 3 we studied how the distortion generated by a static, neutral, external distribution of matter affects the horizon and the exterior and interior regions of a static, vacuum, 5-dimensional Schwarzschild-Tangherlini black hole. The solution is presented in the generalized Weyl form and admits three Killing vectors. Therefore, it represents a static, $U(1) \times U(1)$ symmetric 5-dimensional black hole. We have shown that the 4-dimensional and 5-dimensional distorted black holes exhibit common properties, i.e., their features are similar although different in details. For example, the following properties are observed for both the static, axisymmetric, vacuum, distorted 4-dimensional black hole and the static, $U(1) \times U(1)$ symmetric, vacuum, 5-dimensional distorted black hole. There exist a certain duality relation between the horizon surface of such a distorted black hole and surface of its "stretched" singularity (3.161). The boundary of the "stretched" singularity for the 5-dimensional Shwarzschild-Tangerlini black hole or the distorted 5-dimensional black hole is a surface of constant proper time. Namely, the topology of the boundary of the "stretched" singularity does not change under the effect of the distortion. Also, it is proved that the Kasner-like behavior of the distorted 5-dimensional black is the same as the undistorted Schwarzschild-Tangherlini one. It might be that these properties are due to the specific symmetry of the Weyl form of solutions in both four and five dimensions. We derived an expression for the space-time Kretschmann invariant $\mathcal{K}$ calculated on the horizon of a 5-dimensional, asymmetric, vacuum black hole in terms of the trace of the square of the Ricci tensor $\mathcal{R}_{AB}\mathcal{R}^{AB}$ of the 3-dimensional horizon surface (3.145).

In Chapter 4 we have constructed the solution representing a distorted 5-dimensional electrically charged static, $U(1) \times U(1)$ symmetric black hole. There exists a certain duality relation between the outer horizon surface of such a distorted black hole and the surface of its inner horizon. We derived an expression for the Kretschmann invariant $\mathcal{K}$ calculated on the horizon of a 5-dimensional, asymmetric, electrically charged black hole in terms of the electromagnetic field invariant F^2 calculated on the black hole horizon and the geometric invariants of the horizon surface, i.e., the 3-dimensional Ricci scalar $\mathcal{R}$, and the trace of the square of the Ricci tensor $\mathcal{R}_{AB}\mathcal{R}^{AB}$ (4.114). This is the generalization of relation (3.145) for a vacuum, static, asymmetric 5-dimensional black hole presented in Chapter 3.

In Chapter 5 we have considered the d-dimensional Fisher solution which represents a static, spherically symmetric, asymptotically flat space-time with a massless scalar field. The solution has two parameters, the mass M and the "scalar charge" Σ. The Fisher solution has a naked curvature singularity which divides the spacetime manifold into two disconnected parts. The part which is asymptotically flat we call the *Fisher spacetime*, and another part we call the *Fisher universe*. The d-dimensional Schwarzschild-Tangherlini solution and the Fisher solution belong to the same theory and are dual to each other. The duality transformation acting in the parameter space (M, Σ) maps the exterior region of the Schwarzschild-Tangherlini black hole into the Fisher spacetime which has a naked timelike singularity, and interior region of the black hole into the Fisher universe, which is an anisotropic expanding-contracting universe and which has two spacelike singularities representing its "Big Bang" and "Big Crunch". The Big Bang singularity and the singularity of the Fisher spacetime are *radially weak* in the sense that a 1-dimensional object moving along a timelike radial geodesic can arrive to the singularities intact. At the vicinity of the singularity the Fisher spacetime of nonzero mass has a region where its Misner-Sharp energy is negative. The Fisher universe has a marginally trapped surface corresponding to the state of its maximal expansion in the angular directions. These results and derived relations between geometric quantities of the Fisher spacetime, the Fisher universe, and the Schwarzschild-Tangherlini black hole may suggest that the massless scalar field transforms the black hole event horizon into the naked radially weak disjoint singularities of the Fisher spacetime and the Fisher universe which are "dual to the horizon."

Appendix A

Gaussian curvatures

The Gaussian curvatures (3.100)–(3.102) corresponding to the 3-dimensional horizon surface defined by the metric (3.97) are the following:

$$K_{+\phi} = \mathcal{N}\left(1 + 4w_{+,\theta\theta} - 8w_{+,\theta}^2 - 4u_{+,\theta}w_{+,\theta} - \frac{2\sin\theta}{1+\cos\theta}(u_{+,\theta} + 3w_{+,\theta})\right), \tag{A.1}$$

$$K_{+\chi} = \mathcal{N}\left(1 + 4u_{+,\theta\theta} - 8u_{+,\theta}^2 - 4u_{+,\theta}w_{+,\theta} + \frac{2\sin\theta}{1-\cos\theta}(w_{+,\theta} + 3u_{+,\theta})\right), \tag{A.2}$$

$$K_{+\theta} = \mathcal{N}\left(1 - 4u_{+,\theta}w_{+,\theta} - \frac{2}{\sin\theta}(u_{+,\theta} - w_{+,\theta}) + 2\cot\theta(u_{+,\theta} + w_{+,\theta})\right), \tag{A.3}$$

where $\mathcal{N} = e^{-2(u_+(\theta)+w_+(\theta)+u_1+w_1)}$.

Appendix B

$\widehat{U}$, $\widehat{W}$, and $\widehat{V}$ near the horizon and singularity

To study the behavior of the distortion fields $\widehat{U}$, $\widehat{W}$, and $\widehat{V}$ near the distorted black hole horizon and singularity, it is convenient to use the ψ coordinate. We can expand the distortion fields given by the exact solutions (3.45)–(3.51) of the Einstein equations near the black hole horizon and singularity. However, to derive a simple form of such expansions, it is easy to construct an approximate solutions to the Einstein equations (3.34)–(3.36). Using Eq. (3.62) we present Eq. (3.34) in the following form:

$$D_{\psi}\widehat{X}(\psi,\theta) = D_{\theta}\widehat{X}(\psi,\theta)\,, \ \ \widehat{X} := (\widehat{U},\widehat{W})\,, \tag{B.1}$$

where

$$D_{\sigma} := \partial_{\sigma}^{2} + \cot\sigma\partial_{\sigma}\,, \ \ \sigma := (\psi,\theta)\,. \tag{B.2}$$

The black hole horizon and singularity correspond to $\psi = 0$ and $\psi = \pi$, respectively. To consider both the cases simultaneously we denote $\psi_{+} := \psi - 0 = \psi$ and $\psi_{-} := \pi - \psi$. According to Eq. (3.45), the function $\widehat{X}$ is an even function of $\psi_{\pm}$. Thus, near the horizon and the singularity it has the following expansion:

$$\widehat{X}(\psi,\theta) = \sum_{k=0}^{\infty} X_{\pm}^{(2k)}(\theta)\,\psi_{\pm}^{2k}\,. \tag{B.3}$$

Using the series expansion for $\cot\psi_\pm$ (see, e.g., [39], p. 75)

$$\cot\psi_\pm = \psi_\pm^{-1}\left[1-\sum_{m=1}^{\infty} C_{2m}\psi_\pm^{2m}\right], \tag{B.4}$$

$$C_{2m} = \frac{(-1)^{m-1}2^{2m}B_{2m}}{(2m)!}, \quad |\psi_\pm| < \pi, \tag{B.5}$$

where B_{2m} are the Bernoulli numbers

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42} \dots, \tag{B.6}$$

we derive

$$D_{\psi_\pm}\psi_\pm^{2k} = 4k^2\psi_\pm^{2(k-1)} - 2k\sum_{m=1}^{\infty} C_{2m}\psi_\pm^{2(k+m-1)}. \tag{B.7}$$

Substituting expansion (B.3) into Eq. (B.1) and using Eq. (B.7) we derive the following recurrence relations for $X_\pm^{(2k)}(\theta)$:

$$\begin{aligned} X_\pm^{(0)} &= x_\pm(\theta) + x_0, \\ X_\pm^{(2)} &= \frac{1}{4}(x_{\pm,\theta\theta} + \cot\theta\, x_{\pm,\theta}), \\ &\vdots \\ X_\pm^{(2k+2)} &= \frac{1}{4(k+1)^2}\left[D_\theta X_\pm^{(2k)} + 2\sum_{m=1}^{k}(k-m+1)C_{2m}X_\pm^{(2(k-m+1))}\right], \\ k &= 0, 1, 2, \dots . \end{aligned} \tag{B.8, B.9}$$

Here $x_\pm := (u_\pm, w_\pm)$ and $x_0 := (u_0, w_0)$ (see Eqs. (3.70), (3.71), (3.73), and (3.74)).

The asymptotic expansion of the distortion field $\widehat{V}$, which is an even function of $\psi_\pm$, near the horizon and the singularity can be written in the form

$$\widehat{V}(\psi,\theta) = \sum_{k=0}^{\infty} V_\pm^{(2k)}(\theta)\,\psi_\pm^{2k}. \tag{B.10}$$

Substituting this expansion together with expansion (B.3) of the distortion fields $\widehat{U}$ and $\widehat{W}$ into equation (3.35) (with η replaced by ψ, according to (3.62)) we can determine the functions $V_{\pm}^{(2k)}(\theta)$. The first two of these functions are the following:

$$
\begin{aligned}
V_{+}^{(0)} &= -\frac{3}{2}(u_0 + w_0) - \frac{1}{2}(u_1 + w_1)\,, \\
V_{+}^{(2)} &= \frac{1}{4}(2u_{+,\theta}^2 + u_{+,\theta}w_{,+\theta} + w_{+,\theta}^2) - \frac{u_{+,\theta} - w_{+,\theta}}{4\sin\theta} - \frac{3}{4}\cot\theta(u_{+,\theta} + w_{+,\theta})\,, \\
&\vdots \\
V_{-}^{(0)} &= \frac{1}{2}[u_1 + w_1 - 3(u_0 + w_0)] - 3(u_-(\theta) + w_-(\theta))\,, \\
V_{-}^{(2)} &= \frac{1}{4}(2u_{-,\theta}^2 + u_{-,\theta}w_{-,\theta} + w_{-,\theta}^2) - \frac{u_{-,\theta} - w_{-,\theta}}{4\sin\theta} - \frac{3}{4}(u_{-,\theta\theta} + w_{-,\theta\theta})\,, \\
&\vdots
\end{aligned}
\tag{B.11}
$$

B.1 Geodesics near the singularity

For a free particle moving in a 5-dimensional distorted black hole interior there exist three integrals of motion related to the Killing vectors (3.25), the energy

$$E := -p_T = -\xi^{\alpha}_{(T)} p_\alpha \,, \tag{B.1}$$

and the angular momenta

$$L_\chi := p_\chi = \xi^{\alpha}_{(\chi)} p_\alpha \,, \;\; L_\phi := p_\phi = \xi^{\alpha}_{(\phi)} p_\alpha \,. \tag{B.2}$$

which correspond to the "axes" $\theta = \pi$ and $\theta = 0$, respectively. The other five constants of motion that characterize geodesic motion in the black hole interior are L_0, the limiting value of $L = [r_0 \sin(\psi_-/2)]^2 \dot{\theta}$ at the singularity $\psi_- = 0$ (with $\psi_- = \pi - \psi$), and θ_0, t_0, χ_0, and ϕ_0, the limiting values of θ, t, χ, and ϕ, respectively, at the singularity. For the Schwarzschild-Tangherlini black hole metric (21), $L = r^2\dot{\theta}$ is a constant of motion, but for a distorted black hole it is not. However, it does have a finite limiting value L_0 at the singularity that may be taken to be a characteristic value for the entire geodesic and hence a constant of motion.

Consider an initial point with coordinates $(\psi_{-i}, \theta_i, t_i, \chi_i, \phi_i)$ near the singularity of the distorted black hole ($\psi_{-i} \ll 1$). The proper time τ to fall from this point to the singularity depends on the location of the point and also on the geodesic constants of motion E, L_χ, L_ϕ, and L_0. One can show that the maximal proper time from the point to the singularity corresponds to $E = L_\chi = L_\phi = L_0 = 0$. We shall call the corresponding geodesic "radial". For the "radial" geodesic $(t, \chi, \phi) =$ *const*, along the geodesic, so $t_0 = t_i$, $\chi_0 = \chi_i$, and $\phi_0 = \phi_i$. In the Schwarzschild-Tangherlini black hole, θ would also be constant for a radial geodesic (which has $L = 0$ all along it), so there $\theta_0 = \theta_i$, but for a distorted black hole neither L nor θ is constant, so $\theta_0 \neq \theta_i$, though θ_0 is uniquely determined by the initial point $(\psi_{-i}, \theta_i, t_i, \chi_i, \phi_i)$ and is actually a function only of ψ_{-i} and θ_i for a fixed distorted black hole metric. This "radial" geodesic is a geodesic of the 2-dimensional metric

$$d\gamma^2 = B_-(d\theta^2 - d\psi_-^2) \,, \tag{B.3}$$

obtained by the dimensional reduction $(T, \chi, \phi) =$ *const* of the metric (3.146).

The Christoffel symbols for the metric (B.3) are

$$\begin{aligned} \Gamma^{\psi_-}{}_{\psi_-\psi_-} &= \Gamma^{\theta}{}_{\theta\psi_-} = \Gamma^{\psi_-}{}_{\theta\theta} = \frac{B_{-,\psi_-}}{2B_-} \,, \\ \Gamma^{\theta}{}_{\psi_-\psi_-} &= \Gamma^{\theta}{}_{\theta\theta} = \Gamma^{\psi_-}{}_{\theta\psi_-} = \frac{B_{-,\theta}}{2B_-} \,. \end{aligned} \tag{B.4}$$

Thus, the geodesic equation

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\, \dot{x}^\beta \dot{x}^\gamma = 0 \tag{B.5}$$

for the metric (B.3) takes the following form:

$$2B_- \ddot{\psi}_- + B_{-,\psi_-}(\dot{\psi}_-^2 + \dot{\theta}^2) + 2B_{-,\theta}\dot{\psi}_-\dot{\theta} = 0\,, \tag{B.6}$$

$$2B_- \ddot{\theta} + B_{-,\theta}(\dot{\psi}_-^2 + \dot{\theta}^2) + 2B_{-,\psi_-}\dot{\psi}_-\dot{\theta} = 0\,. \tag{B.7}$$

Here the over dot denotes the derivative with respect to the proper time τ. These equations obey the constraint

$$B_-(\dot{\psi}_-^2 - \dot{\theta}^2) = 1\,, \tag{B.8}$$

that is, the normalization condition $u_\alpha u^\alpha = -1$ for the 5-velocity u^α.

Expansion (3.147) for the metric function B_- near the singularity in the leading order in ψ_- is

$$B_- \approx \frac{\psi_-^2}{16} e^{-4(u_-(\theta)+w_-(\theta)-u_1-w_1)}\,. \tag{B.9}$$

Substituting this expression into the geodesic equations (B.6), (B.7), and the constraint (B.8), we derive

$$\psi_-\ddot{\psi}_- + \dot{\psi}_-^2 + \dot{\theta}^2 - 4(u_{-,\theta} + w_{-,\theta})\psi_-\dot{\psi}_-\dot{\theta} \approx 0, \tag{B.10}$$

$$\psi_-\ddot{\theta} - 2(u_{-,\theta} + w_{-,\theta})\psi_-(\dot{\psi}_-^2 + \dot{\theta}^2) + 2\dot{\psi}_-\dot{\theta} \approx 0, \tag{B.11}$$

$$e^{-4(u_-(\theta)+w_-(\theta)-u_1-w_1)}\psi_-^2(\dot{\psi}_-^2 - \dot{\theta}^2) \approx 16\,. \tag{B.12}$$

According to expression (B.9), the order of approximation in the geodesic equations (B.10)–(B.12) corresponds to the order of approximation of the metric (3.146).

We use the shift freedom of the proper time τ to set $\tau = 0$ at the singularity for each of the "radial" geodesics approaching the singularity (see footnote 3). The point $\tau = 0$ is a singular point of equations (B.10)-(B.12). To find an approximate solution to the geodesic equations near the singular point, one can apply the method of asymptotic splittings described in [2]. A "radial" geodesic approaching the singularity is uniquely determined by the limiting value $\theta = \theta_0$ at $\tau = 0$. The asymptotic expansions of ψ_- and θ near $\tau = 0$ have the following form:

$$\psi_- = 2\sqrt{2}\,\tilde{\tau}^{1/2} + \frac{3}{\sqrt{2}} f_{,\theta}^2(\theta_0)\,\tilde{\tau}^{3/2} + \mathcal{O}(\tilde{\tau}^{5/2})\,, \tag{B.13}$$

$$\theta = \theta_0 + 2f_{,\theta}(\theta_0)\,\tilde{\tau} + \mathcal{O}(\tilde{\tau}^2)\,, \tag{B.14}$$

where $\tilde{\tau} = e^{f(\theta_0)}\tau$ and $f(\theta) = 2(u_-(\theta) + w_-(\theta) - u_1 - w_1)$.

Appendix C

The Einstein and the Klein-Gordon Equations

The Einstein equations (5.3) for a static, spherically symmetric metric of the form

$$ds^2 = -e^A dt^2 + e^B dr^2 + e^C d\Omega^2_{(d-2)}\,, \tag{C.1}$$

where A, B, C are functions of r, reduce to

$$2A_{,rr} + A_{,r}[A_{,r} - B_{,r} + (d-2)C_{,r}] = 0\,, \tag{C.2}$$

$$C_{,r}[2A_{,r} + (d-3)C_{,r}] - 4(d-3)e^{B-C} = \frac{4\varphi^2_{,r}}{d-3}\,, \tag{C.3}$$

$$A_{,rr}C_{,r} - C_{,rr}A_{,r} + 2(d-3)A_{,r}e^{B-C} = 0\,. \tag{C.4}$$

The Klein-Gordon equation (5.4) for the static, spherically symmetric scalar field $\varphi = \varphi(r)$ is

$$\left(e^{\frac{1}{2}[A-B+(d-2)C]}\varphi_{,r}\right)_{,r} = 0\,. \tag{C.5}$$

Integrating this equation with an appropriate constant of integration we derive

$$\varphi_{,r} = \frac{4(d-3)\Gamma(\frac{d-1}{2})\Sigma}{(d-2)\pi^{\frac{d-3}{2}}} e^{-\frac{1}{2}[A-B+(d-2)C]}\,. \tag{C.6}$$

A substitution of Eq. (C.6) into Eq. (C.3) gives a closed system of equations for the metric functions A, B, C.

Appendix D

The Riemann Tensor and the Kretschmann Invariant

The Riemann tensor components for the metric (5.5) defined in a local orthonormal frame are (no summation over $\hat{\alpha}$)

$$R_{\hat{t}\hat{r}\hat{t}}{}^{\hat{r}} = -\frac{S}{1-S^2}\frac{R}{r_o^{d-3}}\left[2r^{d-3}-(1+S)r_o^{d-3}\right] , \tag{D.1}$$

$$R_{\hat{t}\hat{\alpha}\hat{t}}{}^{\hat{\alpha}} = -\frac{R_{\hat{t}\hat{r}\hat{t}}{}^{\hat{r}}}{d-2} , \tag{D.2}$$

$$R_{\hat{r}\hat{\alpha}\hat{r}}{}^{\hat{\alpha}} = -\frac{R\left[2Sr^{d-3}-(1+S)r_o^{d-3}\right]}{(1-S^2)(d-2)r_o^{d-3}} , \tag{D.3}$$

$$R_{\hat{\beta}\hat{\alpha}\hat{\beta}}{}^{\hat{\alpha}} = \frac{R\left[4Sr^{d-3}-(1+S)^2r_o^{d-3}\right]}{(1-S^2)(d-2)(d-3)r_o^{d-3}} , \tag{D.4}$$

where R is the Ricci scalar (5.45) and the indices $\hat{\alpha}, \hat{\beta} = 3, ..., d$ stand for orthonormal components in the compact dimensions of the $(d-2)$-dimensional round sphere. The corresponding Kretschmann invariant is given by

$$\begin{aligned} \mathcal{K} \equiv R_{\hat{a}\hat{b}\hat{c}\hat{d}}R^{\hat{a}\hat{b}\hat{c}\hat{d}} &= \frac{2}{(1-S^2)^2}\frac{R^2}{r_o^{2(d-3)}}\left(\frac{d-1}{d-2}\right)\Bigg\{2S^2\left[2r^{d-3}-(1+S)r_0^{d-3}\right]^2 \\ &+ \frac{2}{(d-1)}\left[2Sr^{d-3}-(1+S)r_0^{d-3}\right]^2 + \frac{1}{(d-1)(d-3)}\left[4Sr^{d-3}-(1+S)^2r_0^{d-3}\right]^2\Bigg\} , \end{aligned} \tag{D.5}$$

where $\{\hat{a}, \hat{b}, \hat{c}, \hat{d}\} = \{\hat{t}, \hat{r}; \hat{\alpha}, \hat{\beta} = 3, ..., d\}$.

Bibliography

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