Amenability of Discrete Semigroup Flows

by

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Abstract

A discrete flow (S, X) is a semigroup S acting on a set X where both S, and X are equipped with the discrete topology. Amenability of semigroups is a topic that explores the existence of measures that are invariant under the semigroup multiplication. The goal of this thesis is to generalize these results to a semigroup acting on a set, i.e. a flow, so that the invariance is with respect to the action.

We start out in Chapter 1 by giving some preliminaries that are important for the results in this thesis.

Chapter 2 generalizes basic theorems characterizing amenability and gives sufficient and necessary conditions for the same. We discuss some relevant topics such as the Hahn-Banach extension theorem and an application of flow amenability - a fixed point theorem.

Next, in Chapter 3, we discuss various Følner conditions - combinatorial properties that characterize aspects of amenability.

Finally, in Chapter 4, we discuss the flow stucture of the Stone-Čech compactification of a flow. We then discuss the concept of density of means and apply some properties of Følner nets.

In Chapter 5 we briefly get into reversible invariance - a property that is equivalent to amenability in groups (and group flows).

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Contents

Introduction

Amenability is a theme that has extended from groups to semigroups, algebras and even quantum groups. A semigroup S is said to be amenable if there exists a finitely additive measure μ on the power set of S, that is invariant under the multiplication of S in some sense, i.e. for any $s \in S$, and $A \subset S$, $\mu(s^{-1}A) = \mu(A)$. Let $m(S)$ denote the set of bounded real-valued functions on S. Then, the existence of an invariant measure is equivalent to the existence of a "mean" - an averaging linear functional on $m(S)$, with invariance dictating how the averaging remains unchanged under translation by elements of S. The general gist of amenability on different structures is the existence of these means or measures that are "invariant" in some sense.

According to Paterson [24], The beginnings of amenability lie in Lebesgue's research in the 1900's, which dealt with the uniqueness of the Lebesgue integral in the absence of the Monotone convergence theorem. Banach proved that the integral was not unique by giving examples on the real line, $\mathbb{R}$. Then, in the 1920's-1930's, Banach and Tarski set out to generalize this notion to a general group G acting on a set X. They characterized groups permitting invariant means, in terms of their actions, finding that G is amenable if and only if X does not have a "G-paradoxical decomposition", i.e. do not permit the famous Banach Tarski paradox. Von Neumann was the first to introduce general amenability of groups and studied this class of groups, making further connections with Banach and Tarski's theorem. In 1950, M. M. Day coined the term "amenable" for groups permitting invariant means, based on a pun with the word "mean" and describing the nice behaviour of such groups. He introduced amenable semigroups and his paper on amenable semigroups from 1970 is widely referenced (see [4]). Ever since, amenability has been a hot topic in abstract harmonic analysis.

In this thesis, our main goal is to extend the concept of invariant means to a flow (S, X), i.e. a semigroup acting on an arbitrary set X. The invariance here, is with respect to the action of S on X. We deal with discrete semigroups and a discrete topology on X, hence touching on combinatorial topics such as the Følner conditions for amenability.

We would like to take note, that upon getting close to the completion of this thesis, we discovered the extensive works on K. Sakai on the topic of amenable transformation semigroups. Sakai has written a series of papers on the topic, [31][32][33][34][35][36][30][28][29], that slipped under our radar. However, due to time constraints, we had no choice but to proceed with our work unmodified. Our work was done entirely independently of Sakai's. Sakai's work focuses more on extreme amenability and as such, the degree of overlap between his work and ours is not major. The overlap occurs in chapters 2 and 3 (Theorems 2.1, 2.18, 2.24, Corollary 3.6, Theorem 3.9, Corollary 3.10). Our approach/proofs sometimes differ from his in these chapters.

Regarding the layout, we first start with some preliminaries for the topic in Chapter 1.

In Chapter 2, we generalize properties of amenable semigroups to semigroup flows and make some useful characterizations. We also talk about Hahn-Banach extension properties and an application of flow amenability- Lau's fixed point theorem [20][19].

In Chapter 3, we generalize the work of Folner [7], Namioka [22] and others on the Følner conditions for amenability. We also briefly generalize and discuss the Følner number introduced by Wong [39].

Chapter 4 deals with introducing flow structure on the Stone-Čech compactification of a flow, which is essentially generalizing the semigroup structure of βS for a semigroup S. It then delves into generalizing the concept of density, as introduced by Hindman and Strauss [13][14], applying some of the results from chapter 3.

In Chapter 5, we briefly discuss reversible invariance, as introduced by Klawe [17].

Lastly, in Chapter 6, we will list some related open questions and problems for flows.

Chapter 1

Preliminaries

In this chapter, we define amenability, and define and give examples of some of the main structures we will be dealing with. To see analogous notions and background in specificity for semigroups, we recommend [4]. We will avoid getting too much into specifics for semigroups.

Throughout this thesis, we will assume all topologies to be Hausdorff unless specified otherwise.

For any set X, we will denote its cardinality by $|X|$.

For any set X, let us denote $\mathscr{P}(X)$ to be the power set of X and $\mathscr{P}_f(X)$ to be all finite subsets of X.

If X is a topological space, for each $A \subset X$, we will denote the closure of A by $cl(A)$.

Given a topological vector space X, we denote by X^*, its continuous dual, i.e. the space of all continuous real-valued linear functions on X. Recall that if X is a normed vector space with norm $\|\cdot\|$, the **operator norm** in X^* is given by $\|f\|_{op} = \sup_{x \in X \setminus \{0\}} \frac{|f(x)|}{\|x\|}$, for each $f \in X^*$. We will omit the subscript "*op*" as context will make the connotation clear.

Given a non-empty set X, let $m(X)$ be the space of bounded real-valued functions on X. Then, $m(X)$ is a Banach space under the supremum norm, i.e. the norm given by: $\|f\|_\infty = \sup_{x \in X} |f(x)|$, for each $f \in m(X)$. Then $l_1(X) \subset m(X)$ is defined to be all $f \in m(X)$ satisfying $\sum_{x \in X} |f(x)| < \infty$, and the norm on $l_1(X)$ is defined to be given by $\|f\|_1 = \sum_{x \in X} |f(x)|$ for all $f \in l_1(X)$.

Definition 1.1. A **semigroup** is a non-empty set S with an associative map $\cdot : S \times S \to S$, called the **semigroup multiplication on** S defined on it. Throughout the paper, we will denote $s \cdot t$ by st for all $s, t \in S$. A set $T \subset S$ is called a **subsemigroup** of S if it is a semigroup under the multiplication on S, i.e. T is closed under the semigroup multiplication of S between its elements.

Examples of semigroups include:

- The real line $\mathbb{R}$ or the complex plane $\mathbb{C}$, under addition or multiplication. Also, $\mathbb{N}$ and $\mathbb{N} \cup \{0\}$ are both subsemigroups of $\mathbb{R}$ under addition or multiplication.
- $M_{m \times n}(\mathbb{R})$, the set of all m-by-n matrices on $\mathbb{R}$, is a semigroup under matrix multiplication or addition.
- Any group in general is a semigroup with its usual multiplication, as is any ring under its multiplication.
- For any set X, we can define an associative multiplication by setting $xy = y$, for all $x \in X$, for each $y \in X$.We can similarly define another associative multiplication by setting $xy = x$, for all $y \in Y$ and for each $x \in X$. Any semigroup with a multiplication satisfying the former (latter) property is called a **right (left) zero semigroup**. If S is a semigroup and $x \in S$ satisfies $sx = x$ ($xs = x$) for all $s \in S$, it is called a **right (left) zero**.
- For any non-empty set X, $\mathscr{P}(X)$ is a semigroup under the multiplication given by $A \cdot B = A \cup B$ for any $A, B \in \mathscr{P}(X)$. It is easy to see that $\mathscr{P}_f(X)$ is a subsemigroup of $\mathscr{P}(X)$.

We will assume X to be an arbitrary non-empty set throughout this chapter.

Definition 1.2. Suppose, and $A \subset X$. Then the **characteristic function of** A is defined to be for each $x \in X$,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.3. Let F be a linear subspace of $m(X)$ that contains all the constant functions, equipped with the supremum norm. A **mean** on F is a linear functional $\mu : F \to \mathbb{R}$ such that $\mu(\chi_X) = 1$ and $\|M\| = 1$.

We shall denote the set of means on $m(X)$ by $\mathscr{M}(X)$. It is a well known fact that $\mathscr{M}(X)$ is non-empty, convex and w^*-compact as a subset of $m(X)^*$.

The simplest examples of means is given by the point measure or point mass, i.e. each $x \in X$, we define

$$\begin{aligned} \delta_x : m(X) &\to \mathbb{R} \\ f &\mapsto f(x) \end{aligned}$$

It is easy to see that for each $x \in X$, δ_x is linear, $\delta_x(\chi_{X}) = 1$ and $\delta_x(f) \leq \|f\|$, for all $f \in m(X)$. It follows that $\|\delta_x\| = 1$ and δ_x is a mean on $m(X)$.

We will use the following proposition without mention. For a proof, see [12].

Proposition 1.4. *The following are properties that every mean $M \in m(S)^*$ satisfies:*

1. $\|M\| = 1$
2. $M(\chi_{X}) = 1$
3. $M(f) \geq 0$ *if* $f \geq 0$ *for each* $f \in m(S)$
4. $\inf_{x \in X} f(x) \leq M(f) \leq \sup_{x \in X} f(x)$

In fact, $M \in m(S)^*$ is a mean if and only if it satisfies any two of the above conditions 1-4 or just 4.

Definition 1.5. Let $\Phi := \{f \in l_1(X) \mid f \geq 0,\ f \text{ has finite support and } \|f\|_1 = 1\}$. An element of Φ is called a **finite mean** on X.

Definition 1.6. Let us define $Q : l_1(X) \to m(X)^*$, where for each $f \in l_1(X)$, $[Q(f)](g) = \sum_{x \in X} f(x)g(x)$. Q is an isometric embedding of $l_1(X)$ into $m(X)^*$.

Remark 1.7. Note that the set of finite means, Φ, is weak* dense in $\mathscr{M}(X)$ via the isometry Q.

Definition 1.8. A **left semigroup flow** (or **left flow**) is a triple (S, X, p) where S is a semigroup, X is a non-empty set and $p : S \times X \to X$ is a map that satisfies $p(st, x) = p(s, p(t, x))$, for all $s, t \in S$, and $x \in X$. Such a function p is called a **left action** of S on X. We can similarly define a **right semigroup flow**, by modifying p to satisfy $p(st, x) = p(t, p(s, x))$, for all $s, t \in S$, and $x \in X$.

Furthermore, if X is a vector space, we say a left flow (right flow) (S, X) is:

1. a **left affine flow (right affine flow)** if $p(s, \cdot)$ is affine for each $s \in S$.

2. a **left-representation (right-representation)** of S on X if $p(s, \cdot)$ is linear for each $s \in S$.

We will use the shorthand notation (S, X) for a flow (S, X, p), denoting $p(s, x)$, for each $s \in S$, $x \in X$, by sx if (S, X) is a left flow and xs if (S, X) is a right flow.

Example 1.9. 1. The simplest example of a flow is of course, a semigroup S acting on itself via left multiplication, (S, S). We may also consider the action via right multiplication and make (S, S) a right flow.

2. Let E be a topological vector space, with the semigroup structure on it given by addition. If we take τ to be the topology on E, E acts on τ via translation, i.e. for any $U \in \tau$, $x \in E$, $x + U \in \tau$. Hence (E, τ) is a flow. In fact, due to the commutativity of addition on E, it is both a left and a right flow.

3. Given any semigroup S, and set X, we can define a trivial flow by taking: $sx = x$, for all $s \in S$, $x \in X$. Of course, this just means that every point in X is a fixed point of S (see Definition 1.11).

4. Any group or semigroup representation constitutes a flow.

5. For any semigroup S, we can consider the action of $(\mathbb{N}, \times)$ on S, via $ns = s^n = \underbrace{s \cdot s \cdot ... \cdot s}_{n \text{ times}}$, for each $n \in \mathbb{N}$, $s \in S$.

We will assume all flows in this paper are left flows unless stated otherwise. Furthermore, for a semigroup S, (S, S) will indicate the left flow given by S acting on itself by left multiplication, unless stated otherwise.

Definition 1.10. Suppose (S, X) is a flow and $A \subset X$ and $s \in S$. We define

$$s^{-1}A = \{x \in X \mid sx \in A\}$$

Note that $s^{-1}A = \varnothing$ if $A \cap sX = \varnothing$.

Definition 1.11. Suppose (S, X) is a flow. We say $x \in X$ is a **fixed point** of S if $sx = x$ for all $s \in S$.

Definition 1.12. Suppose (S, X), (T, Y) are flows. Then, we define the **product flow** of these to be given by the flow $(S \times T, X \times Y)$, where $S \times T$ is a semigroup under $(s, t)(s', t') = (ss', tt')$ and $(s, t)(x, y) = (sx, ty)$ for each $s, s' \in S$, $t, t' \in T$, $x \in X$, $y \in Y$.

Remark 1.13. We can define other flows $(S \times T, X \times Y)$ with a different semigroup multiplication on $S \times T$ and/or a different action of $S \times T$ on the set $X \times Y$. However, in this thesis, we will freely use these notations to indicate the semigroup multiplication, and action defined in Definition 1.12.

Definition 1.14. Given a flow (S, X), we define the following notions:

1. For a subsemigroup T of S, and $Y \subset X$, we call (T, Y) a **subflow** if $tY \subset Y$, for each $t \in T$.

2. If $T = S$ in 1, we call (S, Y) an S**-ideal**

Note that every S-ideal is indeed a subflow of (S, X).

Example 1.15.

1. If S is a semigroup and L is a left ideal of S, (S, L) is an S-ideal of (S, S).

2. If (S, X) is a flow, and $x \in X$ is a fixed point of S, $(S, \{x\})$ is an S-ideal and for any subsemigroup $T \subset S$, $(T, \{x\})$ is a subflow.

3. Consider a semigroup S, and the flow $(\mathbb{N}, S)$ described in Example 1.9. Fix any $n \in \mathbb{N}$. Define $T = \left\{ s \in S \mid \exists t \in S \text{ such that } s = t^n = \underbrace{t \cdot t \cdot \ldots \cdot t}_{n \text{ times}} \right\}$. Then, for each $m \in \mathbb{N}$, $(m\mathbb{N}, T)$ is a subflow of $(\mathbb{N}, S)$.

Definition 1.16. Suppose (S, X) is a flow. We say an element $s \in S$ is S**-cancellable** if the map

$$X \to X$$
$$x \mapsto sx$$

is injective. Similarly, we say an element $x \in X$ is X**-cancellable**, if the map

$$S \to X$$
$$s \mapsto sx$$

is injective. If all the elements of S are S-cancellable, we say that (S, X) is a S**-cancellative** flow, and if all the elements of X are X-cancellable, we say that (S, X) is X**-cancellative**. Note that for a semigroup S, labelling $X = S$, if (S, X) with the action being left multiplication, is S-cancellative, we say that S is left-cancellative, and if it is X-cancellative, we say S is right-cancellative. The notions of left- and right- cancellable elements are similarly defined.

Example 1.17.

1. The flow (E, τ) for a topological vector space E with topology τ, from Example 1.9, is a E-cancellative flow, and a τ-cancellative flow.

2. Let S be a semigroup and $X \neq \varnothing$ be an arbitrary set. Let us define the action of S on X to be given by $sx = x$ for each $s \in S$, $x \in X$. Then the flow (S, X) is S-cancellative.

3. Consider the flow $(\mathbb{N}, \mathbb{R})$ given by $(\mathbb{N}, \times)$ acting on $\mathbb{R}$ via multiplication. Then, $(\mathbb{N}, \mathbb{R})$ is $\mathbb{N}$-cancellative, and every element in $\mathbb{R}\backslash\{0\}$ is $\mathbb{R}$-cancellable.

Let (S, X) be a flow.

Definition 1.18. For each $s \in S$, we define the S**-translation operator** $L_s : m(X) \to m(X)$ to be given by $[L_s(f)](x) = f(sx)$, for each $x \in X$ and $f \in m(X)$. When S acts on itself via left (right) multiplication, this is called a **left-translation (right translation) operator**. We call $L_s(f)$ the S**-translate of** f **by** s for each $s \in S$, $f \in m(X)$. Again, if S acts on itself on the left (right), and $f \in m(S)$, this is called the **left-translate (right translate)** of f by s.

Definition 1.19. We define (S, X) to be **amenable** if there is a mean M on $m(X)$ that is S-invariant, i.e. $M(f) = M(L_s f)$, for all $s \in S$, $f \in m(X)$. If (S, S) is amenable under left multiplication (right multiplication), it is said to be **left-amenable (right-amenable)**

respectively. If it is both left amenable and right amenable, it is simply called **amenable** (see [4]). We will label S-invariant means on $m(X)$ and left-invariant means on $m(S)$ by $\mathscr{M}_l(X)$ and $\mathscr{M}_l(S)$ respectively.

Example 1.20.
1. Any left amenable semigroup S is an amenable left flow (S, S). Examples of left amenable semigroups include abelian semigroups and right-zero semigroups.
2. Any flow (S, X) with S amenable is also amenable. (See Proposition 2.8.)

More examples of amenable flows will follow along the way.

1.1 Action of S on $m(X)$ and $\mathscr{M}(X)$

Let (S, X) be a left flow. Then, S has a natural action on $m(X)$, given for each $s \in S$, by $L_s : m(X) \to m(X)$. For each $s, t \in S$, for all $f \in m(X)$,

$$[L_t(L_s f)](x) = [L_s f](tx) = f(stx) = [L_{st} f](x), \text{ for all } x \in X$$

Hence, $L_t \circ L_s = L_{st}$ and the action of S on $m(X)$ is a linear right action, by the linearity of the S-translation operators L_s, $s \in S$. Thus, $(S, m(X))$ is a right representation of S on $m(X)$. It follows that S induces a left linear action on $m(X)^*$, given for each $s \in S$ by the adjoint of L_s, $L_s^* : m(X)^* \to m(X)^*$, $M \mapsto M \circ L_s$ acting on $m(X)^*$. If we restrict this to $\mathscr{M}(X)$, it is still a left action. Indeed, if $M \in \mathscr{M}(X)$, $s \in S$, then $[L_s^* M](\chi_X) = [M \circ L_s](\chi_X) = M(\chi_X) = 1$ and $\|L_s^* M\| = \|M \circ L_s\| \le \|M\| = 1$ which implies that $L^* M \in \mathscr{M}(X)$. Since $\mathscr{M}(X)$ is a convex subset of $m(X)^*$, $(S, \mathscr{M}(X))$ is a left affine flow.

For any set $X \neq \varnothing$, for any $f \in m(X)$, $M \in m(X)^*$, let us denote $M(f)$ by (M, f). Let (S, X) be a left flow. For each $N \in m(X)^*$, define $\widetilde{N} : m(X) \to m(S)$, $[\widetilde{N} f](s) = (N, L_s f)$. We define $\odot : m(S)^* \times m(X)^* \to m(X)^*$ by $(M \odot N, f) = (M, \widetilde{N} f)$, for $M \in m(S)^*$, $N \in m(X)^*$ and for all $f \in m(X)$. By the linearity of the S-translation operators $\{L_s \mid s \in S\}$ on $m(X)$, and N and M, and due to M being bounded, it is clear that $M \odot N \in m(X)^*$.

If, furthermore, $M \in \mathscr{M}(S)$, $N \in \mathscr{M}(X)$,

- $[\widetilde{N}\chi_X](s) = N(L_s \chi_X) = N(\chi_X) = 1$, for all $s \in S$, and $(M \odot N, \chi_X) = (M, \tilde{N}\chi_X) = (M, \chi_S) = 1$

- $|(M \odot N, f)| = |(M, \widetilde{N}f)| \leq \|M\|\|\widetilde{N}f\| \leq \|M\|\|N\|\|f\| = \|f\|$, for all $f \in m(X)$

It follows that $\mathscr{M}(S) \odot \mathscr{M}(X) \subset \mathscr{M}(X)$.

One can observe that $\mathscr{M}(S)$ and $m(S)^*$ are semigroups under "$\odot$" applied with $X = S$ (See [4]). For simplicity, we will use the same notation for this operation defined on $m(S)^* \times m(S)^*$ versus $m(S)^* \times m(X)^*$, context making the connotation clear.

Let $M, K \in m(S)^*$, $N \in m(X)^*$. For each $s \in S$, $f \in m(X)$, and all $t \in S$,

$$[\widetilde{K}(\widetilde{N}f)](s) = (K, L_s[\widetilde{N}f]), \text{ where, } \{L_s[\widetilde{N}f]\}(t) = [\widetilde{N}f](st) = N(L_{st}f)$$

while

$$[\widetilde{K \odot N}f](s) = (K \odot N, L_s f) = (K, \widetilde{N}(L_s f)), \text{ where, } [\widetilde{N}(L_s f)](t) = N(L_t L_s f) = N(L_{st} f)$$

and thus, by $\widetilde{N}(L_s f) = L_s(\widetilde{N}f)$, we end up with $\widetilde{K}(\widetilde{N}f) = \widetilde{K \odot N}f$, and,

$$\big((M \odot K) \odot N, f\big) = \big(M \odot K, \widetilde{N}f\big) = \big(M, \widetilde{K}(\widetilde{N}f)\big) = \big(M, \widetilde{K \odot N}f\big) = \big(M \odot (K \odot N), f\big)$$

Thus, "$\odot$" on $m(S)^* \times m(X)^*$ is an associative action and $(m(S)^*, m(X)^*)$, $(\mathscr{M}(S), \mathscr{M}(X))$ are flows. In fact, one can easily check that these are affine flows.

Chapter 2

Amenability of Discrete Semigroup Flows

The goal of this chapter is to establish some of the characterizations, necessary and sufficient conditions for amenability in flows, such as Dixmier's condition. We also discuss the concept of a homomorphism of flows. Lastly, we discuss Hahn-Banach extension properties and the Fixed point characterization introduced by Lau [19][20]. In essence, we generalize some of the basic theorems for semigroups amenability to flows. References for this chapter include [4], [12], [24] and [9].

In his papers, Sakai has covered Theorem 2.1, Theorem 2.19 and Theorem 2.26. For Theorem 2.26, our definition of a homomorphism of flows is more general and matches Lau's in [19]. Sakai also has done work on some fixed point properties like in Section 2.4. However, our theorem is a result from [19].

2.1 Sufficient and Necessary Conditions

Let (S, X) be a flow. Let us define $\mathscr{H}_l(\mathrm{X})$ to be the space of all functions on X of the form $\sum_{i=1}^{n} f_i - L_{s_i} f_i$, where each $f_i \in m(X)$, $s_i \in S$ and $n \in \mathbb{N}$. The following is a generalization of Dixmier's theorem.

Theorem 2.1 (Dixmier's condition). *There is a S-invariant mean on $m(X)$, if and only if for all $g \in \mathscr{H}_l$, $\sup_{x\in X} g(x) \geq 0$.*

Proof. Suppose M is a S-invariant mean on $m(X)$. Then, $M(g) \leq \sup_{x\in X} g(x)$ for all $g \in \mathscr{H}_l$. Given $g = \sum_{i=1}^{n} f_i - L_{a_i} f_i$, for $a_i \in S$, $f_i \in m(X)$ and $n \in \mathbb{N}$, M is left invariant implies

$M(\sum_{i=1}^n f_i - L_{a_i} f_i) = \sum_{i=1}^n M(f_i) - M(L_{a_i} f_i) = 0$. Thus, $M(g) = 0 \leq \sup_{x\in X} g(x)$.

Conversely, suppose for all $g \in \mathscr{H}_l$, it is the case that $\sup_{x\in X} g(x) \geq 0$. Then, define a mean N on $\mathscr{H}_l$, $N \equiv 0$. Let P be the sublinear functional on $m(X)$ given by $P(g) = \sup_{x\in X} g(x)$. Since $0 = N(g) \leq \sup_{x\in X} g(x) = P(g)$, by the Hahn-Banach extension theorem, N can be extended to a linear functional M on $m(X)$ while being dominated by P. Thus, $M(-g) \leq P(-g) = \sup_{x\in X} -g(x)$, which gives us $M(g) \geq -\sup_{x\in X} -g(x) = \inf_{x\in X} g(x)$ and

$$\inf_{x\in X} g(x) \leq M(g) \leq \sup_{x\in X} g(x)$$

which implies that M is a mean on $m(X)$. M is clearly S-invariant as $M \equiv 0$ on $\mathscr{H}_l$. ∎

Definition 2.2. We define a flow (S, X) to be **commutative** if it satisfies $stx = tsx$, for all $s, t \in S, x \in X$.

The following is a well known theorem. A proof can be found in [24].

Theorem 2.3 (Markov-Kakutani Fixed Point Theorem). *Suppose K is a compact convex subset of a locally convex space. Then, if S is a commutative semigroup of continuous, affine transforms from K to K, S has a common fixed point in K.*

We will use this theorem to prove the following:

Theorem 2.4. *If (S, X) is commutative, it is amenable.*

Proof. Let us consider for each $s \in S$, the map $L_s^* : \mathscr{M}(X) \to \mathscr{M}(X)$ given by $L_s{}^*(M) = M \circ L_s$. Recall that each $L_s{}^*$ is affine and that $\mathscr{M}$ is a weak* compact convex subset of $m(X)^*$.

Observe that for each $s, t \in S$,

$$[(L_s \circ L_t)f](x) = f(stx) = f(tsx) = [(L_t \circ L_s)f](x)$$

for all $f \in m(X)$, $x \in X$, by the commutativity of the action. Thus, $\{L_s^* \mid s \in S\}$ is a commutative semigroup under composition, i.e. for the left affine flow $(S, \mathscr{M}(X))$, we can consider each $s \in S$ as an affine operator on $\mathscr{M}(X)$, where the order of action of elements of S on $\mathscr{M}(X)$ does not matter.

If $\{M_\alpha\} \subset \mathscr{M}$ converges to $M \in \mathscr{M}$ in the weak* topology of $m(S)^*$, then for each $f \in m(S)$, $s \in S$,

$$[{L_s}^* M_\alpha - {L_s}^* M](f) = [M_\alpha \circ L_s - M \circ L_s](f) = M_\alpha(L_s f) - M(L_s f) \to 0 \text{ in } \alpha$$

It follows that ${L_s}^* M_\alpha \to {L_s}^* M$ in the weak* topology on $\mathscr{M}(X)$. Thus, ${L_s}^*$ is weak*-weak* continuous for each $s \in S$.

$\{{L_s}^* \mid s \in S\}$ is a commutative semigroup of affine transforms from $\mathscr{M}(X)$ to $\mathscr{M}(X)$ and must have a common fixed point by the Markov-Kakutani fixed point theorem. If we let $M \in \mathscr{M}(X)$ be this fixed point, M is a S-invariant mean, due to $M \circ L_s = L_s^* M = M$ for each $s \in S$, and (S, X) is amenable. ∎

Remark 2.5. When we consider $\{{L_s}^* \mid s \in S\}$ as a semigroup of operators on $\mathscr{M}(X)$, this is the same as considering S acting on $\mathscr{M}(X)$. In other words, we showed that the flow $(S, \mathscr{M}(X))$ is weak*-weak* continuous in the second variable, i.e. for each $s \in S$, the map $\mathscr{M}(X) \to \mathscr{M}(X)$, $M \mapsto L_s^* M = sM$ is weak*-weak* continuous.

Corollary 2.6. *Every commutative semigroup S is left and right amenable.*

Note that in Theorem 2.4, we do not require S to be commutative, we only require that action is invariant under the order in which elements of S are applied to elements of X. To observe this, consider the first part of the following example:

Example 2.7.

1. Let $S = M_{n \times n}(\mathbb{R})$, $X = \mathbb{R}^m$, for any $n, m \in \mathbb{N}$, where S is a semigroup under matrix multiplication. We define the action of S on X by $M \cdot v = det(M)v$, for $M, N \in S$. Note that for any $M, N \in S$, $v \in X$,

   $$M \cdot (N \cdot v) = M \cdot det(N)v = det(M)det(N)v = det(MN)v = (MN) \cdot v$$

   and

   $$(MN) \cdot v = det(MN)v = det(M)det(N)v = det(N)det(M)v = det(NM)v = (NM) \cdot v$$

   The associativity of the action is confirmed, as well as (S, X) being commutative, even though S is not commutative. Furthermore, if we consider $n = 2$, $\mathbb{F}_2$, the free semigroup on two generators, is a subsemigroup of S. It is well known that $\mathbb{F}_2$ is a non-amenable semigroup (see [24]). Thus, setting $T = \mathbb{F}_2$, (T, X) is an example of a flow, where T is non-commutative, non-amenable, but (T, X) is still commutative, amenable.

2. For a commutative semigroup S, any flow (S, X) is amenable. As an example, if E is a topological vector space with topology τ, the flow (E, τ) from Example 1.9 is amenable, since E is commutative under addition. Similarly, $(\mathbb{N}, S)$ is amenable for any semigroup S, with the flow action as defined in Example 1.9.

Proposition 2.8. *The following are equivalent:*

1. *S is left amenable*
2. *Every left flow (S, X) is amenable*

Proof. Suppose 1 holds and 2 does not hold. Then Dixmier's condition fails for (S, X). There exist $f_i \in m(X)$, $s_i \in S$, $1 \leq i \leq n$, for some $n \in \mathbb{N}$ such that $\|\sum_{i=1}^{n} f_i - L_{s_i} f_i\|_\infty < 0$. Fix $x \in X$. We construct $g_i \in m(S)$ by taking $g_i(s) = f_i(sx)$, for all $s \in S$, for each i, $1 \leq i \leq n$. Then we have

$$\left\|\sum_{i=1}^{n} g_i - L_{s_i} g_i\right\|_\infty = \sup_{s\in S}\left|\sum_{i=1}^{n} f_i(sx) - [L_{s_i} f_i](sx)\right| \leq \sup_{x\in X}\left|\sum_{i=1}^{n} f_i(x) - [L_{s_i} f_i](x)\right| = \left\|\sum_{i=1}^{n} f_i - L_{s_i} f_i\right\|_\infty < 0$$

which means Dixmier's condition fails for S, which contradicts the amenability of S. Hence, (S, X) must be amenable.

On the other hand if 2 holds, 1 is clear by the amenability of (S, S) via left multiplication. ∎

Remark 2.9. Note that by 1 of Example 2.7, it is clear that if (S, X) is a left flow that is amenable, it does not necessarily imply that S is amenable. In fact, Proposition 2.8 emphasizes how much weaker the amenability of (S, X) is compared to the amenability of S.

Proposition 2.10. *Suppose (S, X) is a flow and X has an element that is fixed by S. Then, (S, X) is amenable.*

Proof. Suppose $x \in X$ is fixed by S. Define $M : m(X) \to \mathbb{R}$ by $M(f) = f(x)$, for all $f \in m(X), x \in X$. M is clearly linear. Furthermore, $|M(f)| = |f(x)| \leq \|f\|_\infty$, for all $f \in m(X)$, and $M(\chi_X) = \chi_X(x) = 1$ and thus M is a mean. Lastly, $M(L_s f) = L_s f(x) = f(sx) = f(x) = M(f)$, for all $f \in m(X), s \in S$. Thus, (S, X) is amenable. ∎

Theorem 2.11. *Suppose (S, X) is amenable. Then for any two cosets sX, tX, $s, t \in S$, $sX \cap tX \neq \varnothing$.*

Proof. Suppose $sX \cap tX = \varnothing$, for some $s, t \in S$. Then, if M is a S-invariant mean on $m(X)$, we have $M(\chi_{sX}) = M(L_s\chi_{sX}) = M(\chi_{s^{-1}(sX)}) = M(\chi_X) = 1$. Similarly, $M(\chi_{tX}) = 1$. Since tX, sX are disjoint, we have, $1 = M(\chi_X) \geq M(X_{tX \cup sX}) = M(X_{tX}) + M(X_{sX}) = 2$, which is a contradiction. Hence, $sX \cap tX \neq \varnothing$, for every $s, t \in S$. ∎

Proposition 2.12. *Suppose (S, X) is an amenable flow. Then if M is a S-invariant mean on $m(X)$, and (T, Y) is a subflow such that $M(\chi_Y) > 0$, (T, Y) is amenable.*

Proof. Let us define a mean N on $m(Y)$ by $f \mapsto \frac{M(\widetilde{f})}{M(\chi_Y)}$, where

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases}$$

Indeed, $|N(f)| = |M(\widetilde{f})| \leq \|M\| \|\widetilde{f}\|_\infty = \|f\|_\infty$, and $N(\chi_Y) = \frac{M(\chi_Y)}{M(\chi_Y)} = 1$.

Now consider the following for all $s \in T$, $x \in X$:

$$[L_s\widetilde{f}](x) = \widetilde{f}(sx) = \begin{cases} f(sx) & \text{if } x \in s^{-1}Y \\ 0 & \text{otherwise} \end{cases}$$

$$[\widetilde{L_s f}](x) = \begin{cases} L_s f(x) & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases} \qquad = \begin{cases} f(sx) & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$[L_s\widetilde{f} - \widetilde{L_s f}](x) = \begin{cases} f(sx) & \text{if } x \in s^{-1}Y \backslash Y \\ 0 & \text{otherwise} \end{cases}$$

Note that $Y \backslash s^{-1}Y = \varnothing$, since $sY \subset Y$.

Now for any $n \in \mathbb{N} \cup \{0\}$, and $x \in X$, if $[L_s\widetilde{f} - \widetilde{L_s f}](s^n x) \neq 0$, then, $s^n x \in s^{-1}Y \backslash Y \implies s^{n+1}x \in Y$, and $s^n x \notin Y$. Then for any $j > n$, $s^j x = s^{j-(n+1)} s^{n+1} x \in s^{j-(n+1)} Y \subset Y$, and thus, $s^j x \notin s^{-1}Y \backslash Y$.

It follows that for any $n \in \mathbb{N}$, $\sum_{i=1}^{n}[L_s\widetilde{f} - \widetilde{L_s f}](s^i x) \leq 1$ and, if we notate $\underbrace{L_s \circ L_s \circ ... \circ L_s}_{n \text{ times}}$ by L_s^n,

$$1 \geq M\left(\sum_{i=1}^{n} L_s^i[L_s\widetilde{f} - \widetilde{L_s f}]\right) = \sum_{i=1}^{n} M\left(L_s^i[L_s\widetilde{f} - \widetilde{L_s f}]\right) = \sum_{i=1}^{n} M\left([L_s\widetilde{f} - \widetilde{L_s f}]\right) = nM\left([L_s\widetilde{f} - \widetilde{L_s f}]\right)$$

Thus, sending $n \to \infty$, $M\left([L_s\widetilde{f} - \widetilde{L_s f}]\right) = 0$ i.e. $N(L_s f) = M[\widetilde{L_s f}] = M[L_s\widetilde{f}] = M(\widetilde{f}) = N(f)$. It follows that N is a T-invariant mean on $m(Y)$ and it follows that (T, Y) is amenable. ∎

To see why the condition $M(\chi_Y) > 0$ is important, consider the following example:

Example 2.13. Suppose S is a left-zero semigroup, so that by Theorem 2.11, S is not left-amenable (and Proposition 2.8 does not apply). Take any set X with $|X| \geq |S|$. Then, there exists an injective map ϕ from S into X. Take for each $s \in S$, $x_s = \phi(s) \in X$. Let us define the flow (S, X) by the action $sx = x_s$ for all $x \in X$, $s \in S$. Indeed, to check associativity, note that for $s, t \in S$, $(st)x = sx = x_s = s(x_t) = s(tx)$. Now if we adjoin an additional element x_0 to S, where x_0 is fixed under S, i.e. $sx_0 = x_0$ for all $s \in S$, $(S, X \cup \{x_0\})$ is a flow containing (S, X) as an S-ideal. By Proposition 2.10, $(S, X \cup \{x_0\})$ is an amenable flow, but (S, X) is not amenable by Theorem 2.11, since $sX \cap tX = \{x_s\} \cap \{x_t\} = \varnothing$, for $s \neq t$ (by the injectivity of ϕ). By the proof of Theorem 2.11, an S-invariant mean on $m(X \cup \{x_0\})$ is given by M, where $M(f) = f(x_0)$ for each $f \in m(X \cup \{x_0\})$. However, note that $M(\chi_X) = \chi_X(x_0) = 0$, as expected.

Proposition 2.14. *Let (S, Y) be a S-ideal of (S, X). If (S, Y) is amenable, so is (S, X).*

Proof. Suppose M is an S-invariant mean on $m(Y)$. Let us define for each $f \in m(X)$, $\widetilde{f} = f \restriction_Y$, and define $N : m(X) \to \mathbb{R}$, $f \mapsto M(\widetilde{f})$. It is clear that N is a linear functional. Furthermore, we have:

- For each $f \in m(X)$, $|N(f)| = |M(\widetilde{f})| \leq \|M\|\|\widetilde{f}\|_\infty \leq \|M\|\|f\|_\infty = \|f\|_\infty$
- $N(\chi_X) = M(\widetilde{\chi_X}) = M(\chi_Y) = 1$

It follows that M is a mean.

Lastly, since Y is closed under the action of S, for any $s \in S$, $f \in m(X)$, and for all $y \in Y$,

$$\widetilde{L_s f}(y) = f(sy) = f \restriction_Y (sy) = \widetilde{f}(sy) = L_s(\widetilde{f})(y)$$

and $\widetilde{L_s f} = L_s(\widetilde{f})$, which gives us

$$N(L_s f) = M(\widetilde{L_s f})) = M(L_s(\widetilde{f})) = M(\widetilde{f}) = N(f)$$

Thus N is an S-invariant mean on $m(X)$ and (S, X) is amenable. ∎

Corollary 2.15. *Suppose $\{X_\lambda\}_{\lambda \in \Lambda}$ is a family of subsets of a set X, and (S, X_λ) is an amenable flow for each $\lambda \in \Lambda$. Then, $(S, \cup_{\lambda \in \Lambda} X_\lambda)$ is amenable.*

Let (S, X) be a flow.

Definition 2.16. For every element $f \in l_1(X)$, and $s \in S$, we define $f \cdot s$ to be given by

$$f \cdot s(x) = \sum_{y = s^{-1}x} f(y) = \begin{cases} 0 & \text{if } s^{-1}x = \varnothing \\ \sum_{sy=x} f(y) & \text{otherwise} \end{cases}$$

Remark 2.17. We note that $f \cdot s \in l_1(X)$ since:

$$\sum_{x \in X} |f \cdot s(x)| = \sum_{x \in X} \left| \sum_{sy=x} f(y) \right| \leq \sum_{x \in X} \sum_{sy=x} |f(y)| = \sum_{x \in X} \sum_{y \in s^{-1}\{x\}} |f(y)| = \sum_{x \in X} |f(x)| = \|f\|_1$$

Note that this is because, firstly, for each $x \in X$, $x \in s^{-1}\{sx\}$, and secondly, if $x \in s^{-1}a$ and $x \in s^{-1}b$, for $a, b, x \in X$, we get $sx = a = b$, i.e. $s^{-1}\{x\}$ are pairwise disjoint sets for $x \in X$.

Definition 2.18. We say a net $\{M_\alpha\}_{\alpha \in A}$ of means on $m(X)$ is **w^*-convergent (norm-convergent)** to S-invariance if

$$w^* \lim_\alpha [M_\alpha \circ L_s - M_\alpha] = 0 \ (\lim_\alpha \|M_\alpha \circ L_s - M_\alpha\| = 0)$$

for each $s \in S$.

Theorem 2.19. *There exists an S-invariant mean M on $m(X)$ if and only if there exists a net $\{\phi_\alpha\}_{\alpha \in A}$ of finite means such that the net $\{Q\phi_\alpha\}_{\alpha \in A}$ is w^*-convergent to S-invariance and M is a limit point of $\{Q\phi_\alpha\}_{\alpha \in A}$.*

Proof. Suppose M is an S-invariant mean on $m(X)$. Since $Q\Phi$ is dense in $\mathscr{M}(X)$, there exists a net $\{\phi_\alpha\}_{\alpha\in A}$ of finite means on X, such that $\{Q\phi_\alpha\}_{\alpha\in A}$ is w^*-convergent to M.

$$w^* \lim_{\alpha\in A}[Q\phi_\alpha \circ L_s - Q\phi_\alpha] = M \circ L_s - M = 0$$

due to M being S-invariant.

Conversely, suppose $\{\phi_\alpha\}_{\alpha\in A}$ is a net of finite means such that the net $\{Q\phi_\alpha\}_{\alpha\in A}$ is w^*-convergent to S-invariance. Since $Q\Phi$ is w^*-dense in $\mathscr{M}(X)$ and $\mathscr{M}(X)$ is compact, there exists a subnet $\{Q\phi_\beta\}_{\beta\in B}$ of $\{Q\phi_\alpha\}_{\alpha\in A}$ that is convergent to some mean M. Now:

$$0 = w^* \lim_{\beta\in B}[Q\phi_\beta \circ L_s - Q\phi_\beta] = M \circ L_s - M$$

implying that M is S-invariant. ∎

Remark 2.20. For any $f \in l_1(S)$, $g \in m(S)$ and $s \in S$,

$$Q[f\cdot s](g) = \sum_{x\in X} f\cdot s(x)g(x) = \sum_{x\in X}\left[\sum_{y\in s^{-1}x} f(y)g(x)\right] = \sum_{x\in X} f(x)g(sx) = \sum_{x\in X} f(x)L_s g(x) = [Qf\circ L_s](g)$$

Recall that convergence of a sequence in the weak topology in $l_1(X)$ is equivalent to weak* convergence in $m(X)^*$ upon applying Q to the sequence.

The following theorem is a generalization of Namioka's elegant proof in [22]:

Theorem 2.21. *The following are equivalent:*

1. (S, X) *is amenable*
2. $\exists\{\psi_\alpha\}_{\alpha\in A} \subset \Phi$ *such that* $\lim_{\alpha\in A} \|Q \circ \psi_\alpha \circ L_s - Q \circ \psi_\alpha\| = 0$,
3. $\exists\{\phi_\alpha\}_{\alpha\in A} \subset \Phi$ *such that* $w^* \lim_{\alpha\in A}[Q \circ \phi_\alpha \circ L_s - Q \circ \phi_\alpha] = 0$,
4. $\exists\{\psi_\alpha\}_{\alpha\in A} \subset \Phi$ *such that* $\lim_{\alpha\in A} \|\psi_\alpha \cdot s - \psi_\alpha\|_1 = 0$,
5. $\exists\{\phi_\alpha\}_{\alpha\in A} \subset \Phi$ *such that* $w \lim_{\alpha\in A}[\phi_\alpha \cdot s - \phi_\alpha] = 0$

Proof. We know by Theorem 2.19 that X has an S-invariant mean if and only if there exists a net $\{\phi_\alpha\}_{\alpha\in A} \subset \Phi$ of finite means such that the net $\{Q\phi_\alpha\}_{\alpha\in A}$ is w^*-convergent to

S-invariance. This gives us the equivalence of 1, 3 and 5.

We define the operator $P : l_1(X) \to l_1(X)^S$ by $[P(f)](s) = f \circ L_s - f$.

P is clearly linear. First, we note that the topology on $l_1(X)^S$ given by the product of weak topologies on $l_1(X)$, is equivalent to the weak topology on $l_1(X)^S$. As a result of this, (S, X) is amenable if and only if there exists a net $\{\phi_\alpha\}_{\alpha \in A} \subset \Phi$ of finite means such that $w \lim_{\alpha \in A}[P(\phi_\alpha)](s) = w \lim_{\alpha \in A}[\phi_\alpha \circ L_s - Q\phi_\alpha] = 0$ for each $s \in S$, if and only if $P(\phi_\alpha) \to 0$ in $\alpha \in A$ in the weak topology on $l_1(X)^S$. It follows that X has an S-invariant mean if and only if 0 is in the weak closure of $P(\Phi)$ in $l_1(X)^S$.

Now, as Φ is a convex subset of $l_1(X)$, $P(\Phi)$ is convex. Since $l_1(X)^S$ is a locally convex space, it follows that the weak closure of $P(\Phi)$ in $l_1(X)^S$ is equivalent to its norm closure. Thus, 0 is in the weak closure of $P(\Phi) \iff$ it is in the norm closure of $P(\Phi)$. It follows that, (S, X) is amenable $\iff$ 0 is in the weak closure of $P(\Phi) \iff$ 0 is in the norm closure of $P(\Phi)$. Hence, the equivalence of 4 and 5 follows. Lastly, the equivalence of 4 and 2 follows from Q being a continuous isometry. ∎

Definition 2.22. We say a flow (S, X) is **transitive**, if for each $x, y \in X$, there exists some $s \in S$, such that $sx = y$.

Theorem 2.23. *Suppose (S, X) is a flow that is transitive (or satisfies $Sy = X$ for an element $y \in X$) and X-cancellative. Then (S, X) is amenable $\implies$ S is amenable.*

Proof. Suppose S is not amenable. Then Dixmier's condition does not hold and $\exists h_i \in m(S)$, $s_i \in S$, $1 \leq i \leq n$, such that $\sup\limits_{t \in S} \sum_{i=1}^{n} h_i(s_i t) - h_i(t) < 0$.

Fix $y \in X$ if (S, X) is transitive (or take $y \in X$ to be the hypothesized element satisfying $Sy = X$). We define a map $\phi : X \to S$ by $\phi(x) \in S$ such that $\phi(x)y = x$. Note that since the action is transitive, $\phi(x)$ exists and due to X-cancellativity, it is unique. Furthermore, for any $s \in S$, $x \in X$,

$$\phi(x)y = x \implies s\phi(x)y = sx = \phi(sx)y \implies s\phi(x) = \phi(sx)$$

by X-cancellativity.

Then we define $g_i : X \to \mathbb{R}$, $g_i(x) = h_i(\phi(x))$, for all $x \in X$. Clearly, $g_i \in m(X)$.

Now note that

$$\begin{aligned}
\sup_{x\in X} \sum_{i=1}^{n} g_i(s_i x) - g_i(x) &= \sup_{x\in X} \sum_{i=1}^{n} h_i(\phi(s_i x)) - h_i(\phi(x)) \\
&= \sup_{x\in X} \sum_{i=1}^{n} h_i(s_i \phi(x)) - h_i(\phi(x)) \\
&\leq \sup_{t\in S} \sum_{i=1}^{n} h_i(s_i t) - h_i(t) < 0
\end{aligned}$$

Thus, Dixmier's condition does not hold for (S, X) which is a contradiction to its amenability. It follows that S must be amenable. ∎

Proposition 2.24. *Suppose (S, X) is a S-cancellative flow with X being a finite set. Then (S, X) is amenable.*

Proof. Let us take $M : m(X) \to \mathbb{R}, f \mapsto \frac{\|f\|_1}{|X|}$. M is clearly linear, and for each $f \in m(X)$,

$$|M(f)| = \frac{\|f\|_1}{|X|} = \sum_{x\in X} \frac{|f(x)|}{|X|} \leq \sum_{x\in X} \frac{\|f\|_\infty}{|X|} = \|f\|_\infty$$

and $M(\chi_X) = \frac{\|\chi_X\|_1}{|X|} = 1$.

It follows that M is a mean on $m(X)$. Now, if $s \in S$ and $f \in m(X)$,

$$M(L_s f) = \frac{\|L_s f\|_1}{|X|} = \sum_{x\in X} \frac{|f(sx)|}{|X|} =^{(*)} \sum_{x\in X} \frac{|f(x)|}{|X|} = M(f)$$

where $(*)$ follows from the fact that the injectivity of the map $X \to X$, $x \mapsto sx$, (due to X-cancellativity) implies its bijectivity. M is thus an S-invariant mean on $m(X)$ and (S, X) is amenable. ∎

2.2 Relation between amenable flows

Definition 2.25. Given semigroups S, T, with a surjective semigroup homomorphism $\phi : S \to T$. We define a **homomorphism of flows** between flows (S, X) and (T, Y) to be a map $T_\phi : X \to Y$ that satisfies $T_\phi(sx) = \phi(s)T_\phi(x)$, for all $x \in X, s \in S$. We call this an **isomorphism** if T_ϕ is bijective.

The notation "T_ϕ" will be used to denote the underlying semigroup homomorphism ϕ between semigroups.

Note that for any semigroup S since the identity map id_S for S is a semigroup homomorphism to itself, any homomorphism of flows $T_{id_S} : X \to Y$ for flows (S, X), (S, Y) is an isomorphism if T_{id_S} is bijective. Given such a flow, we will omit the subscript "id_s" unless further context is required.

Theorem 2.26. *Suppose $\phi : S \to T$ is a surjective semigroup homomorphism and $T_\phi : X \to Y$ is a homomorphism of flows (S, X), (T, Y), then (S, X) amenable $\implies$ (T, Y) amenable. Furthermore, if T_ϕ is an isomorphism, then the converse holds.*

Proof. Suppose (S, X) is amenable. Then there exists an S-invariant mean M on $m(X)$. We define N on $m(Y)$ to be the map given by $g \mapsto M(g \circ T_\phi)$. N is a mean due to the following:

- By the linearity of M, clearly, N is linear.
- N is bounded as $|N(g)| = |M(g \circ T_\phi)| \leq \|M\|\|g \circ T_\phi\| \leq \|M\|\|g\| = \|g\| \implies \|N\| \leq 1$
- $N(\chi_Y) = M(\chi_Y \circ T_\phi) = M(\chi_X) = 1$

Now for any $t \in T$, $g \in m(Y)$, let $s \in \phi^{-1}\{t\}$ be arbitrarily fixed. Then,

$$[(L_t g) \circ T_\phi](x) = (L_t g)(T_\phi(x)) = g(tT_\phi(x)) = g(T_\phi(sx)) = [L_s(g \circ T_\phi)](x)$$

for all $x \in X$.Thus, $(L_t g) \circ T_\phi = L_s(g \circ T_\phi)$ and we have $N(L_t g) = M((L_t g) \circ T_\phi) = M(L_s(g \circ T_\phi)) = M(g \circ T_\phi) = N(g)$. N is thus an T-invariant mean on $m(Y)$.

On the other hand, assume that T_ϕ is bijective and that (T, Y) is amenable. Suppose N is an T-invariant mean on $m(Y)$. Then, we define a mean M on $m(X)$ by $M(f) = N(f \circ T_\phi^{-1})$, for all $f \in m(X)$. M being a mean is easily checked in the same way as above. Now for any $s \in S$, for all $y \in Y$,

$${T_\phi}^{-1}(\phi(s)y) = {T_\phi}^{-1}(\phi(s)T_\phi\{{T_\phi}^{-1}(y)\}) = {T_\phi}^{-1}(T_\phi\{s{T_\phi}^{-1}(y)\}) = s{T_\phi}^{-1}(y)$$

and $(L_s f) \circ {T_\phi}^{-1} = L_t(f \circ T_\phi^{-1})$, for all $f \in m(X)$. Then, M is an S-invariant mean follows similarly to the proof of the converse. ∎

Corollary 2.27. *Suppose $T : (S, X) \to (S, Y)$ is a homomorphism of flows. Then, (S, X) is amenable $\implies$ (S, Y) is amenable. If T is bijective, then the converse holds as well.*

Theorem 2.28. *Suppose (S, X), (T, Y) are flows. The product flow $(S \times T, X \times Y)$ is amenable if and only if (S, X) and (T, Y) are amenable.*

Proof. Suppose M is a S-invariant mean on $m(X)$ and N is a T-invariant mean on $m(Y)$. We define a $K : m(X \times Y) \to \mathbb{R}$ by $f \mapsto N(\widetilde{M}(f))$ where we set $\widetilde{M}(f) : Y \to \mathbb{R}$ to be given by $\widetilde{M}(f)(y) = M(f(., y))$, for all $y \in Y$.

K is clearly linear as N and M are linear. Furthermore, $K(\chi_{X \times Y}) = N(\widetilde{M}(\chi_{X \times Y})) = N(\chi_Y) = 1$, due to $\widetilde{M}(\chi_{X \times Y})(y) = M(\chi_{X \times Y}(., y)) = M(\chi_X) = 1$. Lastly, $|K(f)| = |N(\widetilde{M}(f))| \leq \|N\| \|\widetilde{M}(f)\|_\infty \leq \|N\| \|M\| \|f\|_\infty = \|f\|_\infty$ due to $\|M\| = 1 = \|N\|$. Thus, K is a mean.

Now for any $(s, t) \in S \times T$ and any $f \in m(X \times Y)$, we have, for all $y \in Y$,

$$\widetilde{M}(L_{(s,t)}(f))(y) = M(L_{(s,t)} f(., y)) = M(f(s., ty)) = M(f(., ty)) = \widetilde{M}(f)(y)$$

by the S-invariance of M.

Thus, $K(L_{(s,t)}(f)) = N(\widetilde{M}(L_(s,t)(f)) = N(L_t(\widetilde{M}(f))) = N(\widetilde{M}(f)) = K(f)$ and K is S-invariant. It follows that $(S \times T, X \times Y)$ is amenable.

Conversely, let K be an S-invariant mean on $m(X \times Y)$. Then for every $f \in m(X)$, define $\widetilde{f} \in m(X \times Y)$ to be the map $\widetilde{f}(x, y) = f(x)$, for all $(x, y) \in X \times Y$. $\widetilde{f}$ is clearly bounded.

We define $M : m(X) \to \mathbb{R}$ by $f \mapsto K(\widetilde{f})$. M is clearly linear due to the linearity of K. Furthermore, $M(\chi_X) = K(\widetilde{\chi_X}) = K(\chi_{X \times Y}) = 1$ and for all $f \in m(X)$, $|M(f)| = |K(\tilde{f})| \leq \|K\| \|\tilde{f}\|_\infty = \|\tilde{f}\|_\infty \leq \|f\|_\infty$ (due to $\|K\| = 1$). Thus, M is a mean.

Fix any $t \in T$. Now for any $s \in S$, and $f \in m(X)$, $\widetilde{L_s f}(x, y) = L_s f(x) = f(sx) = \widetilde{f}(sx, ty) = L_{(s,t)} \widetilde{f}(x, y)$, for all $(x, y) \in X \times Y$. Thus, $M(L_s f) = K(\widetilde{L_s f}) = K(L_{(s,t)} \tilde{f}) = K(\widetilde{f}) = M(f)$. M is hence an S-invariant mean and it follows that (S, X) is amenable. (T, Y) being amenable follows similarly. ∎

Definition 2.29. Suppose $\{S_i\}_{i\in I}$ is a family of semigroups. Then, we define **the direct product of** $\{S_i\}_{i\in I}$, $\prod_{i\in I} S_i$ to be the semigroup with semigroup multiplication given by pointwise multiplication, i.e for $s, t \in \prod_{i\in I} S_i$, $st(i) = s(i)t(i)$ for all $i \in I$.

Definition 2.30. Given a family $\{S_i\}_{i\in I}$ of semigroups with identity (denoted by e_i for each $i \in I$), their **weak direct product** $S = \prod_{i\in I}^{w} S_i$ is the subsemigroup of $\prod_{i\in I} S_i$ given by all elements $s = \{s_i\}_{i\in I}$ that satisfy $s_i = e_i$ for all but finitely many $i \in I$.

Proposition 2.31. *Suppose $\{(S_i, X_i)\}_{i\in I}$ is a family of amenable flows, where for each $i \in I$, S_i has an identity element e_i. Then so is $(S, X) = (\prod_{i\in I}^{w} S_i, \prod_{i\in I} X_i)$.*

Proof. First, note that if I is a finite set, we are done by Theorem 2.28. We may thus assume, without loss of generality, that I is an infinite set.

Suppose (S, X) is not amenable. By Dixmier's condition, there exist $s_k \in S$, $f_k \in m(X)$, $1 \le k \le n$ such that $\inf_{x\in X} \sum_{k=1}^{n} f_k(x) - f_k(s_k x) > 0$.

For each s_k, we have that $s_k(i) = e_i$ for all but finitely many $i \in I$. We define $A = \cup_{k=1}^{n}\{i \in I \mid s_k(i) \neq e_i\}$. A is clearly a finite set. Fix $c_i \in X_i$, for all $i \in I\backslash A$ (Note $I\backslash A \neq \varnothing$, since I is infinite and A is finite). Let us label elements of A as $a_1, ..., a_p$, where $p = |A|$. We define the following maps:

$$v_k : \prod_{a\in A} X_a \to \mathbb{R}$$
$$y = (x_{a_1}, ..., x_{a_p}) \mapsto f_k(x_y)$$

where we define

$$x_y(i) = \begin{cases} c_i & \text{if } i \in I\backslash A \\ x_{a_l} & \text{if } i = a_l \in A. \end{cases}$$

Since $f_k \in m(X)$, $v_k \in m(X)$, for all $1 \le k \le n$.

Furthermore, define $t_k \in \prod_{a\in A} S_a$ to be the element $t_k(a) = s_k(a)$, for all $a \in A$, i.e. $t_k = s_k \restriction_{\prod_{a\in A} X_a}$.

Now we have:

$$\begin{aligned}
\inf_{y\in\prod_{a\in A}X_a}\sum_{k=1}^n v_k(y)-v_k(t_ky) &= \inf_{y\in\prod_{a\in A}X_a}\sum_{k=1}^n f_k(x_y)-f_k(x_{t_ky})\\
&=^{(*)}\inf_{y\in\prod_{a\in A}X_a}\sum_{k=1}^n f_k(x_y)-f_k(s_kx_y)\\
&\geq \inf_{x\in X}\sum_{k=1}^n f_k(x)-f_k(s_kx)>0
\end{aligned}$$

where at $(*)$, we used the fact that $s_k(i)=e_i$, for all $i\notin A$ gives us, for all $i\in I$,

$$x_{t_ky}(i)=\begin{cases} c_i & \text{if } i\in I\backslash A\\ [t_ky](a_l) & \text{if } i=a_l\in A\end{cases} = \begin{cases} c_i & \text{if } i\in I\backslash A\\ s_k(a_l)y(a_l) & \text{if } i=a_l\in A\end{cases} = s_kx_y(i)$$

i.e $x_{t_ky}=s_kx_y$. This means that Dixmier's condition fails for $\left(\prod_{a\in A}S_a,\prod_{a\in A}X_a\right)$ and it is not amenable. This is a contradiction to Theorem 2.28. Thus, (S,X) has to be amenable. ∎

Proposition 2.32. *Suppose $\{S_\lambda\}_{\lambda\in\Lambda}$ is a directed family of sub-semigroups of a semigroup S, and (S_λ,X) are amenable flows for all $\lambda\in\Lambda$. Then, $(\cup_{\lambda\in\Lambda}S_\lambda,X)$ is an amenable flow.*

Proof. Assume $(\cup_{\lambda\in\Lambda}S_\lambda,X)$ is not amenable. Then there exist functions $h_i\in m(\cup_{\lambda\in\Lambda}X_\lambda)$ and $s_i\in\cup_{\lambda\in\Lambda}S_\lambda$, with $1\leq i\leq n$, for some $n\in\mathbb{N}$ such that

$$\sup_{x\in\cup_{\lambda\in\Lambda}X_\lambda}\sum_{x\in X}h_i(x)-h_i(s_ix)<0$$

But since, there exists some $S_{\lambda_0},\lambda_0\in\Lambda$, such that $s_i\in S_{\lambda_0}$, for all i, $1\leq i\leq n$ (due to the union being directed), Dixmier's condition fails for (S_{λ_0},X), which is a contradiction to its amenabilility. Hence, $(\cup_{\lambda\in\Lambda}S_\lambda,X)$ must be amenable. ∎

2.3 Hahn-Banach Extension Property

Definition 2.33. If (S,X) is a flow and $f:X\to\mathbb{R}$, f is said to be S-**invariant** if $L_sf=f$ for all $s\in S$.

Definition 2.34. Suppose (S,X) is a left or right representation. We say that a subspace $Y\subset X$ is S-**invariant** if (S,Y) is an S-ideal.

Definition 2.35. Suppose (S, X) is a left-representation. We say that (S, X) has the **Hahn-Banach Extension property (HBEP)** if the following holds:

If Y is a S-invariant subspace of X, and

1. $\phi : Y \to \mathbb{R}$ is a S-invariant linear functional;
2. $p : X \to \mathbb{R}$ is a sublinear functional;
3. $p(sx) \leq p(x)$ for all $s \in S$, $x \in X$;
4. $\phi(y) \leq p(y)$ for all $y \in Y$;

hold, then there exists a S-invariant linear functional $\psi : X \to \mathbb{R}$ such that $\psi(x) \leq p(x)$ for all $x \in X$, and $\psi \restriction_Y = \phi$.

The following theorem was proven by Lau in [20]:

Theorem 2.36. *Let S be a semigroup. The following are equivalent:*

1. *S is left amenable*
2. *Every right linear representation (S, X) has the HBEP.*

Proof. Suppose S is amenable and M is a left-invariant mean on $m(S)$. Let us denote the left translation operator on $m(S)$ by λ_s for each $s \in S$.

Let Y be a S-invariant subspace of X and $f : Y \to \mathbb{R}$ be a linear map bounded by a sublinear functional $p : X \to \mathbb{R}$. By the Hahn-Banach extension theorem, there exists an extension F of f to X, that is still bounded by p. For each $x \in X$, define

$$\begin{aligned} g_x : S &\to \mathbb{R} \\ s &\mapsto F(xs) \end{aligned}$$

Note that $g_x(s) = \sup_{s \in S} F(xs) \leq p(xs) \leq p(x)$, for each $x \in X$, $s \in S$, which gives us that $g_x \in m(S)$. Then, define

$$\begin{aligned} G : X &\to \mathbb{R} \\ x &\mapsto M(g_x) \end{aligned}$$

Consider the following:

- For each $x, y \in X$, $a \in \mathbb{R}$ and for all $s \in S$,

$$[g_x + ag_y](s) = F(xs) + aF(ys) = F(xs + ays) = F((x + ay)s) = g_{x+ay}(s)$$

and M is linear, thus, G is linear.

- If $y \in Y$, $g_y(s) = F(ys) = f(ys) = f(y)$, for all $s \in S$, which gives us $G(y) = M(g_y) = M(f(y)\chi_{_S}) = f(y)$, i.e. $G \restriction_Y = f$.

- For each $x \in X$, $G(x) = M(g_x) \leq \sup_{s \in S} g_x(s) \leq p(x)$.

- For each $t \in S$, $x \in X$, $[\lambda_t g_x](s) = g_x(ts) = F(xts) = g_{xt}(s)$, so that $\lambda_t g_x = g_{xt}$, and $L_t G(x) = G(xt) = M(g_{xt}) = M(\lambda_t g_x) = M(g_x) = G(x)$.

Thus, G is an extension of f that is bounded by p and the HBEP is satisfied.

On the contrary, suppose every right linear representation of S satisfies the HBEP. We know that $(S, m(S))$ is a right linear representation. Consider $Y = \{\text{Constant functions on S}\}$, which is a S-invariant subspace of $m(S)$. Let us define a mean N on this space by $N(a\chi_{_S}) = a$, for each $a \in \mathbb{R}$. It is clear that N is an left-invariant mean by default. Let us take $p : m(S) \to \mathbb{R}$, $f \to \|f\|_\infty$. p is clearly a sublinear functional and $M(f) \leq p(f)$, for all $f \in Y$. By our hypothesis, N can be extended to a left-invariant linear function M on $m(S)$ while still being bounded by p. Thus, $M(f) \leq p(f) = \|f\|_\infty$, for all $f \in m(S)$ and $M(\chi_{_S}) = N(\chi_{_S}) = 1$, which gives us that M is a mean on $m(S)$ that is left-invariant. It follows that S is left amenable. ∎

Proposition 2.37. *If $T_\phi : (S, X) \to (T, Y)$ is a linear isomorphism of flows (isomorphism of representations), then (S, X) has the HBEP if and only if (T, Y) has the HBEP.*

Proof. Suppose $A \subset Y$ is a T-invariant linear subspace of Y and f is a T-invariant linear functional on A, that is dominated by a sublinear functional $p : Y \to \mathbb{R}$ on A, satisfying $p(ty) \leq p(y)$ for each $t \in T$, and all $y \in Y$. Then, $T_\phi^{-1}(A)$ is a linear subspace of X by the linearity of T_ϕ, f, $f \circ T_\phi : T_\phi^{-1}(A) \to \mathbb{R}$ is linear and $f \circ T_\phi(x) \leq p \circ T_\phi(x)$, for all $x \in X$. Note that $p \circ T_\phi : X \to \mathbb{R}$ is a sublinear functional and for any $s \in S$, $p \circ T_\phi(sx) = p(\phi(s)T_\phi(x)) \leq p(T_\phi(x)) = p \circ T_\phi(x)$, for all $x \in X$

For each $s \in S$, for all $x \in X$,

$$L_s(f \circ T_\phi)(x) = f \circ T_\phi(sx) = f(\phi(s)T_\phi(x)) = f(T_\phi(x)) = f \circ T_\phi(x)$$

so that $f \circ T_\phi$ is a S-invariant linear functional on $T_\phi^{-1}(A)$.

Then, since (S, X) satisfies the HBEP, it follows that there exists S-invariant functional g on X, such that g extends $f \circ T_\phi$, and $g \leq p \circ T_\phi$ on X. Then, consider $g \circ T_\phi^{-1} : Y \to \mathbb{R}$. Since g, T_ϕ are linear, it follows that $g \circ T_\phi^{-1} : Y \to \mathbb{R}$ is linear. Furthermore, $g \circ T_\phi^{-1} \leq p \circ T_\phi \circ T_\phi^{-1} = p$, and for all $a \in A$,

$$g \circ T_\phi^{-1}(a) = g(T_\phi^{-1}(a)) = f \circ T_\phi(T_\phi^{-1}(a)) = f(a)$$

Lastly, for any $t \in T$, if $s \in \phi^{-1}\{t]\}$, then, for all $y \in Y$,

$$\begin{aligned} L_t(g \circ T_\phi^{-1})(y) = g \circ T_\phi^{-1}(ty) &= g \circ T_\phi^{-1}(\phi(s) T_\phi T_\phi^{-1} y) \\ &= g \circ T_\phi^{-1}(T_\phi(s T_\phi^{-1} y)) \\ &= g(s T_\phi^{-1} y) \\ &= g(T_\phi^{-1} y) \\ &= g \circ T_\phi^{-1}(y) \end{aligned}$$

so that $g \circ T_\phi^{-1}$ is T-invariant.

It follows that $g \circ T_\phi^{-1} : Y \to \mathbb{R}$ is a T-invariant linear extension of f that is dominated by p. Thus, (T, Y) satisfies the HBEP.

The converse follows similarly. ∎

Let X, Y be non-empty sets. Suppose $T : X \to Y$ is an arbitrary map. Let us define the map $S_T : m(Y) \to m(X)$, $f \mapsto f \circ T$. S_T is clearly linear. Hence, we define ${S_T}^* : m(X)^* \to m(Y)^*$ to be the adjoint of S_T.

Lemma 2.38. *If $T : X \to Y$ is surjective, then S_T is a linear isometry and ${S_T}^*$ is a surjective linear map of norm 1 that maps $\mathscr{M}(X)$ onto $\mathscr{M}(Y)$.*

Proof. Suppose $T : X \to Y$ is surjective. For any $f \in m(Y)$, since T is onto, $\|S_T f\| = \sup_{x \in X} |S_T f(x)| = \sup_{y \in Y} |f(y)| = \|f\|$, and hence, S_T is an isometry.

Now suppose $M \in m(Y)^*$. Define N on $S_T(m(Y))$ which is a subspace of $m(X)$, by $N(f) = M({S_T}^{-1} f)$. The map is well defined due to S_T being injective and is linear by the linearity of S_T and M. Since S_T is an isometry, we have $|N(f)| \leq \|M\| \|{S_T}^{-1} f\|_\infty = \|M\| \|f\|_\infty$, which means $\|N\| \leq \|M\|$ and $N \in S_T(m(Y))^*$. We may extend N to a map $\widetilde{N} \in m(Y)^*$ by the Hahn-Banach extension theorem, with $\|N\| = \|\widetilde{N}\|$. Then, we have, for all $f \in m(Y)$, $[S_T^*\widetilde{N}](f) = \widetilde{N}(S_T f) = N(S_T f) = M({S_T}^{-1} S_T f) = M(f)$. Thus, $S_T^*\widetilde{N} = M$ and ${S_T}^*$ is surjective.

S_T^* being of norm one follows from S_T being of norm one and $\|S_T\| = \|{S_T}^*\|$ due to $m(X)$, $m(Y)$ being Banach spaces. Suppose $M \in \mathscr{M}(X)$. $[{S_T}^*M](\chi_X) = M(S_T\chi_X) = M(\chi_X \circ T) = M(\chi_Y) = 1$ and $\|{S_T}^*M\| \leq \|{S_T}^*\|\|M\| = 1$; thus, ${S_T}^*M \in \mathscr{M}(Y)$. ${S_T}^*$ hence maps $\mathscr{M}(X)$ into $\mathscr{M}(Y)$.

Now suppose $M \in \mathscr{M}(Y)$. Then, taking N as before, $N(\chi_X) = N(S_T(\chi_Y)) = M(\chi_Y) = M(\chi_Y) = 1$. It follows that $\|N\| = 1$ and by the Hahn-Banach theorem, we can extend N to a function $\widetilde{N} \in m(X)^*$, with $\|N\| = \|\widetilde{N}\|$ and thus $\widetilde{N} \in \mathscr{M}(X)$. Then, again, ${S_T}^*(\widetilde{N}) = M$ and this conclude the proof. ∎

Theorem 2.39. *Let (S, X) and (T, Y) be flows and $T_\phi : X \to Y$ be a surjective homomorphism of flows. Then, if S is amenable, ${S_{T_\phi}}^*$ maps $\mathscr{M}_L(X)$ onto $\mathscr{M}_L(Y)$.*

Proof. Now suppose $M \in \mathscr{M}_L(X)$. Then, for any $t \in T$, and $f \in m(Y)$, taking any $s \in \phi^{-1}(\{t\})$, we have for all $x \in X$,

$$[S_{T_\phi}(L_t f)](x) = [(L_t f) \circ T](x) = L_t f(T(x)) = f(tT(x)) = f(T(sx)) = S_{T_\phi} f(sx) = [L_s(S_{T_\phi} f)](x)$$

and it follows that

$$S_{T_\phi}(L_t f) = L_s(S_{T_\phi} f) \qquad (*)$$

Thus, $[{S_{T_\phi}}^*M](L_t f) = M(S_{T_\phi}(L_t f)) = M(L_s(S_{T_\phi} f)) = M(S_{T_\phi} f) = [{S_{T_\phi}}^*M](f)$, i.e ${S_{T_\phi}}^*$ maps $\mathscr{M}_L(X)$ into $\mathscr{M}_L(Y)$.

Now suppose $M \in \mathscr{M}_L(Y)$. Let us define again, $N : S_{T_\phi}(m(Y)) \to \mathbb{R}$ by $g \mapsto M({S_{T_\phi}}^{-1} g)$. First note that $S_{T_\phi}(m(Y))$ is an S-invariant subspace of $m(X)$ by $(*)$ and for every $t \in T$, $g \in m(X)$, if $s \in \phi^{-1}(\{t\})$,

$$S_{T_\phi}(L_t({S_{T_\phi}}^{-1} g)) = L_s(S_{T_\phi} {S_{T_\phi}}^{-1} g) = L_s g = S_{T_\phi}({S_{T_\phi}}^{-1}(L_s g))$$

by $(*)$. It follows by the injectivity of S_{T_ϕ} (Lemma 2.38) that $L_t \circ {S_{T_\phi}}^{-1} = {S_{T_\phi}}^{-1} \circ L_s$, for all $s \in S$, i.e. ${S_{T_\phi}}^{-1}$ is a homomorphism of flows on $S_{T_\phi}(m(Y))$, where S acts on $S_{T_\phi}(m(Y))$ via the right action $S \times Im(S_{T_\phi}) \to Im(S_{T_\phi})$, $(s, f) \mapsto L_s f$. Hence, N is a S-invariant mean on $S_{T_\phi}(m(Y))$ since, for any $s \in S$, $N \circ L_s = M \circ {S_{T_\phi}}^{-1} \circ L_s = M \circ L_s \circ {S_{T_\phi}}^{-1} = M \circ {S_{T_\phi}}^{-1}$ by the S-invariance of M.

Now we refer back to Theorem 2.36, and since S is amenable, we extend can N to an invariant linear functional on $\widetilde{N}$ on $m(Y)$ that is invariant under L_t^*, for all $t \in T$ and has the same norm. Clearly, $\widetilde{N}$ is a T-invariant mean. ∎

2.3.1 An application - Generalized Banach Limits for Bounded Nets

Suppose $(X, \leq)$ is a directed set, and (S, X) is a flow that satisfies $sx \geq x$, for each $s \in S$, $x \in X$. Let us denote $m_c(X)$ to be all elements of $m(X)$ that are convergent in X, i.e. for each $f \in m_c(X)$, the limit $\lim_{x\in X} f(x)$ exists.

Proposition 2.40. *If S is left amenable, there exists a S-invariant mean M on $m(X)$, that satisfies $M(f) = \lim\limits_{x\in X} f(x)$ for all $f \in m_c(X)$.*

Proof. $m_c(X)$ is a subspace of X since for any $f, g \in m_c(X)$ and $\alpha \in \mathbb{R}$, $f + \alpha g$ is also convergent and bounded.

For each $f \in m_c(X)$, since $sx \geq x$ for all $s \in S$, $x \in X$, $\{f(sx)\}_{x\in X}$ is a subnet of $\{f(x)\}$; thus, $\lim_{x\in X} f(sx) = \lim_{x\in X} f(x)$. Thus, $m_c(X)$ is also an S-invariant subspace of $m(X)$.

Now we define a function $l : m_c(X) \to \mathbb{R}$ by $l(f) = \lim_{x\in X} f(x)$. l is clearly linear and $l \circ L_s = l$, for each $s \in S$. We also have for each $f \in m_c(X)$, $|l(f)| = |\lim_{x\in X} f(x)| \leq \|f\|$, and $l(\chi_X) = 1$, which gives us $\|l\| = 1$.

Note that $\|\cdot\|_\infty : m(X) \to \mathbb{R}$ is a sublinear functional and $l(f) \leq \|f\|_\infty$, for all $f \in m(X)$.

By Theorem 2.36, there exists a linear functional $M : m(X) \to \mathbb{R}$ such that:

- M is S-invariant, i.e. $M(f) = M(L_s f)$, for all $s \in S$, $f \in m(X)$
- $M \restriction_{m_c(X)} = l$, i.e. for all $f \in m_c(X)$, $M(f) = \lim_{x\in X} f(x)$
- $M(f) \leq \|f\|_\infty$, for all $f \in m(X)$

Since $M(f) \leq \|f\|_\infty$ and $M(\chi_X) = l(\chi_X) = 1$, $\|M\| = 1$ and M is a S-invariant mean. ∎

Now if we set $S = (N, +)$ and $X = (N, \leq)$, where "$\leq$" is the usual order on N, so that N acts on itself via addition, the requirement $n + m \geq m$, for all $n, m \in \mathbb{N}$. Moreover, $m_c(\mathbb{N}) = \{\text{All convergent sequences in } \mathbb{R}\}$ since all convergent sequences in $\mathbb{R}$ are bounded. As $(N, +)$ is a commutative semigroup, by Theorem 2.4 it is amenable and all the conditions in Proposition 2.40 are satisfied. We end up with the following corollary:

Corollary 2.41. *Let $c(\mathbb{N})$ be the set of all convergent sequences in $\mathbb{R}$. Then, there exists a linear functional $M : m(\mathbb{N}) \to \mathbb{R}$, such that*

- $\|M\| = 1$
- $M(x) = \lim_{n\to\infty} x(n)$, *for each $x \in c(\mathbb{N})$*
- *For each $x \in m(\mathbb{N})$, $n \in \mathbb{N}$, $M(x) = M(L_n x)$, where $L_n x$ is the sequence given by $(x(n+1), x(n+2), ...)$*

2.4 Fixed Point Characterization

For any topological space X, let $C(X)$ denote the continuous real-valued functions on X and let $\mathscr{A}(X)$ denote the set of affine continuous real-valued functions on X. If X is compact, $C(X)$ is a Banach space with the supremum norm $\|\cdot\|_\infty$.

The following is a lemma from [25]:

Lemma 2.42. *Suppose X is a locally convex space and Y is a compact convex subset of X. Then, The subspace of $C(Y)$ given by $X^* \restriction_Y + \mathbb{R} = \{f \restriction_Y + c\chi_Y \mid f \in X^*, c \in \mathbb{R}\}$ is uniformly (i.e. $\|\cdot\|$-norm) dense in $\mathscr{A}(Y)$. Any mean on $\mathscr{A}(Y)$ is a point measure.*

The following theorem is a slightly modified version of Lau's [19]:

Theorem 2.43. *Suppose Y is a compact convex subset of a locally convex space X and (S, Y) is an affine flow. Then, (S, Y) is amenable (as a discrete flow) if and only if Y has a fixed point for S.*

Proof. Suppose (S, Y) is amenable and M is an S-invariant mean on $m(Y)$. Let us define $T : \mathscr{A}(Y) \to m(Y)$ to be the inclusion map. Then if $T^* : m(Y)^* \to \mathscr{A}(Y)^*$ is the adjoint map of T, $T^*(K) = K \restriction_{\mathscr{A}(Y)}$, for each $K \in m(Y)^*$, and it can be easily checked that T^* carries means on $m(Y)$ to means on $\mathscr{A}(Y)$ and T^*M is a mean on $\mathscr{A}(Y)$. Furthermore, T^*M is S-invariant since M is S-invariant and T^*M is simply the restriction of M to $\mathscr{A}(Y)^*$. By Lemma 2.42 however, $T^*M = \delta_y$ for some $y \in Y$. Thus, for any $f \in \mathscr{A}(Y)$, and $s \in S$,

$$f(sy) = \delta_y(L_s f) = T^*M(L_s f) = T^*M(f) = f(y)$$

.

However, since $\mathscr{A}(Y) \supset Y^*$, we have that $\mathscr{A}(Y)$ separates points of Y and it must be that $sy = y$, for all $s \in S$. Thus, y is a fixed point of S.

On the other hand, if Y has a fixed point for S, by Proposition 2.10, (S, Y) is amenable. ∎

Chapter 3

Følner Conditions and the Følner Number

The Følner conditions are combinatorial properties that were initially introduced by Følner [7] for characterizing group amenability. Namioka [22] generalized these to semigroups, giving different types of Følner type conditions that act as sufficient and necessary conditions for semigroup amenability. We generalize some of Namioka's theorems and Wong's [39] concept of the Følner number to flows. However, just as Følner conditions get difficult to generalize from groups to semigroups, they get more difficult to generalize from semigroups to flows. we are unsure if some of the theorems of Namioka [22] and Yang [40] generalize to flows. Open questions regarding this are listed in chapter 6.

Sakai has covered Corollary 3.6, Theorem 3.8 and Corollary 3.9. We believe that our proof for Corollary 3.6 is a bit different from Sakai's, while the proofs for Theorem 3.8 and Corollary 3.9 are the same generalizations of Namioka's proof from [22].

Definition 3.1. We define the different Følner conditions for (S, X) as follows:

- (S, X) is said to satisfy the **Weak Følner condition (WFC)** if $\exists k$, $0 < k < 1$, such that for any $n \in \mathbb{N}$, $s_1, ..., s_n \in S$, there exists $A \subset X$ finite such that $n^{-1} \sum_{i=1}^{n} |A \backslash s_i A| \leq k|A|$.
- (S, X) is said to satisfy the **Strong Følner condition (SFC)** if for any $\varepsilon > 0$,and $F \subset S$ finite, there exists $A \subset X$ finite such that $|A \backslash sA| \leq \varepsilon |A|$ for each $s \in F$.
- (S, X) is said to satisfy the **Følner condition (FC)** if for any $\varepsilon > 0$, and $F \subset S$ finite, there exists $A \subset X$ finite such that $|sA \backslash A| \leq \varepsilon |A|$ for each $s \in F$.

- (S, X) is said to satisfy the **Weak Namioka-Følner condition (WNFC)** if $\exists k$, $0 < k < 1$, such that for any $s_1, ..., s_n, t_1, ..., t_n \in S$, there exists $A \subset X$ finite such that $n^{-1} \sum_{i=1}^{n} |t_i A \cap s_i A| \geq k|A|$.
- (S, X) is said to satisfy the **Strong Namioka-Følner condition (SNFC)** if $\exists k$, $0 < k < 1/2$, such that for any $s_1, ..., s_n \in S$, there exists $A \subset X$ finite such that $n^{-1} \sum_{i=1}^{n} |A \backslash s_i A| \leq k|A|$.

We say that S satisfies a particular left or right Følner condition, if (S, S) satisfies it with left or right multiplication respectively.

The following is the relation between the Følner conditions:

$$SFC \implies SNFC \implies WNFC \implies WFC$$

We also introduce the notion of a Følner net:

Definition 3.2. Suppose (S, X) is a semigroup flow and $\{F_\alpha\}_{\alpha \in A} \subset \mathscr{P}_f(x)$ is a net. Then we call $\{F_\alpha\}_{\alpha \in A}$ a **Følner net**, if for every $s \in S$,

$$\lim_{\alpha \in A} \frac{|sF_\alpha \Delta F_\alpha|}{|F_\alpha|} = 0$$

Proposition 3.3. (S, X) *has a Følner net if and only if it satisfies the SFC.*

Proof. Suppose (S, X) has Følner net $\{F_\alpha\}_{\alpha \in A} \subset \mathscr{P}_f(x)$. Let $F \in \mathscr{P}_f(S)$, and $\varepsilon > 0$ be given. Since

$$\lim_{\alpha \in A} \frac{|sF_\alpha \Delta F_\alpha|}{|F_\alpha|} = 0$$

for each $s \in S$, consider for each $s \in F$, $\alpha_s \in A$, that satisfy $\frac{|sF_\alpha \Delta F_\alpha|}{|F_\alpha|} < \varepsilon$ for all $\alpha \geq \alpha_s$. Then, take $\beta = \max \alpha_{s s \in F}$, where F_β satisfies $\frac{|sF_\beta \Delta F_\beta|}{|F_\beta|} < \varepsilon$, for all $s \in F$. SFC thus holds.

On the contrary, if (S, X) satisfies the SFC, we define a net $\{F_{(n,A)}\}_{\mathbb{N} \times \mathscr{P}_f(S)} \subset \mathscr{P}_f(X)$, where for each $(n, A) \in \mathbb{N} \times \mathscr{P}_f(S)$,$F_{(n,A)}$ satisfies

$$\frac{|F_{(n,A)} \backslash sF_{(n,A)}|}{|F_{(n,A)}|} < \frac{1}{2^n}$$

Note than the existence of $F_(n, A)$ for each $n \in \mathbb{N}$, $A \in \mathscr{P}_f(S)$ is guaranteed by the SFC.

Then, for any $F \in \mathscr{P}_f(S)$, $m \in \mathbb{N}$, if we take $A \supset F$, i.e. for any $(A,n) \geq (F,m)$

$$\begin{aligned}
\frac{|sF_{(n,A)}\Delta F_{(n,A)}|}{|F_{(n,A)}|} &= \frac{|(sF_{(n,A)}\backslash F_{(n,A)})\bigcup(F_{(n,A)}\backslash sF_{(n,A)})|}{|F_{(n,A)}|} \\
&\leq \frac{|sF_{(n,A)}\backslash F_{(n,A)}|}{|F_{(n,A)}|} + \frac{|F_{(n,A)}\backslash sF_{(n,A)}|}{|F_{(n,A)}|} \\
&= \frac{|sF_{(n,A)}| - |F_{(n,A)} \cap sF_{(n,A)}|}{|F_{(n,A)}|} + \frac{|F_{(n,A)}\backslash sF_{(n,A)}|}{|F_{(n,A)}|} \\
&\leq \frac{|F_{(n,A)}| - |F_{(n,A)} \cap sF_{(n,A)}|}{|F_{(n,A)}|} + \frac{|F_{(n,A)}\backslash sF_{(n,A)}|}{|F_{(n,A)}|} \\
&= 2\frac{|F_{(n,A)}\backslash sF_{(n,A)}|}{|F_{(n,A)}|} \\
&< \frac{1}{2^{n-1}}
\end{aligned}$$

Taking $n \in \mathbb{N}$ to be arbitrarily large, it follows that

$$\lim_{(n,A)\in\mathbb{N}\times\mathscr{P}_f(S)} \frac{|sF_{(n,A)}\backslash F_{(n,A)}|}{|F_{(n,A)}|} = 0$$

and that $\{F_{(n,A)}\}_{(n,A)\in\mathbb{N}\times\mathscr{P}_f(S)}$ is a Følner net in X. ∎

Definition 3.4. Let X be a non-empty set. Then, for each $A \in \mathscr{P}_f(X)$, we define $\mu_A = \frac{\chi_A}{|A|}$.

Theorem 3.5. *Let (S,X) be a flow. If $\mathcal{F} = \{F\}_{\alpha\in A} \subset \mathscr{P}_f(X)$ is a Følner net, every weak* limit point of $\{Q\mu_{F_\alpha}\}_{\alpha\in A}$ in $m(X)^*$ is a S-invariant mean.*

Proof. For each $f \in m(X)$, and $s \in S$, we have, for all $\alpha \in A$:

$$\begin{aligned}
\left|[Q\mu_{F_\alpha} \circ L_s](f)(x) - Q\mu_{F_\alpha}(f)(x)\right| &= \left|Q\mu_{F_\alpha}(L_s f)(x) - Q\mu_{F_\alpha}(f)(x)\right| \\
&= \left|\sum_{x\in X}\mu_{F_\alpha}(x)L_s f(x) - \sum_{x\in X}\mu_{F_\alpha}(x)f(x)\right| \\
&= \left|\sum_{x\in X}\mu_{F_\alpha}(x)f(sx) - \sum_{x\in X}\mu_{F_\alpha}(x)f(x)\right| \\
&= |F_\alpha|^{-1}\left|\sum_{x\in F_\alpha}(f(sx) - f(x))\right| \qquad (1) \\
&\leq 2|F_\alpha|^{-1}\big(|F_\alpha| - |F_\alpha \cap sF_\alpha|\big)\|f\|_\infty \\
&= 2|F_\alpha|^{-1}|F_\alpha\backslash sF_\alpha|\|f\|_\infty \leq 2\|f\|_\infty|F_\alpha\Delta sF_\alpha||F_\alpha|^{-1}
\end{aligned}$$

Here, at (1), we note that if $x \in F_\alpha \cap sF_\alpha$, then $x = sz$ for some $z \in F_\alpha$ and $f(sz)$ cancels out $-f(x)$. Thus a total of $2|F_\alpha \cap sF_\alpha|$ terms cancel out of $2|F_\alpha|$ terms.

Since $\mathcal{F}$ is a Følner net, $|F_\alpha \Delta sF_\alpha||F_\alpha|^{-1} \to 0$ in α, and hence, $\big|[Q\mu_{F_\alpha} \circ L_a](f)(x) - Q\mu_{F_\alpha}(f)(x)\big| \to 0$ in α, i.e. $\{Q\mu_\alpha\}_{\alpha \in A}$ w^*-converges to S-invariance. By Theorem 2.19, every weak* limit point of $\{Q\mu_{F_\alpha}\}_{\alpha \in A}$ is a S-invariant mean. ∎

Corollary 3.6. *If (S, X) satisfies the SFC, it is amenable.*

Proof. By Proposition 3.3, (S, X) has a Følner net $\mathcal{F} = \{F_\alpha\}_{\alpha \in A}$. Now $\{QF_\alpha\}_{\alpha \in A} \subset \mathscr{M}(X)$ has a weak* convergent subnet in $\mathscr{M}(X)$, since $\mathscr{M}(X)$ is weak* compact i.e. a w^*-limit point exists in $\mathscr{M}(X)$ for $\{QF_\alpha\}_{\alpha \in A} \subset \mathscr{M}(X)$ and the rest follows from Theorem 3.5. ∎

Lemma 3.7. *Any member f of $\Phi(X)$ can be written as $f = \sum_{i=1}^n \lambda_i |A_i|^{-1} \chi_{A_i}$ where: each A_i is non empty and finite and $A_i \supset A_{i+1}$, $\lambda_i > 0$ and $\sum_{i=1}^n \lambda_i = 1$. Furthermore, in this case, for each $s \in S$, $\|f \cdot s - f\|_1 \geq \sum_{i=1}^n \lambda_i \frac{|sA_i \setminus A_i|}{|A_i|}$*

Proof. Let f take on the distinct values $0 < a_1 < a_2 < ... < a_n$. Then, we define $A_i = \{x \in X \mid a_i \leq f(x)\}$, for $1 \leq i \leq n$. It is clear that $A_i \supset A_{i+1}$ and that

$$f = a_1 \chi_{A_1} + (a_2 - a_1)\chi_{A_2} + ... + (a_n - a_{n-1})\chi_{A_n} = \sum_{i=1}^n \lambda_i |A_i|^{-1} \chi_{A_i}$$

with and $\lambda_1 = a_1$, and $\lambda_i = (a_i - a_{i-1})|A_i|$, for all $2 \leq i \leq n$; and along with $\lambda_i > 0$ for all $1 \leq i \leq n$, we have

$$1 = \sum_{x \in X} f(x) = \sum_{i=1}^n \lambda_i |A_i|^{-1} \sum_{x \in X} \chi_{A_i}(x) = \sum_{i=1}^n \lambda_i$$

This concludes the proof to our first statement.

Now for any finite subset $A \in \mathscr{P}_f(X)$, and any $s \in S$, we have:

$$\mu_A \cdot s(x) = \sum_{y \in s^{-1}\{x\}} \mu_A(y) = |A|^{-1}|A \cap s^{-1}\{x\}| = \begin{cases} |A|^{-1}|A \cap s^{-1}\{x\}| & \text{if } x \in sA \\ 0 & \text{if } x \notin sA \end{cases}$$

and

$$\mu_A(x) = \begin{cases} |A|^{-1} & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Then, we have

$$(*) \qquad \mu_A \cdot s - \mu_A(x) = \begin{cases} |A|^{-1}|A \cap s^{-1}\{x\}| & \text{if } x \in sA \backslash A \\ -|A|^{-1} & \text{if } x \in A \backslash sA \\ |A|^{-1}(|A \cap s^{-1}\{x\}| - 1) & \text{if } x \in sA \cap A \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} \geq |A|^{-1} & \text{if } x \in sA \backslash A \\ < 0 & \text{if } x \in A \backslash sA \\ \geq 0 & \text{if } x \in sA \cap A \\ = 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} \geq |A|^{-1} & \text{if } x \in sA \backslash A \\ < 0 & \text{if } x \in A \backslash sA \\ \geq 0 & \text{otherwise} \end{cases}$$

Since

$$f \cdot s - f = \sum_{i=1}^{n} \lambda_i |A_i|^{-1} \left(\chi_{A_i} \cdot s - \chi_{A_i} \right) = \sum_{i=1}^{n} \lambda_i \left(\mu_{A_i} \cdot s - \mu_{A_i} \right)$$

we have

$$\begin{aligned} \|f \cdot s - f\|_1 &= \sum_{x \in X} \sum_{i=1}^{n} \lambda_i \left| \mu_{A_i} \cdot s(x) - \mu_{A_i}(x) \right| \\ &= \sum_{i=1}^{n} \lambda_i \sum_{x \in X} \left| \mu_{A_i} \cdot s(x) - \mu_{A_i}(x) \right| \\ &\geq \sum_{i=1}^{n} \lambda_i \sum_{x \in X \backslash A_i} \left| \mu_{A_i} \cdot s(x) - \mu_{A_i}(x) \right| \\ &\geq \sum_{i=1}^{n} \lambda_i |A_i|^{-1} |sA_i \backslash A_i| \end{aligned}$$

where we used $(*)$ at the last step. This completes the proof. ∎

Theorem 3.8. *If (S, X) is amenable, it satisfies the FC.*

Proof. Let $s_1, ..., s_n \in S$, and $\varepsilon > 0$ be given. By Theorem 2.21, there exists $\phi \in \Phi$, such that $\|\phi \cdot s - \phi\| < \varepsilon/n$, for all $s \in S$. Since $\phi \in \Phi$, by Lemma 3.7, $\phi = \sum_{n=1}^{k} \lambda_i |A_i|^{-1} \chi_{A_i}$, for some $\lambda_i \in \mathbb{R}$ and $A_i \subset X$ satisfying the properties in Lemma 3.7. We claim that $\exists i_0$, $1 \leq i_0 \leq k$, such that $|s_j A_{i_0} \backslash A_{i_0}| < \varepsilon |A_{i_0}|$, for all j with $1 \leq j \leq n$.

For each $1 \leq j \leq n$, define $K_j = \{1 \leq i \leq n \mid |s_j A_i \backslash A_i| < \varepsilon |A_i|\}$, so that for all $i \notin K_j$, we have $|s_j A_i \backslash A_i| \geq \varepsilon |A_i|$.

Then, by Lemma 3.7, we have for each s_j, $1 \leq j \leq n$,

$$\varepsilon/n > \|\phi \cdot s_j - \phi\| \geq \sum_{i=1}^{n} \lambda_i \frac{|s_j A_i \backslash A_i|}{|A_i|} \geq \sum_{i \in \{1,...,n\} \backslash K_j} \lambda_i \frac{|s_j A_i \backslash A_i|}{|A_i|} \geq | \sum_{i \in \{1,...,n\} \backslash K_j} \lambda_i \varepsilon$$

so that $\sum\limits_{i \in \{1,...,n\} \backslash K_j} \lambda_i < 1/n$

Define the weighted counting measure μ on $\mathscr{P}(\{1, ..., n\})$ given by

$$\mu(K) = \begin{cases} 0 & \text{if } K = \varnothing \\ \sum_{i \in K} \lambda_i \neq 0 & \text{otherwise} \end{cases}$$

for every $K \subset \{1, ..., n\}$.

Then,

$$\begin{aligned} 1 - \mu\left(\bigcap_{j=1}^{n} K_j\right) = \mu(\{1, ..., n\} \backslash \cap_{j=1}^{n} K_j) &= \mu\left(\bigcup_{j=1}^{n} \{1, ..., n\} \backslash K_j\right) \\ &\leq \sum_{j=1}^{n} \mu\left(\{1, ..., n\} \backslash K_j\right) \\ &= \sum_{i \in \{1,...,n\} \backslash K_j} \lambda_i < 1/n \end{aligned}$$

Thus, $\mu(\bigcap_{j=1}^{n} K_j) > 1 - 1/n > 0$, and we have that $\bigcap_{j=1}^{n} K_j \neq \varnothing$. Taking any $i_0 \in \bigcap_{j=1}^{n} K_j$, by the definition of K_j, for each $1 \leq j \leq n$, A_{i_0} satisfies the required properties. ∎

Corollary 3.9. *If (S, X) is S-cancellative, it is amenable if and only if it satisfies the SFC.*

We conclude with the following diagram, which has been adapted from [40]:

$$
\begin{tikzcd}
SFC \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dd, Leftrightarrow, bend right=60, "S\text{-cancellative}"'] & SNFC \arrow[r, Rightarrow] \arrow[d, Rightarrow] & WNFC \arrow[r, Rightarrow] & WFC \\
\text{Amenability} \arrow[d, Rightarrow] & \text{Amenability in special cases} & & \\
FC & & &
\end{tikzcd}
$$

Figure 3.1: Relations between Følner conditions and Amenability for a flow (S, X)

Definition 3.10. A subsemigroup T of a semigroup S is said to be **finitely generated** if there exists $F \in \mathscr{P}_f(S)$, such that for any $t \in T$, there exists $n \in \mathbb{N}$, $t_i \in F$ such that $t = t_1 \cdot t_2 \cdot ... \cdot t_n$.

Theorem 3.11. *Suppose (S, X) is an S-cancellative flow. Then, (S, X) is amenable if every subflow (T, Y) of (S, X) with a finitely generated semigroup T is amenable.*

Proof. Suppose every finitely generated subflow of (S, X) is amenable. Then, let $F \in \mathscr{P}_f(S)$, and consider the subsemigroup T_F generated by F. Since (T_F, X) is amenable, for any fixed $\varepsilon > 0$, there exists a finite subset A of X such that $|sA \backslash A| \leq \varepsilon|A|$, for all $s \in F$. Since (S, X) is S-cancellative, the FC and SFC are equivalent and (S, X) is amenable. ∎

Proposition 3.12. *Let (S, X) and (T, Y) be left flows that satisfy the SFC. Then, $(S \times T, X \times Y)$ also satisfies the SFC.*

Proof. Let $\mathcal{F} = \{F_\alpha\}_{\alpha \in A} \subset \mathscr{P}_f(X)$, $\mathcal{G} = \{G_\beta\}_{\beta \in B} \subset \mathscr{P}_f(Y)$ be Følner nets. Then, consider the product net $\{F_\alpha \times G_\beta\}_{(\alpha,\beta) \in A \times B}$. Let $(s, t) \in S \times T$, and $\varepsilon > 0$ be given.

Choose $\xi \in A$, such that for all $\alpha \geq \xi$, $\frac{|F_\alpha \Delta s F_\alpha|}{|F_\alpha|} < \sqrt{\frac{\varepsilon}{2}}$, and $\zeta \in B$ such that for all $\beta \geq \zeta$, $\frac{|G_\beta \Delta t G_\beta|}{|G_\beta|} < \sqrt{\frac{\varepsilon}{2}}$. Then, for any $(\alpha, \beta) \geq (\xi, \zeta)$,

$$
\begin{aligned}
\frac{|(F_\alpha \times G_\beta)\Delta(s,t)(F_\alpha \times G_\beta)|}{|F_\alpha \times G_\beta|} &= \frac{|(F_\alpha \times G_\beta)\backslash(s,t)(F_\alpha \times G_\beta)|}{|F_\alpha \times G_\beta|} + \frac{|(s,t)(F_\alpha \times G_\beta)\backslash(F_\alpha \times G_\beta)|}{|F_\alpha \times G_\beta|} \\
&= \frac{|F_\alpha\backslash sF_\alpha||G_\beta\backslash tG_\beta|}{|F_\alpha||G_\beta|} + \frac{|sF_\alpha\backslash F_\alpha||tG_\beta\backslash G_\beta|}{|F_\alpha||G_\beta|} \\
&\leq 2\frac{|F_\alpha\backslash sF_\alpha \bigcup F_\alpha\backslash sF_\alpha|\,|G_\beta\backslash tG_\beta \bigcup tG_\beta\backslash G_\beta|}{|F_\alpha||G_\beta|} \\
&\leq 2\frac{|F_\alpha\Delta sF_\alpha||G_\beta\Delta tG_\beta|}{|F_\alpha||G_\beta|} \\
&= 2\frac{|F_\alpha\Delta sF_\alpha|}{|F_\alpha|}\frac{|G_\beta\Delta tG_\beta|}{|G_\beta|} \\
&< \sqrt{\frac{\varepsilon}{2}}\sqrt{\frac{\varepsilon}{2}} = \varepsilon
\end{aligned}
$$

It follows that for each $(s,t) \in S \times T$, $\lim\limits_{(\alpha,\beta)\in A\times B} \frac{|(F_\alpha\times G_\beta)\Delta(s,t)(F_\alpha\times G_\beta)|}{|F_\alpha\times G_\beta|} = 0$ and $\{F_\alpha \times G_\beta\}_{(\alpha,\beta)\in A\times B} \subset \mathscr{P}_f(X \times Y)$ is a Følner net. It follows from Proposition 3.3, $(S \times T, X \times Y)$ satisfies the SFC. ∎

Definition 3.13. We say that (S, X) has the property P_α for $\alpha > 0$, if for any finite subset $\{s_1, ..., s_n\} \subset S$, we can find a finite set $E \subset X$, such that $\frac{1}{n}\sum_{i=1}^n |E\backslash s_iE| \leq \alpha|E|$.

Definition 3.14. We define the **Følner number** of (S, X) to be given by

$$F(S,X) = \inf\{\alpha \geq 0 \mid S \text{ has the property } P_\alpha\}$$

Proposition 3.15.

$$F(S,X) = \sup_{K\in\mathscr{P}_f(S)} \inf_{E\in\mathscr{P}_f(X)} \frac{1}{|K|}\sum_{s\in K}\frac{|E\backslash sE|}{|E|}$$

Proof. Suppose (S, X) has the property P_α for some $\alpha > F(S,X)$. Then, for any $F \in \mathscr{P}_f(S)$, there exists some $E \in \mathscr{P}_f(X)$ such that $\frac{1}{|F|}\sum_{s\in F}|E\backslash sE| \leq \alpha|E|$. It follows that for any $F \in \mathscr{P}_f(S)$,

$$\inf_{E\in\mathscr{P}_f(X)} \frac{1}{|F|}\sum_{s\in K}\frac{|E\backslash sE|}{|E|} \leq \alpha|E|$$

and hence $G(S,X) \geq \alpha$. Taking $\alpha \to F(S,X)$, $G(S,X) \geq F(S,X)$.

Now suppose $G(S,X) > F(S,X)$. Then, consider any $\alpha \in \mathbb{R}$ such that $G(S,X) < \alpha < F(S,X)$. Then, for any $F \in \mathscr{P}_f(S)$,

$$\inf_{E\in\mathscr{P}_f(X)} \frac{1}{|F|}\sum_{s\in K}\frac{|E\backslash sE|}{|E|} \leq \alpha$$

which means that $F(S,X)$ satisfies property P_α. But this gives us $F(S,X) \leq \alpha$, which is a contradiction. It follows that $G(S,X) = F(S,X)$. ∎

Theorem 3.16. *Suppose (S,X) is a flow. Then,*

1. $F(S,X) = 0 \implies (S,X)$ *is amenable*
2. (S,X) *is amenable and S-cancellative* $\implies F(S,X) = 0$

Proof. 1. Let $\varepsilon > 0$ be arbitrary. Then, for any finite number of elements $s_1, ..., s_n \in S$, since $F(S,X) = 0 \leq \frac{\varepsilon}{n}$, there exists $E \subset X$ finite such that $\frac{1}{n}\sum_{i=1}^n |E\backslash s_iE| \leq \frac{\varepsilon}{n}$. This implies that $\sum_{i=1}^n |E\backslash s_iE| \leq \varepsilon$ which gives us $|E\backslash s_iE| < \varepsilon$, for all $1 \leq i \leq n$. Hence, (S,X) satisfies the SFC and by Corollary 3.6, it is amenable.

2. This follows from the fact that the FC and SFC are equivalent if (S,X) is S-cancellative and by Theorem 3.8. ∎

Proposition 3.17. *Suppose (S,X) has n pairwise disjoint cosets $s_1X, ..., s_nX$. Then $F(S,X) \geq 1 - \frac{1}{n}$.*

Proof. Suppose E is a finite subset of X. Then, $s_1E, ..., s_nE$ are pairwise disjoint give at $(*)$,

$$\frac{1}{n}\sum_{i=1}^{n}|E\backslash s_iE|\frac{1}{n}\sum_{i=1}^{n}|E|-|E\cap s_iE| = \frac{1}{n}(\sum_{i=1}^{n}|E|-\sum_{i=1}^{n}|E\cap s_iE| \geq^{(*)} \frac{1}{n}(\sum_{i=1}^{n}|E|-|E|) = \left(1-\frac{1}{n}\right)|E|$$

It follows by Proposition 3.15, that $F(S,X) \geq 1 - \frac{1}{n}$. ∎

In Theorem 2.26, amenability is preserved by a homomorphism of flows. We may want to ask if, similarly, a homomorphism of flows preserves the Følner number of a flow. Yang [40] showed that this is not the case. He showed that there exists a semigroup S with $F(S,S) = 0$ and a surjective semigroup homomorphism ϕ from S to a semigroup T, where $F(T,T) = 1$. Note that since ϕ is a semigroup homomorphism, the map $T_\phi : (S,S) \to (T,T)$ given by $T_\phi(s) = \phi(s)$ for each $s \in S$, is a homomorphism a flows. We can conclude that the SFC is not in general preserved by a homomorphism of flows.

Chapter 4

The Stone-Čech Compactification and Density of Means

We begin with some preliminaries on the theory of ultrafilters. We will skip the proofs as these can be found on any textbook on ultrafilters - one we recommend is - [15]. We however, used [18] to study the structure of ultrafilters in the setting of semigroups. In the second section, we discuss the flow structure of the Stone-Čech compactification of a flow (S, X) and how this is a flow, analogous to βS being a semigroup for a semigroup S. In the second section, we discuss density of means for flows. Hindman and Strauss [13][14] generalized the existing concept of upper and lower asymptotic densities on $\mathbb{N}$ to semigroups, showing interesting properties that exist for densities defined using Følner nets. We generalize the results of Hindman and Strauss to flows.

4.1 Some Preliminaries

Let X be a non-empty set.

Definition 4.1. A **filter on** X is a subset w of $\mathscr{P}(X)$ that satisfies the following:

- $S \in w$ and $\varnothing \notin w$
- If $A \in \mathcal{F}$, and $A \subset B \subset X$, then $B \in w$
- For any $A, B \in w$, $A \cap B \in w$

Example 4.2. Examples of filters include:

- For any topological space X, and $x \in X$, the set of all neighbourhoods of x forms a filter on X called the **neighbourhood filter of** x.

- In general, for any non-empty set $A \subset X$, we can define the filter, $\mathcal{F}_A$ to be the filter that contains all supersets of A, i.e. $w_A = \{B \subset S \mid A \subset B\}$. This is called the **principal filter generated by** A. For singletons $x \in X$, let us denote the principal filter generated by $\{x\}$ by $\widehat{x}$.

- Lastly, for any non-empty set, define $\mathcal{F}$ to be the collection of all co-finite subsets of X. This is called the **Fréchet filter on** X.

Remark 4.3. A family $\mathscr{A} \subset \mathscr{P}(X)$ of subsets for $X \neq \varnothing$ can be extended to a filter if and only if it has the finite intersection property. Clearly, this can be done by considering supersets of all sets in $\mathscr{A}$.

Definition 4.4. A filter w on X is said to be an **ultrafilter** if it is maximal, i.e. if $w \subset v$, for some filter v on X, $v = w$.

By an application of Zorn's lemma, one can easily see that:

Proposition 4.5. *Every filter on X is contained in some ultrafilter.*

Definition 4.6. A filter w on X is said to be a **prime filter** if $A \cup B \in w$ implies $A \in w$ or $B \in w$, for each $A, B \in \mathscr{P}(X)$.

Theorem 4.7. *The following are equivalent for a filter w on X.*

1. *w is an ultrafilter*

2. *For each $A \in \mathscr{P}(X)$, either $A \in w$ or $S \backslash A \in w$*

3. *w is prime*

Example 4.8.
- For each $x \in X$, the principal filter $\widehat{x}$ is an ultrafilter. In fact, this ultrafilter is the unique ultrafilter containing $\{x\}$. Furthermore, *a principal filter is an ultrafilter if and only if it is generated by a singleton.*

- For a topological space X, if $x \in X$, the neighbourhood filter of x is an ultrafilter if and only if $\{x\}$ is open, i.e. x is an isolated point of X.
- The Fréchet filter on a space X is an ultrafilter if and only if X is finite.

Definition 4.9. A filter w on X is said to be **fixed** if $\bigcap w \neq \varnothing$, and **free** otherwise.

Note that a filter is fixed if and only if it is a principal filter generated by some set:

Proposition 4.10. *An ultrafilter w on X is fixed if and only if it contains a finite subset of X, or equivalently if and only if, for some $x \in X$, $\widehat{x} \subset w$.*

By default, this means that every free ultrafilter on X contains only infinite sets, and if X is finite, every ultrafilter on X is fixed. Furthermore by 2 of Theorem 4.7, this means that every free ultrafilter on X contains the Fréchet filter on X.

Definition 4.11. Let Y be a topological space and X be a non-empty set. Consider a sequence $\{y_x\}_{x \in X} \subset Y$, and an ultrafilter p on X. Then, $y \in Y$ is said to be a w**-limit** of $\{y_x\}_{x \in X}$ if and only if, for each neighbourhood U of y, there exists $A \in w$, such that $\{y_x \mid x \in A\} \subset U$, or equivalently, $\{x \in X \mid y_x \in U\} \in w$.

Observe that if $v \supset w$ is a filter, then, every w-limit is a v-limit.

Example 4.12.

1. If we consider the principal ultrafilter w_{x_0}, for $x_0 \in X$, then w_{x_0}-limit of $\{y_x\}_{x \in X}$ is y_{x_0}.
2. For a metric space Y, consider a sequence $\{y_n\}_{n \in \mathbb{N}}$. Then, if w is the Fréchet filter on Y, a w-limit of $\{y_n\}_{n \in \mathbb{N}}$ is just the usual limit of $\{y_n\}_{n \in \mathbb{N}}$ in the metric topology.

Like usual limits in topological spaces, for an arbitrary topological space, the limit in Definition 4.11 need not exist or be unique if it does. To this end, we have:

Theorem 4.13. *Suppose Y is a topological space and $\{y_x\}_{x \in X} \subset Y$ is a sequence and w is a filter on X.*

1. *If Y is Hausdorff, and a w-limit of $\{y_x\}_{x \in X}$ exists, it is unique.*
2. *If $A \in w$, every w-limit of $\{y_x\}_{x \in X}$ is in $cl\{y_x \mid x \in A\}$. If w is an ultrafilter, every point in $\bigcap_{A \in w} cl\{y_x \mid x \in A\}$ is a w-limit of $\{y_x\}_{x \in X}$.*

3. *If X is compact and w is an ultrafilter, a w-limit always exists for $\{y_x\}_{x\in X}$.*

In some situations, continuity preserves ultrafilter limits like in the usual case:

Theorem 4.14. *Suppose Y and Z are compact hausdorff topological spaces, and w is an ultrafilter on a non-empty set X. Then, if $f : Y \to Z$ is a continuous function and $\{y_x\}_{x\in X} \subset Y$ is a sequence,*

$$w - \lim_{x\in X} f(y_x) = f(w - \lim_{x\in X} y_x)$$

.

Since we assume all topologies are Hausdorff (unless indicated otherwise), we may assume ultrafilter limits are unique.

Now let us denote βX to be the set of all ultrafilters on X.

Definition 4.15. For each $A \in \mathscr{P}(X)$, we define $\widehat{A} = \{\mathcal{F} \in \beta X \mid A \in F\}$. $\widehat{A}$ is known as the **Stone set corresponding to** A. Note that $\widehat{x}$ is the unique ultrafilter in $\widehat{\{x\}}$ (from Example 4.2), and hence, we will shorten and refer to $\{\widehat{x}\}$ by $\widehat{x}$, with context making connotation clear. Note that $\widehat{S} = \beta X$ and $\widehat{\varnothing} = \varnothing$.

It is easy to make the following observation using Theorem 4.7:

Proposition 4.16. *For any $A, B \in \mathscr{P}(X)$, the following hold,*

1. $\widehat{A \cap B} = \widehat{A} \cap \widehat{B}$
2. $\widehat{A \cup B} = \widehat{A} \cup \widehat{B}$
3. $\widehat{X \backslash A} = \beta X \backslash \widehat{A}$

Then, we define the following:

Definition 4.17. We define the **Stone topology** on βX to be the topology generated by $\mathcal{B} = \{\widehat{A} \mid A \in \mathscr{P}(X)\}$ as a base of open sets. $\mathcal{B}$ is called the **Stone base of** βX.

It follows that a subset of βX is open if and only if it is a union of a family of stone sets $\widehat{A_i}$ for some $A_i \in \mathscr{P}(X)$, $i \in I$. Furthermore, note that by 3 of Proposition 4.16, for any $A \in \mathscr{P}(X)$, $\widehat{A}$ is closed, and that $\mathcal{B}$ is a family of clopen sets in the Stone topology.

Definition 4.18. A topological space is called a **Boolean space** if it is Hausdorff, compact and has a base of clopen sets.

Definition 4.19. Let us define the canonical embedding map:

$$\begin{aligned} \phi : X &\to \beta X \\ x &\mapsto \widehat{x} \end{aligned}$$

If X is given the discrete topology, ϕ is a continuous injective map, and hence, X is embedded in βX. If X is finite, by Proposition 4.10, ϕ is a bijection.

From now on, we will identify X as a subset of βX and denote for each $x \in X$, $\widehat{x} \in \beta X$ by $x \in \beta X$. We may use the hat notation for enhancing clarity when using $\widehat{x}$ as a set.

Definition 4.20. For $X \neq \varnothing$, βX along with the Stone topology is known as the **Stone-Čech compactification of** X.

This is due to the following theorems:

Theorem 4.21. *βX with the Stone topology is a Boolean space. Moreover, $\phi(X)$ is dense in βX in the Stone topology.*

The Stone-Cech compactification satisfies an important universal property:

Theorem 4.22 (**Universal property of** βX)**.** *Suppose Y is a compact Hausdorff space and $f : X \to Y$ is any arbitrary function. Then, there exists a unique function $\tilde{f} : \beta X \to Y$ that is continuous with respect to the stone topology on βX and satisfies $\tilde{f} \circ e = f$, given pointwise by $\tilde{f}(w) = w - \lim_{x \in X} f(x)$*

In other words, if we consider X to be a topological subspace of βX, $\tilde{f}$ is the extension of f to βX. We call $\tilde{f}$ the **Stone-Čech extension of** f.

Theorem 4.23. *The following is true for any set $X \neq \varnothing$:*

1. *For each $A \subset X$, $cl(\phi(A)) = \widehat{A}$.*
2. *$\widehat{A}$ is topologically isomorphic to βA*

Recall the following:

Definition 4.24. A family $\mathcal{F}$ of subsets of X is said to be **partition regular**, if for any finite partition $\{U_i\}_{1\leq i\leq n}$ of X, there exists some $1\leq j\leq n$, $F\in\mathcal{F}$ such that $F\subset U_j$.

Theorem 4.25. *Let X be a non-empty set. Suppose $\mathscr{C}\subset\mathscr{P}(X)$, is nonempty and $\varnothing\notin\mathscr{C}$. Then, if we take $\mathscr{B}=\{Y\in\mathscr{P}(X)\mid Y\subset Z,\ \text{for some } Z\in\mathscr{C}\}$, the following statements are equivalent:*

1. *$\mathscr{C}$ is partition regular.*
2. *If $\mathscr{A}\subset\mathscr{P}(X)$ has the property: for any $n\in\mathbb{N}$ and $\mathscr{A}_i\in\mathscr{A}$, $1\leq i\leq n, \cap_{i=1}^n\mathscr{A}_i\in\mathscr{B}$; then, there is an ultrafilter w on X, such that $\mathscr{A}\subset w\subset\mathscr{C}$.*
3. *Whenever $A\in\mathscr{C}$, there exists an ultrafilter w on X, such that $A\in w\subset\mathscr{C}$.*

4.2 The Stone-Čech compactification of a Flow

Let (S,X) be a discrete flow. It is well known that βS is a right topological semigroup with the multiplication defined by: if $p,q\in\beta S$, $A\subset S$, $A\in pq$ if and only if $\{s\in S\mid\{s^{-1}A\in q\}\}\in p$. For each $s\in S$, let us define the map

$$\begin{aligned}\lambda_s:\beta S&\to\beta S\\ p&\mapsto sp\end{aligned}$$

and for each $q\in\beta S$, let us define the map

$$\begin{aligned}\gamma_q:\beta S&\to\beta S\\ p&\mapsto pq\end{aligned}$$

We recall that these maps are continuous with respect to the Stone topology on βS.

Like the semigroup structure on βS, we wish to define a flow structure for $(\beta S,\beta X)$. Let us denote the stone topology on βS and βX by τ_S and τ_X respectively.

For each $s\in S$, let us define

$$\begin{aligned}L_s:X&\to\beta X\\ x&\mapsto sx\end{aligned}$$

Since $(\beta X, \tau_X)$ is a compact Hausdorff space, by the universal property Theorem 4.22, there exists a unique continuous map $\tilde{L}_s : (\beta X, \tau_X) \to (\beta X, \tau_X)$ that extends L_s.

Then, we can define, for each $w \in \beta X$, the map

$$\begin{aligned} R_w : S &\to \beta X \\ s &\mapsto \tilde{L}_s(w) \end{aligned}$$

Again, by the universal property Theorem 4.22, there exists a unique continuous extension $\tilde{R}_w : (\beta S, \tau_S) \to (\beta X, \tau_X)$.

We can thus define $pw = \tilde{R}_w(p)$, for each $p \in \beta S$, $w \in \beta X$. However, is $(\beta S, \beta X)$ a flow with this definition?

Theorem 4.26. *$(\beta S, \beta X)$ is a flow with the action map $(p, w) \mapsto \tilde{R}_w(p)$, for each $p \in \beta S$, $w \in \beta X$. Furthermore, this action map is continuous in the first variable.*

Proof. We need to show that the map defines an action, i.e. associativity. First, note that for any $s, t \in S \subset \beta S$, $x \in X \subset \beta X$, $st(x) = (st)x$ in βX.

For any fixed $s, t \in S$, the functions

$$\begin{aligned} \beta X &\to \beta X \\ w &\mapsto (st)w = \tilde{L}_{st}(w) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \beta X &\to \beta X \\ w &\mapsto s(tw) = [\tilde{L}_s \circ \tilde{L}_t](w) \end{aligned}$$

are both continuous and coincide on X. Since X is dense in βX, it follows that these functions also coincide on βX. Hence, for each $s, t \in S$, $w \in \beta X$, $(st)w = s(tw)$, i.e. $[\tilde{R}_w \circ \lambda_s](t) = [\lambda_s \circ \tilde{R}_w](t)$.

Now suppose $s \in S$, and $w \in \beta X$ are fixed. Consider the continuous maps

$$\begin{aligned} \beta S &\to \beta X \\ p &\mapsto (sp)w = [\tilde{R}_w \circ \lambda_s](p) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \beta S &\to \beta X \\ p &\mapsto s(pw) = [\lambda_s \circ \tilde{R}_w](p) \end{aligned}$$

Again, these maps coincide on S, and by the density of S in βS, it follows that they coincide on βS. It follows that, for each $p \in \beta S$, $w \in \beta X$, $(sp)w = s(pw)$, or $[\tilde{R}_w \circ \gamma_p](s) = \tilde{R}_{pw}(s)$.

For the last step, let us fix $q \in \beta S$, $w \in \beta X$. Consider the following continuous maps:

$$\begin{array}{lcl} \beta S \to \beta X & & \beta S \to \beta X \\ \quad p \mapsto (pq)w = [\tilde{R}_w \circ \gamma_q](p) & \text{and} & \quad p \mapsto p(qw) = \tilde{R}_{qw}(p) \end{array}$$

As goes the pattern, these functions coincide on S, and as S is dense in βS, must coincide on βS.

It follows that for any $p, q \in \beta S$, $w \in \beta X$, $(pq)w = p(qw)$, and $(\beta S, \beta X)$ is a flow.

Continuity in the first variable of the action map follows from the definition of $\tilde{R}_w$ for each $w \in \beta X$ and the discussion above. ∎

It is a natural question to ask what the sets in pw, for $p \in \beta S$, $w \in \beta X$ look like. To this end, we introduce the following:

Corollary 4.27. *Let (S, X) be a flow. For any $s \in S$, $p \in \beta S$, $w \in \beta X$, the following hold.*

- $\widehat{s}w = w - \lim\limits_{x \in X} \widehat{sx}$
- $pw = p - \lim\limits_{s \in S}(w - \lim\limits_{x \in X} \widehat{sx})$

Proof. The proof follows directly from the definition of the flow $(\beta S, \beta X)$ and Theorem 4.22. ∎

Furthermore, we have the following:

Theorem 4.28. *Let (S, X) be a flow. For any $x \in X$, $p \in \beta S$, $w \in \beta X$, $A \subset X$, the following hold:*

1. $A \in \widehat{sx} \iff sx \in A \iff x \in s^{-1}A \iff s^{-1}A \in \widehat{x} \iff \{x\} \in s^{-1}A$
2. $A \in \widehat{s}w \iff s^{-1}A = \{x \in X \mid sx \in A\} \in w$
3. $A \in pw \iff \Big\{ s \in S \,\Big|\, \{x \in X \mid sx \in A\} \in w \Big\} \in p$

Proof. 1. This simply follows from the definition of $\widehat{sx}$ in βX.

2. By Corollary 4.27 $A \in \widehat{s}w$ if and only if $A \in w - \lim_{x \in X} \widehat{sx}$.
   Suppose $s^{-1}A = \{x \in X \mid sx \in A\} \in w$. Then, by Theorem 4.13, Theorem 4.23, $\widehat{s}w \in cl\{\widehat{sx} \mid x \in s^{-1}A\} = cl\{sX \cap A\} = \widehat{sX \cap A}$, i.e $sX \cap A \in \widehat{s}w$, and since $sX \cap A \subset A$, $A \in \widehat{s}w$. On the other hand, if $A \in \widehat{s}w$, $A \in w - \lim_{x \in X} \widehat{sx}$. Then, since $\widehat{A}$ is a neighbourhood of $\widehat{s}w$, $\{x \in X \mid sx \in A\} \in w$ by definition of the limit.

3. By Corollary 4.27, $A \in pw$ if and only if $A \in p - \lim\limits_{p \in S}(w - \lim\limits_{x \in X} \widehat{sx})$. Suppose $A \in pw$. Then, since $\widehat{A}$ is a neighbourhood of pw, by Theorem 4.13, $\{s \in S \mid w - \lim\limits_{x \in X} \widehat{sx} \in \widehat{A}\} \in p$. However by 2, for any $s \in S$, $w - \lim\limits_{x \in X} \widehat{sx} \in \widehat{A}$ if and only if $\{x \in X \mid sx \in A\} \in w$. It follows that $\left\{s \in S \,\middle|\, \{x \in X \mid sx \in A\} \in w\right\} \in p$. ∎

Proposition 4.29. *Let (S, X), (T, Y) be flows. If $T_\phi : X \to Y$ is a homomorphism of flows, then its Stone-Čech extension $\tilde{T}_{\tilde{\phi}} : \beta X \to \beta Y$ is also a homomorphism of flows.*

Proof. We first observe that for the semigroup homomorphism $\phi : S \to \beta T$ can be extended continuously to the semigroup homomorphism $\tilde{\phi} : \beta S \to \beta T$, given by $\tilde{\phi}(p) = p - \lim\limits_{s \in S} \phi(s)$, for each $p \in \beta S$, by Theorem 4.22.

Again, by Theorem 4.22, we can extend $T_\phi : X \to \beta Y$ to βX by considering for each $w \in \beta X$, $\tilde{T}_{\tilde{\phi}}(w) = w - \lim\limits_{x \in X} T_\phi(x)$. Thus, for any $p \in \beta S$, $w \in \beta X$, by Corollary 4.27, we have the iterated limits:

$$\begin{aligned}
\tilde{T}_{\tilde{\phi}}(pw) &= pw - \lim_{y \in X} T_\phi(y) \\
&= p - \lim_{s \in S} \Big[w - \lim_{x \in X} \big(sx - \lim_{y \in X} T_\phi(y) \big) \Big] \\
&= p - \lim_{s \in S} \Big[w - \lim_{x \in X} T_\phi(sx) \Big] \\
&= p - \lim_{s \in S} \Big[w - \lim_{x \in X} \big\{ \phi(s) T_\phi(x) \big\} \Big] \\
&= p - \lim_{s \in S} \Big[w - \lim_{x \in X} \big\{ \tilde{L}_{\phi(s)}(T_\phi(x)) \big\} \Big] \\
&= p - \lim_{s \in S} \Big[\tilde{L}_{\phi(s)} \big(w - \lim_{s \in S} T_\phi(x) \big) \Big] \\
&= p - \lim_{s \in S} \Big[\tilde{L}_{\phi(s)} \big(\tilde{T}_{\tilde{\phi}}(w) \big) \Big] \\
&= p - \lim_{s \in S} \Big[\tilde{R}_{\tilde{T}_{\tilde{\phi}}(w)} \big(\phi(s) \big) \Big] \\
&= \tilde{R}_{\tilde{T}_{\tilde{\phi}}(w)} \big(p - \lim_{s \in S} \phi(s) \big) \\
&= \tilde{R}_{\tilde{T}_{\tilde{\phi}}(w)} \big(\tilde{\phi}(p) \big) \\
&= \tilde{\phi}(p) \tilde{T}_{\tilde{\phi}}(w)
\end{aligned}$$

where we used the continuity theorem, Theorem 4.14.

Thus, $\tilde{T}_{\tilde{\phi}}$ is indeed a homomorphism of flows. ∎

4.3 Density of Means

The concept of density for semigroups was inspired by the existing concept for the natural numbers, $\mathbb{N}$.

Definition 4.30. The two notions of density on $\mathbb{N}$ are defined as follows, for each $A \subset N$:

1. **Upper asymptotic density**: $\overline{d}(A) = \limsup\limits_{n \to \infty} \frac{|A \cap \{1,\ldots,n\}|}{n}$
2. **Lower asymptotic density**: $\underline{d}(A) = \liminf\limits_{n \to \infty} \frac{|A \cap \{1,\ldots,n\}|}{n}$

The upper asymptotic density has many nice properties. It is partition regular, i.e. for each $A, B \subset \mathbb{N}$, $\overline{d}(A \cup B) > 0$ implies $\overline{d}(A) > 0$ or $\overline{d}(B) > 0$. It is also translation invariant, where for each $n \in \mathbb{N}$, $A \subset \mathbb{N}$, $\overline{d}(n + A) = \overline{d}(A) = \overline{d}(-n + A)$. Lastly, it is additive for translations of sets, i.e. for each $m, n \in \mathbb{N}$, and $A \subset \mathbb{N}$, $\overline{d}(m + A \cup n + A) = \overline{d}(m + A) + \overline{d}(n + A)$. The lower asymptotic density does not in general such nice properties. However, under certain conditions, there exist sets whose upper and lower asymptotic densities are equal.

One might question as to why density is a useful concept. Consider the intrinsic sizes of $\mathbb{N}$ versus $A = \{n^2 \mid n \in \mathbb{R}\}$. Both these sets have the same cardinality, however, this is a bit counterintuitive, considering how consecutive elements of $\mathbb{N}$ are evenly spaced apart, while the consecutive elements of A get sparser and sparser. As such, one might want a notion of size that distinguishes between the sizes of these two sets. Density is one such example. Indeed, $\overline{d}(\mathbb{N}) = 1$ while $\overline{d}(A) = 0$.

Hindman and Strauss generalized these concepts of density to a general semigroup S in [13]. However, unlike the $\mathbb{N}$ case, these use nets in their definition, i.e. the upper and lower density is defined with respect to a specific net (or a sequence if S is countable). Not all nets give nice properties for their corresponding upper density. This is where the SFC comes in. It turns out that Følner nets in particular give the nice properties that we have in the case of $\mathbb{N}$. In fact, seeing as $\mathbb{N}$ is amenable, for each $m \in \mathbb{N}$, if we take $n \geq m$, we have:

$$\begin{aligned}
\frac{|\{1,...,n\} \backslash (m + \{1,...,n\})|}{|\{1,...,n\}|} &= \frac{|\{1,...,n\} \backslash \{1+m,...,n+m\})|}{n} \\
&= \frac{n-(n-m)}{n} \\
&= \frac{m}{n} \qquad \to 0 \text{ as } n \to \infty
\end{aligned}$$

In other words, $\{1, ..., n\}_{n\in\mathbb{N}}$ is a Følner net and the "niceness" of the upper asymptotic density is not unfounded.

There is a third notion of density, which we will not be dealing with, as it does not make sense in our case (unless our flow (S, X) is both a left flow and a right flow in a nice associative manner). To learn about this density, see [13].

Let X be a non-empty set.

Definition 4.31. Let $\mathcal{F} = \{F_\alpha\}_{\alpha\in A} \subset \mathscr{P}_f(X)$ be a net. We define the notions of **upper** and **lower density corresponding to** $\mathcal{F}$, for each $A \in \mathscr{P}(X)$ as follows:

1. $\underline{d}_{\mathcal{F}}(Y) = \sup\left\{\lambda \geq 0 \middle| \text{ There exists } \alpha \in A, \text{ such that, for all } \beta \geq \alpha, |Y \cap F_\beta| \geq \lambda|F_\beta|\right\}$
2. $\overline{d}_{\mathcal{F}}(Y) = \sup\left\{\lambda \geq 0 \middle| \text{ For each } \alpha \in A, \text{ there exists } \beta \geq \alpha \text{ such that } |Y \cap F_\beta| \geq \lambda|F_\beta|\right\}$

Observe that $\overline{d}_{\mathcal{F}}(A) \geq \underline{d}_{\mathcal{F}}(A)$ for every $A \subset X$.

Definition 4.32. Given a flow (S, X) and a net $\mathcal{F} \subset \mathscr{P}_f(X)$,

$$\mathcal{D}_{\mathcal{F}}(X) = \{w \in \beta X \mid \text{ For each } C \in w, \overline{d}_{\mathcal{F}}(C) > 0\}$$

We will shorten this to $\mathcal{D}_{\mathcal{F}}$ for the purpose of this paper. Let us also define $X^+ = \{B \in \mathscr{P}(X) \mid \overline{d}_{\mathcal{F}}(B) > 0\}$.

Lemma 4.33. *Let (S, X) be a flow and $\mathcal{F} \subset \mathscr{P}_f(X)$ be a net. Then, for every $Y \subset X$, we have:*

$$\overline{d}_{\mathcal{F}}(Y) = \inf_{\alpha\in A} \sup_{\beta\geq\alpha} \frac{|F_\beta \cap Y|}{|F_\beta|}$$

Proof. Let us label for every $Y \subset X$, $N(Y) = \inf\limits_{\alpha\in A} \sup\limits_{\beta\geq\alpha} \frac{|F_\beta\cap Y|}{|F_\beta|}$.

Now suppose $\overline{d}_{\mathcal{F}}(Y) > \lambda$. Then, given $\alpha \in A$, there exists $\beta \geq \alpha$ such that $\frac{|Y\cap F_\beta|}{|F_\beta|} \geq \lambda$. It follows that $\sup\limits_{\beta\geq\alpha} \frac{|F_\beta\cap Y|}{|F_\beta|} \geq \lambda$, and since $\alpha \in A$ was arbitrary, $N(Y) \geq \lambda$. Sending λ to $\overline{d}_{\mathcal{F}}(Y)$, we have that $N(Y) \geq \lambda$.

On the other hand, if $N(Y) > \lambda$. Then, for every $\alpha \in A$, there exists some $\beta \geq \alpha$ such that $\frac{|F_\beta \cap Y|}{|F_\beta|} \geq \lambda$. Thus, $\overline{d}_{\mathcal{F}}(Y) \geq \lambda$. Sending λ to N(Y), we have that $\overline{d}_{\mathcal{F}}(Y) \geq N(Y)$.

Thus, $N(Y) = \overline{d}_{\mathcal{F}}(Y)$. ∎

Proposition 4.34. *Let (S, X) be a flow and $\mathcal{F} \subset \mathscr{P}_f(X)$ be a net. Then, for any $B, C \in \mathscr{P}(X)$, $\overline{d}_{\mathcal{F}}(B \cup C) \leq \overline{d}_{\mathcal{F}}(B) + \overline{d}_{\mathcal{F}}(C)$. It follows that the set $X^+ = \{B \in \mathscr{P}(X) \mid \overline{d}_{\mathcal{F}}(B) > 0\}$ is partition regular, and for any $B \in X^+$, $\mathcal{D}_{\mathcal{F}} \cap cl(B) \neq \varnothing$.*

Proof. Suppose $B, C \subset X$. Note that if $\overline{d}_{\mathcal{F}}(C) + \overline{d}_{\mathcal{F}}(B) = 0$, the inequality is trivial, since by Lemma 4.33, there is some $\alpha, \xi \in A$ such that for any $\beta \geq \alpha, \xi$,

$$0 = \frac{|F_\beta \cap B| + |F_\beta \cap C|}{|F_\beta|} \geq \frac{|F_\beta \cap (B \cup C)|}{|F_\beta|}$$

and we have $\overline{d}_{\mathcal{F}}(B \cup C) = 0$.

So taking $\overline{d}_{\mathcal{F}}(C) + \overline{d}_{\mathcal{F}}(B) > 0$, assume the converse, i.e. suppose $\overline{d}_{\mathcal{F}}(B \cup C) > \overline{d}_{\mathcal{F}}(B) + \overline{d}_{\mathcal{F}}(C)$. Then, by Lemma 4.33, we have

$$\overline{d}_{\mathcal{F}}(B \cup C) = \inf_{\alpha \in A} \sup_{\beta \geq \alpha} \frac{|F_\beta \cap (B \cup C)|}{|F_\beta|} > \inf_{\alpha \in A} \sup_{\beta \geq \alpha} \frac{|F_\beta \cap B|}{|F_\beta|} + \inf_{\alpha \in A} \sup_{\beta \geq \alpha} \frac{|F_\beta \cap C|}{|F_\beta|} = \overline{d}_{\mathcal{F}}(B) + \overline{d}_{\mathcal{F}}(C)$$

Now, suppose $\overline{d}_{\mathcal{F}}(B \cup C) - (\overline{d}_{\mathcal{F}}(B) + \overline{d}_{\mathcal{F}}(C)) = \lambda > 0$. Then, $\overline{d}_{\mathcal{F}}(B \cup C) - \frac{\lambda}{2} > \overline{d}_{\mathcal{F}}(B) + \overline{d}_{\mathcal{F}}(C)$.

Let $\varepsilon > 0$. By Lemma 4.33, using the approximation to the infimum, and totality of A, there exists some $\xi \in A$, such that

$$\sup_{\beta \geq \xi} \frac{|F_\beta \cap B|}{|F_\beta|} + \sup_{\beta \geq \xi} \frac{|F_\beta \cap C|}{|F_\beta|} \leq \overline{d}_{\mathcal{F}}(B) + \overline{d}_{\mathcal{F}}(C) - \varepsilon$$

so that we have

$$\begin{aligned}
\overline{d}_{\mathcal{F}}(B \cup C) &= \inf_{\alpha \in A} \sup_{\beta \geq \alpha} \frac{|F_\beta \cap (B \cup C)|}{|F_\beta|} \\
&\leq \inf_{\alpha \in A} \left(\sup_{\beta \geq \alpha} \frac{|F_\beta \cap B|}{|F_\beta|} + \sup_{\beta \geq \alpha} \frac{|F_\beta \cap C|}{|F_\beta|} \right) \\
&\leq \overline{d}_{\mathcal{F}}(B) + \overline{d}_{\mathcal{F}}(C) - \varepsilon \\
&< \overline{d}_{\mathcal{F}}(B \cup C) - \varepsilon - \frac{\lambda}{2}
\end{aligned}$$

This is clearly a contradiction. It hence follows that $\overline{d}_{\mathcal{F}}(B \cup C) \leq \overline{d}_{\mathcal{F}}(B) + \overline{d}_{\mathcal{F}}(C)$.

Now for any partition $\{U_i\}_{1 \leq i \leq n}$ of X, $1 = \overline{d}_{\mathcal{F}}(X) \leq \sum_{i=1}^{n} \overline{d}_{\mathcal{F}}(U_i)$, implies that for some $1 \leq j \leq n$, $\overline{d}_{\mathcal{F}}(U_j) > 0$ and $U_j \in X^+$. Thus, X^+ is partition regular. For any $B \in X^+$, by Theorem 4.25, there exists $p \in \beta S$, such that $A \in p$, and $p \subset X^+$. Thus, $p \in \widehat{A} \cap \mathcal{D}_{\mathcal{F}} = cl(A) \cap \mathcal{D}_{\mathcal{F}}$. ∎

Definition 4.35. Let (S, X) be a left flow. We say (S, X) is b**-weakly** S**-cancellative**, for $b \in \mathbb{N}$, if for each $s \in S$, and $x \in X$, $|s^{-1}\{x\}| \leq b$. We say that (S, X) is **weakly** S**-cancellative** if for each $s \in S$, and $x \in X$, $|s^{-1}\{x\}| < \infty$.

Definition 4.36. Given a flow (S, X), and $\mathcal{F} = \{F_\alpha\}_{\alpha \in A} \subset \mathscr{P}_f(X)$, let us define the following properties for $\mathcal{F}$:

(P1) For each $\varepsilon > 0$, $s \in S$, there exist $n \in \mathbb{N}$, $\alpha \in A$, such that for each $\beta \geq \alpha$, there exists $\xi \geq \beta$ that satisfies $|sF_\beta \backslash F_\xi| < \varepsilon |F_\beta|$ and $|F_\xi| \leq n|F_\beta|$

(P2) For each $T \in \mathscr{P}_f(S)$, there exists $n \in \mathbb{N}$, $\alpha \in A$, such that for all $\beta \geq \alpha$, $|F_\beta| \leq n \left| \bigcap_{s \in T} s^{-1} F_\beta \right|$

Theorem 4.37. *Let (S, X) be a flow and $\mathcal{F} = \{F_\alpha\}_{\alpha \in A} \subset \mathscr{P}_f(X)$ be a net. If (S, X) is b-weakly S-cancellative for some $b \in \mathbb{N}$, and $\mathcal{F}$ satisfies (P1), then for any $Y \subset X$, $s \in S$, $\overline{d}_{\mathcal{F}}(s^{-1}Y) > 0$ implies $\overline{d}_{\mathcal{F}}(Y) > 0$, i.e. $(S, \mathcal{D}_{\mathcal{F}})$ is a subflow of $(\beta S, \beta X)$.*

Proof. Suppose (S, X) is b-weakly S-cancellative, for some $b \in \mathbb{N}$ and $\mathcal{F}$ satisfies (P1). Let $Y \subset X$, and $s \in S$ satisfy $\overline{d}_{\mathcal{F}}(s^{-1}Y) > 0$. Let us fix some $c > 0$ such that $\overline{d}_{\mathcal{F}}(s^{-1}Y) > c$, and define $\varepsilon = (\overline{d}_{\mathcal{F}}(s^{-1}Y) - c)(2b)^{-1}$. Using (P1), let $n \in \mathbb{N}$, $\alpha \in A$ satisfy: for each $\beta \geq \alpha$, there exists $\xi \geq \beta$ that satisfies $|sF_\beta \backslash F_\xi| < \varepsilon |F_\beta|$ and $|F_\xi| \leq n|F_\beta|$.

Suppose $\tau \in A$.

Take some $\beta \geq \alpha, \tau$, so that there exists $\xi \geq \beta$ (by the definition of $\overline{d}_{\mathcal{F}}(s^{-1}Y)$ such that

$$|s^{-1}Y \cap F_\xi| \geq \frac{\overline{d}_{\mathcal{F}}(s^{-1}Y) + c}{2} |F_\xi| \tag{4.1}$$

Then, there exists $\zeta \in A$, such that $\zeta \geq \xi$ and

$$|sF_\xi \backslash F_\zeta| < \varepsilon |F_\xi| \tag{4.2}$$

and

$$|F_\zeta| \le n|F_\xi| \tag{4.3}$$

For each $x \in Y \cap sF_\xi$, $|s^{-1}\{x\}| \le b$, and $s^{-1}Y \cap F_\xi \subset s^{-1}(Y \cap sF_\xi) = \bigcup_{x \in Y \cap sF_\xi} s^{-1}\{x\}$, gives us $|s^{-1}Y \cap F_\xi| \le |s^{-1}(Y \cap sF_\xi)| \le \sum_{x \in Y \cap sF_\xi} |s^{-1}\{x\}| \le b|Y \cap sF_\xi|$, i.e.

$$|Y \cap sF_\xi| \ge \frac{1}{b}|s^{-1}Y \cap F_\xi| \tag{4.4}$$

By

$$|Y \cap sF_\xi| = \left|(Y \cap sF_\xi \cap F_\zeta) \bigcup (Y \cap sF_\xi \backslash F_\zeta)\right| \le |Y \cap sF_\xi \cap F_\zeta| + |Y \cap sF_\xi \backslash F_\zeta| \le |Y \cap F_\zeta| + |sF_\xi \backslash F_\zeta|$$

it follows that,

$$\begin{aligned}
|Y \cap F_\zeta| &\ge |Y \cap sF_\xi| - |sF_\xi \backslash F_\zeta| \\
&> \frac{1}{b}|s^{-1}Y \cap F_\xi| - \varepsilon|F_\xi| && \text{by Equation (4.4) and Equation (4.2)} \\
&\ge \frac{\overline{d}_{\mathcal{F}}(s^{-1}Y) + c}{2b}|F_\xi| - \varepsilon|F_\xi| && \text{by Equation (4.1)} \\
&= \frac{\overline{d}_{\mathcal{F}}(s^{-1}Y) + c}{2b}|F_\xi| - \frac{\overline{d}_{\mathcal{F}}(s^{-1}Y) - c}{2b}|F_\xi| \\
&= \frac{c}{2b}|F_\xi|
\end{aligned}$$

Thus, for an arbitrary $\tau \in A$, we found $\zeta \ge \tau$, such that $|Y \cap F_\zeta| \ge \frac{c}{2b}|F_\zeta|$, giving us $\overline{d}_{\mathcal{F}}(Y) > 0$.

Suppose $w \in \mathcal{D}_{\mathcal{F}}$, $p \in \beta S$. Then, by Theorem 4.28, if $A \in pw$, $\left\{s \in S \,\middle|\, s^{-1}A \in w\right\} \in p$ which means that there exists $s \in S$, such that there exists $s^{-1}A \in w$. Then, since $w \in \mathcal{D}_{\mathcal{F}}$, $\overline{d}_{\mathcal{F}}(s^{-1}A) > 0$, it follows that $\overline{d}_{\mathcal{F}}(A) > 0$, by our first claim. Hence, $pw \in \mathcal{D}_{\mathcal{F}}$. ∎

Definition 4.38. Let (S, X) be a flow. We define a $A \subset X$ to be **syndetic** if there exists $T \in \mathscr{P}_f(S)$ such that $X = \bigcup_{s \in T} s^{-1}A$.

Theorem 4.39. *Suppose (S, X) is a b-weakly S-cancellative flow, for some $b \in \mathbb{N}$, and $\mathcal{F} = \{F_\alpha\}_{\alpha \in A} \subset \mathscr{P}_f(X)$ is a net. Then the following hold for any $Y \subset X$:*

1. *If $\mathcal{F}$ satisfies (P2) and Y is syndetic, then $\underline{d}_{\mathcal{F}}(Y) > 0$*

2. *If $\mathcal{F}$ satisfies (P1) and Y is syndetic, then $\overline{d}_{\mathcal{F}}(Y) > 0$*

Proof. 1. Suppose $\mathcal{F}$ satisfies (P2) and $Y \subset X$ is syndetic, with $X = \bigcup_{s\in T} s^{-1}Y$ for $T \in \mathscr{P}_f(S)$. By (P2), there exist $n \in \mathbb{N}$, and $\alpha \in A$, such that for any $\beta \geq \alpha$, $|F_\beta| \leq n \left|\bigcap_{s\in T} s^{-1}F_\beta\right|$.

Since $X = \bigcup_{s\in T} t^{-1}Y$, for any given $x \in \bigcap_{s\in T} s^{-1}F_\beta$, there exists, some $t \in T$, such that $tx \in Y$, and $tx \in t(\bigcap_{s\in T} s^{-1}F_\beta) \subset F_\beta$, i.e $tx \in Y \cap F_\beta$. Let us define a function $\phi : \bigcap_{s\in T} s^{-1}F_\beta \to (Y \cap F_\beta) \times T$, given by $x \mapsto (tx, t)$, where for each $x \in \bigcap_{s\in T} s^{-1}F_\beta$, $t \in T$ is chosen as discussed. Now, for each $(tx, t) \in \phi(\bigcap_{s\in T} s^{-1}F_\beta)$, by b-weak S-cancellativity, $|\phi^{-1}(tx, t)| = |(t^{-1}\{tx\}) \bigcap (\cap_{s\in T} s^{-1}F_\beta)| \leq b$ and thus,

$$|\phi^{-1}((Y \cap F_\beta) \times T)| = |\bigcap_{s\in T} s^{-1}F_\beta| \leq b|(Y \cap F_\beta) \times T)| = b|Y \cap F_\beta||T|$$

It follows that, given $\zeta \in A$, choosing any $\xi \geq \zeta, \alpha$, for all $\beta \geq \xi$,

$$|F_\beta| \leq n \left|\cap_{s\in T} s^{-1}F_\beta\right| \leq nb|Y \cap F_\beta||T|$$

i.e. $|Y \cap F_\beta| \geq \frac{1}{nb|T|}|F_\beta|$ and $\underline{d}_{\mathcal{F}}(Y) \geq \frac{1}{nb|T|} > 0$.

2. Suppose $\mathcal{F}$ satisfies (P1), and $Y \subset X$ is syndetic. Suppose $X = \bigcup_{s\in T} s^{-1}Y$ for $T \in \mathscr{P}_f(S)$. Then, $1 = \overline{d}_{\mathcal{F}}(\chi_{X}) = \overline{d}_{\mathcal{F}}(\bigcup_{s\in T} s^{-1}Y) > 0$, implies by the regularity of X^+ (see Proposition 4.34), that for some $t \in T$, $\overline{d}_{\mathcal{F}}(t^{-1}Y) > 0$, which by Theorem 4.37, implies that $\overline{d}_{\mathcal{F}}(Y) > 0$. ∎

So far, we have considered densities with respect to arbitrary nets $\mathcal{F} \subset \mathscr{P}_f(X)$. However, as mentioned earlier, there are some nice properties to be found if we restrict $\mathcal{F}$ to be a Følner net, in the case that S satisfies the SFC (see Proposition 3.3).

Lemma 4.40. *Let (S, X) be a S-cancellative flow. Then, for any $F \in \mathscr{P}_f(X)$, $Y \in \mathscr{P}(X)$ and $s \in S$,*

$$||Y \cap F| - |s^{-1}Y \cap F|| \leq |s^{-1}F \backslash F| + |sF \backslash F| = |s^{-1}F \Delta F|$$

Proof. Suppose $F \in \mathscr{P}_f(X)$, $Y \in \mathscr{P}(X)$ and $s \in S$. Then, by S-cancellativity,

$$\begin{aligned}|s^{-1}Y \cap F| = |s(s^{-1}Y \cap F)| = |Y \cap sF| &= \left|(Y \cap F \cap sF) \bigcup (Y \cap sF \backslash F)\right| \\ &\leq |Y \cap F \cap sF| + |Y \cap sF \backslash F| \\ &\leq |Y \cap F| + |sF \backslash F|\end{aligned}$$

and

$$\begin{aligned}|Y \cap F| = |s^{-1}(Y \cap F)| = |s^{-1}Y \cap s^{-1}F| &= \left|(s^{-1}Y \cap s^{-1}F \cap F) \bigcup (s^{-1}Y \cap s^{-1}F \backslash F)\right| \\ &\leq |s^{-1}Y \cap s^{-1}F \cap F| + |s^{-1}Y \cap s^{-1}F \backslash F| \\ &\leq |s^{-1}Y \cap F| + |s^{-1}F \backslash F|\end{aligned}$$

so that,

$$\begin{aligned}||Y \cap F| - |s^{-1}Y \cap F|| \leq |s^{-1}F \backslash F| + |sF \backslash F| &= |s^{-1}F \backslash F| + |sF| - |F \cap sF| \\ &= |s^{-1}F \backslash F| + |F| - |s^{-1}(F \cap sF)| \\ &= |s^{-1}F \backslash F| + |F| - |s^{-1}F \cap F| \\ &= |s^{-1}F \backslash F| + |F \backslash s^{-1}F| \\ &= |s^{-1}F \Delta F|\end{aligned}$$

as desired. ∎

Theorem 4.41. *Suppose (S, X) is a S-cancellative flow and $\mathcal{F} = \{F_\alpha\}_{\alpha \in A} \subset \mathscr{P}_f(X)$ is a Følner net. Then, for each $Y \subset X$ and $s \in S$, the following holds:*

1. $\underline{d}_{\mathcal{F}}(s^{-1}Y) = \underline{d}_{\mathcal{F}}(Y) = \underline{d}_{\mathcal{F}}(sY)$
2. $\overline{d}_{\mathcal{F}}(s^{-1}Y) = \overline{d}_{\mathcal{F}}(Y) = \overline{d}_{\mathcal{F}}(sY)$

Proof. Suppose $Y \subset X$ and $s \in S$.

1. Let $\lambda > 0$ such that $0 < \lambda \leq \underline{d}_{\mathcal{F}}(Y)$. Then, suppose $\alpha \in A$ is such that for all $\beta \geq \alpha$, $|Y \cap F_\beta| \geq \lambda |F_\beta|$. Fix any $\varepsilon > 0$, and since $\mathcal{F}$ is a Følner net, choose $\xi \in A$ such that for all $\beta \geq \xi$, $|F_\alpha \Delta s F_\alpha| = |s^{-1}F_\alpha \Delta F_\alpha| \leq \varepsilon |F_\alpha|$. Then, letting $\zeta \geq \xi, \alpha$, for all $\beta \geq \zeta$, by Lemma 4.40,

$$|s^{-1}Y \cap F_\beta| \geq |Y \cap F_\beta| - |s^{-1}F_\beta \Delta F_\beta| \geq (\lambda - \varepsilon)|F_\beta|$$

Taking the supremum over all $\varepsilon > 0$, and then over $\lambda \geq \underline{d}_{\mathcal{F}}(Y)$, of

$$\left\{\lambda - \varepsilon \geq 0 \middle| \text{ There exists } \alpha \in A, \text{ such that, for all } \beta \geq \alpha, |Y \cap F_\beta| \geq \lambda - \varepsilon |F_\beta| \right\},$$

it follows that $\underline{d}_{\mathcal{F}}(s^{-1}Y) \geq \underline{d}_{\mathcal{F}}(Y)$.

Similarly, using Lemma 4.40, we can show that $\underline{d}_{\mathcal{F}}(Y) \geq \underline{d}_{\mathcal{F}}(s^{-1}Y)$, so that $\underline{d}_{\mathcal{F}}(Y) = \underline{d}_{\mathcal{F}}(s^{-1}Y)$.

To show that $\underline{d}_{\mathcal{F}}(sY) = \underline{d}_{\mathcal{F}}(Y)$, we simply note that $s^{-1}sY = Y$, and use the above result with sY in place of Y.

2. The proof for this is very similar to the first part and will hence be omitted. ∎

For each $Y \subset X$, we can use density with respect to a Følner net to obtain a countably additive S-invariant measure on the Borel subsets $\mathcal{C}$ of βX, as follows:

Theorem 4.42. *Suppose (S, X) be a S-cancellative flow with a Følner net $\mathcal{F} = \{F_\alpha\}_{\alpha \in A} \subset \mathscr{P}_f(X)$. For each $Y \subset X$, there exists a probability measure μ on $\mathcal{C}$ such that:*

1. $\mu(\widehat{Y}) = \overline{d}_{\mathcal{F}}(Y)$
2. *For each $Z \subset X$, $\mu(\widehat{Z}) \leq \overline{d}_{\mathcal{F}}(Z)$*
3. *For each $P \in \mathcal{C}$, $s \in S$, $\mu(s^{-1}P) = \mu(P) = \mu(sP)$*

Proof. Let $Y \subset X$. We start out by giving the set $\mathbb{N} \times A$ the product order. Then, for each $n \in \mathbb{N}$, and $\alpha \in A$, there exists $\beta \in A$, $\beta \geq \alpha$, such that $|Y \cap F_\beta| \geq (\overline{d}_{\mathcal{F}}(Y) - \frac{1}{2^n})|F_\beta|$. Let us define $F_{(n,\alpha)} = F_\beta$, and do this analogously for each $(n, \alpha) \in \mathbb{N} \times A$. Note that

$$\lim_{(n,\alpha) \in \mathbb{N} \times A} \frac{|Y \cap F_{(n,\alpha)}|}{|F_{(n,\alpha)}|} = \overline{d}_{\mathcal{F}}(Y)$$

Now, let $C(\beta X) \subset m(\beta X)$ be the set of all continuous functions on βX with the supremum norm. For each $(n, \alpha) \in \mathbb{N} \times A$, we define a linear functional:

$$\begin{aligned} T_{(n,\alpha)} : C(\beta X) &\to \mathbb{R} \\ f &\mapsto \frac{1}{|F_{(n,\alpha)}|} \sum_{x \in F_{(n,\alpha)}} f(x) \end{aligned}$$

For each $f \in C(\beta X)$, let us define the closed interval $I_f = [-\|f\|_\infty, \|f\|_\infty] \subset \mathbb{R}$. Since for $(n,\alpha) \in \mathbb{N} \times A$, $f \in C(\beta X)$, $|T_{(n,\alpha)}(f)| \leq \|f\|_\infty$, it follows that $T_{(n,\alpha)} \subset \prod_{f \in C(\beta X)} I_f \subset \mathbb{R}^{C(\beta X)}$, where $\prod_{f\in C(\beta X)} I_f$ is a compact space under the product topology on $\mathbb{R}$, by Tychanoff's theorem. Hence, the net $\{T_{(n,\alpha)}\}_{(n,\alpha)\in\mathbb{N}\times A}$ has a pointwise-convergent subsequence $\{T_{(n_k,\beta)}\}_{(k,\beta)\in\mathbb{N}\times B}$, which converges pointwise to some $T \in \prod_{f\in C(\beta X)} I_f$.

Since each element of the net $\{T_{(n,\alpha)}\}_{(n,\alpha)\in\mathbb{N}\times A}$ is a positive linear functional, T is a positive linear functional, clearly bounded due to its range. Then, as βX is a compact Hausdorff space, by the Reisz Representation theorem, there exists a unique regular measure μ on $\mathcal{B}$ such that $\mu(\beta X) = T(\chi_{\beta X})$ and $T(f) = \int f d\mu$ for each $f \in C(\beta X)$.

Observe that for each $Z \subset X$, $\chi_{\widehat{Z}} \in C(\beta X)$, since we have that, for any $U \subset \mathbb{R}$ open, $\chi_{\widehat{Z}}$ is either $\widehat{Z}$, $\varnothing$, or βX, all of which are open. Hence,

$$\mu(\widehat{Z}) = T(\chi_{\widehat{Z}}) = \lim_{(k,\beta)\in\mathbb{N}\times A} T_{(n_k,\beta)}(\chi_{\widehat{Z}}) = \lim_{(k,\beta)\in\mathbb{N}\times A} \frac{1}{|F_{(n_k,\beta)}|} \sum_{x\in F_{(n_k,\beta)}} \chi_{\widehat{Z}}(x) = \lim_{(k,\beta)\in\mathbb{N}\times A} \frac{|Z \cap F_{(n_k,\beta)}|}{|F_{(n_k,\beta)}|}$$

This gives us the following properties:

1. $\mu(\widehat{Y}) = \lim_{(k,\beta)\in\mathbb{N}\times A} \frac{|Y\cap F_{(n_k,\beta)}|}{|F_{(n_k,\beta)}|} = \lim_{(n,\alpha)\in\mathbb{N}\times A} \frac{|Y\cap F_{(n,\alpha)}|}{|F_{(n,\alpha)}|} = \overline{d}_{\mathcal{F}}(Y)$
2. For any $Z \subset X$,
$$\mu(\widehat{Z}) = \lim_{(k,\beta)\in\mathbb{N}\times A} \frac{|Z \cap F_{(n_k,\beta)}|}{|F_{(n_k,\beta)}|} \leq \overline{d}_{\mathcal{F}}(Z)$$
3. Let $s \in S$ be arbitrary. By Lemma 4.40, for any $Z \subset X$,
$$\begin{aligned}\left|\mu(\widehat{Z}) - \mu(\widehat{s^{-1}Z})\right| &= \left|T(\chi_Z) - T(\chi_{s^{-1}Z})\right| \\ &= \lim_{(k,\beta)\in\mathbb{N}\times A} \left|\frac{|Z \cap F_{(n_k,\beta)}|}{|F_{(n_k,\beta)}|} - \frac{|s^{-1}Z \cap F_{(n_k,\beta)}|}{|F_{(n_k,\beta)}|}\right| \\ &\leq \lim_{(k,\beta)\in\mathbb{N}\times A} \frac{|F_{(n_k,\beta)} \Delta s F_{(n_k,\beta)}|}{|F_{(n_k,\beta)}|} = 0\end{aligned}$$
gives us $\mu(\widehat{Z}) = \mu(s^{-1}\widehat{Z}) = \mu(s\widehat{Z})$

   Let $\mathcal{A} = \{\sum_{i=1}^n a_i \chi_{\widehat{Z}} \mid Z \subset S, n \in \mathbb{N}, a_i \in \mathbb{R}\}$. $\mathcal{A}$ is a subalgebra of $C(\beta X)$ that contains the constant function 1, and separates points of βX, and satisfies 3. Hence, by the Stone-Weierstrass theorem, it is uniformly dense in $C(X)$. By the continuity of T, and of $\tilde{L}_s$ it follows that $\mu(P) = \mu(s^{-1}P) = \mu(s\widehat{P})$, for each $p \in \mathcal{C}$.

Finally, $\mu(\beta X) = \mu(\widehat{S}) = T(\chi_{\beta X}) = 1$ since

$$\frac{1}{|F_{(n,\alpha)}|}\sum_{x\in F_{(n,\alpha)}} \chi_{\beta X}(x) = \frac{1}{|F_{(n,\alpha)}|}\sum_{x\in F_{(n,\alpha)}} 1 = 1$$

for each $(n, \alpha) \in \mathbb{N} \times A$. ∎

Corollary 4.43. *Suppose (S, X) is a S-cancellative flow, and $\mathcal{F} \subset \mathscr{P}_f(X)$ is a Følner net. Then, the following hold for each $Y \subset X$.*

1. *If $T \in \mathscr{P}_f(S)$ satisfies $\overline{d}_{\mathcal{F}}(s^{-1}A \cap t^{-1}A) = 0$ for each $s \neq t, s, t \in T$, then, $\overline{d}_{\mathcal{F}}(\bigcup_{s\in T} s^{-1}A)) = |T|d(A)$*

2. *If $T \in \mathscr{P}_f(S)$ satisfies $\overline{d}_{\mathcal{F}}(sA \cap tA) = 0$ for each $s \neq t$, $s, t \in T$, then, $\overline{d}_{\mathcal{F}}(\bigcup_{s\in T} sA)) = |T|d(A)$*

This proof of this is a straightforward generalization of Corollary 4.8 in [13].

Proposition 4.44. *Let (S, X) is a flow. If $\mathcal{F} = \{F_\alpha\}_{\alpha\in A} \subset \mathscr{P}_f(X)$ is a Følner net, it satisfies (P1) and (P2)*

Proof. Note that to show that $\mathcal{F}$ satisfies (P1), it is enough to observe that for each $\varepsilon > 0$, $s \in S$, taking $n = 1$, there exists $\alpha \in A$, such that for any $\beta \geq \alpha$,

$$|sF_\beta \backslash F_\beta| \leq |sF_\beta \backslash F_\beta| + |F_\beta \backslash sF_\beta| = |sF_\beta \Delta F_\beta| < \varepsilon |F_\beta|$$

and $|F_\beta| \leq n|F_\beta|$.

To show that $\mathcal{F}$ satisfies (P2), suppose $T \in \mathscr{P}_f(S)$. For any $F \in \mathscr{P}_f(X)$,

$$\begin{aligned}
|F| &= \left|\left(F \cap \bigcap_{s\in T} s^{-1}F\right) \bigcup \left(F \cap \left\{F \backslash \bigcap_{s\in T} s^{-1}F\right\}\right)\right| \\
&\leq \left|F \cap \bigcap_{s\in T} s^{-1}F\right| + \left|F \backslash \bigcap_{s\in T} s^{-1}F\right| \\
&\leq \left|\bigcap_{s\in T} s^{-1}F\right| + \left|\bigcup_{s\in T} F \backslash s^{-1}F\right| \\
&\leq \left|\bigcap_{s\in T} s^{-1}F\right| + \sum_{s\in T} |F \backslash s^{-1}F|
\end{aligned}$$

However, we also have for each $s \in T$

$$|F| = \left|(F \cap sF) \bigcup (F \backslash sF)\right| \leq |F \cap sF| + |F \backslash sF| \leq |s^{-1}F \cap F| + |F \backslash sF| = |F| - |F \backslash s^{-1}F| + |F \backslash sF|$$

so that $|F \backslash s^{-1}F| \leq |F \backslash sF|$, and $|F| \leq | \cap_{s\in T} s^{-1}F| + |T||F \backslash sF|$

Let $\varepsilon = \frac{1}{2|T|}$, and choose $\alpha \in A$, such that for all $\beta \geq \alpha$ and $s \in T$, $|F_\beta \Delta sF_\beta| \leq \varepsilon |F_\beta|$ so that $|F_\beta \backslash sF_\beta| \leq \varepsilon |F_\beta|$ (see Proposition 3.3).

It follows that for any $\beta \geq \alpha$, $|F_\beta| \leq | \cap_{s\in T} s^{-1}F_\beta| + |T||F_\beta \backslash sF_\beta| \leq | \cap_{s\in T} s^{-1}F_\beta| + \frac{1}{2}|F_\beta|$, so that $|F_\beta| \leq 2| \cap_{s\in T} s^{-1}F_\beta|$. ∎

We conclude that Theorem 4.37, Theorem 4.39 hold without the additional requirements (P1), (P2) being imposed, for Følner nets. We now generalize the notion of Følner density:

Definition 4.45. Suppose (S, X) is a left flow that satisfies SFC. Then, for any $Y \subset X$, define **Følner density** as follows:

$$d(Y) = \sup \left\{ \lambda \geq 0 \;\middle|\; \begin{array}{c} \text{For each } T \in \mathscr{P}_f(X), \text{ and } \varepsilon > 0, \text{ there exists } F \in \mathscr{P}_f(X) \\ \text{such that } |T \cap F| \geq \lambda |F| \text{ and for all } s \in T, |sF \Delta F| < \varepsilon |F| \end{array} \right\}$$

Theorem 4.46. *Suppose (S, X) is a flow that satisfies SFC. Then, for any $Y \subset X$,*

$$d(Y) = \sup\{\overline{d}_{\mathcal{F}}(Y) \mid \mathcal{F} \textit{ is a Følner net in } X\}$$

Moreover, for each $Y \subset X$, there exists a Følner net $\mathcal{F}$ such that $d(Y) = \overline{d}_{\mathcal{F}}(Y)$.

Proof. Let $Y \subset X$ and consider $\mathbb{N} \times \mathscr{P}_f(X)$ with the product ordering. For each $(n, T) \in \mathbb{N} \times \mathscr{P}_f(X)$, define $F_{(n,T)} \in \mathscr{P}_f(X)$ to be a set that satisfies $|T \cap F_{(n,T)}| \geq \left(d(Y) - \frac{1}{n}\right) |F_{(n,T)}|$ and for all $s \in T$, $|sF_{(n,T)} \Delta F_{(n,T)}| \leq \frac{1}{n} \left|F_{(n,T)}\right|$. Then, it is clear that $\mathcal{F} = \{F_{(n,T)}\}_{(n,T) \in \mathbb{N} \times \mathscr{P}_f(X)}$ is a Følner net. It follows that $\sup\{\overline{d}_{\mathcal{F}}(Y) \mid \mathcal{F}$ is a Følner net in $X\} \geq d(Y)$.

Let $\lambda > 0$ be any real number that satisfies $\lambda < \sup\{\overline{d}_{\mathcal{F}}(Y) \mid \mathcal{F}$ is a Følner net in $X\}$. Pick a Følner net $\mathcal{F} = \{F_\alpha\}_{\alpha \in A}$, such that $\overline{d}_{\mathcal{F}}(Y) > \lambda$. Since $\mathcal{F}$ is a Følner net, choose $\xi \in A$ be such that for any $\beta \geq \xi$, and $s \in T$, $|F_\beta \Delta sF_\beta| < \varepsilon |F_\beta|$. Then, using the definition of $\overline{d}_{\mathcal{F}}(Y)$, take $\beta \geq \xi$ such that $|Y \cap F_\beta| \geq \lambda |F_\beta|$. Now we have that for $\beta \in A$,

$|F_\beta \Delta s F_\beta| < \varepsilon |F|$ and $|Y \cap F_\beta| \geq \lambda |F_\beta|$ are true, so that $d(Y) \geq \lambda$. Taking the supremum over all $\lambda < \sup\{\overline{d}_{\mathcal{F}}(Y) \mid \mathcal{F} \text{ is a Følner net in } X\}$, $d(Y) \geq \sup\{\overline{d}_{\mathcal{F}}(Y) \mid \mathcal{F} \text{ is a Følner net in } X\}$.

Hence, the first claim is true. To see that second claim, we simply observe that

$$d(Y) = \overline{d}_{\{F_{(n,T)}\}_{(n,T)\in\mathbb{N}\times\mathscr{P}_f(X)}}(Y) \quad \blacksquare$$

Corollary 4.47. *If (S, X) is a flow that satisfies the SFC, then for any $Y \subset X$, there exists a Følner net $\mathcal{F} = \{F_\alpha\}_{\alpha\in A} \subset \mathscr{P}_f(X)$ such that*

$$d(Y) = \lim_{\alpha\in A} \frac{|Y \cap F_\alpha|}{|F_\alpha|}$$

Proof. By Theorem 4.46, we have that $d(Y) = d_{\mathcal{G}}(Y)$ for some Følner net $\mathcal{G} = \{G_\alpha\}_{\alpha\in A} \subset \mathscr{P}_f(X)$. Recall that

$$d_{\mathcal{G}}(Y) = \sup\left\{\lambda \geq 0 \middle| \text{ For each } \alpha \in B, \text{ there exists } \beta \geq \alpha \text{ such that } |Y \cap G_\beta| \geq \lambda |G_\beta|\right\}$$

For any $\lambda < d_{\mathcal{G}}(Y)$, and $\alpha \in A$, $\beta \geq \alpha$ such that $\frac{|Y\cap G_\beta|}{|G_\beta|} \geq \lambda$. Let us order $[0, d(Y))\times A \subset \mathbb{R}\times A$ with the product order and take $F_{(\lambda,\alpha)} = G_\beta$, as discussed for each $(\lambda, \alpha) \in [0, d(Y)) \times A$. Then, we have

$$\lim_{(\lambda,\alpha)\in[0,d(Y))\times A} \frac{|Y \cap F_{(\lambda,\alpha)}|}{|F_{(\lambda,\alpha)}|} \geq \lim_{(\lambda,\alpha)\in[0,d(Y))\times A} \lambda = d(Y)$$

However, since $\mathcal{F} = \{F_{(\lambda,\alpha)}\}_{(\lambda,\alpha)\in[0,d(Y))\times A}$ is a subnet of $\mathcal{G}$, it follows that $\mathcal{F}$ is a Følner net. Hence, by Theorem 4.46,

$$\lim_{(\lambda,\alpha)\in[0,d(Y))\times A} \frac{|Y \cap F_{(\lambda,\alpha)}|}{|F_{(\lambda,\alpha)}|} = d(Y)$$

and our claim is true. $\blacksquare$

Corollary 4.48. *If (S, X) is a flow that satisfies SFC and is S-cancellative, then, for each $Y \subset X$, and $s \in S$,*

$$d(s^{-1}Y) = d(Y) = d(sY)$$

Proof. By Theorem 4.46, there exists Følner nets $\mathcal{F} = \{F_\alpha\}_{\alpha\in A}, \mathcal{G} = \{G\}_{\beta\in B}, \mathcal{H} = \{H\}_{\lambda\in\Lambda} \subset \mathscr{P}_f(X)$ such that $d(Y) = \overline{d}_{\mathcal{F}}(F)$, $d(s^{-1}Y) = \overline{d}_{\mathcal{G}}(s^{-1}Y)$, $d(sY) = \overline{d}_{\mathcal{H}}(sY)$.

By Theorem 4.41, for $\mathcal{M} = \mathcal{F}, \mathcal{G}, \mathcal{H}$,

$$\overline{d}_{\mathcal{M}}(s^{-1}Y) = \overline{d}_{\mathcal{M}}(Y) = \overline{d}_{\mathcal{M}}(sY)$$

and we have by Theorem 4.46

$$d(Y) = \overline{d}_{\mathcal{F}}(Y) = \overline{d}_{\mathcal{F}}(s^{-1}Y) \leq d(s^{-1}Y) = \overline{d}_{\mathcal{G}}(s^{-1}Y) = \overline{d}_{\mathcal{G}}(sY) \leq d(sY) = \overline{d}_{\mathcal{G}}(sY)$$

and similarly, $d(sY) \leq d(s^{-1}Y) \leq d(Y)$ so that $d(sY) = d(s^{-1}Y) = d(Y)$. ∎

Definition 4.49. Let us define

$$\mathscr{M}_0(X) = \{N \in m(X)^* \mid \text{There exists a Følner net } \{F_\alpha\}_{\alpha\in A} \subset \mathscr{P}_f(X) \text{ such that } w^* \lim_{\alpha\in A} Q\mu_{F_\alpha} = N\}$$

Note that by Theorem 3.5, $\mathscr{M}_0(X) \subset \mathscr{M}_l(X)$.

Theorem 4.50. *If (S, X) satisfies the SFC, then, for each $Y \subset X$,*

$$d(Y) \leq \sup_{M\in\mathscr{M}_l(X)} M(\chi_Y)$$

Furthermore, if $\mathscr{M}(X)$ is the weak closure of the convex hull of $\mathscr{M}_0(X)$, equality holds.*

Proof. Suppose (S, X) satisfies the SFC and $Y \subset X$. Then, by Corollary 4.47, there exists a Følner net $\mathcal{F} = \{F_\alpha\}_{\alpha\in A} \subset \mathscr{P}_f(X)$, such that

$$d(Y) = \lim_{\alpha\in A} \frac{|Y \cap F_\alpha|}{|F_\alpha|} = \lim_{\alpha\in A} Q\mu_{F_\alpha}(\chi_Y)$$

Since $\mathscr{M}(X)$ is weak* compact, $\{Q\mu_{F_\alpha}\}_{\alpha\in A}$ has a limit point N in $\mathscr{M}_0(X)$ by Theorem 3.5. Then, if $\{Q\mu_{F_\beta}\}_{\beta\in B}$ is a subnet of $\{Q\mu_{F_\alpha}\}_{\alpha\in A}$ that weak* converges to N,

$$d(Y) = \lim_{\beta\in B} \frac{|Y \cap F_\beta|}{|F_\beta|} = \lim_{\beta\in B} Q\mu_{F_\beta}(\chi_Y) = N(\chi_Y) \leq \sup_{M\in\mathscr{M}_l(X)} M(\chi_Y) \quad (*)$$

Now suppose $\mathscr{M}(X)$ is the weak* closure of the convex hull of $\mathscr{M}_0(X)$. Then, consider the set $C = \{M \in \mathscr{M}_l(X) \mid M(\chi_Y) \leq d(Y)\}$. C contains $\mathscr{M}_0(X)$ by Theorem 4.46. Furthermore, it is easily checked that C is convex and weak* closed. It follows that if $\mathscr{M}(X)$ is the weak* closure of the convex hull of $\mathscr{M}_0(X)$, it is equal to C, and equality holds in $(*)$. ∎

Lemma 4.51. *Supose (S, X) and (T, Y) are flows that satisfy the SFC. If $C \times D \subset X \times Y$, then $d(C \times D) \geq d(C)d(D)$.*

Proof. By Corollary 4.47, there exist Følner nets $\mathcal{F} = \{F_\alpha\}_{\alpha \in A}$, $\mathcal{G} = \{G_\beta\}_{\beta \in B}$, such that $d(C) = \lim_{\alpha \in A} \frac{|C \cap F_\alpha|}{|F_\alpha|}$, $d(D) = \lim_{\beta \in B} \frac{|D \cap G_\beta|}{|G_\beta|}$.

Then, equipping $A \times B$ with the product ordering,

$$\begin{aligned} d(C)d(D) &= \lim_{\alpha \in A} \frac{|C \cap F_\alpha|}{|F_\alpha|} \lim_{\beta \in B} \frac{|D \cap G_\beta|}{|G_\beta|} \\ &= \lim_{\alpha \in A} \lim_{\beta \in B} \frac{|C \cap F_\alpha||D \cap G_\beta|}{|F_\alpha||G_\beta|} \\ &= \lim_{(\alpha,\beta) \in A \times B} \frac{|(C \times D) \cap (F_\alpha \times G_\beta)|}{|F_\alpha \times G_\beta|} \leq d(C \times D) \end{aligned}$$

where at the last step, we used the fact that $\{F_\alpha \times G_\beta\}_{(\alpha,\beta) \in A \times B}$ is a Følner net by Proposition 3.12, and Theorem 4.46. ∎

Suppose we label the set of characteristic functions on X by $char(X)$ and set Y to be the S-invariant subspace of $m(X)$ consisting of linear combinations of elements of $char(X)$.

Lemma 4.52. *Suppose (S, X) is a flow and $T : char(X) \to [0, 1]$, such that*

1. $T(\chi_{X}) = 1$
2. $T(\chi_{A \cup B}) = T(\chi_{A}) + T(\chi_{B})$, *if $A, B \subset X$ are disjoint*
3. $T(\chi_{s^{-1}A}) = T(\chi_{A})$ *for each $s \in S$, and $A \subset X$*

Then, T can be extended to a S-invariant mean on $m(X)$.

Proof. It is easy to see that T can be extended to a linear functional N on Y by taking for any $n \in \mathbb{N}$, $a_i \in \mathbb{R}$, $A_i \subset X$, $N(\sum_{i=1}^n a_i \chi_{A_i}) = \sum_{i=1}^n a_i T(\chi_{A_i})$.

Also, without loss of generality, taking $\{A_i\}_{1 \leq i \leq n}$ to be pairwise disjoint, by 2,

$$\left| N\left(\sum_{i=1}^n a_i \chi_{A_i}\right)\right| = \left|\sum_{i=1}^n a_i T(\chi_{A_i})\right| \leq \left(\sup_{1 \leq j \leq n} |a_j|\right) \left|\sum_{i=1}^n T(\chi_{A_i})\right|$$

$$
\begin{aligned}
&= \left(\sup_{1\le j\le n} |a_j|\right) \left| T\left(\bigcup_{i=1}^{n} A_i\right)\right| \\
&\le \left(\sup_{1\le j\le n} |a_j|\right) \left| T\left(\chi_{X}\right)\right| \\
&= \sup_{1\le j\le n} |a_j| \\
&= \left\| \sum_{i=1}^{n} a_i \chi_{A_i} \right\|_\infty
\end{aligned}
$$

where we use the fact that for each $A \subset X$, $1 = T(\chi_X) = T(\chi_{X\setminus A\cup A}) = T(\chi_{X\setminus A}) + T(\chi_A)$, which gives us $T(\chi_A) \le 1$. Hence $\|N\| \le 1$, and further by 1, $\|N\| = 1$.

Now, by 3, and the linearity of the L_s operator for each $s \in S$, it follows that N is S-invariant on Y. Then, by the Hahn-Banach extension theorem, we can extend N to a continuous functional M on $m(X)$ of norm 1. Then, since M is S-invariant on Y, and Y is dense in $m(X)$, it follows by the continuity of M and L_s, for each $s \in S$, that M is S-invariant on all of X. Hence, M is an S-invariant mean on $m(X)$ that is an extension of T. ∎

In the following special case, we achieve equality in Lemma 4.51:

Theorem 4.53. *Let (S, X) and (T, Y) be flows that satisfy the SFC. If the closed convex hulls of $\mathscr{M}_0(X)$ and $\mathscr{M}_0(Y)$ are $\mathscr{M}_l(X)$ and $\mathscr{M}_l(Y)$ respectively, then for any $C \subset X$, $D \subset Y$,*

$$d(C \times D) = d(C)d(D)$$

Proof. One side of the inequality is given by Lemma 4.51. To see the other side, first we note that by Theorem 4.50, $d(C \times D) \le \sup_{M\in\mathscr{M}_l(X\times Y)} M(\chi_{C\times D})$. We wish to show that for each $M \in \mathscr{M}_l(X \times Y)$, $M(\chi_{C\times D}) \le d(C)d(D)$.

Suppose $M \in \mathscr{M}_l(X \times Y)$. WLOG, $M(C \times D) > 0$. We define:

$$
\begin{aligned}
N : char(X) &\to [0, 1] \\
\chi_A &\mapsto M(\chi_{A\times Y})
\end{aligned}
$$

and

$$\begin{aligned} P : char(Y) &\to [0,1] \\ \chi_B &\mapsto \frac{M(\chi_{C\times B})}{M(\chi_{C\times Y})} \end{aligned}$$

Note that $N(\chi_X) = M(\chi_{X\times Y}) = 1$, $P(\chi_Y) = \frac{M(\chi_{C\times Y})}{M(\chi_{C\times Y})} = 1$. For any disjoint sets $A, A' \subset X$,

$$\begin{aligned} N(\chi_{A\cup A'}) = M(\chi_{(A\cup A')\times Y}) = M(\chi_{(A\times Y)\cup(A'\times Y)}) &= M(\chi_{(A\times Y)} + \chi_{(A'\times Y)}) \\ &= M(\chi_{(A\times Y)}) + M(\chi_{(A'\times Y)}) \\ &= N(\chi_A) + N(\chi_{A'}) \end{aligned}$$

Similarly for disjoint sets $B, B' \subset Y$, $P(\chi_{B\cup B'}) = P(\chi_B) + P(\chi_{B'})$.

Lastly, for any $s \in S$, $A \subset X$, taking any fixed $t \in T$, we have, by the $S \times T$-invariance of M,

$$N(\chi_{s^{-1}A}) = M(\chi_{s^{-1}A\times Y}) = M(\chi_{s^{-1}A\times t^{-1}Y}) = M(\chi_{(s,t)^{-1}(A\times Y)}) = M(L_{(s,t)}\chi_{A\times Y}) = M(\chi_{A\times Y}) = N(\chi_A)$$

and similarly, for any $q \in T$, $B \subset X$, taking any fixed $s \in S$, $t \in T$,

$$\begin{aligned} P(\chi_{q^{-1}B}) = \frac{M(\chi_{C\times q^{-1}B})}{M(\chi_{C\times Y})} = \frac{M(L_{(s,t)}\chi_{C\times q^{-1}B})}{M(\chi_{C\times Y})} = \frac{M(\chi_{s^{-1}C\times t^{-1}q^{-1}B})}{M(\chi_{C\times Y})} &= \frac{M(\chi_{s^{-1}C\times (qt)^{-1}B})}{M(\chi_{C\times Y})} \\ &= \frac{M(L_{(s,qt)}\chi_{C\times B})}{M(\chi_{C\times Y})} \\ &= \frac{M(\chi_{C\times B})}{M(\chi_{C\times Y})} = P(\chi_B) \end{aligned}$$

By Lemma 4.52, we can extend N, P to S- and T-invariant means, $\tilde{N}$ and $\tilde{P}$, on $m(X)$ and $m(Y)$ respectively.

Then, by Theorem 4.50,

$$M(\chi_{C\times D}) = M(\chi_{C\times Y})\left(\frac{M(\chi_{C\times D})}{M(\chi_{C\times Y})}\right) = \tilde{N}(\chi_C)\tilde{P}(\chi_D) \le d(C)d(D)$$

It follows that $d(C \times D) = d(C)d(D)$. ∎

Chapter 5

Reversible Invariance

Reversible invariance, called "left/right-measurability" for semigroups, was initially introduced by Klawe in [17]. For a group G, the existence of a mean satisfying $M(\chi_A) = M(\chi_{gA})$, for each $g \in G$, $A \subset G$, is equivalent to the existence of a mean satisfying $M(\chi_A) = M(\chi_{g^{-1}A})$, for each $g \in G$, $A \subset G$, and both define amenability. However, since inverses do not necessarily exist for semigroups, amenability is defined using the nicer, latter condition. Reversible invariance defines the first condition, and was explored by Klawe in comparison to amenability. We generalize some of Klawe's results to flows. However, Klawe showed that reversible invariance interestingly implies SFC. We have been unable to generalize this and it remains an open question.

Definition 5.1. A semigroup S is said to be **left-measurable** if there exists a mean $M \in m(S)^*$ such that $M(\chi_{tA}) = M(\chi_A)$, for all $A \in \mathscr{P}(S)$, $t \in S$. M is called **left-reversible invariant**.

Definition 5.2. Let (S, X) be a flow. A mean $M \in \mathscr{M}(X)$ is said to be **S-reversible invariant** if it satisfies $M(\chi_A) = M(\chi_{sA})$ for all $s \in S$ and all $A \subset X$. We say (S, X) is **reversible invariant** if it has a reversible invariant mean.

Theorem 5.3. *A mean M on (S, X) is S-reversible invariant if and only if it is S-invariant and for all $s \in S$, $M(\chi_{Z_s}) = 1$ where $Z_s = \{x \in X \mid s^{-1}(sx) = \{x\}\}$.*

Proof. Suppose M is reversible invariant. Clearly it is S-invariant since for all $s \in S$, $A \subset X$, $m(\chi_{s^{-1}A}) = m(\chi_{s\{s^{-1}A\}}) = m(\chi_A)$.

Suppose $s \in S$. Using the axiom of choice, we can write $X \backslash Z_s = A_1 \cup A_2$, where $A_1 \cap A_2 = \varnothing$ and $sA_1 = sA_2 = s(X \backslash Z_s)$ (This is because, for each $x \in X$, for each distinct element in

$s^{-1}\{sx\}$, we can choose one fixed element y_x and set A_1 to be the set of all these elements and set A_2 to be the union of the sets of all elements $s^{-1}x\backslash\{y_x\}$.)

Now, we have that

$$M(\chi_{s(X\backslash Z_s)}) = M(\chi_{X\backslash Z_s}) = M(\chi_{A_1\cup A_2}) = M(\chi_{A_1})+M(\chi_{A_2}) = M(\chi_{sA_1})+M(\chi_{sA_2}) = 2M(\chi_{s(X\backslash Z_s)})$$

Thus, $M(\chi_{s(X\backslash Z_s)}) = 0 \implies M(\chi_{Z_s}) = 1$.

On the other hand, suppose M is an S-invariant mean and $M(\chi_{Z_s}) = 1$ for each $s \in S$. Suppose $A \subset X$, $M(\chi_{sA}) = M(\chi_{s^{-1}(sA)}) = M(\chi_{s^{-1}(sA)\backslash A}) + M(\chi_{A})$.

However, $s^{-1}(sA)\backslash A \subset X\backslash Z_s$ and $\mu(\chi_{X\backslash Z_s}) = 0$, thus, we have $M(\chi_{sA}) = M(\chi_{A})$ and M is S-reversible invariant. ∎

Corollary 5.4. *If (S, X) is an S-cancellative amenable flow, it is reversible invariant.*

Proof. For each $s \in S$, $Z_s = \{x \in X \mid s^{-1}x = \{x\}\} = X$, since $sa = sx$, gives us, by S-cancellativity, $a = x$. Hence, for any S-invariant mean M on $m(X)$, $M(\chi_{Z_s}) = M(\chi_{X}) = 1$ for each $s \in S$. The rest follows from Theorem 5.3. ∎

Example 5.5.

1. Consider the flow $(\mathbb{N}, \mathbb{R})$ where the action of $(\mathbb{N}, +)$ on $\mathbb{R}$ is via addition. Then, since $\mathbb{N}$ is abelian, it follows that the flow is amenable. However, it is also $\mathbb{N}$-cancellative, and thus reversible invariant.

2. Any group flow is reversible invariant.

3. For a vector space E with topology τ, the aforementioned flow (E, τ) is E-cancellative and amenable, and thus reversible invariant.

4. Consider the flow, (S, X) from 3 of Example 1.9. Fix any $x \in X$; the S-invariant mean M, given by $f \mapsto f(x)$ is reversible invariant, since, for any $A \subset X$, $sA = A$, gives $M(\chi_{A}) = M(\chi_{sA})$.

Proposition 5.6. *Suppose (S, X), (T, Y). Then $(S \times T, X \times Y)$ is reversible invariant if and only if (S, X) and (T, Y) are reversible invariant.*

Proof. Suppose M is a reversible invariant mean on $m(X \times Y)$. By Theorem 5.3, M is S-invariant. Let us define the $N : m(X) \to \mathbb{R}$, $f \mapsto M(\widetilde{f})$, where $\widetilde{f}(x, y) = f(x)$. By the proof of Theorem 2.28, we know that N is a S-invariant mean on $m(X)$.

Now, let us consider χ_{Z_s} for each $s \in S$. For any $t \in T$, we have, if $(x, y) \in Z_{(s,t)}$, then, $x \in Z_s$, for otherwise there exists $z \in X$ such that $z \neq x$ and $sx = sz$, which means $(s, t)(x, y) = (s, t)(z, y)$ and $(x, y) \notin Z_{(s,t)}$ which would be a contradiction.

Suppose $(x, y) \in Z_{(s,t)}$. Then, since $x \in Z_s$, $\widetilde{\chi_{Z_s}}(x, y) = \chi_{Z_s}(x) = 1$. Thus $\widetilde{\chi_{Z_s}} \geq \chi_{Z_{(s,t)}}$, and we have:

$$1 = M(\chi_{Z_{(s,t)}}) \leq M(\widetilde{\chi_{Z_s}}) \leq M(\chi_{X \times Y}) = 1$$

where the first equality follows from Theorem 5.3. It follows that $N(\chi_{Z_s}) = 1$. Since $s \in S$ was arbitrary, by Theorem 5.3, (S, X) is reversible invariant. The proof for (T, Y) follows similarly.

Conversely, suppose now that (S, X) and (T, Y) are reversible invariant flows. Let M, N be reversible invariant means on $m(X)$ and $m(Y)$ respectively. Again, by Theorem 5.3, these are S and T invariant means respectively. Let us define $K : m(X \times Y) \to \mathbb{R}$, $f \mapsto N(\widetilde{M}(f))$ where we set $\widetilde{M}(f) : Y \to \mathbb{R}$ to be given by $\widetilde{M}(f)(y) = M(f(., y))$. By Theorem 2.28, K is an $(S \times T)$-invariant mean.

Suppose $(s, t) \in S \times T$. If $x \in Z_s$, $y \in Z_t$, then $(x, y) \in Z_{(s,t)}$ since if $(s, t)(x, y) = (s, t)(x', y')$, for $(x', y') \neq (x, y)$, either $x \notin Z_s$, or $y \notin Z_t$ which would be a contradiction. It follows that $\chi_{Z_{(s,t)}} \geq \chi_{Z_s \times Z_t}$.

Now, for all $y \in Y$,

$$\widetilde{M}(\chi_{Z_s \times Z_t})(y) = M(\chi_{Z_s \times Z_t}(., y)) = M(\chi_{Z_s} \chi_{Z_t}(y)) = \chi_{Z_t}(y) M(\chi_{Z_s}) = \chi_{Z_t}(y)$$

where we used $M(\chi_{Z_s}) = 1$ by Theorem 5.3, and we end up with $\widetilde{M}(\chi_{Z_s \times Z_t}) = \chi_{Z_t}$. Then, $K(\chi_{Z_s \times Z_t}) = N(\widetilde{M}(\chi_{Z_s \times Z_t})) = N(\chi_{Z_t}) = 1$ by Theorem 5.3.

Since K is a mean, we conclude $1 = K(\chi_{X \times Y}) \geq K(\chi_{Z_{(s,t)}}) \geq K(\chi_{Z_s \times Z_t}) = 1$ and $K(\chi_{Z_{(s,t)}}) = 1$.

Since $(s, t) \in S \times T$ was arbitrary and we showed that K is a $S \times T$-invariant mean; by

Theorem 5.3 $(S \times T, X \times Y)$ is a reversible invariant flow. ∎

Proposition 5.7. *Suppose S transitively acts on itself on the right and the action (S, X) is S-cancellative, then, S is reversible invariant $\implies$ (S, X) is reversible invariant.*

Proof. Fix $x \in X$. Suppose M is a left-reversible mean on S. Define N on $m(X)$ by $N(f) = M(\widetilde{f})$, where $\widetilde{f}(s) = f(sx)$, for all $s \in S$.

M is a mean:

- Suppose $f, g \in m(X)$ and $a \in \mathbb{R}$. Then,

$$\widetilde{f + ag}(s) = (f + ag)(sx) = f(sx) + ag(sx) = \widetilde{f}(s) + a\widetilde{g}(s)$$

  Thus, $N(f + ag) = M(\widetilde{f + ag}) = M(\widetilde{f}) + aM(\widetilde{g}) = N(f) + aN(g)$

- $N(\chi_X) = M(\widetilde{\chi_X}) = M(\chi_X) = 1$

- For all $f \in m(S)$, $|N(f)| = |M(\widetilde{f})| \leq \|M\| \|\widetilde{f}\|_\infty \leq \|M\| \|f\|_\infty = \|f\|_\infty$

Now suppose $A \subset X$. Then, $\widetilde{\chi_{tA}} = \chi_B$, where $B = \{s \in S \mid sx \in tA\}$ and $\widetilde{\chi_A} = \chi_C$, where $C = \{s \in S \mid sx \in A\}$. Clearly $tC \subset B$ as $sx \in A \implies (ts)x \in tA$.

Suppose $s \in t^{-1}B$ which is non-empty due to the transitivity of the action. Then, $ts \in B \implies tsx \in tA \implies sx \in A$ (by S-cancellativity) which gives us $s \in C$. Thus, $t^{-1}B \subset C$. This implies that $B \subset tC$ which gives us $tC = B$. Thus, $N(\widetilde{\chi_{tA}}) = M(\chi_B) = M(\chi_{tC}) = M(\chi_C) = M(\widetilde{\chi_A}) = N(\chi_A)$, and we conclude that N is S-reversible. We conclude that (S, X) is reversible invariant. ∎

Consider Theorem 2.26; it is natural to ask if a similar property is satisfied by reversible invariant flows. The answer, in general, is no. This was shown by Sorenson [38], as he proved that the homomorphic image of a left measurable semigroup is not necessarily left measurable. If we consider semigroups S, T, with $\phi : S \to T$ being a surjective semigroup homomorphism, the map $T_\phi : (S, S) \to (T, T)$ given by $T(s) = \phi(s)$, for each $s \in S$, is a surjective homomorphism of flows. However, as Sorenson showed, (T, T) need not be reversible invariant if (S, S) is.

As such, reversible invariance, not in general as nicely behaved as amenability.

Chapter 6

Conclusion, Open Questions and Future Work

The results obtained in Chapters 2, 3 and 4 show that amenability generalizes quite nicely from semigroups to flows. Important characterizations, such as Dixmier's condition (Theorem 2.1), existence of an invariant net of finite means (Theorem 2.19) and Følner's conditions (Corollary 3.9) in the left-cancellative case, are well generalized. Moreover, one has the concept of a homomorphism of flows (Definition 2.25), similar to the concept of a semigroup homomorphism that preserves expected properties, such as amenability (Theorem 2.26) and HBEP (Proposition 2.37). However, not everything works smoothly in generalizing the concepts involved.

6.1 Lack of a semigroup structure on X

Firstly, generalizing X to be an arbitrary set, with no multiplication defined on it, restricts us a lot. Namioka [22] showed that any semigroup S satisfying the SNFC is amenable. It is still an open question as to whether this holds for general flows. Namioka's proof took advantage of the multiplicative structure on S which we cannot use for a general set X.

Suppose S is a semigroup that is amenable. Then, one can define a relation on S as follows: for $s, t \in S$, $s \sim t$ if and only if there exists $x \in S$, such that $sx = tx$. The relation satisfies reflexivity and symmetry, and by Theorem 2.11, satisfies transitivity. Consider S quotiented by this relation, which we will denote by S'. S' is a well defined semigroup with the natural semigroup multiplication defined. Using this quotiented semigroup, Argabright and Wilde [2] proved that commutative semigroups satisfy SFC. Furthermore, Klawe [17] proved a variety of results including the following progression:

1. A left amenable right cancellative semigroup satisfies SFC if and only if it is left-cancellative.
2. If S is left measurable and right cancellative, it is left cancellative.
3. Every left measurable semigroup satisfies the SFC.

We question if it is possible to generalize this to a left flow (S, X):

1. An amenable X-cancellative flow (S, X) satisfies SFC if and only if it is S-cancellative.
2. If (S, X) is reversible invariant and X-cancellative, it is S-cancellative.
3. Every reversible invariant flow (S, X) satisfies the SFC.
4. Every commutative flow satisfies SFC.

Since the main tool used here was the quotienting of the semigroup, it is hard to generalize this to an arbitrary flow. This is because, in general, transitivity fails when considering the analogous relation on (S, X), due to the lack of a multiplication on X.

Yang [40] used Klawe's [17] work to obtain results on the Følner number. He used the quotient structured of S' to show that if the Følner number $F(S, S) \neq 0$, then $F(S, S) \geq 1/6$. We wish to see if we can generalize this to general flows. The hurdle again, is that a similar relation between elements of flows is not in general, an equivalence relation, and it is not possible to quotient the flow as a result.

We also wonder if we can remove the transitive condition on the action in Proposition 5.7, like we have for Proposition 2.8.

6.2 Inability to easily generalize fundamental concepts on S

We do not know how well we can generalize some of the fundamental concepts existing on semigroups to flows. Numakura [23] showed the existence of a unique minimal two-sided ideal for a compact semigroup (finite semigroup in our discrete case). Using this, Rosen

[26] proved that a finite discrete semigroup is left amenable if and only if it has exactly one minimal left ideal. Yang used this result to obtain equality in Proposition 3.17 for finite semigroups, and to show that if S is a semigroup with a semigroup homomorphism mapping S onto a finite semigroup, the Følner number of S is greater than the Følner number of the finite semigroup. We do not know whether it is possible to get a "kernel-like" structure for a flow. Since this allows characterization of amenability in the compact semigroup case using minimal ideals, we want to know if it is possible to obtain a similar result for flows. We also want to know if Yang's results can be generalized.

6.3 Lack of embeddability in groups

Another tool that is missing with flows (that is essentially due to the lack of a semigroup structure on X), is being able to take advantage of the additional structure on a group by embedding a semigroup in a group. A result of Yang [40] that we are interested in generalizing is: If S is a cancellative semigroup, then its Følner number is either 0 or 1 according to whether S is amenable or not. Is it true, that for a S- and X-cancellative flow (S, X), then $F(S, X)$ is either 0 or 1 according to whether (S, X) is amenable or not? The proof of Yang's result uses Dubriel's theorem, which allows a cancellative semigroup S to be embedded into a group if every two right ideals of S have a non-empty intersection.

Another interesting result, is that of Luthar [21], who showed that a commutative semigroup has a unique invariant mean if and only if it contains a finite ideal. This was done by using the fact that the finite ideal is actually a group and using its group structure. Granirer [10][11] used this to show that, for any semigroup S, there is a connection between the dimension of $\mathcal{M}(S)$ and the number of left ideals in S that are groups. We wish to be able to obtain some of these connections for flows, i.e. - what kind of structures in a flow influence the dimension of the set of means on it? Can we relate finite S-ideals or finite subflows of a flow to the uniqueness of means?

6.4 What is the advantage of working with a flow?

The main advantage of working with flows is that we no longer need to restrict ourselves to one set (a semigroup S). Even natural examples, such as the action of $(\mathbb{N}, +)$ on $\mathbb{R}$, $(\mathbb{N}, \mathbb{R})$ via multiplication or addition, is not taken into consideration by the concept of amenability of a semigroup. Seeing structures such as $(S, m(X))$, $(S, \mathcal{M}(X))$, $(S, \mathcal{M}_l(X))$, in terms of flows allows a cleaner approach and better understanding of results, and the theorems from Chapter 2, such as the Hahn-Banach extension property (Proposition 2.37) can be applied to these structures. Moreover, the concept of a homomorphism of flow can be used to transfer the nice properties of one flow to another, such as Theorem 2.26.

Given a flow (S, X), Proposition 2.8 really highlights how much stronger the amenability of S is in relation to the amenability of (S, X). As a result, it is not unexpected that difficulties occur in obtaining results for (S, X) from its amenability, that usually follow in the case of (S, S) when S is amenable. However, on the contrary, it is much easier to achieve the amenability of (S, X) (see Example 2.7). As a result, perhaps a nice property that may follow from the amenability of (S, X), (such as Theorem 2.43) may be overlooked when only considering the amenability of S. Hence, when we consider for example, applications of amenability in topological dynamics and differential equations, looking at flows as a whole instead of only the semigroup, may be more wholesome and useful.

We have yet to fully discover how much we can obtain from the concept of the amenability of flows, in terms of applications. For our future work, we would like to develop the theory further, solve the aforementioned open problems, and state some applications of the theory in practicality.

List of Symbols

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Bibliography

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